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WKB eigenmode construction for analytic Toeplitz operators

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Abstract

We provide almost eigenfunctions for Toeplitz operators with real-analytic symbols, at the bottom of non-degenerate wells. These almost eigenfunctions follow the WKB ansatz; the error is $O(e^{-cN})$, where $c > 0$ and $N \rightarrow +\infty$ is the inverse semiclassical parameter.

1 Introduction

This paper is concerned with Berezin-Toeplitz quantization. We associate, to a real-valued function f on a compact Kähler manifold M , a sequence of self-adjoint operators $(T_N(f))_{N \geq 1}$ acting on spaces of sections over M . These operators are called *Toeplitz operators*. Examples of Toeplitz operators are spin systems (where M is a product of two-spheres), which are indexed by the total spin $S = \frac{N}{2}$. Motivated by questions arising in the physics literature about the behaviour of spin systems at low temperature, we wish to study the eigenvalues and eigenvectors of Toeplitz operators in the limit $N \rightarrow +\infty$. In this paper we specifically study exponential estimates, that is, approximate expressions with $O(e^{-cN})$ remainder for some $c > 0$.

We provide, in the special case where f is real-analytic and reaches a non-degenerate minimum, a construction of almost eigenfunctions for $T_N(f)$: we build (Theorem A) a sequence of normalised sections $(u(N))_{N \geq 1}$ and a real sequence $(\lambda(N))_{N \geq 1}$, with asymptotic expansions in decreasing powers of N , such that

$$T_N(f)u(N) = \lambda(N)u(N) + O(e^{-cN}).$$

The sequence $u(N)$ takes the form of a Wentzel-Kramers-Brillouin (WKB) ansatz: it is written as

$$u(N) : x \mapsto CN^d e^{N\varphi(x)}(u_0 + N^{-1}u_1 + \dots).$$

Since $T_N(f)$ is self-adjoint, the existence of an almost eigenfunction implies that $\lambda(N)$ is exponentially close to the spectrum of $T_N(f)$, but not necessarily that $u(N)$ is exponentially close to an eigenfunction. In Theorem A, we also prove that, if f is Morse, the eigenvectors associated with the lowest eigenvalue of $T_N(f)$ are exponentially close to a finite sum of almost eigenvectors $u(N)$ constructed above.

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1.1 Bergman kernels and Toeplitz operators

Let (M, ω) be a compact symplectic manifold. Berezin-Toeplitz quantization associates, to a function f , a sequence of Toeplitz operators $(T_N(f))_{N \geq 1}$. To this end, we have to provide a supplementary geometrical information: a complex structure J , which encodes the notion of holomorphic functions, and which is compatible with ω .

Definition 1.1. Let (M, ω, J) be a Kähler manifold. Let L be a complex line bundle over M , and let h be a Hermitian metric on L , such that $\text{curv } h = 2i\pi\omega$. (The couple (L, h) exists if and only if the integral of ω over each closed surface in M is an integer multiple of 2π . We then say that M is *quantizable*.) Let $N \in \mathbb{N}$.

The *Bergman projector* S_N is the orthogonal projector, from the space of square-integrable sections $L^2(M, L^{\otimes N})$ to the subspace of holomorphic sections $H_0(M, L^{\otimes N})$.

Let $f \in C^\infty(M, \mathbb{R})$. The Toeplitz operator $T_N(f)$ associated with f is the following operator:

$$\begin{aligned} T_N(f) : H_0(M, L^{\otimes N}) &\rightarrow H_0(M, L^{\otimes N}) \\ u &\mapsto S_N(fu). \end{aligned}$$

The space $H_0(M, L^{\otimes N})$ is always finite-dimensional. Given a Hilbert basis (s_1, \dots, s_{d_N}) of $H_0(M, L^{\otimes N})$, the Bergman projector S_N admits the following integral kernel:

$$S_N(x, y) = \sum_{i=1}^{d_N} s_i(x) \otimes \overline{s_i(y)}.$$

The study of the Bergman kernel as $N \rightarrow +\infty$ lies at the core of the semiclassics of Toeplitz quantization. In a previous article [3], we developed a semiclassical machinery in real-analytic regularity, in order to give asymptotic formulas for S_N , and Toeplitz operators, in the case where the symplectic form ω is real-analytic on the complex manifold (M, J) .

Definition 1.2. Let (M, ω, J) be a compact quantizable Kähler manifold and $(S_N)_{N \geq 1}$ be the associated sequence of Bergman projectors. Let $x \in M$ and $N \in \mathbb{N}$. The *coherent state* ψ_x^N at x is the element of $H_0(M, L^{\otimes N}) \otimes \overline{L}_x$ given by freezing the second variable of the Bergman kernel: for every $y \in M$, one has

$$\psi_x^N(y) = S_N(y, x).$$

Theorem A. *Let M be a quantizable compact real-analytic Kähler manifold. Let f be a real-analytic function on M with $\min(f) = 0$.*

1. *Let $P_0 \in M$ be a non-degenerate minimal point of f . Then there exist*

- *positive constants c, c', R ,*
- *a neighbourhood V of P_0 ,*
- *a holomorphic function φ on V with $|\varphi(x)| \leq \frac{d(x, P_0)^2}{2}$,*
- *a sequence of holomorphic functions $(u_k)_{k \geq 0}$, with $u_0(P_0) = 1$ and $u_k(P_0) = 0$ for $k \neq 0$,*
- *a real sequence $(\lambda_k)_{k \geq 0}$, where λ_0 is the ground state energy of the Hessian of f at P_0 (see [2]),*

such that, if $\psi_{P_0}^N$ denotes the coherent state at P_0 , then with

$$u(N) = \psi_{P_0}^N e^{N\varphi} \left(\sum_{k=0}^{cN} N^{-k} u_k \right),$$

one has

$$\left\| T_N(f)u(N) - N^{-1} \left(\sum_{k=0}^{cN} N^{-k} \lambda_k \right) u(N) \right\|_{L^2(M, L^{\otimes N})} \leq C e^{-c'N},$$

and

$$\begin{aligned} |\lambda_k| &\leq CR^k k! \\ \sup_U |u_k| &\leq CR^k k!, \end{aligned}$$

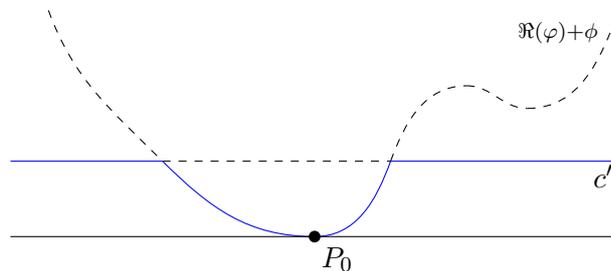
2. If the minimal set of f consists in a finite number of non-degenerate minimal points, then any eigenfunction of $T_N(f)$ with minimal eigenvalue is exponentially close to a linear combination of the functions constructed in item 1 at each minimal point.

The pseudodifferential equivalent of this result is claimed in [6], using the Sjöstrand analytic classes [9], but all details are not given.

Pseudodifferential operators with real-analytic symbols can be written *exactly* as Toeplitz operators, with $M = \mathbb{C}^n$, so that Theorem A also contains (modulo some hypotheses on f at infinity) a complete proof for the result stated in [6]. This point of view on pseudodifferential operators is pertinent for WKB eigenmode construction and exponential estimates, both from the perspective of physics [11] and from mathematics (all related proofs use the Fourier-Bros-Iagolnitzer transformation, which relates pseudodifferential operators to Toeplitz operators). In addition, the Toeplitz setting contains other semiclassical quantum operators such as spin systems, on which tunnelling estimates are widely studied in the physics community [7], although not always in a rigorous way.

Remark 1.3. If the minimal set of f consists in several non-degenerate wells, then applying the Part 1 of Theorem A at every well yields that the actual ground state, which is exponentially close to an orthogonal linear combination of almost eigenfunctions as above, has Agmon-type exponential decay in a neighbourhood of the minimal set, as in [5].

Even if the function φ can be defined and yields, formally, exponential decay far from the minimal point, this rate of decay is blurred, not only by the error terms in the expression of the Bergman kernel (Proposition 2.3) but also by the fact that we can only sum up to cN with c small when summing analytic symbols (see Proposition 2.2), which yields a fixed error of order $e^{-c'N}$ with $c' > 0$ small. This yields an upper bound on the decay rate, as a function of the position, which follows the blue, continuous line in the following picture:



Near P_0 , the rate of decay is sharp, but we have no explicit control on the constant c' .

Theorem A has applications to tunnelling in spin systems. In Proposition 5.1 we prove that, if f has two symmetrical wells, and λ_0, λ_1 denote the two first eigenvalues of $T_N(f)$ (with multiplicity), then

$$\lambda_1 - \lambda_0 \leq C e^{-c'N},$$

where c' is as in Theorem A. In the physics community, the tunnelling rate $-N^{-1} \log(\lambda_1 - \lambda_0)$ is often estimated using the degree zero approximation φ in the WKB ansatz, which solves a Hamilton-Jacobi equation (see Proposition 3.3). However, in Proposition 5.2, we provide a series of examples which illustrate that the tunnelling rate is not given by φ , let alone by the best possible constant c' in Theorem A.

1.2 Outline

In Section 2 we briefly present the tools which we developed in [3] to tackle problems from semiclassical analysis in real-analytic regularity. We then proceed to the proof of Theorem A. Section 3 contains the geometrical ingredients required in order to build a *formal* WKB ansatz, that is, for every $K \in \mathbb{N}$, an approximate eigenstate of the form

$$x \mapsto \psi_{P_0}(x) e^{\varphi(x)} (a_0(x) + N^{-1} a_1(x) + \dots + N^{-K} a_K(x)).$$

In Section 4, we identify the formal sequences $(a_k)_{k \geq 0}$ and $(\lambda_k)_{k \geq 0}$ corresponding to a candidate for the smallest eigenvalue and associated eigenvector, and prove that these sequences belong to an analytic class; this allows us to construct an approximate eigenstate of the form

$$x \mapsto \psi_{P_0}(x) e^{\varphi(x)} \sum_{k=0}^{cN} N^{-k} a_k(x),$$

which satisfies the eigenvalue equation for $T_N(f)$ up to $O(e^{-c'N})$, with $c > 0$ and $c' > 0$. A standard analysis of the distribution of low-lying eigenvalues of $T_N(f)$ allows us to conclude the proof in Section 5, where we also discuss the constant c' in the statement of Theorem A.

2 Calculus of analytic Toeplitz operators

The core of the proof of Theorem A consists in Propositions 3.5 and 4.2, where we prove that the sequences $(\lambda_k)_{k \geq 0}$ and $(u_k)_{k \geq 0}$ can be built by induction and satisfy the growth control

$$\begin{aligned} |\lambda_k| &\leq CR^k k! \\ |u_k| &\leq CR^k k!. \end{aligned}$$

To this end, we use the framework developed in our previous paper [3], which allowed us to study Toeplitz operators with real-analytic regularity.

For some real parameters $r > 0, m$, we say that a function on a smooth open set U of \mathbb{R}^d belongs to the space $H(m, r, U)$ when there exists $C > 0$ such that, for every $j \geq 0$, one has

$$\|u\|_{C^j(U)} \leq C \frac{r^j j!}{(j+1)^m}.$$

The minimal C such that the control above is true is a Banach norm for the space $H(m, r, U)$. Such functions are real-analytic. Reciprocally, for all $V \subset\subset U$, every real-analytic function on U belongs to $H(m, r, V)$ for some m, r .

Generalising this notion leads to the definition of *analytic (formal) symbols*.

Definition 2.1.

- Let X be a compact manifold (with smooth boundary). We fix a finite set $(\rho_V)_{V \in \mathcal{V}}$ of local charts on open sets V which cover X .

Let $j \geq 0$. The C^j norm of a function $f : X \mapsto \mathbb{C}$ which is continuously differentiable j times is defined as

$$\|f\|_{C^j(X)} = \max_{V \in \mathcal{V}} \sup_{x \in V} \sum_{|\mu|=j} |\partial^\mu (f \circ \rho_V)(x)|.$$

- Let X be a compact manifold (with boundary), with a fixed set of covering local charts.

Let r, R, m be positive real numbers. The space of analytic symbols $S_m^{r,R}(X)$ consists of sequences $(a_k)_{k \geq 0}$ of real-analytic functions on X , such that there exists $C \geq 0$ such that, for every $j \geq 0, k \geq 0$, one has

$$\|a_k\|_{C^j(X)} \leq C \frac{r^j R^k (j+k)!}{(j+k+1)^m}.$$

The norm of an element $a \in S_m^{r,R}(X)$ is defined as the smallest C as above; then $S_m^{r,R}(X)$ is a Banach space.

These analytic classes, which we defined and studied in [3], are well-behaved with respect to standard manipulations of functions (multiplication, change of variables, ...) and, most importantly, with respect to the stationary phase lemma. Another important property is the summation of such symbols: if \hbar is a semiclassical parameter (here $\hbar = N^{-1}$), then for $c > 0$ small depending on R , the sum

$$\sum_{k=0}^{c\hbar^{-1}} \hbar^k u_k$$

is uniformly bounded as $\hbar \rightarrow 0$; in this sum, terms of order $k = \hbar^{-1}$ are exponentially small, so that the precise choice of c has an exponentially small influence on the sum.

Proposition 2.2.

