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From deterministic to stochastic surrender risk models: impact of correlation crises on economic capital

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Abstract

In this paper we raise the matter of considering a stochastic model of the surrender rate instead of the classical S-shaped deterministic curve (in function of the spread), still used in almost all insurance companies. For extreme scenarios, due to the lack of data, it could be tempting to assume that surrenders are conditionally independent with respect to a S-curve disturbance. However, we explain why this conditional independence between policyholders decisions, which has the advantage to be the simplest assumption, looks particularly maladaptive when the spread increases. Indeed the correlation between policyholders' decisions is most likely to increase in this situation. We suggest and develop a simple model which integrates those phenomena. With stochastic orders it is possible to compare it to the conditional independence approach qualitatively. In a partially internal Solvency II model, we quantify the impact of the correlation phenomenon on a real life portfolio for a global risk management strategy.

Keywords : Risk management, Applied probability, Life insurance, Surrender risk, Correlation risk.

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Surrender risk represents one of the main dangers faced by a life insurance institution. It corresponds to the risk that many policyholders surrender their contract earlier than expected and choose to reinvest their money in another product or in some project. Because fees are charged throughout the duration of the contract, the insurer may not have enough time to charge the fees in the case of early surrenders. Massive early surrenders might also cause important liquidity issues and of course a loss of market share. Many policyholders surrender their life insurance contract every year, mainly to finance a project (building a new house, purchasing a new car, ...) or because the tax incentive delay (8 years in France) or the penalty relief delay has been reached. Insurers are used to forecasting lapse rates, which may be explained by different factors like age, wealth and so on. In Solvency II, internal or partially internal models are being developed by many companies (see Devineau & Loisel (2009) for a description and comparison of some internal models with the standard formula). They have to go from a deterministic model, often based on a so-called S-shaped lapse rate curve to a stochastic model. The S-shaped curve corresponds to the lapse rate expressed as a function of the difference Δr between the interest rate given by the contract and the one that the policyholder could obtain somewhere else in the market. The idea that practitioners have followed is that even if Δr is very small, some policyholders are going to surrender their contract for tax reasons or to fund a personal project, that the lapse rate is increasing in Δr , and that even if Δr is very large, some policyholders are going to stay in the portfolio because they do not really pay attention to the market evolution. The problem with this S-shaped curve is that one has not observed policyholders' behavior in the extreme situation in which Δr is very large.

To build a stochastic model, given this lack of information, one must more rely on thought experiments than on statistical data (which simply does not exist). It could be tempting for internal model designers to use a Gaussian distribution around the value of the lapse rate in the S-shaped, deterministic curve to describe stochastic surrender risk. In this paper, we explain why it may be preferable to use a bi-modal distribution, due to the likely change in the correlation between policyholders' decisions in extreme scenarios. This change of correlation in extreme situation, called correlation crisis in Biard et al. (2008), Loisel (2010) and Loisel (2008) prevents us from applying the classical Gaussian approximation based on the central limit theorem. This theorem holds when decisions of policyholders are independent. Here, this would be the case only given a certain factor that would incorporate the level of information of policyholders and the reputation of the company and of the insurance sector. This factor should be a key element in the internal model to understand the correlation of surrender risks with other risks like market risk or default risk. Correlation crises have followed the sub-prime crisis in both stock markets and credit derivatives market: in many cases, correlation increases in adverse scenarios. For surrender risk, it is likely that an extreme situation in interest rates markets would lead either to massive surrenders, or to almost normal

lapse rates, depending on political declarations and on other factors: for example, one of the first things that leaders of developed countries said at the beginning of the last crisis was: *We guarantee bank deposits and classical savings products*. This leads to anticipate policyholders' behavior more like a 0–1 law than according to a bell-shaped unimodal distribution. In this paper, we propose a basic model that takes into account correlation crises: as Δr increases, correlation between policyholders' decisions increases, and one goes (continuously) from a bell-shaped distribution in the classical regime to a bimodal situation when Δr is large. The model is proposed in Section 1 and interpreted in Section 2. In Section 3, we explain how to compute surrender rate distributions, with closed formulas and with simulations. In Section 4, we make use of stochastic orderings in order to study the impact of correlation on the surrender rate distribution from a qualitative point of view. In Section 5, we quantify this impact on a real-life portfolio for a global risk management strategy based on a Solvency II partial internal model.

1 The model

Assume that when Δr is zero, policyholders behave independently with average lapse rate $\mu(0)$, whereas when Δr is very large (15%, say), the average lapse rate is $1 - \epsilon$ with ϵ very small, and correlation between individual decisions is $1 - \eta$, with η very small. The following model captures these simple features: let I_k be the random variable that takes value 1 if the k^{th} policyholder surrenders her contract, and 0 otherwise. Assume that

$$I_k = J_k I_0 + (1 - J_k) I_k^\perp,$$

where J_k corresponds to the indicator that the k^{th} policyholder follows the market consensus (copycat behavior). The random variable J_k follows a Bernoulli distribution whose parameter p_0 is increasing in Δr , and $I_0, I_1^\perp, I_2^\perp, \dots$ are independent, identically distributed random variables, whose parameter p is also increasing in Δr . This means that the surrender probability increases with Δr , and that the correlation (Kendall's τ or Spearman ρ) between I_k and I_l (for $k \neq l$) is equal to $P(J_k = 1 \mid \Delta r = x)$ when $\Delta r = x$, and that in general (without conditioning) the correlation between I_k and I_l (for $k \neq l$) is equal to

$$\int_0^{+\infty} P(J_k = 1 \mid \Delta r = x) dF_{\Delta r}(x).$$

This is because given that $\Delta r = x$, I_k and I_l (for $k \neq l$) admit a Mardia copula (linear sum of the independent copula and of Fréchet upper bound)¹. For a

¹ Here the copula of I_k and I_l (for $k \neq l$) is not unique as their distributions are not continuous.

portfolio of 20000 policyholders, the Gaussian assumption is not too bad for the case where $\Delta r = 0$. We show here with realistic values of the S-shaped curve how this bell-shaped curve progressively evolves as Δr increases and at some point $\Delta r = x_0$ becomes bi-modal. McNeil et al. (2005) perfectly illustrates the problem of correlation risk and its consequences on tail distribution in a general context.