Summation *Let X be a compact Riemannian manifold with boundary and let $f \in S_m^{r,R}(X)$. Let $c_R = \frac{e}{3R}$. Then*

1. *The function*

$$f(N) : x \mapsto \sum_{k=0}^{c_R N} N^{-k} f_k(x)$$

is bounded on X uniformly for $N \in \mathbb{N}$.

2. *For every $0 < c_1 < c_R$, there exists $c_2 > 0$ such that*

$$\sup_{x \in X} \left| \sum_{k=c_1 N}^{c_R N} N^{-k} f_k(x) \right| = O(e^{-c_2 N}).$$

Cauchy product *There exists $C_0 \in \mathbb{R}$ and a function $C : \mathbb{R}^2 \mapsto \mathbb{R}$ such that the following is true.*

Let X be a compact Riemannian manifold (with boundary) and with a fixed finite set of covering charts. Let $r, R \geq 0$ and $m \geq 4$. For $a, b \in S_m^{r,R}(X)$, let us define the Cauchy product of a and b as

$$(a * b)_k = \sum_{i=0}^k a_i b_{k-i}.$$

1. The space $S_m^{r,R}(X)$ is an algebra for this Cauchy product, that is,

$$\|a * b\|_{S_m^{r,R}} \leq C_0 \|a\|_{S_m^{r,R}} \|b\|_{S_m^{r,R}},$$

Moreover, there exists $c > 0$ depending only on R such that as $N \rightarrow +\infty$, one has

$$(a * b)(N) = a(N)b(N) + O(e^{-cN}).$$

2. Let r_0, R_0, m_0 positive and $a \in S_{m_0}^{r_0, R_0}(X)$ with a_0 nonvanishing. Then, for every m large enough depending on a , for every $r \geq r_0 2^{m-m_0}, R \geq R_0 2^{m-m_0}$, a is invertible (for the Cauchy product) in $S_m^{r,R}(X)$, and its inverse a^{*-1} satisfies:

$$\|a^{*-1}\|_{S_m^{r,R}(X)} \leq C(\|a\|_{S_{m_0}^{r_0, R_0}(X)}, \min(|a|)).$$

This summation property, together with the stationary phase lemma, allows us to study Toeplitz operators up to an exponentially small error. One of the main results of [3] is an expansion of the Bergman kernel on a real-analytic Kähler manifold, with error $O(e^{-c'N})$, in terms of an analytic symbol.

Proposition 2.3. (See [3], Theorem A) Let M be a quantizable compact real-analytic Kähler manifold of complex dimension d . There exists positive constants r, R, m, c, c', C , a neighbourhood U of the diagonal in $M \times M$, a section Ψ of $L \boxtimes \bar{L}$ over U , and an analytic symbol $a \in S_m^{r,R}(U)$, holomorphic in the first variable, anti-holomorphic in the second variable, such that the Bergman kernel S_N on M satisfies, for each $x, y \in M \times M$ and $N \geq 1$:

$$\left\| S_N(x, y) - \Psi^{\otimes N}(x, y) \sum_{k=0}^{cN} N^{d-k} a_k(x, y) \right\|_{h^{\otimes N}} \leq C e^{-c'N}.$$

Similar ideas appear in the literature, and have been successfully applied to the theory of pseudodifferential operators with real-analytic symbols. Early results [1] use a special case of our analytic classes, when $m = 0$; from there, a more geometrical theory of analytic Fourier Integral operators was developed [9], allowing one to gradually forget about the parameters r and R . It is surprising that the introduction of the parameter m , which mimics the definition of the Hardy spaces on the unit ball, was never considered, although it simplifies the manipulation of analytic functions (the space $H(m, r, V)$ is stable by product if and only if $m \geq 3$). In the paper [3] and the present article, it is crucial that we are able to choose m arbitrary large.

3 Geometry of the WKB Ansatz

In this section we provide the geometric ingredients for the proof of Theorem A. We formally proceed as in the case of a Schrödinger operator [4]. If a real-analytic, real-valued function f has a non-degenerate minimum at $P_0 \in M$, we seek for a sequence of eigenfunctions of the form

$$\psi_{P_0}^N e^{N\varphi}(u_0 + N^{-1}u_1 + \dots),$$

where $\psi_{P_0}^N$ denotes the coherent state at P_0 . If the value of f at the bottom of the well is 0, then the associated sequence of eigenvalues should be of order $O(N^{-1})$, that is to say, follow the asymptotic expansion:

$$N^{-1}\lambda_0 + N^{-2}\lambda_1 + \dots$$

When solving the eigenvalue problem, the terms of order 0 in

$$e^{-N\varphi} T_N(f) \psi_{P_0}^N e^{N\varphi}(u_0 + N^{-1}u_1 + \dots)$$

yield an equation on φ . In the case of a Schrödinger operator this is the eikonal equation $|\nabla\varphi|^2 = V$, which is solved using the Agmon metric. In our more general case, we are in presence of a form of the Hamilton-Jacobi equation (1) which we solve in Proposition 3.3 using a geometric argument based on the existence of a stable manifold, in the spirit of [10]. Associated with f and φ are transport equations which we must solve in order to recover the sequence of functions $(a_k)_{k \geq 0}$. In Proposition 3.5 we study this transport equation under the point of view of symbol spaces of Definition 2.1. Then, in Proposition 4.2, we perform an analytic summation of the a_k 's in order to find an exponentially accurate eigenfunction for $T_N(f)$, with exponential decay away from P_0 .

The plan of this section is as follows: we begin in Subsection 3.1 with the study of an analytic phase which will be a deformation of the phase Φ_1 considered above. We then define and study the Hamilton-Jacobi equation associated with a real-analytic function near a non-degenerate minimal point, and the associated transport equations, in Subsections 3.2 and 3.3 respectively.

In the rest of this article,

- M is a quantizable real-analytic compact Kähler manifold;
- f is a real-valued function on M with real-analytic regularity, such that $\min(f) = 0$ and all minimal points are non-degenerate;
- $U \subset M$ is an open set on which f vanishes at exactly one point. U is identified with a neighbourhood of 0 in \mathbb{C}^d , with $f(0) = 0$ (in particular, $P_0 = 0$);
- ϕ is a Kähler potential on U such that

$$\phi(y) = \frac{|y|^2}{2} + O(|y|^3);$$

- $\tilde{\phi}$ is the function on $U \times U$, holomorphic in the first variable, anti-holomorphic in the second variable, such that $\tilde{\phi}(x, x) = \phi(x)$ (holomorphic extension or polarisation of ϕ);
- More generally, $\tilde{}$ represents holomorphic extension of real-analytic functions: for instance, \tilde{f} is the extension of f and is defined on $U \times U$;
- Φ_1 is the phase associated with the composition of two Bergman kernels, that is,

$$\Phi_1 : (x, y, \bar{w}, \bar{z}) \mapsto 2\tilde{\phi}(x, \bar{w}) - 2\tilde{\phi}(y, \bar{w}) + 2\tilde{\phi}(y, \bar{z}) - 2\tilde{\phi}(x, \bar{z}).$$

Here, and in all this article, we write $\Phi_1(x, y, \bar{w}, \bar{z})$ to indicate that Φ_1 has anti-holomorphic dependence in its two last variables.

The section Ψ of Proposition 2.3 satisfies the following equation, for all x, y, z close enough from each other:

$$\langle \Psi^{\otimes N}(x, y), \Psi^{\otimes N}(y, z) \rangle_{L_y^{\otimes N}} = \Psi^{\otimes N}(x, z) \exp(N\Phi_1(x, y, \bar{y}, \bar{z})).$$

Here, \bar{y} is merely the complex conjugate of y .

3.0 Formal identification of the WKB ansatz

We search for an eigenfunction of $T_N(f)$ of the form

$$x \mapsto e^{N\varphi(x)}(u_0(x) + N^{-1}u_1(x) + \dots)\psi_0^N(x),$$

where ψ_0^N is the coherent state at 0 (see Definition 1.2), and ϕ, u_0, u_1, \dots are holomorphic functions on U .

This construction is local. Indeed, the holomorphic functions ϕ, u_0, u_1, \dots can only be extended to the whole of M if they are constant. However, if φ does not grow too fast (see Definition 3.1), then the trial function above is exponentially small outside any fixed neighbourhood of zero.

In particular, applying $T_N(f)$ yields

$$T_N(f)(e^{N\varphi}(u_0 + N^{-1}u_1 + \dots)\psi_0^N) : \\ x \mapsto \psi_0^N(x)e^{N\varphi(x)} \int_U e^{N\Phi_1(x,y,\bar{y},0)+N\varphi(y)-N\varphi(x)} f(y) \left(\sum_{k=0}^{cN} N^{d-k} a_k(x,y) \right) \left(u_0(y) + N^{-1}u_1(y) + \dots \right) dy \\ + O(e^{-cN}).$$

If the function appearing in the exponential is a positive phase function (see Proposition 3.2), one can apply the stationary phase lemma. If $y_*(x)$ is the critical point of this phase (which belongs to $U \times U$), at dominant order, one has

$$T_N(f)(e^{N\varphi}u_0\psi_0^N)(x) = \psi_0^N(x)e^{N\varphi(x)} \tilde{f}(y_*(x))a_0(x, y_*(x))u_0(y_*(x))J(x) + O(N^{-1}).$$

where J is a non-vanishing Jacobian.

Since we search for an eigenfunction with eigenvalue close to zero, we want this principal term to vanish. As J and a_0 do not vanish, this yields

$$\tilde{f}(y_*(x)) = 0,$$

which boils down to a particular PDE on φ , the *Hamilton-Jacobi equation*. We provide a geometric solution to this equation in Proposition 3.3.

At next order, the eigenvalue equation reads, for all $x \in U$,

$$N^{-1}\lambda_0 u_0(x) + O(N^{-2}) = T_N(f)(e^{N\varphi}(u_0 + N^{-1}u_1)\psi_0^N)(x) + O(N^{-2}) \\ = N^{-1}\psi_0^N(x)e^{N\varphi(x)} \left(\tilde{f}J(a_0u_1 + a_1u_0)(y_*(x)) + \tilde{\Delta}(x)(\tilde{f}a_0u_0J)(y_*(x)) \right) + O(N^{-2}).$$

Here $\tilde{\Delta}(x)$ is the Laplace operator conjugated with a change of variables (this change of variables acts on (y, \bar{y}) and is parametrized by x : it conjugates the initial phase with $v \mapsto -|v|^2$).

Since $\tilde{f}(y_*(x)) = 0$, there is no contribution from u_1 at this order. Moreover, one can distribute

$$\tilde{\Delta}(\tilde{f}a_0u_0J) = \tilde{f}a_0J\tilde{\Delta}u_0 + u_0\tilde{\Delta}(\tilde{f}a_0J) + \tilde{\nabla}(\tilde{f}a_0J) \cdot \tilde{\nabla}(u_0).$$

Then, the first term of the right-hand side is zero when evaluated at $y_*(x)$ since $\tilde{f}(y_*(x)) = 0$. The second term, evaluated at zero will yield the associated eigenvalue at first order. Hence, it remains to solve

$$\left(\tilde{\nabla}(x)(\tilde{f}a_0J) \right) (y_*(x)) \cdot \left(\tilde{\nabla}(x)u_0 \right) (y_*(x)) = u_0(x) \left(\lambda_0 - \tilde{\Delta}(x)(\tilde{f}a_0J)(y_*(x)) \right).$$

Observe that \tilde{f} , as the complex extension of f , has a critical point at $x = 0$, so that, as long as $y_*(0) = 0$ (which is proved in Proposition 3.2), there holds $\tilde{\nabla}(0)(\tilde{f}a_0J)(y_*(0)) = 0$. Hence, the equation above implies

$$\lambda_0 = \tilde{\Delta}(0)(\tilde{f}a_0J)(0).$$

We will see in Proposition 4.1 that λ_0 indeed corresponds to the ground state energy of the Hessian of f at zero.

It remains to solve an equation of the form

$$\left(\tilde{\nabla}(x)(\tilde{f}a_0J) \right) (y_*(x)) \cdot \left(\tilde{\nabla}(x)u_0 \right) (y_*(x)) = u_0(x)h(x),$$

where h vanishes at zero. We solve this equation in Proposition 3.5.

Similar equations are satisfied by the successive terms u_k . This family of equations is solved (with a convenient control on the size of the solution) in Proposition 3.5. Then, in Section 4 we prove that the sequence u_k indeed forms an analytic symbol and that the eigenvalue equation admits a solution up to an $O(e^{-cN})$ error.

3.1 A family of phase functions

In this subsection we study a family of analytic phases (in the sense of Definition 3.11 in [3]) given by a WKB ansatz at the bottom of a well. To begin with, we describe the conditions on a holomorphic function φ at a neighbourhood of zero, such that $e^{N\varphi}\psi_{P_0}^N$ is a convenient first-order candidate for the ground state of $T_N(f)$.

Definition 3.1. A holomorphic function φ on U is said to be *admissible* under the following conditions:

$$\begin{aligned}\varphi(0) &= 0 \\ d\varphi(0) &= 0 \\ \exists t < 1, \forall x \in U, |\varphi(x)| &< \frac{t}{2}|x|^2.\end{aligned}$$

Proposition 3.2. Let φ be an admissible function. The function from $U \times U$ to \mathbb{R} defined by:

$$(x, y) \mapsto \Phi_1(x, y, \bar{y}, 0) + \varphi(y) - \varphi(x)$$

is, for all x in a small neighbourhood of zero, a positive phase function of y .