2 Interpretation of the model

The S-shaped curve of the surrender rate in function of Δr on Figure 1 shows that the less attractive the contract is, the more the policyholder tends to surrender it. Obviously the surrender rate average is quite low in a classical economical regime (Region 1, low Δr on Figure 1), but is significantly increasing as Δr increases. Indeed when interest rates rise, equilibrium premiums decrease and a newly acquired contract probably provides the same coverage at a lower price: the investor acts as the opportunity to exploit higher yields available on the market. On the contrary, if the interest rates drop then the guaranteed credited rate of the contract may be (when it is possible) lowered by the insurer (for financial reasons or to stimulate the policyholder to surrender).

By consequence, Region 1 in Figure 1 illustrates the case corresponding to independent decisions of policyholders (here the correlation tends to 0) whereas Region 2 corresponds to much more correlated behaviors (correlation tends to 1 in this situation) because of a crisis for instance. The underlying idea of the paper is that as long as the economy remains in “good health”, the correlation between policyholders is quasi nonexistent and thus the surrender rate (independent individual decisions) can be modeled thanks to the Gaussian distribution whose mean and standard deviation are those observed. Indeed the suitable distribution in Region 1 is the classical Normal distribution represented in Figure 2.

On the contrary the sharp rise of the surrender rate at some level Δr in Fig-

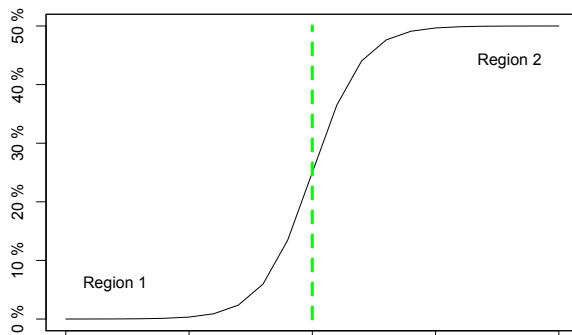


Figure 1. Surrender rate versus Δr .

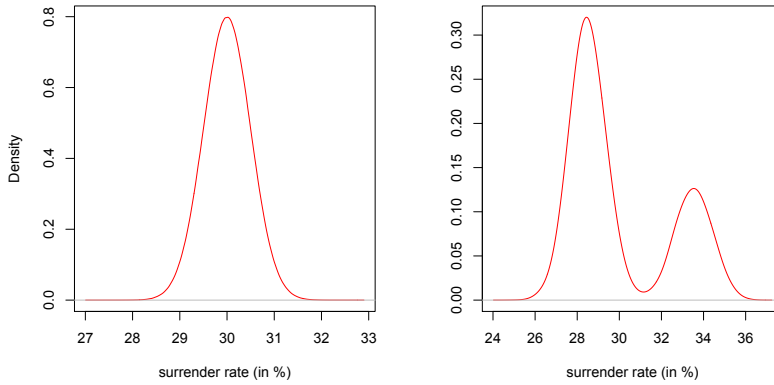


Figure 2. On the left, the density of a Gaussian distribution and on the right a Bi-modal density (mean equals 30).

ure 1, followed by a flat plateau which is the maximum reachable surrender rate (this bound is often suggested by an expert since we consider that we have never observed it), reflects that economical conditions are deteriorating. The crucial point is to realize that in such a situation the assumption of independent behaviors can become strongly erroneous: the correlation between policyholders' decisions makes the surrender rate distribution change. This is the consequence of two different behaviors or scenarios, either almost all policyholders surrender their contract or they do not. The more suitable distribution to explain it is the so-called *Bi-modal* distribution illustrated in Figure 2. The main difference with the Gaussian model is that the average surrender rate results from two peaks of the density.

Note that irrational behavior of policyholders could also lead to correlation crises between their decisions even if Δr is small. We shall see that this situation is the one that has the strongest impact on economic capital needs. Irrational behavior must be understood here as atypical with respect to the historical records of the insurance company, following some rumor or some recommendation from journalists or brokers. From a financial perspective, irrational behavior corresponds to the one of policyholders who do not surrender their contract even if it would pay to surrender it. As life insurance contracts feature more and more complex embedded options or guarantees, and as tax incentives are at stake, it might be difficult for the policyholder to use them optimally. However, it can be noticed in the US life insurance market (in which many variable annuities are present) that policyholders seem to become more and more rational. This uncertainty on future policyholder's rationality is somehow partly captured by our correlation crisis model.

3 Distribution of surrender rates

3.1 Combinatorial approach

Consider a portfolio of $n \geq 2$ policyholders. Denote by

$$N = \sum_{k=1}^n J_k$$

be the number of copycat policyholders, and by

$$M = \sum_{k=1}^n I_k$$

the number of policyholders who surrender their contracts. Recall that

$$I_k = J_k I_0 + (1 - J_k) I_k^\perp,$$

where J_k corresponds to the indicator that the k^{th} policyholder follows the market consensus (copycat behavior), where the random variable J_k follows a Bernoulli distribution whose parameter p_0 , and where $I_0, I_1^\perp, I_2^\perp, \dots$ are independent, identically distributed random variables with parameter p (and independent from the $(J_l)_{l \geq 1}$). If copycat behavior is surrender (when $I_0 = 1$), then for M to be equal to some integer $k \in [0, n]$, the number N of copycat policyholders has to be smaller or equal to k , otherwise one would have $M \geq N > k$. Similarly, if copycat behavior corresponds to no surrender ($I_0 = 0$), then for M to be equal to some integer $k \in [0, n]$, the number N of copycat policyholders has to be smaller or equal to $n - k$, otherwise one would have $M \leq n - N < n - (n - k) = k$. From the total probability formula, for $0 \leq k \leq n$, one has

$$\begin{aligned} P(M = k) &= P(M = k \mid I_0 = 0)P(I_0 = 0) + P(M = k \mid I_0 = 1)P(I_0 = 1) \\ &= \sum_{i=0}^k P(M = k \mid I_0 = 1, N = i) P(I_0 = 1, N = i) \\ &\quad + \sum_{j=0}^{n-k} P(M = k \mid I_0 = 0, N = j) P(I_0 = 0, N = j). \end{aligned}$$