The complex critical point is $y_*(x) = (x, \bar{y}_c(x))$, where the holomorphic function $x \mapsto \bar{y}_c(x)$ satisfies

$$-2\partial_1\tilde{\phi}(x, \bar{y}_c(x)) + 2\partial_1\tilde{\phi}(x, 0) = -\partial\varphi(x).$$

In particular, $\bar{y}_c(0) = 0$.

Proof. Near $y = \bar{w} = 0$, there holds

$$\Phi_1(0, y, \bar{w}, 0) = -y \cdot \bar{w} + O(|y, \bar{w}|^3).$$

In particular, for $x = 0$, the function $(y, \bar{w}) \mapsto \Phi_1(0, y, \bar{w}, 0) + \varphi(y)$ has a critical point at $(0, 0)$ with non-degenerate, real negative Hessian (because $|\varphi(y)| \leq \frac{t|y|^2}{2}$). In particular, for x small enough, the function $(y, \bar{w}) \mapsto \Phi_1(x, y, \bar{w}, 0) + \varphi(y) - \varphi(x)$ has exactly one critical point near 0, with non-degenerate, negative real Hessian. The critical point (y, \bar{w}) satisfies the two equations

$$\begin{aligned}\bar{\partial}_{\bar{w}}\tilde{\phi}(x, \bar{w}) - \bar{\partial}_{\bar{w}}\tilde{\phi}(y, \bar{w}) &= 0 \\ -2\partial_y\tilde{\phi}(y, \bar{w}) + 2\partial_y\tilde{\phi}(y, 0) &= -\partial\varphi(y).\end{aligned}$$

The first equation yields $y = x$, then the second equation has only one solution $\bar{w} := \bar{y}_c(x)$, so that the phase at this critical point is equal to

$$2\tilde{\phi}(x, \bar{y}_c(x)) - 2\tilde{\phi}(x, \bar{y}_c(x)) + 2\tilde{\phi}(x, 0) - 2\tilde{\phi}(x, 0) + \varphi(x) - \varphi(x) = 0.$$

This concludes the proof. □

3.2 Hamilton-Jacobi equation

Let φ be an admissible function. For every $x \in M$ close to 0, there exists one $\overline{y}_c(x)$ in \overline{U} such that $(x, \overline{y}_c(x))$ is a critical point for the phase of Proposition 3.2.

In order to find the phase of the WKB ansatz, we want to solve, in a neighbourhood of 0, the following system of equations on φ and \overline{y}_c , where φ is an admissible function:

$$\begin{cases} \tilde{f}(x, \overline{y}_c(x)) = 0. \\ -2\partial_1 \tilde{\phi}(x, \overline{y}_c(x)) + 2\partial_1 \tilde{\phi}(x, 0) = -\partial\varphi(x). \end{cases} \quad (1)$$

This will be called the *Hamilton-Jacobi equation*. This equation is non-trivial already at the formal level: for fixed x the equation $\tilde{f}(x, y) = 0$ defines (a priori) a manifold of complex codimension 1, which has a singularity at $x = 0$. On the other hand, we need to ensure that $\partial_1 \tilde{\phi}(x, \overline{y}_c(x))$ is a closed 1-form in order to solve for φ .

Proposition 3.3. *The Hamilton-Jacobi equation (1) admits a solution near 0. It is given by the stable manifold of the Hamiltonian flow of \tilde{f} , with respect to a particular symplectic form.*

Proof. Since the Taylor expansion of ϕ at zero is

$$\phi(x) = \frac{1}{2}|x|^2 + O(|x|^3),$$

the map

$$\overline{w} \mapsto 2\partial_1 \tilde{\phi}(x, \overline{w}) = \overline{w} + O(|x, \overline{w}|^2)$$

is a biholomorphism in a neighbourhood of zero, for x small. Let γ_x denote its inverse, then γ_x is tangent to identity at $x = \overline{w} = 0$.

Let

$$\tilde{f}_1 : (x, z) \mapsto \tilde{f}(x, \gamma_x(z)),$$

then the Hamilton-Jacobi equation (1) is equivalent to the modified system:

$$\begin{cases} \tilde{f}_1(x, z_c(x)) = 0 \\ -z_c(x) + 2\partial_1 \tilde{\phi}(x, 0) = -\partial\varphi(x). \end{cases}$$

Let Q be the Hessian of f at zero and \tilde{Q} its holomorphic extension. Then $\tilde{f}_1(x, z) = \tilde{Q}(x, z) + O(|x, z|^3)$ since γ_x is tangent to identity at $x = \overline{w} = 0$.

In the modified system, there holds $z_c(x) = \partial(2\tilde{\phi}(x, 0) + \varphi(x))$, so that finding $x \mapsto z_c(x)$ amounts to finding a holomorphic Lagrange submanifold $L = \{x, z_c(x)\}$ of $\mathbb{C}^d \times \mathbb{C}^d$ near 0, for the standard symplectic form $\Im(\sum dx_j \wedge dz_j)$ (which extends the symplectic form $\sum d\Re(x_j) \wedge d\Im(x_j)$), such that L is contained in $\{\tilde{f}_1 = 0\}$ and is transverse to x . Then, near 0, one has $L = \{x, \partial F(x)\}$ for some holomorphic F , and it will only remain to check that $\varphi = F - 2\tilde{\phi}(\cdot, 0)$ is admissible. As in [10], from f and the standard symplectic form, the Lagrangean L will be constructed as the stable manifold of the fixed point 0 for the symplectic flow of \tilde{f}_1 .

Let us first focus on the special case where \tilde{f}_1 is quadratic. Then $\tilde{f}_1 = \tilde{Q}$.

The quadratic form Q admits a symplectic diagonalisation with respect to the (real) symplectic form $\sum d\Re(x_j) \wedge d\Im(x_j)$: there exists a symplectic matrix S , and positive numbers $\lambda_1, \dots, \lambda_d$, such that

$$Q = S^T \text{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \dots, \lambda_d, \lambda_d)S.$$

Let us study how this symplectic change of variables behaves under complexification. From the *KAK* decomposition of the semisimple Lie group $Sp(2d)$ (or, more practically, using a Singular Value Decomposition), the matrix S can be written as $U_1 D U_2$, where U_1 and U_2 belong to $Sp(2d) \cap O(2d) \simeq U(d)$, and $D = \text{diag}(\mu_1, \mu_1^{-1}, \dots, \mu_d, \mu_d^{-1})$.

The complexified actions of U_1 and U_2 are straightforward: for $j = 1, 2$ one has $\tilde{U}_j(x, z) = (U_j x, U_j^{-1} z)$. The action of D is diagonal: $D = \text{diag}(D_1, \dots, D_d)$, with

$$D_j(\Re(x_j), \Im(x_j)) = \mu_j \Re(x_j) + \mu_j^{-1} \Im(x_j).$$

Hence, the action of \tilde{D} is block-diagonal, with

$$\tilde{D}_j(x_j, z_j) = \left(\frac{\mu_j + \mu_j^{-1}}{2} x_j - \frac{\mu_j - \mu_j^{-1}}{2} z_j, \frac{\mu_j - \mu_j^{-1}}{2} x_j + \frac{\mu_j + \mu_j^{-1}}{2} z_j \right).$$

After applying successively the changes of variables $\tilde{U}_1, \tilde{D}, \tilde{U}_2$, in the new variables, the quadratic form becomes

$$\tilde{f}_1 \circ \tilde{S} : (q, p) \mapsto \sum_{j=1}^d \lambda_j q_j p_j.$$

Among the zero set of this form, a space of particular interest is $\{p = 0\}$. It is a holomorphic Lagrangean subspace, which is preserved by the symplectic gradient flow of \tilde{f}_1 , and such that every solution starting from this subspace tends to zero for positive time. This subspace $\{p = 0\}$ is the *stable manifold of zero* for the symplectic gradient of \tilde{f}_1 . Let us show that, in the starting coordinates (x, z) , the stable manifold of \tilde{f}_1 has the requested properties for the solution of the Hamilton-Jacobi equation.

- The inverse change of variables \tilde{U}_2^{-1} leaves $\{p = 0\}$ invariant.
- The inverse change of variables \tilde{D}^{-1} sends $\{p = 0\}$ to $\{z = Ax\}$, with $\|Ax\|_{\ell^2} \leq t\|x\|_{\ell^2}$ for some $t < 1$. Indeed, the matrix A has diagonal entries $\frac{\mu_j - \mu_j^{-1}}{\mu_j + \mu_j^{-1}}$.
- The inverse change of variables \tilde{U}_1^{-1} sends $\{z = Ax\}$ to $\Lambda_0 = \{z = U_1 A U_1^{-1} x\}$, with a similar property: for some $t < 1$, there holds $\|U_1 A U_1^{-1} x\|_{\ell^2} \leq t\|x\|_{\ell^2}$.

Then Λ_0 is a linear space of the form $\{z = \partial F_0(x)\}$, where F_0 is the holomorphic function

$$F_0 : x \mapsto \frac{1}{2} \langle x, U_1 A U_1^{-1} x \rangle.$$

Then $\varphi : x \mapsto F_0(x) - 2\tilde{\phi}(x, 0) = F_0(x) + O(|x|^3)$ is a solution to the Hamilton-Jacobi equations.

If \tilde{f}_1 is quadratic, we just identified a holomorphic Lagrange submanifold transverse to $\{x = 0\}$ and contained in $\{\tilde{f}_1 = 0\}$, as the stable manifold of 0 for the Hamiltonian flow of \tilde{f}_1 . In the general case, \tilde{f}_1 is a small perturbation of its quadratic part in a small neighbourhood of 0, so that, by the stable manifold Theorem ([8], Theorem 6.1), the stable subspace Λ_0 is deformed into a stable manifold L which has the same properties: L is Lagrangean (since it is a stable manifold of a symplectic flow, it must be isotropic, and L has maximal dimension), and it is transverse to x a small neighbourhood of zero since $T_0 L$ is the linear Lagrangean subspace Λ_0 described above. Moreover, the Hamiltonian flow of \tilde{f}_1 preserves \tilde{f}_1 so that L is contained in $\{\tilde{f}_1 = 0\}$.

We finally let F be a holomorphic function such that $L = \{x, \partial F(x)\}$. With $\varphi : x \mapsto F(x) - 2\tilde{\phi}(x, 0)$, and $z_c(x) = \partial F(x)$, we obtain a solution to the modified Hamilton-Jacobi equation

$$\begin{cases} \tilde{f}_1(x, z_c) = 0 \\ -z_c + \partial_1 \tilde{\phi}(x, 0) = -\partial \varphi(x). \end{cases}$$

Since $\tilde{\phi}(x, 0) = O(|x|^3)$, one has $\varphi(x) = F(x) + O(|x|^3) = F_0(x) + O(|x|^3)$, so that

$$|\varphi(x)| = |F_0(x)| + O(|x|^3) < \frac{t}{2}|x|^2$$

for some $t < 1$ on a neighbourhood of 0. This concludes the proof. \square

Remark 3.4 (Uniqueness). In general, the solution to the Hamilton-Jacobi is non-unique. Once written in the form

$$\sum \lambda_j q_j p_j,$$

we might have chosen another solution than $\{p = 0\}$. Among these linear spaces, only $\{p = 0\}$ corresponds to an admissible solution.

Proposition 4.3 implies that there can be only one admissible solution to the Hamilton-Jacobi equation. Indeed, from two different admissible solutions, one can build two different approximate eigenfunctions corresponding to the same ground state energy.

3.3 Transport equations

Given an admissible function φ which solves the Hamilton-Jacobi equation (1) associated with f , the function

$$(x, y) \mapsto \Phi_1(x, y, \bar{y}, 0) + \varphi(y) - \varphi(x)$$

is a positive phase function of y , with parameter x , by Proposition 3.2; one can apply the holomorphic Morse lemma to reduce this function, after a change of variables, to the holomorphic extension of the phase $(x, v) \mapsto -|v|^2$. The Laplace operator and the standard gradient, conjugated by the change of variables above and which appear in the stationary phase lemma (we use the notation convention of Proposition 3.13 in [3]), are associated with a family of transport equations, which we solve now.

Proposition 3.5. *Let $f' : U \times \tilde{U} \mapsto \mathbb{C}$ be holomorphic and such that*

$$f'(x, y, \bar{w}) = \tilde{f}(y, \bar{w}) + O(|x, y, \bar{w}|^3),$$

and let φ be an admissible solution of the Hamilton-Jacobi equation (1). Let $x \in U$ and let $\tilde{\nabla}(x)$ denote the modified gradient in the stationary phase lemma associated with the phase

$$(y, \bar{w}) \mapsto \Phi_1(x, y, \bar{w}, 0) + \varphi(y) - \varphi(x).$$

That is, if κ_x is a biholomorphism $(y, \bar{w}) \mapsto v(x, y, \bar{w})$ which conjugates the phase above with the holomorphic extension of $-|v|^2$, the operator $\tilde{\nabla}(x)$ acts on functions defined on $U \times \tilde{U}$ by

$$(\tilde{\nabla}(x)a) : (x, y, \bar{w}) \mapsto \left(\frac{\partial a(x, \kappa_x^{-1}(v))}{\partial v_j} (x, \kappa_x(y, \bar{w})) \right)_{1 \leq j \leq 2d}.$$

Let also \bar{y}_c be the holomorphic function of x such that $(x, \bar{y}_c(x))$ is the critical point of the phase above.