From the mutual independence of the $(J_k)_{k \geq 1}$ and $(I_l^\perp)_{l \geq 1}$, for $0 \leq k \leq n$ we have

$$P(M = k) = p \sum_{i=0}^k a_{i,k} + (1 - p) \sum_{j=0}^{n-k} b_{j,k},$$

where for $0 \leq i \leq k$,

$$a_{i,k} = C_n^i p_0^i (1 - p_0)^{n-i} C_{n-i}^{k-i} p^{k-i} (1 - p)^{n-k},$$

and for $0 \leq j \leq n - k$ we have

$$b_{j,k} = C_n^j p_0^j (1 - p_0)^{n-j} C_{n-j}^k p^k (1 - p)^{n-j-k}.$$

Note that for fixed k , the $a_{i,k}$, $0 \leq i \leq k$ and the $b_{j,k}$, $0 \leq j \leq n - k$ can be computed than to the following recursive formulas: for $0 \leq i \leq k$, we have

$$\frac{a_{i+1,k}}{a_{i,k}} = \frac{C_n^{i+1}}{C_n^i} \frac{p_0}{p(1-p_0)} \frac{C_{n-i-1}^{k-i-1}}{C_{n-i}^{k-i}} = \frac{k-i}{i+1} \frac{p_0}{p(1-p_0)}$$

and for $0 \leq j \leq n - k$, we have

$$\frac{b_{j+1,k}}{b_{j,k}} = \frac{C_n^{j+1}}{C_n^j} \frac{p_0}{(1-p)(1-p_0)} \frac{C_{n-j-1}^k}{C_{n-j}^k} = \frac{n-j-k}{j+1} \frac{p_0}{(1-p)(1-p_0)}.$$

Note that as

$$a_0 = b_0 = (1 - p_0)^n C_n^k p^k (1 - p)^{n-k}$$

is generally quite small, it is often preferable to start by a_{i_0} and b_{j_0} such that a_{i_0} and b_{j_0} are large enough in order to minimize round-off errors. Some efficient algorithms and associated accuracy to make this kind of computations can be found in Viquerat (2010).

3.2 Simulation approach

As surrender risk is only one of the numerous risks considered in a complex internal risk model, the simulation approach is almost always necessary in risk management applications. The number of simulations is of course a key factor to get an accurate and consistent approximation of the surrender rate distribution whatever the socio-economic context (see for example Vidyasagar (2001) for a comprehensive analysis of randomized algorithms and the VC dimension approach to determine this number). The number of policyholders in the portfolio is somewhat important because the range of surrender rate values narrows (less dispersion), although it does not really affect the shape of the distribution. Figure 3 corroborates these observations and that is why hereafter one sets the number of simulations to 1000 000 and the number of policyholders to 10 000. Of course the main effect is the one of the individual surrender probability (p), because it directly impacts the mean of the distribution and breeds a worse risk profile.

Let us now focus on the consequences of varying the correlation, the core of this paper. The correlation parameter represented by the probability p_0 to make the same decision (surrender or not) plays the main role: Figure 4 shows that increasing the correlation considerably reshapes the distribution of the

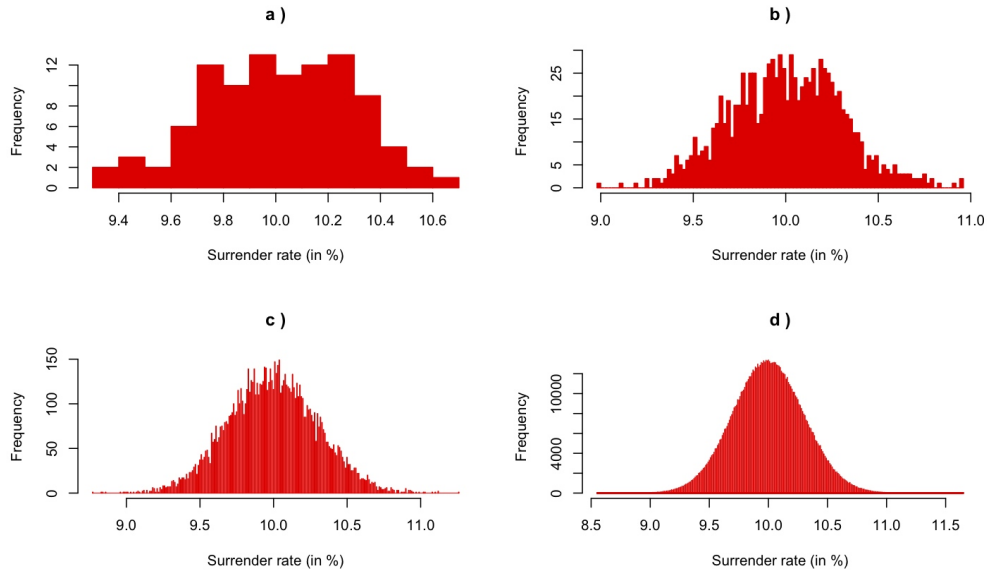


Figure 3. Effect of the number of simulations on the surrender rate distribution: a) 100, b) 1 000, c) 10 000 and d) 1 000 000. No copycat behavior, individual surrender probability equals 10%, 10 000 policyholders in the portfolio.

surrender rate.

Actually both the surrender probability and the correlation increase during an economic crisis, and this is no good news for the insurer who is faced to a much more frightening situation where the bumps become well-balanced (high risk in both economic situations) and the modes move away (see Figures 4 and 8).