Then, for every $g : U \rightarrow \mathbb{C}$ holomorphic with $g(0) = 0$, and every $h : U \rightarrow \mathbb{C}$ holomorphic with $h(0) = 0$, there exists a unique holomorphic function $u : U \rightarrow \mathbb{C}$ with $u(0) = 0$ which solves the following transport equation:

$$(\tilde{\nabla}(x)f')(x, x, \bar{y}_c(x)) \cdot (\tilde{\nabla}(x)[(x, y, \bar{w}) \mapsto u(y)])(x, x, \bar{y}_c(x)) = h(x)u(x) + g(x).$$

Moreover, up to a fixed linear change of variables, there exist $r_0(h, f', \varphi), m_0(h, f', \varphi), C(h, f', \varphi) > 0$ such that, for every

$$k \geq 0, \quad m \geq m_0(h, f', \varphi), \quad r \geq r_0(h, f', \varphi)(3/2)^{m-m_0(h, f', \varphi)}, \quad C_g > 0,$$

for every g as above which satisfies, for every $j \geq 0$,

$$\sum_{|\mu|=j} |\partial^\mu g(0)| \leq C_g \frac{r^j (j+k+1)!}{(1+j+k+1)^m}.$$

one has, for every $j \geq 0$,

$$\sum_{|\mu|=j} |\partial^\mu u(0)| \leq C(h, f', \varphi) C_g \frac{r^j (j+k)!}{(1+j+k)^m}.$$

Proof. We let X be the vector field on U such that

$$(\tilde{\nabla}(x)f')(x, x, \bar{y}_c(x)) \cdot (\tilde{\nabla}(x)[(x, y, \tilde{w}) \mapsto u(y)])(x, x, \bar{y}_c(x)) = X \cdot u(x).$$

The proof consists in three steps. In the first step we prove that all trajectories of X converge towards 0 in negative time, so that there is no dynamical obstruction to the existence of u (if X had wandering or closed trajectories, solving $X \cdot u = fu + g$ would require conditions on f and g). In the second step, we identify the successive terms of a formal power expansion of u , which allows us to control successive derivatives of u at 0. In the third step, we prove that the solution u is well-defined on U .

First step

We study the dynamics of the vector field X in a neighbourhood of zero. To this end, we relate κ to the linear change of variables which appeared in the proof of Proposition 3.3 in the case where f is quadratic.

We first note that, as the Taylor expansion of f' is

$$f' = \tilde{f} + O((x, y, \bar{w})^3) = O((x, y, \bar{w})^2),$$

one has $X(0) = 0$. The Hessian of φ at zero is determined by the Hessian of f at zero; it then determines the linear part of κ at 0, hence the linear part of X at 0. Up to a linear unitary change of variables, there exists a diagonal matrix A , a unitary matrix U , and positive $\lambda_1, \dots, \lambda_d$, such that

$$f : x \mapsto \sum_{j=1}^d \lambda_j |(UAx)_j|^2 + O(|x|^3).$$

Then $\varphi(x) = \frac{1}{2}x \cdot UAU^{-1}x + O(|x|^3)$, so that the phase reads

$$\Phi_1(x, y, \bar{w}, 0) + \varphi(y) - \varphi(x) = 2(x - y) \cdot \left(\bar{w} - \frac{1}{4}UAU^{-1}(x + y) \right) + O(|x, y, \bar{w}|^3).$$

In particular, at first order, one can write

$$\kappa_x(y, \bar{w}) = \left(y - x, \bar{w} - \frac{1}{4}U^{-1}AU(y + x) \right) + O(|(x, y, \bar{w})|^2).$$

Hence, the inverse change of variables is of the form

$$\kappa_x^{-1}(v, \bar{v}) = \left(v + x, \bar{v} + \frac{1}{4}U^{-1}AU(v + 2x) \right) + O(|(x, v, \bar{v})|^2),$$

so that

$$u \circ \kappa_x^{-1}(v, \bar{v}) = u(v + x + O(|(x, v, \bar{v})|^2))$$

is holomorphic with respect to v , at first order.

We then wish to compute

$$\tilde{\nabla}(x)f' \cdot \tilde{\nabla}(x)u := \bar{\partial}_v(f' \circ \kappa_x^{-1}) \cdot \partial_v(u \circ \kappa_x^{-1}) + \partial_v(f' \circ \kappa_x^{-1}) \cdot \bar{\partial}_v(u \circ \kappa_x^{-1})$$

which is equal, at first order, to the opposite symplectic flow (for the symplectic form $\Im(dv \wedge d\bar{v})$) of f applied to u :

$$\tilde{\nabla}(x)f' \cdot \tilde{\nabla}(x)u := \bar{\partial}_v(\tilde{f} \circ \kappa_x^{-1}) \cdot \partial_v(u \circ \kappa_x^{-1}) - \partial_v(\tilde{f} \circ \kappa_x^{-1}) \cdot \bar{\partial}_v(u \circ \kappa_x^{-1}) + O(|x|^2).$$

As seen in the proof of Proposition 3.3, the critical manifold $\{v = \bar{v} = 0\}$ is the stable manifold for the Hamiltonian flow of \tilde{f} , so that each trajectory of the vector field above is repulsed from zero in a non-degenerate way; this concludes the first part of the proof.

Second step.

Since X has 0 as non-degenerate repulsive point, it can be diagonalised: there exists a linear change of variables on \mathbb{C}^d after which

$$X = \sum_{i=1}^d \lambda_i x_i \partial_{x_i} + O(|x|^2),$$

for positive λ_i . From now on we apply this linear change of variables and we will control $\|\nabla^j u(0)\|_{\ell^1}$ in these coordinates. Let us expand

$$\begin{aligned} X \cdot u(x) &= \sum_{i=1}^d \left(\lambda_i x_i + \sum_{|\nu| \geq 2} \frac{a_{i,\nu}}{\nu!} x^\nu \right) \frac{\partial}{\partial x_i} u(x) \\ h(x) &= \sum_{|\nu| \geq 1} \frac{h_\nu}{\nu!} x^\nu \\ g(x) &= \sum_{|\nu| \geq 1} \frac{g_\nu}{\nu!} x^\nu. \end{aligned}$$

Then, for some $V \subset\subset U$ which contains 0, for some positive r_0, m_0 , one has $a_i \in H(m_0, r_0, V)$ and $h \in H(m_0, r_0, V)$, so that, for all ν such that $|\nu| \geq 1$,

$$\begin{aligned} |h_\nu| &\leq C_h \frac{r_0^{|\nu|} \nu!}{(1 + |\nu|)^{m_0}} \\ |a_{i,\nu}| &\leq C_a \frac{r_0^{|\nu|-1} \nu!}{(|\nu|)^{m_0}} \quad \text{if } |\nu| \geq 2. \end{aligned}$$

Let $m \geq m_0$ and $r \geq r_0 2^{m-m_0}$, to be fixed later on. Then, one has

$$\begin{aligned} |h_\nu| &\leq C_h \frac{r^{|\nu|} \nu!}{(1 + |\nu|)^m} \\ |a_{i,\nu}| &\leq C_a \frac{r^{|\nu|-1} \nu!}{(|\nu|)^m}. \end{aligned}$$

Let us suppose that, for some $k \geq 0$, for every $j \geq 0$, one has

$$\sum_{|\nu|=j} |g_\nu| \leq C_g \frac{r^j (j+k+1)!}{(1+k+j+1)^m}.$$

We will solve the transport equation with

$$u : x \mapsto \sum_{|\nu| \geq 1} \frac{u_\nu}{\nu!} x^\nu,$$

and prove by induction on $j \geq 0$ that

$$\sum_{|\mu|=j} |u_\mu| \leq C(h, f', \phi) C_g \frac{r^j (j+k)!}{(1+k+j)^m},$$

as long as m is large enough with respect to C_a and C_h , and r is large enough accordingly.

For $j = 0$, one has $u(0) = 0$ by hypothesis. The transport equation is equivalent to the following family of equations indexed by μ with $|\mu| \geq 1$:

$$u_\mu \frac{\sum_{i=1}^d \lambda_i \mu_i}{\mu!} = \sum_{|\nu| \geq 1} \frac{h_\nu u_{\mu-\nu}}{\nu! (\mu-\nu)!} + \frac{g_\mu}{\mu!} - \sum_{i=1}^d \sum_{|\nu| \geq 2} \frac{a_{i,\nu} u_{\mu-\nu+\eta_i}}{\nu! (\mu-\nu+\eta_i)!}.$$

Here, as in the rest of the proof, η_i denotes the base polyindex with coefficients $(0, 0, \dots, 0, 1, 0, \dots, 0)$ where the 1 is at the site i .

Observe that u_μ appears only on the left-hand side of the equation above, while the right-hand side contains coefficients u_ρ with $\rho < \mu$. As the eigenvalues λ_i are all positive, one can solve for u_μ by induction. Indeed, there exists $C_\lambda > 0$ such that, for every $|\mu| \neq 0$ there holds

$$\sum_{i=1}^d \lambda_i \mu_i \geq C_\lambda^{-1} (|\mu| + 1).$$

In particular,

$$|u_\mu| \leq \frac{C_\lambda}{|\mu| + 1} \left(|g_\mu| + \left| \sum_{|\nu| \geq 1} \frac{h_\nu u_{\mu-\nu} \mu!}{\nu! (\mu-\nu)!} \right| + \left| \sum_{i=1}^d \sum_{|\nu| \geq 2} \frac{a_{i,\nu} u_{\mu-\nu+\eta_i} \mu!}{\nu! (\mu-\nu+\eta_i)!} \right| \right).$$

One has

$$\begin{aligned} \sum_{|\mu|=j} \left| \sum_{|\nu| \geq 1} \frac{h_\nu u_{\mu-\nu} \mu!}{\nu! (\mu-\nu)!} \right| &= \sum_{\ell=1}^{j-1} \sum_{|\rho|=\ell} |u_\rho| \sum_{\substack{|\mu|=j \\ \mu \geq \rho}} \frac{|h_{\mu-\rho}| \mu!}{(\mu-\rho)! \rho!} \\ &\leq C_h \sum_{\ell=1}^{j-1} r^{j-\ell} \sum_{|\rho|=\ell} |u_\rho| \sum_{\substack{|\mu|=j \\ \mu \geq \rho}} \frac{\mu!}{\rho! (1+j-\ell)^m}. \end{aligned}$$

For $|\rho| = \ell$ there holds

$$\sup_{\substack{|\mu|=j \\ \mu \geq \rho}} \frac{\mu!}{\rho!} \leq \frac{j!}{\ell!},$$

since if ρ_M denotes the largest index of ρ the supremum above is $(\rho_M + 1)(\rho_M + 2) \dots (\rho_M + j - \ell)$.

Moreover, there are less than $(j - \ell + 1)^d$ polyindices μ such that $|\mu| = j$ and $\mu \geq \rho$ with $|\rho| = \ell$. Hence,

$$\begin{aligned} \sum_{|\mu|=j} \left| \sum_{|\nu| \geq 1} \frac{h_\nu u_{\mu-\nu} \mu!}{\nu! (\mu-\nu)!} \right| &\leq C_h \sum_{\ell=1}^{j-1} r^{j-\ell} \frac{j! (1+j-\ell)^d}{\ell! (1+j-\ell)^m} \sum_{|\rho|=\ell} |u_\rho| \\ &\leq C_h C(h, f', \phi) C_g \frac{r^j}{(1+k+j)^m} \sum_{\ell=1}^{j-1} \frac{j! (\ell+k)!}{\ell!} \frac{(1+j-\ell)^d (1+k+j)^m}{(1+j-\ell)^m (1+k+\ell)^m}. \end{aligned}$$

First, $\frac{(\ell+k)!}{\ell!} = (\ell+k)(\ell+k-1)\dots(\ell+1)$ is increasing with respect to ℓ , so that

$$\frac{j!(\ell+k)!}{\ell!} \leq \frac{j!(k+j)!}{j!} = (k+j)!.$$

Second, from Lemma 2.12 in [3], if $m \geq \max(d+2, 2d)$, there holds

$$\sum_{\ell=1}^{j-1} \frac{(1+j-\ell)^d(1+k+j)^m}{(1+j-\ell)^m(1+k+\ell)^m} \leq C(d) \frac{3^m}{4^m}.$$

In particular,

$$\sum_{|\mu|=j} \left| \sum_{|\nu| \geq 1} \frac{h_\nu u_{\mu-\nu} \mu!}{\nu! (\mu-\nu)!} \right| \leq C_h C(d) \frac{3^m}{4^m} C(h, f', \varphi) C_g \frac{r^j (j+k)!}{(1+k+j)^m}.$$

For m large enough with respect to $C_h C(d) C_\lambda$, and $r \geq r_0 2^{m-m_0}$, one has

$$\sum_{|\mu|=j} \left| \sum_{|\nu| \geq 1} \frac{h_\nu u_{\mu-\nu} \mu!}{\nu! (\mu-\nu)!} \right| \leq \frac{1}{3C_\lambda} C(h, f', \varphi) C_g \frac{r^j j!}{(1+k+j)^m}.$$

Similarly, one can control, for $1 \leq i \leq d$, the quantity

$$\begin{aligned} \left| \sum_{|\nu| \geq 2} \frac{a_{i,\nu} u_{\mu-\nu+\eta_i} \mu!}{\nu! (\mu-\nu+\eta_i)!} \right| &= \sum_{\ell=1}^{j-1} \sum_{|\rho|=\ell} |u_\rho| \sum_{\substack{|\mu|=j \\ \mu \geq \rho - \eta_i}} \frac{|a_{i,\mu-\rho+\eta_i}| \mu!}{(\mu-\rho+\eta_i)! \rho!} \\ &\leq C_a \sum_{\ell=1}^{j-1} \sum_{|\rho|=\ell} |u_\rho| \sum_{\substack{|\mu|=j \\ \mu \geq \rho - \eta_i}} r^{j-\ell} \frac{\mu!}{\rho! (1+j-\ell)^m} \end{aligned}$$

Letting ρ_M denote again the large index of ρ , and ρ_m its smallest non-zero index, then

$$\max_{|\mu| \geq |\rho|-1} \frac{\mu!}{\rho!} = \frac{(\rho_M + j - \ell + 1)!}{\rho_M! \rho_m} \leq \frac{(j+1)!}{\ell!}.$$

In particular, since

$$\sum_{|\rho|=\ell} |u_\rho| \leq C(h, f', \varphi) C_g \frac{r^\ell (k+\ell)!}{(k+\ell+1)^m},$$

one has, since $(k+\ell)!/\ell! = (k+\ell)(k+\ell-1)\dots(\ell+1) \leq (k+j+1)!/(j+1)!$, that

$$\begin{aligned} \left| \sum_{|\nu| \geq 2} \frac{a_{i,\nu} u_{\mu-\nu+\eta_i} \mu!}{\nu! (\mu-\nu+\eta_i)!} \right| &\leq C_a C(h, f', \varphi) C_g r^j \sum_{\ell=1}^{j-1} \frac{(k+\ell)!(j+1)!}{\ell!} \frac{(1+j-\ell)^d}{(1+j-\ell)^m (1+k+\ell)^m} \\ &\leq C_a C(h, f', \varphi) C_g \frac{r^j (j+k+1)!}{(1+j+k)^m} \sum_{\ell=1}^{j-1} \frac{(1+j-\ell)^d (1+j+k)^m}{(1+j-\ell)^m (1+k+\ell)^m} \\ &\leq C_a C(d) \frac{3^m}{4^m} C(h, f', \varphi) C_g \frac{r^j (j+k+1)!}{(1+j+k)^m}. \end{aligned}$$

Hence, for m and r large enough, one has, for every $1 \leq i \leq d$,

$$\left| \sum_{|\nu| \geq 2} \frac{a_{i,\nu} u_{\mu-\nu+\eta_i} \mu!}{\nu! (\mu-\nu+\eta_i)!} \right| \leq \frac{1}{3dC_\lambda} C(h, f', \varphi) C_g \frac{r^j (j+k+1)!}{(1+k+j)^m}.$$

To conclude, if $C(h, f', \varphi) \geq 3$, then

$$\sum_{|\mu|=j} |u_\mu| \leq \frac{1}{j+1} \left(\frac{1}{3} C(h, f', \varphi) + \frac{1}{3} C(h, f', \varphi) + \frac{1}{3} C(h, f', \varphi) \right) C_g \frac{r^j (j+k)!}{(1+k+j)^m},$$

which concludes the induction.

Third step

It only remains to prove that u is well-defined and holomorphic on U . Since the sequence of derivatives of u at 0 is well-controlled, the associated power series converges on some small neighbourhood V of 0. Then, from the knowledge of u on V one can build u on U using the geometric structure of the transport equation. Indeed, we recall that 0 is a repulsive fixed point for X . In particular, letting $(\Phi_t)_{t \in \mathbb{R}}$ denote the flow of $-X$, there exists $T > 0$ such that $\Phi_T(U) \subset V$. Then the transport equation on u is equivalent to

$$u(x) = u(\Phi_T(x)) + \int_0^T g(\Phi_t(x)) dt + \int_0^T u(\Phi_t(x)) h(\Phi_t(x)) dt.$$

By the analytic Picard-Lindelöf theorem, the unique solution of this degree 1 differential equation, where the initial data $u(\Phi_T(x))$ and the coefficients have real-analytic dependence on $\Phi_T(x) \in V$, is well-defined and real-analytic. Then u is well-defined on U , and holomorphic since the derived equation on $\bar{\partial}u$ is $\bar{\partial}u = 0$. This concludes the proof. \square

4 Construction of almost eigenvectors

Solving the Hamilton-Jacobi equation then controlling successive transport equations allows us to prove the first part of Theorem A, which is the object of this section.

The strategy of proof is the following: we first exhibit sequences $(u_i)_{i \geq 0}$ and $(\lambda_i)_{i \geq 0}$ such that the eigenvalue equation (3) is valid up to $O(N^{-\infty})$, and we control these sequences in analytic spaces. Then we prove that one can perform an analytic summation in (3).

Before proceeding, we note that, if φ is admissible and $u(N)$ is the summation of an analytic symbol, then $e^{N\varphi} u(N) \psi_0^N$ concentrates at 0, and moreover, by Proposition 2.2 and the stationary phase lemma, there exists $C > 0$ such that, for every $N \in \mathbb{N}$, there holds

$$\frac{1}{C} N^{-n} \|e^{N\varphi} u(N) \psi_0^N\|_{L^2} \leq C N^{-n}.$$

In particular, if

$$\|(T_N(f) - \lambda_N) e^{N\varphi} u(N) \psi_0^N\|_{L^2} \leq C e^{-c'N}$$

then λ_N will be exponentially close to the spectrum of $T_N(f)$. Thus, through Proposition 4.3 we are indeed providing almost eigenstates of $T_N(f)$ which concentrate on 0.

Proposition 4.1. *Let φ denote an admissible solution to the Hamilton-Jacobi equations (1), and let ψ_0^N denote the sequence of coherent states at 0. There exists $V \subset\subset U$ containing zero, a sequence $(u_k)_{k \geq 0}$ of holomorphic functions on U , and a sequence $(\lambda_k)_{k \geq 0}$ of real numbers, such that for every $K \geq 0$ there holds*

$$\left\| \left(T_N(f) - \sum_{k=0}^K N^{-k} \lambda_k \right) \psi_0^N e^{N\varphi} \sum_{k=0}^N N^{-k} u_k \right\|_{L^2(V)} = O(N^{-n-K-1}).$$

One has

$$\lambda_0 = \min \operatorname{Sp}(T_1(\operatorname{Hess}(f)(0))).$$

Proof. By Proposition 2.3, if a denotes the symbol of the Bergman kernel, then there exists $c' > 0$ such that, for all $x \in U$,

$$\begin{aligned} T_N(f) \left(\psi_0^N e^{N\varphi} u(N) \right) (x) = \\ \psi_0^N(x) e^{N\varphi(x)} \int_{y \in M} e^{N(\Phi_1(x, y, \bar{y}, 0) + \varphi(y) - \varphi(x))} \frac{a(N)(x, \bar{y})}{a(N)(x, 0)} a(N)(y, 0) f(y, \bar{y}) u(N)(y) dy + O(e^{-c'N}). \end{aligned}$$

We are able to apply the stationary phase Lemma. Let $*$ denote the Cauchy product of symbols, and let b be the analytic symbol such that

$$b(x, y, \bar{w}) = f(y, \bar{w}) a(x, \bar{w}) * a^{*-1}(x, 0) * a(y, 0) J(x, y, \bar{w}),$$

where J is the Jacobian of the change of variables κ_x mapping

$$(y, \bar{w}) \mapsto \Phi_1(x, y, \bar{w}, 0) + \varphi(y) - \varphi(x)$$

to the holomorphic extension of $v \mapsto -|v|^2$. Let also $\tilde{\Delta}(x) = \kappa_x^{-1} \circ \Delta \circ \kappa_x$ (where κ_x acts on functions by a change of variables). Then

$$e^{-N\varphi(x)} T_N(f) \left(\psi_0^N e^{N\varphi} u \right) (x) = \psi_0^N(x) \sum_{k=0}^{+\infty} N^{-k} \sum_{n=0}^k \frac{\tilde{\Delta}^n(x)}{n!} (u(y) b_{k-n}(x, y, \bar{w})) \Big|_{(y, \bar{w})=(x, \bar{y}_c(x))} + O(N^{-\infty}). \quad (2)$$

The modified Laplace operator $\tilde{\Delta}(x)$ depends on x and acts on y, \bar{w} . Using Proposition 3.5 with

$$f' : (x, y, \bar{w}) \mapsto b_0(x, y, \bar{w}),$$

which indeed coincides with f up to $O(|x, y, \bar{w}|^3)$, we will construct by induction a sequence of holomorphic functions u_i and a sequence of real numbers λ_i such that

$$T_N(f) \left(\psi_0^N e^{N\varphi} \sum_{k=0}^{+\infty} N^{-k} u_k \right) (x) = \psi_0^N(x) e^{N\varphi(x)} \left(\sum_{j=0}^{+\infty} N^{-j-1} \lambda_j \right) \left(\sum_{k=0}^{cN} N^{-k} u_k(x) \right) + O(N^{-\infty}). \quad (3)$$

We further require that

$$u_k(0) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{else.} \end{cases}$$

In the right-hand side of (3), there are no terms of order 0. In the left-hand side, the term of degree 0 is given by the term $k = 0$ in (2), so that one needs to solve

$$f(x, \bar{y}_c(x)) u_0(x) \frac{a_0(x, 0)}{a_0(y, 0)} a_0(x, \bar{y}) J(x, x, \bar{y}_c(x)) = b_0(x, x, \bar{y}_c(x)) u_0(x) = 0.$$

Since $f(x, \bar{y}_c(x)) = 0$, this equation is always satisfied.

By the stationary phase lemma (2), the order 1 in (3) reads

$$\lambda_0 u_0(x) - (\tilde{\Delta}(x) b_0)(x, x, \bar{y}_c(x)) u_0(x) - (\tilde{\nabla}(x) b_0)(x, x, \bar{y}_c(x)) \cdot \tilde{\nabla}(x) u_0(x) = 0. \quad (4)$$

The equation (4) allows us to solve for u_0 with the supplementary condition $u_0(0) = 1$. Indeed, as $\tilde{\nabla}(x)b_0(0) = 0$, at $x = 0$, the order 1 reads

$$\lambda_0 - (\tilde{\Delta}(x)b_0)(0, 0, 0) = 0,$$

so that we set

$$\lambda_0 = (\tilde{\Delta}(x)b_0)(0, 0, 0).$$

We now prove that λ_0 coincides with the ground state energy of the associated quadratic operator $T_N(\text{Hess}(f)(0))$. Indeed, λ_0 depends only on the Hessian of f and ϕ at zero (which together determine the Hessian of φ at zero as seen in Proposition 3.3, thus they determine the linear part of the change of variables κ in the stationary phase lemma, which in turn determines $\tilde{\Delta}$ and J at 0). If f and ϕ are quadratic, then the solution φ of the Hamilton-Jacobi equation is also quadratic as constructed in Proposition 3.3, so that $u_0 = 1$ satisfies (3) exactly. Thus, λ_0 is an eigenvalue of $T_N(\text{Hess}(f)(0))$ which depends continuously on $\text{Hess}(f)(0)$. Moreover, if $\text{Hess}(f)(0) : y \mapsto |y|^2$, then $\text{Hess}(\varphi) = 0$ so that the eigenvector of $T_N(\text{Hess}(f)(0))$ associated with λ_0 is the coherent state (in \mathbb{C}^d) ψ_0^N , which is the ground state of $T_N(|y|^2)$; thus in this case λ_0 is the ground state energy. Since the set of positive definite quadratic forms in \mathbb{R}^{2d} is connected, and since there is always a gap between the ground state energy and the first excitation, then λ_0 is always the ground state energy of $T_N(\text{Hess}(f)(0))$.

We wish now to find u_0 such that $u_0(0) = 1$. Setting $v_0 = u_0 - 1$ yields

$$\tilde{\nabla}(x)v_0(x) \cdot (\tilde{\nabla}(x)b_0)(x, x, \bar{y}_c(x)) = v_0(x) \left[(\tilde{\Delta}(x)b_0)(x, x, \bar{y}_c(x)) - (\tilde{\Delta}(x)b_0)(0, 0, 0) \right].$$

We then solve for v_0 using Proposition 3.5 with $f' = b_0$, which indeed yields $v_0(0) = 0$.

Let us now find the remaining terms of the sequences $(u_k)_{k \geq 0}$ and $(\lambda_k)_{k \geq 0}$ by induction. For $k \geq 1$, the term of order $k + 1$ in (3) is given again by the stationary phase lemma (2): at this order, the equation is

$$\begin{aligned} \lambda_k u_0(x) + \lambda_0 u_k(x) - (\tilde{\Delta}(x)b_0)(x, x, \bar{y}_c(x))u_k(x) - (\tilde{\nabla}(x)b_0)(x, x, \bar{y}_c(x)) \cdot \tilde{\nabla}(x)u_k(x) \\ = - \sum_{j=1}^{k-1} \lambda_j u_{k-j}(x) + \sum_{n=2}^{k+1} \sum_{l=0}^{k+1-n} \frac{\tilde{\Delta}^n(x)}{n!} (u_l(y)b_{k+1-n-l}(x, y, \bar{w})) \Big|_{(y, \bar{w})=(x, \bar{y}_c(x))}. \end{aligned} \quad (5)$$

In this equation, we have put to the left-hand side all terms involving λ_k or u_k , and all terms involving λ_l and u_l with $l < k$ to the right-hand side. We can apply Proposition 3.5 to solve for u_k, λ_k once $(u_l, \lambda_l)_{0 \leq l \leq k-1}$ are known.