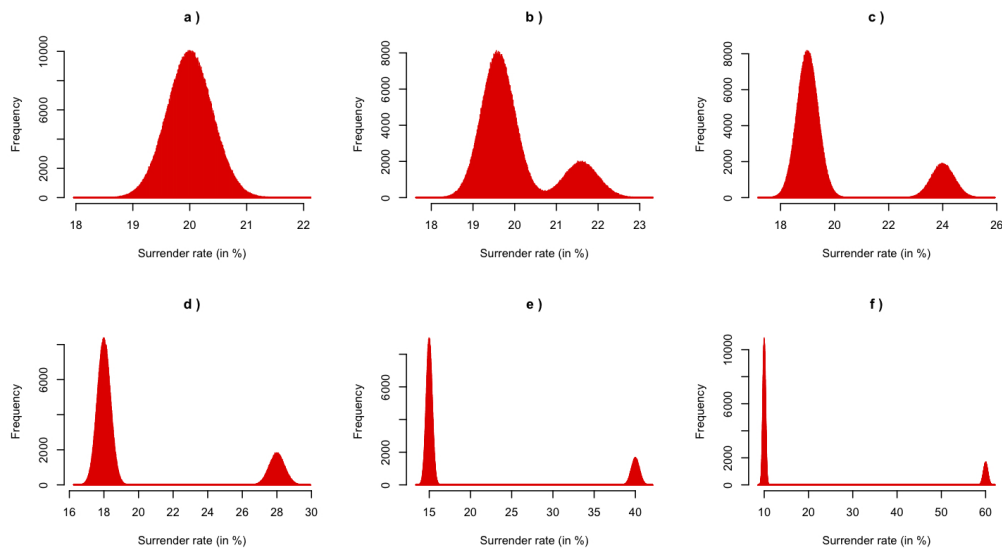


Figure 4. Reshaping the surrender rate distribution (Gaussian to Bi-modal). From top left to bottom right, copycat behavior probability equals: a) 0%, b) 2%, c) 5%, d) 10%, e) 25% and f) 50%. Individual surrender probability equals 20%, 1 000 000 simulations and 10 000 policyholders.

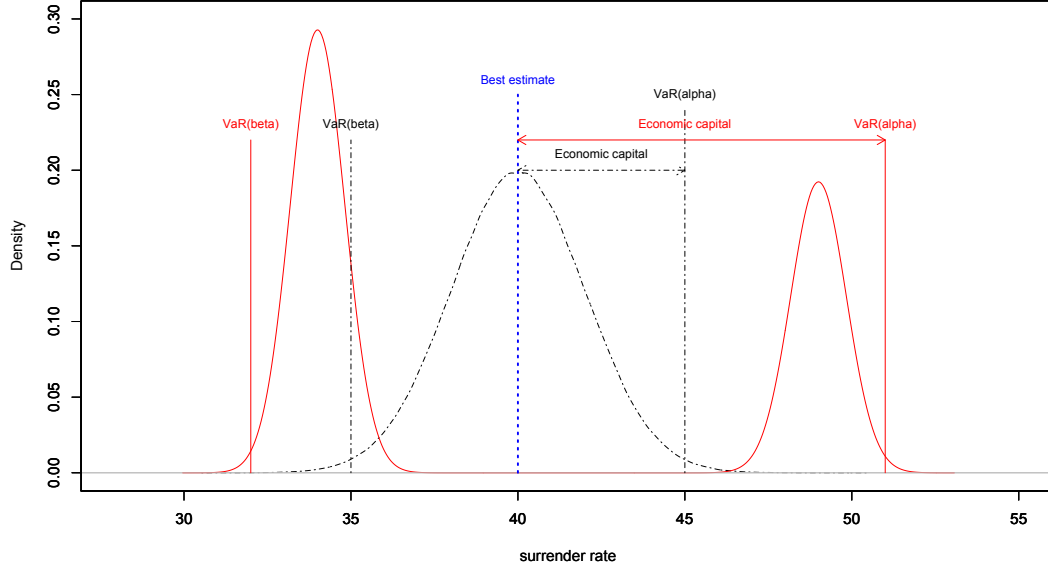


Figure 5. Density of surrender rate for independent behaviors (dotted, black curve) and for correlated behaviors (solid, red curve): the associated economic capital (which depends on the difference between the VaR_α and the best estimate) increases in the correlation.

Figures 4 and 5 confirm the theory developed in the previous section. We may notice that for a given Δr (and thus for a given p_0 in theory), the bigger the correlation parameter p_0 is, the more the surrender density shape becomes bi-modal. The insurer is interested in quantifying the difference between these distributions in terms of behavior risk. Risk measures will help us to get indicators of the gap between these distributions. More precisely, a quantity of interest for the insurer is the well-known *Value-at-Risk* or VaR , defined for random variable X and threshold α as follows:

$$VaR_\alpha(X) = \inf\{x \in X, F_X(x) \geq \alpha\}.$$

In our application, the random variable X represents the surrender rate. This means that the insurer expects to experiment a surrender rate lower than VaR with $\alpha\%$ confidence. In life insurance, the classical threshold is set to $\alpha = 99.5\%$.

Sometimes the insurer also fears that the surrender rate suddenly drops, because the size of its portfolio may become too large with respect to its capital constraints (since death benefit guarantees for instance have been priced with some risk exposure predictions based on experienced surrenders), or because interest rate levels might be unfavorable to the insurer. In this case, the configurations defined above have to be adapted, as illustrated on Figure 5 where the risk measures that are interesting for the insurer are VaR_α (right-hand side, risk of massive surrenders) and VaR_β (left-hand side, risk that policyholders surrender their contract much more seldom than expected).

4 Qualitative comparison with stochastic orders

Denote by $M_{(p,p_0)}$ the number of policyholders who surrender their contract in the above model when $P(J_1 = 1) = p_0$ and when $P(I_1 = 1) = p$. Let us examine how the values of p and p_0 affect the distribution of the conditional surrender rate. We use the concept of s -convex stochastic ordering (see Lefèvre & Utev (1996) and Denuit et al. (1998)). Given two random variables Y and Z , for any $s = 1, 2, \dots$, we have

$$X \leq_{s-cx}^{\mathcal{D}} Y \text{ if } E[\phi(Y)] \leq E[\phi(Z)] \text{ for all } s\text{-convex function } \phi : \mathcal{D} \rightarrow \mathbb{R}, \quad (1)$$

i.e. in short, for any function ϕ on \mathcal{D} whose s -th derivative exists and satisfies $\phi^{(s)} \geq 0$. Note that the first $s - 1$ moments of Y and Z are then necessarily equal. The order $\leq_{1-cx}^{\mathcal{D}}$ is just the usual stochastic order \leq_1 , $\leq_{2-cx}^{\mathcal{D}}$ is the usual convex order \leq_2 (which implies in particular $Var(Y) \leq Var(Z)$). In addition, X is said to be smaller than Y in the increasing convex order (denoted by \leq_{icx}) if

$$E(f(X)) \leq E(f(Y))$$

for every increasing convex function such that the expectation is defined.