Observe that (5), at order $k + 1$, takes the form

$$(\tilde{\nabla}(x)b_0)(x, x, \bar{y}_c(x)) \cdot \tilde{\nabla}u_k(x) = g_k(x) + h(x)u_k(x),$$

with $h(x) = \tilde{\Delta}(x)b_0(x, x, \bar{y}_c(x)) - \lambda_0$ and

$$g_k(x) = - \sum_{l=1}^{k-1} \lambda_l u_{k-l}(x) - \lambda_k u_0 + \sum_{n=2}^{k+1} \sum_{l=0}^{k+1-n} \frac{\tilde{\Delta}^n(x)}{n!} (u_l(y)b_{k+1-n-l}(x, y, \bar{w})) \Big|_{(y, \bar{w})=(x, \bar{y}_c(x))}.$$

By construction of λ_0 , one has $h(0) = 0$; moreover,

$$g_k(0) = \sum_{n=2}^{k+1} \sum_{l=0}^{k+1-n} \frac{\tilde{\Delta}^n(0)}{n!} (u_l(y)b_{k+1-n-l}(0, y, \bar{w})) \Big|_{(y, \bar{w})=(0,0)} - \lambda_k.$$

Thus, one can solve for λ_k by setting $g_k(0) = 0$, then solve for u_k using Proposition 3.5 (indeed, g_k is a holomorphic function, so that it belongs to some analytic space $H(m, r, V)$). This concludes the proof. \square

Proposition 4.2. *Let $(u_k)_{k \geq 0}$ and $(\lambda_k)_{k \geq 0}$ be the sequences constructed in the previous proposition, corresponding to an $O(N^{-\infty})$ eigenfunction of $T_N(f)$. Then there exist $C > 0$, $R > 0$, $r > 0$, $m \in \mathbb{R}$ and an open set $V \subset\subset U$ containing 0 such that, for all $k \geq 0, j \geq 0$, one has*

$$\|u_k\|_{C^j(X)} \leq C \frac{r^j R^k (j+k)!}{(j+k+1)^m}$$

$$|\lambda_k| \leq C \frac{R^k (k+1)!}{(k+2)^m}.$$

Before the proof, let us recall that we fixed the following convention for the C^j norm in Definition 2.1 (in the case of an open set of \mathbb{C}^d):

$$\|f\|_{C^j(V)} = \sup_{x \in V} \sum_{|\mu|=j} |\partial^\mu f(x)|.$$

In particular, this corresponds to the control in Proposition 3.5 (which we performed only at zero).

Proof. Let us prove that λ_k and u_k are controlled in an analytic way as k grows. The proof consists in three steps. In the first step, we show that in equation (5) (that is, in the definition of g_k), no derivatives of u_l of order larger than n appear. The second step is an induction: we suppose some control on all derivatives of u_l at zero, for $0 \leq l \leq k-1$, and we apply the Lemma 4.6 of [3] to deduce that the derivatives of g_k at zero are well-behaved. We then apply Proposition 3.5 to obtain a control on the derivatives of u_k at zero. In the last step, we deduce, from a control of the derivatives of u_k at zero, a control of the same nature on a small open neighbourhood.

First step.

Let g be a holomorphic function near 0 in M . Then $T_N(g)$ is, locally, a multiplication operator, so that

$$e^{-N\varphi} T_N(g)(\psi_0^N e^{N\varphi} u) = \psi_0^N g u + O(e^{-c'N}).$$

In this particular case, no derivative of u of order ≥ 1 appear in (2), hence in (5).

We then decompose any real-analytic function g as

$$g(y, \bar{y}) = g(y, \bar{y}_c(x)) + (g(y, \bar{y}) - g(y, \bar{y}_c(x))).$$

In the right-hand side, the second term vanishes when $\bar{y} = \bar{y}_c(x)$, so that, with

$$\Phi : (x, y, \bar{w}) \mapsto \Phi_1(x, y, 0) + \varphi(y) - \varphi(x),$$

there exists a smooth vector-valued function g_1 such that

$$g(y, \bar{y}) = g(y, \bar{y}_c(x)) + \partial_y \Phi(x, y, \bar{y}) \cdot g_1(x, y, \bar{y}).$$

Now S_N acts as the identity on holomorphic functions and \bar{y}_c is a holomorphic function of x so that, by integration by parts:

$$\begin{aligned} & \int e^{-N\Phi(x, y, \bar{y})} a(N)(x, \bar{y}) g(y, \bar{y}) u(y) dy \\ &= g(x, \bar{y}_c(x)) u_0(x) + \int e^{-N\Phi(x, y, \bar{y})} a(N)(x, \bar{y}) \partial_y \Phi(x, y, \bar{y}) \cdot g_1(x, y, \bar{y}) u(y) dy + O(e^{-c'N}) \\ &= g(x, \bar{y}_c(x)) u_0(x) + N^{-1} \int e^{-N\Phi(x, y, \bar{y})} a(N)(x, \bar{y}) \partial_y [g_1(x, y, \bar{y}) u(y)] dy + O(e^{-c'N}). \end{aligned}$$

By induction, the terms of order N^{-k} in the expansion (2) only contain derivatives of u of order smaller than k .

Second step.

Let us prove by induction that the sequences $(u_k)_{k \geq 0}$ and $(\lambda_k)_{k \geq 0}$ are analytic symbols. We will make use of the precise controls obtained in Proposition 3.5. Since $(b_k)_{k \geq 0}$ is an analytic symbol and u_0 is holomorphic, by Proposition 2.2 there exists a small open neighbourhood V of zero and $r_0, R_0, m_0, C_b, C_0 > 0$ such that, after the Morse change of variables,

$$\|b_k\|_{C^j(V)} \leq C_b \frac{r_0^j R_0^k (j+k)!}{(j+k+1)^{m_0}}$$

$$\|u_0\|_{C^j(V)} \leq C_0 \frac{r_0^j j!}{(j+1)^{m_0}}.$$

In particular, for any $m \geq m_0$, for any $r \geq 2^{m+1-m_0} r_0$ and $R \geq 2^{m+1-m_0} R_0$, one has, after the Morse change of variables,

$$\|b_k Jf\|_{C^j(V)} \leq C_b \frac{(r/3)^j (R/3)^k (j+k)!}{(j+k+1)^m}$$

$$\|u_0\|_{C^j(V)} \leq C_0 \frac{r^j j!}{(j+1)^m}.$$

In equation (5), let us isolate the terms involving u_0 . There holds

$$\begin{aligned} & \lambda_k u_0(x) + \lambda_0 u_k(x) - \tilde{\Delta}(x) b_0(x, x, \bar{y}) u_k(x) - \tilde{\nabla}(x) b_0(x, x, \bar{y}_c(x)) \cdot \tilde{\nabla}(x) u_k(x) \\ &= \sum_{n=2}^{k+1} \frac{\tilde{\Delta}^n(x)}{n!} (u_0(y) b_{k+1-n}(x, y, \bar{y})) (x, \bar{y}_c(x)) \\ & \quad - \sum_{j=1}^{k-1} \lambda_j u_{k-j}(x) + \sum_{n=2}^{k+1} \sum_{l=1}^{k+1-n} \frac{\tilde{\Delta}^n(x)}{n!} (u_l(y) b_{k+1-n-l}(x, y, \bar{y})) (x, \bar{y}_c(x)). \end{aligned}$$

Let m, r, R, C_u, C_λ be large enough (they will be fixed in the course of the induction), and suppose that, for all $0 \leq l \leq k-1$ and all $j \geq 0$, one has

$$|\lambda_l| \leq C_\lambda \frac{R^l (l+1)!}{(l+2)^m}$$

$$\|\nabla^j u_l(0)\|_{\ell^1} \leq C_u \frac{r^j R^l (j+l)!}{(j+l+1)^m}.$$

To begin with, we estimate how the iterated modified Laplace operator $\tilde{\Delta}^n(x)$ acts on u_ℓ using the fact that it differentiates it at most n times.

After a change of variables $\kappa_x : (y, \bar{w}) \mapsto v(x, y, \bar{y})$ for which the phase is the holomorphic extension of the standard quadratic form $-|v|^2$, one has simply

$$\tilde{\Delta}(x) = \Delta_v = \sum_{i=1}^{2d} \frac{\partial^2}{\partial v_i^2}.$$

Hence,

$$\begin{aligned} & \Delta_v^n [u_l(y(v))b_{k+1-n-l}(x, y(v), \bar{y}(v))]_{v=0} \\ &= \sum_{|\mu|=n} \sum_{\nu \leq 2\mu} \frac{n!(2\mu)!}{\mu! \nu! (2\mu - \nu)!} \partial_v^\nu u_l(x, y(x, v))_{v=0} \partial_v^{2\mu - \nu} b_{k+1-n-l}(x, y(x, v), \bar{y}(x, v))_{v=0}. \end{aligned}$$

Since Δ_v^n differentiates at most n times on u_l , in the expression above, the differential operator

$$\partial_v^\nu u_l(x, y(x, v))_{v=0}$$

can be replaced with its truncation into a differential operator of degree less or equal to n , which we denote by $(\partial_\kappa^\nu)^{[\leq n]} u_l(x)$ as in [3], Lemma 4.6 and Proposition 4.7. In particular, for every $\rho \in \mathbb{N}^d$,

$$\begin{aligned} & \partial_x^\rho \Delta_v^n [u_l(y(v))b_{k+1-n-l}(x, y(v), \bar{y}(v))]_{v=0} = \\ & \sum_{|\mu|=n} \sum_{\nu \leq 2\mu} \sum_{\rho_1 \leq \rho} \frac{n!(2\mu)! \rho!}{\mu! \nu! (2\mu - \nu)! \rho_1! (\rho - \rho_1)!} \partial_x^{\rho_1} (\partial_\kappa^\nu)^{[\leq n]} u_l(x) \partial_x^{\rho - \rho_1} \partial_v^{2\mu - \nu} b_{k+1-n-l}(x, y(x, v), \bar{y}(x, v))_{v=0}. \end{aligned}$$

Moreover, if $|\mu| = n$ then

$$\frac{n!}{\mu!} \leq (2d)^n,$$

and if $\nu \leq 2\mu$ then, by Lemma 5.2 in [3],

$$\frac{(2\mu)! \rho!}{\nu! (2\mu - \nu)! \rho_1! (\rho - \rho_1)!} = \binom{2\mu}{\nu} \binom{\rho}{\rho_1} \leq \binom{2n}{|\nu|} \binom{|\rho|}{|\rho_1|}.$$

Hence,

$$\begin{aligned} & \|\nabla_x^j \Delta_v^n [u_l(y(v))b_{k+1-n-l}(x, y(v), \bar{y}(v))]_{v=x=0}\|_{\ell^1} \\ & \leq (2d)^n \sum_{i_1=0}^{2n} \sum_{j_1=0}^j \binom{2n}{i_1} \binom{j}{j_1} \|\nabla_x^{j_1} (\nabla_\kappa^{i_1})^{[\leq n]} u_l\|_{\ell^1} \|b_{k+1-n-l}(x, y(v), \bar{y}(v))\|_{C^{j-j_1+2n-i_1}(V)}. \end{aligned}$$

By the induction hypothesis, one has

$$\|\nabla_v^j u_l(0)\|_{\ell^1} \leq C_u \frac{r^j R^k (j+l)!}{(j+l+1)^m},$$

then, by Lemma 4.6 in [3],

$$\begin{aligned} & \|\nabla_x^{j_1} (\nabla_\kappa^{i_1})^{[\leq n]} u_l(y(v))_{x=v=0}\|_{\ell^1} \\ & \leq i_1^{d+1} j_1^{d+1} C_u \frac{r^{j_1+i_1} R^l}{(i_1+j_1+l+1)^m} (C'' r_0)^{i_1} \times \begin{cases} \max((n+j_1+l)!(i_1-n)!, (j_1+l)!i_1!) & \text{if } i_1 \geq n \\ (i_1+j_1+l)! & \text{otherwise.} \end{cases} \quad (6) \end{aligned}$$

In the case $l = 0$, the constant C_u can be replaced with the smaller constant C_0 .