Proposition 1 *When the correlation parameter is fixed, the number of surrenders is stochastically increasing in p : for fixed $p_0 \in (0, 1)$, if $p < p'$ then we have*

$$M_{(p,p_0)} \leq_1 M_{(p',p_0)}.$$

Proof: The result follows from elementary, classical results about stochastic orderings of Bernoulli and Binomial distributions. \square

Proposition 2 *When the individual surrender probability p is fixed, the correlation parameter induces a 2-convex ordering of numbers of surrenders: for fixed $p \in (0, 1)$, if $p_0 < p'_0$ then we have*

$$M_{(p,p_0)} \leq_2 M_{(p,p'_0)}.$$

Proof: Given that the number of copycat policyholders N is equal to k , the total number of surrenders reads

$$M_{(p,k)} = k.I_0 + 0.I_1^\perp + 0.I_2^\perp + \dots + 0.I_k^\perp + 1.I_{k+1}^\perp + \dots + 1.I_n^\perp.$$

Given that $N = k'$ with $k \leq k'$, we have

$$M_{(p,k')} = k'.I_0 + 0.I_1^\perp + \dots + 0.I_k^\perp + \dots + 0.I_{k'}^\perp + 1.I_{k'+1}^\perp + \dots + 1.I_n^\perp.$$

The two random variables $M_{(p,k)}$ and $M_{(p,k')}$ may be compared with respect to the majorization order (see e.g. Marshall & Olkin (1979)). For any vector Z ,

denote by $Z^\downarrow = (z_1^\downarrow, \dots, z_K^\downarrow)$ the vector with the same components, but sorted in decreasing order. Given two vectors $Y = (y_1, \dots, y_K)$ and $Z = (z_1, \dots, z_K)$ of size $K \geq 1$ such that

$$\sum_{i=1}^K y_i = \sum_{i=1}^K z_i,$$

recall that Z is said to majorize Y if for all $j \leq K$,

$$\sum_{i=1}^j y_i^\downarrow = \sum_{i=1}^j z_i^\downarrow.$$

From Marshall & Olkin (1979), if the vector $\alpha = (\alpha_0, \dots, \alpha_n)$ is lower than the vector $\beta = (\beta_0, \dots, \beta_n)$ in the Majorization partial order, and if the X_i are independent and identically distributed, then we get the following convex ordering:

$$\sum_i \alpha_i X_i \leq_2 \sum_i \beta_i X_i.$$

Here we set $X_i = I_i^\perp$ and

$$(\alpha_0, \dots, \alpha_n) = (k, \underbrace{0, \dots, 0}_{k \text{ times}}, 1, \dots, 1) \text{ and } (\beta_0, \dots, \beta_n) = (k', \underbrace{0, \dots, 0}_{k' \text{ times}}, 1, \dots, 1).$$

For $k \leq k'$, the vector $(\beta_0, \dots, \beta_n)$ clearly majorizes the vector $(\alpha_0, \dots, \alpha_n)$. Moreover, the random variable $(N \sim \text{Bin}(n, p_0))$ is stochastically increasing in p_0 . So we can conclude that for $p_0 \leq p'_0$,

$$M_{(p, p_0)} \leq_2 M_{(p, p'_0)},$$

where $M_{(p, p_0)}$ denotes the number of policyholders who surrenders when the copycat probability is p_0 and when the individual surrender probability is p . \square

Proposition 3 *In the model where both p and p_0 increase with Δr , Δr induces an increasing-convex ordering of numbers of surrenders: denote by M (resp. M') the number of surrenders when $\Delta r = x$ (resp. $\Delta r = x'$). If $x < x'$ then we have*

$$M \leq_{icx} M'.$$

Proof: We use the same arguments as in the proof of Propositions 1 and 2. A combination of these arguments yields to this proposition because when Δr increases, both p and p_0 increase. \square

Proposition 1 implies that the expected value, the Value-at-Risk of any level $\alpha \in (0, 1)$ and the stop-loss premiums $E[(M - m)_+]$ for $0 \leq m \leq n$ are increasing in p and in Δr . Proposition 2 implies that the variance and the

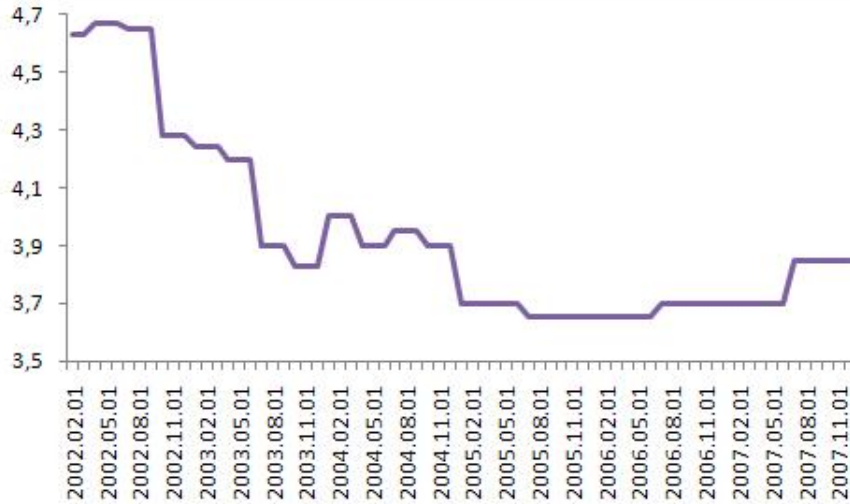


Figure 6. Evolution of the difference between the credited rate and the risk-free interest rate on the market (universal).

stop-loss premiums $E[(M - m)_+]$ for $0 \leq m \leq n$ are increasing in p_0 (when p is fixed). It also shows that if $p_0 < p'_0$, there exists some level $\alpha_0 \in (0, 1)$ such that for $\alpha > \alpha_0$,

$$VaR_\alpha(M_{(p,p_0)}) < VaR_\alpha(M_{(p,p'_0)})$$

because the Karlin-Novikov cut criterion says in that case that the cumulative distribution functions of $M_{(p,p_0)}$ and $M_{(p,p'_0)}$ cross only once.