The control (6) allows us to conclude the induction. We first solve for λ_k using equation (5) at $x = 0$:

$$\lambda_k = \sum_{n=2}^{k+1} \sum_{l=0}^{k+1-n} \frac{\tilde{\Delta}^n}{n!} (u_l(y) b_{k+1-n-l}(x, y, \bar{y}))(0, 0).$$

Then, by the induction hypothesis and (6),

$$|\lambda_k| \leq C_u C_b \sum_{n=2}^{k+1} \frac{R^k (k+1)!}{(k+2)^m} (2d)^n R \left(\frac{C'' r^2}{R} \right)^n \times \\ \sum_{i_1=0}^{2n} \sum_{l=0}^{k+1-n} \frac{(2n)! A(i_1, 0, l, n) (k+1-n-l+2n-i_1)!}{3^{k+1-n-l+2n-i_1} i_1! (2n-i_1)! n! (k+1)!} \frac{(k+2)^m}{(i_1+l+1)^m (k+2+n-l-i_1)^m},$$

with

$$A(i_1, j_1, l, n) = \begin{cases} \max((n+j_1+l)!(i_1-n)!, (j_1+l)!i_1!) & \text{if } i_1 \geq n \\ (i_1+j_1+l)! & \text{otherwise.} \end{cases}$$

Let us prove that

$$\frac{(2n)! A(i_1, 0, l, n) (k+1-n-l+2n-i_1)!}{3^{k+1-n-l+2n-i_1} i_1! (2n-i_1)! n! (k+1)!} \leq 2^n.$$

If $i_1 \leq n$, since

$$(k+1-n-l+2n-i_1)! \leq (2n-i_1)! (k+1-n-l)! 2^{k+1-n-l-2n-i_1},$$

one has

$$\frac{(2n)! (i_1+l)! (k+1-n-l+2n-i_1)!}{3^{k+1-n-l+2n-i_1} i_1! (2n-i_1)! n! (k+1)!} \leq \frac{(2n)! (i_1+l)! (k+1-n-l)!}{n! i_1! (k+1)!}.$$

The right-hand side is increasing with respect to i_1 as it can be written $C(i_1+l)(i_1+l-1)\dots(i_1+1)$, so that

$$\frac{(2n)! (i_1+l)! (k+1-n-l+2n-i_1)!}{3^{k+1-n-l+2n-i_1} i_1! (2n-i_1)! n! (k+1)!} \leq \frac{(2n)! (n+l)! (k+1-n-l)!}{n! n! (k+1)!} \leq 2^n \binom{k+1}{n+l}^{-1} \leq 2^n.$$

If $i_1 \geq n$, in one case, since

$$\frac{(k+1-n-l+2n-i_1)!}{(2n-i_1)!} \leq \frac{(k+1-l)!}{n!},$$

one has

$$\frac{(2n)! i_1! l! (k+1-n-l+2n-i_1)!}{3^{k+1-n-l+2n-i_1} i_1! (2n-i_1)! n! (k+1)!} \leq \frac{(2n)! l! (k+1-l)!}{n! n! (k+1)!} \leq 2^n \binom{k+1}{l}^{-1} \leq 2^n.$$

In the other case,

$$\frac{(2n)! (n+l)! (i_1-n)! (k+1-n-l+2n-i_1)!}{3^{k+1-n-l+2n-i_1} i_1! (2n-i_1)! n! (k+1)!} \leq \frac{(2n)! (n+l)! (i_1-n)! (k+1-n-l)!}{i_1! n! (k+1)!}.$$

The right-hand term is maximal at $i_1 = n$, so that

$$\frac{(2n)! (n+l)! (i_1-n)! (k+1-n-l+2n-i_1)!}{3^{k+1-n-l+2n-i_1} i_1! (2n-i_1)! n! (k+1)!} \leq \frac{(2n)! (n+l)! (k+1-n-l)!}{n! n! (k+1)!} \leq 2^n.$$

In particular,

$$|\lambda_k| \leq C_u C_b \sum_{n=2}^{k+1} \frac{R^k (k+1)!}{(k+2)^m} (2d)^n R \left(\frac{2C'' r^2}{R} \right)^n \sum_{i_1=0}^{2n} \sum_{l=0}^{k+1-n} \frac{(k+2)^m}{(i_1+l+1)^m (k+2+n-l-i_1)^m},$$

Since $(k+2)^m \leq (k+2+n)^m$, one has

$$|\lambda_k| \leq C_u C_b \sum_{n=2}^{k+1} \frac{R^k (k+1)!}{(k+2)^m} (2d)^n R \left(\frac{2C''r^2}{R} \right)^n \sum_{i_1=0}^{2n} \sum_{l=0}^{k+1-n} \frac{(k+n+2)^m}{(i_1+l+1)^m (k+2+n-l-i_1)^m},$$

Then, by Lemma 2.12 in [3], there holds

$$|\lambda_k| \leq C_u C_b \frac{R^k (k+1)!}{(k+2)^m} R \sum_{n=2}^{k+1} \left(\frac{\kappa r^2}{R} \right)^n.$$

For R large enough (once r, m, C_u, C_λ are fixed), this is smaller than $C_\lambda \frac{R^k (k+1)!}{(k+2)^m}$.

We now pass to the control on u_k . We recall that u_k solves an equation of the form

$$X \cdot u_k = h u_k + g_k,$$

with X and h independent on k and

$$g_k : x \mapsto - \sum_{l=1}^{k-1} \lambda_l u_{k-l}(x) - \lambda_k u_0(x) + \sum_{n=2}^{k+1} \sum_{l=0}^{k+1-n} \frac{\tilde{\Delta}^n(x)}{n!} (u_l(y) b_{k+1-n-l}(x, y, \bar{y})) \Big|_{(y, \bar{y})=(x, \bar{y}_c(x))}.$$

Let us control the derivatives of g_k at zero, in order to apply Proposition 3.5. One has first

$$\|\lambda_k \nabla^j u_0(0)\|_{\ell^1} \leq C_\lambda C_0 \frac{r^j R^k (j+k+1)!}{(j+k+2)^m}.$$

Once C_λ is fixed, this is smaller than ϵC_u for C_u large enough.

Moreover,

$$\left\| \sum_{l=1}^{k-1} \lambda_l \nabla^j u_{k-l}(0) \right\|_{\ell^1} \leq C_\lambda C_u \frac{r^j R^k (j+k+1)!}{(j+k+2)^m} \sum_{l=1}^{k-1} \underbrace{\frac{l!(j+k-l+1)!}{(j+k+1)!}}_{= \binom{j+k+1}{l}^{-1} \leq 1} \frac{(k+j+2)^m}{(l+2)^m (k-l+j+1)^m}.$$

Hence, by Lemma 2.12 in [3],

$$\left\| \sum_{l=1}^{k-1} \lambda_l \nabla^j u_{k-l}(0) \right\|_{\ell^1} \leq C C_\lambda C_u \frac{3^m r^j R^k (j+k+1)!}{4^m (j+k+2)^m}.$$

Once C_λ and C_u are fixed, the constant $C C_\lambda C_u \frac{3^m}{4^m}$ is smaller than ϵC_u for m large enough.

It remains to estimate

$$\left\| \nabla^j \left[x \mapsto \sum_{n=2}^{k+1} \sum_{l=0}^{k+1-n} \frac{\tilde{\Delta}^n(x)}{n!} (u_l(y) b_{k+1-n-l}(x, y, \bar{y})) \Big|_{(y, \bar{y})=(x, \bar{y}_c(x))} \right]_{x=0} \right\|_{\ell^1}.$$

By (6), one has

$$\begin{aligned}
& \left\| \nabla^j \left[x \mapsto \sum_{n=2}^{k+1} \sum_{l=0}^{k+1-n} \frac{\widetilde{\Delta}^n(x)}{n!} (u_l(y) b_{k+1-n-l}(x, y, \bar{y})) \Big|_{(y, \bar{y})=(x, \bar{y}_c(x))} \right]_{x=0} \right\|_{\ell^1} \\
& \leq C_u C_b \frac{r^j R^k (j+k+1)!}{(j+k+2)^m} \sum_{n=2}^{k+1} R \left(\frac{\kappa r^2}{R} \right)^n \sum_{l=0}^{k+1-n} \sum_{i_1=0}^{2n} \sum_{j_1=0}^j \\
& \quad \frac{(2n)! j! A(i_1, j_1, l, n) (k+1-n-l+2n-i_1+j-j_1)!}{3^{k+1-n-l+2n-i_1+j-j_1} i_1! (2n-i_1)! j_1! (j-j_1)! n! (k+j+1)!} \\
& \quad \times \frac{(k+j+2)^m}{(i_1+l+j_1+1)^m (k+2+n-l-i_1+j-j_1)^m}.
\end{aligned}$$

Let us prove, similarly to the control on λ_k , that

$$\frac{(2n)! j! A(i_1, j_1, l, n) (k+1-n-l+2n-i_1+j-j_1)!}{3^{k+1-n-l+2n-i_1+j-j_1} i_1! (2n-i_1)! j_1! (j-j_1)! n! (k+j+1)!} \leq 2^n.$$

Again,

$$\frac{(k+1-n-l+2n-i_1+j-j_1)!}{(2n-i_1)! (j-j_1)! (k+1-n-l)!} \leq 3^{k+1-n-l+2n-i_1+j-j_1},$$

so that

$$\frac{(2n)! j! A(i_1, j_1, l, n) (k+1-n-l+2n-i_1+j-j_1)!}{3^{k+1-n-l+2n-i_1+j-j_1} i_1! (2n-i_1)! j_1! (j-j_1)! n! (k+j+1)!} \leq \frac{(2n)! j! A(i_1, j_1, l, n) (k+1-n-l)!}{i_1! j_1! n! (k+j+1)!}.$$

If $i_1 \leq n$, then $A(i_1, j_1, l, n) = (i_1 + j_1 + l)!$ so that

$$\frac{(2n)! j! (i_1 + j_1 + l)! (k+1-n-l)!}{i_1! j_1! n! (k+j+1)!}$$

is increasing with respect to i_1 and j_1 . Thus, it is maximal at $i_1 = n$ and $j_1 = j$, so that

$$\begin{aligned}
\frac{(2n)! j! A(i_1, j_1, l, n) (k+1-n-l+2n-i_1+j-j_1)!}{3^{k+1-n-l+2n-i_1+j-j_1} i_1! (2n-i_1)! j_1! (j-j_1)! n! (k+j+1)!} & \leq \frac{(2n)! (n+j+l)! (k+1-n-l)!}{n! n! (k+j+1)!} \\
& = \binom{2n}{n} \binom{k+j+1}{n+j+l}^{-1} \leq 2^n.
\end{aligned}$$

If $i_1 \geq n$, then $A(i_1, j_1, l, n) = \max((n+j_1+l)!(i_1-n)!, (j_1+l)!i_1!)$. On one hand,

$$\frac{(2n)! j! (n+j_1+l)! (i_1-n)! (k+1-n-l)!}{n! i_1! j_1! (k+j+1)!}$$

is increasing with respect to j_1 and decreasing with respect to i_1 , and at $i_1 = n, j_1 = j$, it is equal to

$$\frac{(2n)! (n+j+l)! (k+1-n-l)!}{n! n! (k+j+1)!} = \binom{2n}{n} \binom{k+j+1}{n+j+l}^{-1} \leq 2^n.$$

On the other hand,

$$\frac{(2n)! j! (j_1+l)! i_1! (k+1-n-l)!}{n! i_1! j_1! (k+j+1)!} = \frac{(2n)! j! (j_1+l)! (k+1-n-l)!}{n! j_1! (k+j+1)!}$$

is increasing with respect to j_1 . At $j_1 = j$, it is equal to

$$\frac{(2n)!(j+l)!(k+1-n-l)!}{n!(k+j+1)!} \leq \frac{(2n)!(j+l)!(k+1-l)!}{n!n!(k+j+1)} = \binom{2n}{n} \binom{k+j+1}{j+l}^{-1} \leq 2^n.$$

Thus,

$$\begin{aligned} & \left\| \nabla^j \left[x \mapsto \sum_{n=2}^{k+1} \sum_{l=0}^{k+1-n} \frac{\tilde{\Delta}^n(x)}{n!} (u_l(y) b_{k+1-n-l}(x, y, \bar{y})) \Big|_{(y, \bar{y})=(x, \bar{y}_c(x))} \right]_{x=0} \right\|_{\ell^1} \\ & \leq C_u C_b \frac{r^j R^k (j+k+1)!}{(j+k+2)^m} \sum_{n=2}^{k+1} R \left(\frac{2\kappa r^2}{R} \right)^n \sum_{l=0}^{k+1-n} \sum_{i_1=0}^{2n} \sum_{j_1=0}^j \frac{(k+j+2)^m}{(i_1+l+j_1+1)^m (k+2+n-l-i_1+j-j_1)^m} \\ & \leq C_u C_b \frac{r^j R^k (j+k+1)!}{(j+k+2)^m} \sum_{n=2}^{k+1} R \left(\frac{2\kappa r^2}{R} \right)^n \sum_{l=0}^{k+1-n} \sum_{i_1=0}^{2n} \sum_{j_1=0}^j \frac{(k+j+n+2)^m}{(i_1+l+j_1+1)^m (k+2+n-l-i_1+j-j_1)^m}. \end{aligned}$$

By Lemma 2.12 in [3], there exists $C > 0$ such that, for m large enough, one has

$$\begin{aligned} & \left\| \nabla^j \left[x \mapsto \sum_{n=2}^{k+1} \sum_{l=0}^{k+1-n} \frac{\tilde{\Delta}^n(x)}{n!} (u_l(y) b_{k+1-n-l}(x, y, \bar{y})) \Big|_{(y, \bar{y})=(x, \bar{y}_c(x))} \right]_{x=0} \right\|_{\ell^1} \\ & \leq C C_u C_b \frac{r^j R^k (j+k+1)!}{(j+k+2)^m} \sum_{n=2}^{k+1} R \left(\frac{2\kappa r^2}{R} \right)^n. \end{aligned}$$

Thus, for R large enough,

$$\left\| \nabla^j \left[x \mapsto \sum_{n=2}^{k+1} \sum_{l=0}^{k+1-n} \frac{\tilde{\Delta}^n(x)}{n!} (u_l(y) b_{k+1-n-l}(x, y, \bar{y})) \Big|_{(y, \bar{y})=(x, \bar{y}_c(x))} \right]_{x=0} \right\|_{\ell^1} \leq \epsilon C_u \frac{r^j R^k (j+k+1)!}{(j+k+2)^m}.$$

To conclude, for every $\epsilon > 0$, there exists C_u, C_λ, m, r, R , such that one can proceed in the induction with

$$\|\nabla^j g_k(0)\|_{\ell^1} \leq \epsilon C_u \frac{r^j R^k (j+k+1)!}{(j+k+2)^m}.$$

Then, one can apply Lemma 3.5 since u_k is given by the transport equation

$$\tilde{\nabla}(x) b_0(x, x, \bar{y}_c(x)) \cdot \tilde{\nabla} u_k(x) = g_k(x) + h(x) u_k(x),$$

where again $h : x \mapsto \tilde{\Delta}(x) b_0(x, x, \bar{y}_c(x)) - \lambda_0$. Hence, there exists $C(b_0, \varphi)$ such that

$$\|\nabla^j u_k(0)\|_{\ell^1} \leq \epsilon C(b_0, \varphi) C_u \frac{r^j R^k (j+k)!}{(j+k+1)^m}.$$

If ϵ is chosen such that $\epsilon < C(b_0, \varphi)^{-1}$, one can conclude the induction.