It is not surprising that, in this model, increasing the correlation parameter and/or the marginal surrender probability leads to a higher measure of risk in general. But the purpose of this paper is also to determine the importance of the impact of correlation on economic capital requirements and on the so-called account value introduced in the next section, for a real-world life insurance portfolio.

5 Application on a real life insurance portfolio

This section does not aim at giving accurate results that are valid for all insurance portfolios, but it illustrates how to implement in practical terms all the theory developed previously. To quantify exactly the impact of correlated behaviors on final economic capital of the company requires a good calibration of all the parameters used in the following, and thus a rather complete and reliable database. This is why all statements and computations of this section have to be moderated.

Products in the scope of this study are savings products extracted from the Spanish reinsurance portfolio of a global life reinsurer.

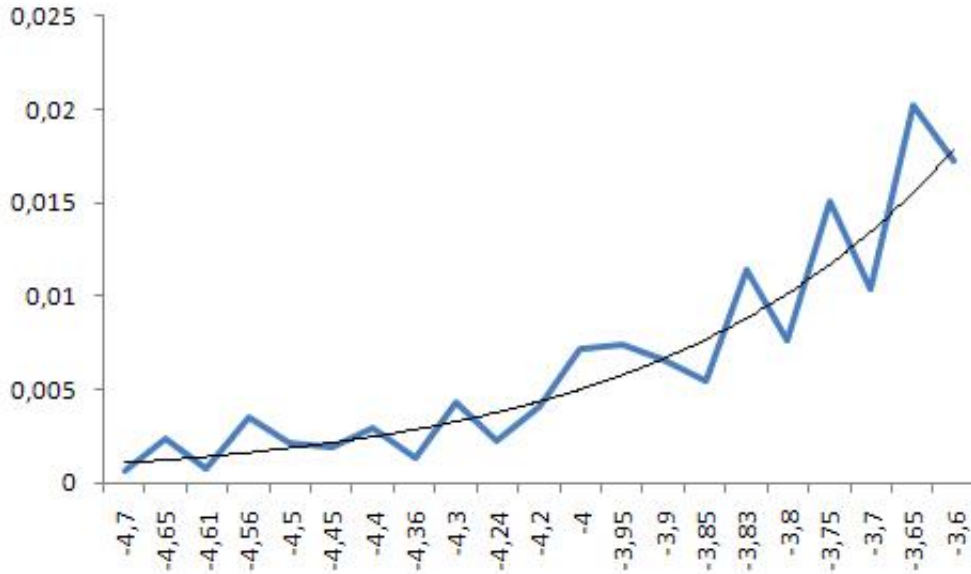


Figure 7. Monthly surrender rate versus Δr , the difference between the risk-free rate and the credited rate.

Monthly data cover the period ranging from February 2002 to December 2007. The number of policyholders in the portfolio varies from 291 at the beginning of the observation period to 25766 in July 2006. On average, there are 17657 policyholders in the portfolio each month. Practitioners consider that Δr should in principle stand for the credited rate of the competitors on the same business line minus the one of the contract (Milhaud et al. (2010)). This information is difficult to obtain. To overcome this problem, some experts agree to say that the credited rate of the competitors somewhat equals to the risk-free rate plus a constant c , so that

$$\Delta r = (\text{risk-free rate} + c) - \text{credited rate}.$$

In our application, we set $c = 0$ and this is the way we define Δr . As you can see on Figure 6, the credited rate of the contract is greater than the risk-free rate during all the period, which yields to $\Delta r < 0$. An exponential smoothing of the surrender rate in function of Δr on this (proprietary) data leads Figure 7: the smaller this difference, the higher the surrender rate. This is quite normal since it means that when the credited rate decreases, Δr tends to 0 and the contract becomes less advantageous. The exponential smoothing has no importance except that it is useful to recognize the S-shaped curve.

As discussed in the first section, one only observes market conditions that correspond to Region 1 in Figure 1. Here we assume that the maximum reachable monthly surrender rate (flat plateau in Region 2 of Figure 1) corresponding to the “expert opinion” is equal to 3.5% a month. Notice that this is another source of error if the expert is wrong. Of course, the calibration is arbitrary here but we are mainly interested in the impact of correlation between policyholders. That is why we take this model, used by some insurance companies,

as a realistic illustrative example. To build a better model to forecast surrender rates is beyond the scope of this paper.

Denote by Y the random variable that represents the surrender rate. The empirical mean of the surrender rate in **Region 1**, denoted by \tilde{Y}_{n_1} , equals

$$\tilde{Y}_{n_1} = \frac{1}{n_1} \sum_{i=1}^{n_1} Y_{i,1}, \text{ where } n_1 \text{ is the number of observations } Y_{i,1} \text{ in Region 1.}$$

The empirical standard deviation, denoted by $\tilde{\sigma}_{n_1}$, is given as usual by the following unbiased estimator:

$$\tilde{\sigma}_{n_1} = \sqrt{\frac{1}{(n_1 - 1)} \sum_{i=1}^{n_1} (Y_{i,1} - \tilde{Y}_{n_1})^2}.$$

Here the numerical application gives $\tilde{Y}_{n_1} = 6.73 \times 10^{-3}$ and $\tilde{\sigma}_{n_1} = 4.28 \times 10^{-3}$.

The variability coefficient, denoted by $\tilde{\nu}_{n_1}$, is defined by $\tilde{\nu}_{n_1} = \frac{\tilde{\sigma}_{n_1}}{\tilde{Y}_{n_1}}$.

In Region 1, $\tilde{\nu}_{n_1}$ equals 63.5%, which reflects overdispersion. Anyway, these statistics are clearly not tolerable as they are based on a very low number of observations but this dispersion also appears because of cohort effects conjugated to tax constraints. The effect of the date with time dependence can also play a role here.

Because data are unavailable for very large surrender rates, we assume that

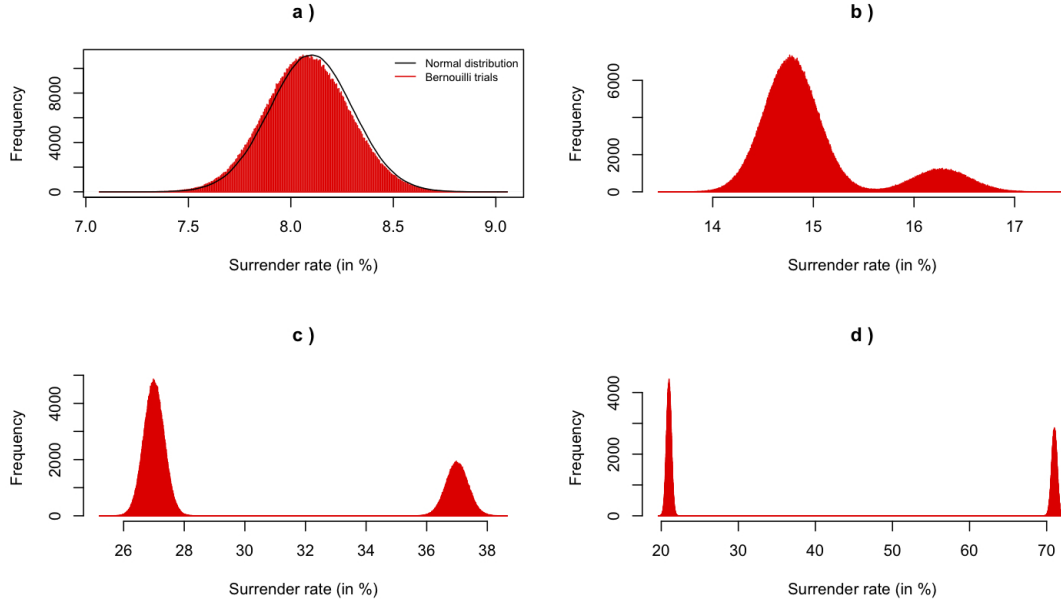


Figure 8. Evolution of surrender rate distribution depending on the economic context (1 000 000 simulations, 17657 policyholders). Individual surrender probabilities and copycat behavior probabilities: a) 8.08% and 0% (theoretical Normal distribution in black), b) 15% and 1.5%, c) 30% and 10% and d) 42% and 50%.

the variability coefficient remains constant whatever the surrender rate average. This assumption enables us to compute an approximation of the standard deviation in **Region 2**, which is $\tilde{\sigma}_{n_2} = \frac{\tilde{Y}_{n_2}}{\tilde{Y}_{n_1}} \tilde{\sigma}_{n_1} \simeq 0.022$. This choice of value is clearly arbitrary, but it would be difficult to come up with a more objective choice.

Thus we simulate in Region 1 Bernoulli trials with parameter p_1 equal to \tilde{Y}_{n_1} and check that the distribution of the surrender rate is close to a Normal distribution with mean \tilde{Y}_{n_1} and standard deviation $\tilde{\sigma}_{n_1}$. Finally, the k^{th} policyholder makes the decision $I_k \sim \text{Bernoulli}(p_1)$, and the Bernoulli indicator J_k that she follows the market consensus J_k has parameter $p_0 = 0$ (independent decisions because we are in Region 1).

In Region 2, $I_k \sim \text{Bernoulli}(p_2)$ (with $p_2 = \tilde{Y}_{n_2} = 0.035$) and the existing correlation impacts the parameter p_0 of J_k . Indeed one sets $J_k \sim \text{Bernoulli}(p_0)$, with $p_0 = 0.5$. The number of simulations in all this section is set to 1000 000 and the number of policyholders is set to 17657. Figure 8 shows the results of these simulations: on the top left, the simulation of behaviors confirms the Gaussian approximation when considering independent Bernoulli trials. The Normal distribution $\mathcal{N}(m = \tilde{Y}_{n_1}, \sigma = \tilde{\sigma}_{n_1})$ fits well and the calibration of the standard deviation does not look so bad. On the bottom right, it is clear that the Gaussian approximation is not at all appropriated to model policyholders' behavior in that case! Other graphs highlight the transition between these two caricatural situations.

Remark: we have converted monthly surrender rates to yearly ones because VaR is usually estimated for a 1-year period in insurance. Time dependence could also be considered and may have a strong impact on the overall surrender risk faced by the company. Nevertheless we focus here on a 1-period framework and keep the multi-period extension for further research. This yields to $\tilde{Y}_{n_1} \simeq 0.0808 = 8.08\%$, $\tilde{\sigma}_{n_1} \simeq 0.05$, $\tilde{Y}_{n_2} \simeq 0.42 = 42\%$ and $\tilde{\sigma}_{n_2} \simeq 0.264$.

The impact of the economical context on VaR is quite impressive: for instance if the correlation increases from 0% to 50% in a "soft" context, the $VaR_{99.5\%}$ increases by 514% (difference denoted by $\Delta VaR_{99.5\%}$). It means that if the insurer wants to be covered against a very high surrender rate and takes correlation crises into account, the cost of capital might be about 510% higher than expected !