Third step.

We successfully constructed and controlled the sequences $(\lambda_k)_{k \geq 0}$ and $(u_k)_{k \geq 0}$ which satisfy (3) at every order. Let us now prove that u_k is controlled on a small neighbourhood of 0.

In the second step, we controlled the functions u_k as follows, *at zero*:

$$\|\nabla^j u_k(0)\|_{\ell^1} \leq C_u \frac{(2r)^j (2R)^k j! k!}{(j+k+1)^m}.$$

Since u_k is real-analytic, in a small neighbourhood of zero, it is given by the power series

$$u(y) = \sum_{\nu} \frac{\partial^{\nu} u(0)}{\nu!} y^{\nu}.$$

Since

$$\frac{\partial^{\nu} u(0)}{\nu!} \leq C_u (2R)^k k! \frac{|\nu|!}{\nu!} (2r)^{|\nu|} \leq C_u (2R)^k k! (2rd)^{|\nu|},$$

the power series above converges for $y \in P(0, (2rd)^{-1})$, the polydisk centred at zero with radius $(2rd)^{-1}$. Moreover, for every $a < 1$, there exists $C(a)$ such that

$$\sup_{P(0, a(2rd)^{-1})} |u_k| \leq C(a) C_u (2R)^k k!.$$

In particular, by Proposition 2.14 in [3], for every $a < \frac{1}{2}$, there exists $C(a)$ such that

$$\|a\|_{H\left(-d, \frac{2d^2 r}{a}, P(0, \frac{a}{2rd})\right)} \leq C(a) C_u (2R)^k k!.$$

In other terms, letting $V = P(0, a(2rd)^{-1})$, for every $j \geq 0$, one has

$$\|a\|_{C^j(V)} \leq C(a) C_u \frac{(2R)^k \left(\frac{2d^2}{a} r\right)^j j! k!}{(j+1)^{-d}}.$$

In particular, u is an analytic symbol on V . □

We are now in position to perform an analytic summation.

Proposition 4.3. *For $c > 0$ and $c' > 0$ small, one has*

$$\left\| \left(T_N(f) - \sum_{j=0}^{cN} N^{-j-1} \lambda_j \right) \left(\psi_0^N e^{N\varphi} \sum_{k=0}^{cN} N^{-k} u_k \right) \right\|_{L^2} = O(e^{-c'N}).$$

Proof. Let $c > 0$. By construction, in a small neighbourhood V of zero (outside of which the result is trivial), there holds

$$\begin{aligned} & \left(T_N(f) - \sum_{j=0}^{cN} N^{-j-1} \lambda_j \right) \left(\psi_0^N e^{N\varphi} \sum_{k=0}^{cN} N^{-k} u_k \right) (x) \\ &= - \sum_{j=0}^{cN} \sum_{k=cN-j}^{cN} N^{-1-j-k} \psi_0^N(x) e^{N\varphi(x)} \lambda_j u_k(x) + \sum_{j+k \leq cN} N^{-1-j-k} \psi_0^N(x) e^{N\varphi(x)} R(j, k, N)(x), \end{aligned}$$

where $R(j, k, N)$ is the remainder at order $cN - k - j$ in the stationary phase Lemma applied to

$$N^{2d} \lambda_j e^{-N\varphi(x)} \int_{y \in M} e^{-N\Phi_1(x, y, \bar{y}, 0) + N\varphi(y)} (u * b)_k(x, y, \bar{y}) dy.$$

Since $\lambda * u$ is an analytic symbol by Proposition 2.2, we have, for $c > 0$ and $c' > 0$ small enough,

$$\left\| \sum_{j=0}^{cN} \sum_{k=cN-j}^{cN} N^{-1-j-k} \lambda_j u_k \right\|_{L^\infty(V)} \leq C e^{-c'N},$$

so that

$$\left\| \left(\sum_{j=0}^{cN} \sum_{k=cN-j}^{cN} N^{-1-j-k} \lambda_j u_k(x) \right) \psi_0^N(x) e^{N\varphi(x)} \right\|_{L^2(V)} \leq C e^{-c'N}.$$

The remainder $R(j, k, N)$ can be estimated using Proposition 3.13 in [3]. Indeed, let $r > 0$ and $R > 0$ be such that $u \in S_4^{r, R}(V)$ and $b \in S_4^{r, R}(V)$. By Proposition 2.2, $u * b$ is an analytic symbol of the same class, so that

$$\|(u * b)_k\|_{C^j(V)} \leq C C_u C_b R^k r^j (j+k)! \leq (C C_u C_b (2R)^k k!) (2r)^j j!.$$

In particular, $(u * b)_k$ admits a holomorphic extension to a k -independent complex neighbourhood \tilde{V} of V , with

$$\sup_{\tilde{V}} |(u * b)_k| \leq C C_u C_b (2R)^k k!.$$

In particular, by Proposition 3.13 in [3], one has, for some $c_1 > 0$, that the remainder at order $c_1 N$ in the stationary phase Lemma applied to

$$N^{2d} \lambda_j e^{-N\varphi(x)} \int_{y \in M} e^{-N\Phi_1(x, y, \bar{y}, 0) + N\varphi(y)} (u * b)_k(x, y, \bar{y}) dy$$

is smaller than $C C_u C_b (2R)^k (2R)^j j! k! e^{-c'N}$. In particular,

$$\left(\frac{1}{n!} \tilde{\Delta}(x)^n ((u * b)_k J)(y_c) \right)_n$$

is an analytic symbol in a fixed class, with norm smaller than $C(2R)^k k!$.

If $j + k < \frac{1}{2}cN$, we will compare $R(j, k, N)$ to the remainder at order $c_1 N$. If $j + k \geq \frac{1}{2}cN$, we will compare $R(j, k, N)$ to the remainder at order 0.

Without loss of generality, $c < c_1$. Then, for all j, k such that $j + k < \frac{1}{2}cN$, since the expansion in the stationary phase

$$\sum_{n=cN-j-k}^{c_1 N} (n! N^{d+n})^{-1} \tilde{\Delta}(x)^n ((u * b)_k J)(y_c)$$

corresponds to an analytic symbol, then by Lemma 2.2 this sum is $O(e^{-c'N})$; thus if $j + k < c/2$ one has

$$R(j, k, N) \leq C e^{-c'N}.$$

If $\frac{1}{2}cN < j + k < cN$, then, on one hand

$$N^{-1-j-k} \left| N^{2d} \lambda_j e^{-N\varphi(x)} \int_{y \in M} e^{-N\Phi_1(x, y, \bar{y}, 0) + N\varphi(y)} (u * b)_k(x, y, \bar{y}) dy \right| \leq C \left(\frac{2R}{N} \right)^{j+k} (j+k)!$$

is smaller than $C e^{-c'N}$ if c is small enough; on the other hand, again

$$\left(\frac{1}{n!} \tilde{\Delta}(x)^n ((u * b)_k J)(y_c) \right)_n$$

is an analytic symbol in a fixed class (with norm smaller than $C(2R)^k k!$), so that, by Proposition 2.2, if c is small enough,

$$N^{d-1-j-k} \lambda_j \sum_{n=0}^{cN-j-k} \frac{1}{n! N^n} \tilde{\Delta}(x)^n ((u * b)_k J)(y_c) < C \left(\frac{2R}{N} \right)^{j+k} (j+k)! \leq C e^{-c'N}.$$

This concludes the proof. \square

5 Spectral estimates at the bottom of a well

5.1 End of the proof of Theorem A

We now prove part 2 of Theorem A. Suppose that $\min(f) = 0$ and that the minimal set of f consists in a finite-number of non-degenerate minimal points P_1, \dots, P_j . At each of these points P_i with $1 \leq i \leq j$, one can construct (see Proposition 4.3) a sequence $v_i(N)$ of $O(e^{-c'N})$ -eigenfunctions of $T_N(f)$. From Proposition 4.1, if μ denotes the Melin value (see Section 3.3 of [2]), then, for every $1 \leq i \leq j$ one has

$$T_N(f)v_i(N) = N^{-1}\mu(P_i)v_i(N) + O(N^{-2}).$$

Moreover, from Theorem B in [2], for $\epsilon > 0$ small, the number of eigenvalues of $T_N(f)$ in the interval $[0, \min_{1 \leq i \leq j} \mu(P_i) + N^{-1}\epsilon]$ is exactly the number of i 's such that P_i minimises μ .

Hence, any normalised sequence of ground states of $T_N(f)$ is $O(Ne^{-c'N}) = O(e^{-(c'-\epsilon)N})$ -close to a linear combination of those $v_i(N)$ whose associated well P_i minimises μ (as the spectral gap is of order N^{-1} and the $v_i(N)$'s are $O(e^{-c'N})$ -eigenvectors). This concludes the proof.

5.2 Tunnelling

The main physical application of Theorem A is the study of the spectral gap for Toeplitz operators which enjoy a local symmetry. Let us formulate a simple version of this result.

Proposition 5.1. *Suppose that $\min(f) = 0$ and that the minimal set of f consists of two non-degenerate critical points P_0 and P_1 . Suppose further that these wells are symmetrical: there exist neighbourhoods U_0 of P_0 and U_1 of P_1 , and a ω -preserving biholomorphism $\sigma : U_0 \mapsto U_1$, such that $\sigma \circ f = f$.*

Then there exists $c > 0$ and $C > 0$ such that, for every $N \geq 1$, the gap between the two first eigenvalues of $T_N(f)$ is smaller than Ce^{-cN} .

Proof. Near P_0 , one can build a sequence of $O(e^{-c'N})$ -eigenvectors as in Proposition 4.3, with $c > 0$; near P_1 one can build another sequence of $O(e^{-c'N})$ -eigenvectors. Since M and f are equivalent near P_0 and near P_1 , the associated sequences of eigenvalues are identical up to $O(e^{-c'N})$, and the approximate eigenvectors are orthogonal with each other since they have disjoint support, so that there are at least two eigenvalues in an exponentially small window near the approximate eigenvalue. As above (see Theorem B in [2]), there are no more than two eigenvalues in the window $[\min Sp(T_N(f)), \min Sp(T_N(f)) + \epsilon N^{-1}]$, for ϵ small; hence the claim. \square

Unfortunately, the actual spectral gap between two symmetrical wells cannot be recovered from Proposition 4.2 or the solution φ of the Hamilton-Jacobi equation.

Proposition 5.2. *Suppose that $\min(f) = 0$ and that the minimal set of f consists of two symmetrical wells. Let λ_0 and λ_1 denote the two first eigenvalues of $T_N(f)$ (with multiplicity), and let*

$$\sigma = \liminf_{N \rightarrow +\infty} \left(-N^{-1} \log(\lambda_1 - \lambda_0) \right).$$

Then σ is unrelated to the best possible constant c' in Proposition 4.2, and unrelated to the solution φ of the Hamilton-Jacobi equation.

Proof. We first let $\chi : [-1, 1] \mapsto \mathbb{R}$ be an even smooth function; we suppose that χ reaches its minimum only at -1 and 1 , with $\chi(-1) = 0$ and $\chi'(-1) > 0$. We consider the associated function f on \mathbb{S}^2 which is the composition of χ with the height function. Then f is invariant under a rotation around the vertical axis, so that $T_N(f)$ is diagonal in the natural spin basis (which consists of the eigenfunctions for the Toeplitz operator associated with the height function). Among this basis, the states which minimise the energy are the coherent states at the North and South poles, respectively; they have the same energy. In this setting the first eigenvalue is degenerate, and shared between two states which localise at one of the two non-degenerate wells.

Let us give a formal solution to the Hamilton-Jacobi equation. In stereographic coordinates near one of the poles, the symbol reads $g(|r|^2) = g(r\bar{r})$ for some $g \in C^\infty(\mathbb{R}, \mathbb{R})$. The expression $g(rs)$ does not make sense if rs is not a real number, but taking $s = 0$ yields $g(r \times 0) = 0$. A formal solution of $\tilde{g}(x, \partial\varphi) = 0$ is then given by $\varphi = 0$. This corresponds indeed to the exponential decay of the exact ground states: $\varphi = 0$ means that the ground state decays as fast as the coherent state (they actually coincide).

In the system above, the formal solution of the Hamilton-Jacobi equation yields the correct decay rate. However, from the point of view of Proposition 4.2, one has $c' = 0$: if χ is not real-analytic near 1 we cannot hope to perform an analytic summation for the sequence λ_i as in Proposition 4.3.

We consider now a smooth perturbation of the function χ above: let $\chi_1 : \mathbb{R} \mapsto [0, 1]$ be a smooth, non-zero function supported on $[0, 1/2]$. If we replace χ with $\chi + \chi_1$ in the previous discussion, we still get a symbol invariant under vertical rotation, which is diagonal in the spin basis. The Hamilton-Jacobi equation has the same formal solution. However, the two candidates for the ground state now have different energies, with an exponentially small but non-zero gap e^{-cN} . Here, c can be made arbitrarily small by moving the support of χ_1 close to 1 . The spectral gap is then not determined by the solution φ of the Hamilton-Jacobi equation. \square

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