Table 1 summarizes it for different economical contexts, levels of correlation and levels of VaR . It also gives the results on another risk measure, the *Tail-Value-at-Risk* (or $TVaR$) which takes into account the tail of the distribution. The $TVaR$ is defined by

$$TVaR_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 VaR_\beta(X) d\beta$$

Eco. context	p_0	$VaR_{95\%}$	$VaR_{99.5\%}$	$\Delta VaR_{99.5\%}$	$TVaR_{90\%}$	$TVaR_{99\%}$	$\Delta TVaR_{99\%}$
S	0	8.823	8.993	0 %	8.844	8.996	0 %
	0.01	9.565	9.746	+8.4 %	9.576	9.758	+8.5 %
	0.02	10.471	10.658	+18.5 %	10.481	10.686	+18.8 %
	0.05	13.297	13.501	+50.1 %	13.314	13.514	+50.2 %
	0.15	22.659	22.886	+154.5 %	22.666	22.867	+154.2 %
	0.3	36.557	36.874	+310 %	36.577	36.909	+310.3 %
	0.5	54.992	55.218	+514 %	55.01	55.233	+514 %
M	0	21.028	21.204	0 %	21.046	21.22	0 %
	0.01	21.668	21.945	+3.5 %	21.688	21.939	+3.4 %
	0.02	22.416	22.693	+7 %	22.433	22.713	+7 %
	0.05	24.879	25.219	+18.9 %	24.905	25.214	+18.8 %
	0.15	32.933	33.301	+57 %	32.96	33.33	+57.1 %
	0.3	45.013	45.341	+113.8 %	45.038	45.368	+113.8 %
	0.5	61.018	61.301	+189.1 %	61.042	61.348	+189.1 %
H	0	43.201	43.563	0 %	43.226	43.513	0 %
	0.01	43.693	43.977	+1 %	43.719	44.054	+1.2 %
	0.02	44.231	44.503	+2.2 %	44.245	44.522	+2.3 %
	0.05	45.959	46.378	+6.5 %	45.983	46.374	+6.6 %
	0.15	51.735	52.053	+19.5 %	51.748	52.071	+19.7 %
	0.3	60.435	60.735	+39.4 %	60.449	60.768	+39.7 %
	0.5	71.965	72.237	+65.8 %	71.977	72.265	+66.1 %

Table 1

Estimates of *Value-at-Risk* (VaR) and *Tail-Value-at-Risk* (TVaR) for the surrender rate with different correlations. S denotes a “soft” economic context ($p = 8.08\%$), M a “Medium” context ($p = 20\%$) and H is a “Hard” economic context ($p = 42\%$). p_0 is the correlation value, the number of simulations is 100 000.

and corresponds in the continuous case to the conditional expectation given that the associated VaR level is overflowed. One can notice that results on the increase of *TVaR* by introducing correlation are very similar to those of *VaR*.

Detailed analysis of the VaR deviations: Let us focus on the 99.5%-*VaR* because this is the *VaR* of interest in Solvency II. Figure 9 shows how *VaR* is affected by an increase in correlation between policyholders’ decisions. For a given individual probability to surrender, 1% say, a shift in correlation from 0 to 1% increases $VaR_{99.5\%}$ by 30 to 50%.

Notice that large **positive** deviations are mainly concentrated for small variations in correlation when considering low surrender propensity. This result is quite interesting and we decided to define classes in terms of *sensitivity* (with

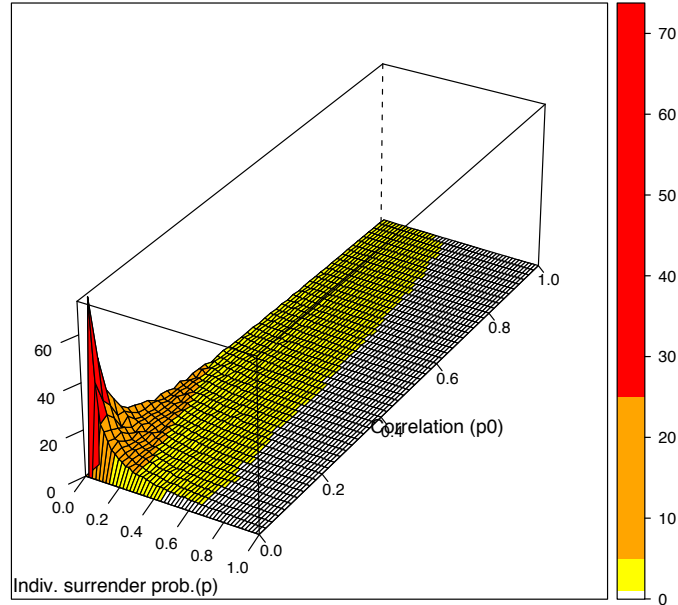


Figure 9. Relative deviation (in %) of the VaR versus p_0 and p .

respect to correlation):

- *very sensitive configuration* (red area on Figure 9): $p \in]0, 0.05]$ and $p_0 \in]0, 0.1]$;
- *sensitive configuration* (orange area): $p \in]0, 0.05]$ and $p_0 \in [0.1, 0.4]$, or $p \in]0.05, 0.2]$ and $p_0 \in]0, 0.3]$;
- *low sensitivity configuration* (yellow area): other situations.

In the *very sensitive configuration*, the $VaR_{99.5\%}$ can increase by up to 70%! In the *sensitive configuration*, the insurer may face less adverse movements but is faced to a potential increase by 5 to 25 % of $VaR_{99.5\%}$. Finally, the *low sensitivity configuration* allows the insurer to be confident because the VaR that she has chosen is often high enough.

6 Gap between standard assumptions and a more realistic model

How do the surrender assumptions affect the funds reserves ? We focus in this section on the consequences on insurer reserves when considering erroneous surrender rate distribution, where policyholders' behaviors are considered to be quite homogeneous (around an "average" behavior) and independent, instead of introducing correlation. In some insurance groups, each entity may define how to integrate the surrender assumptions and modeling into the pricing of the product, and the estimations of its future profitability. Sometimes several definitions are quite different even within a single entity,

Scenario	BE	EC / AV($VaR_{99.5\%}^{Normal}$)	Correlation	EC ($VaR_{99.5\%}^{Bi-modal}$)	ΔEC	ΔAV
Soft ($p = 8.08\%$)	(0)	0.84% / 1 091 029 (0)	$p_0 = 0.05$	5.45%	4.61%	5506.7
			$p_0 = 0.15$	14.85%	14.01%	14857.1
			$p_0 = 0.3$	28.7%	27.86%	28743.9
			$p_0 = 0.5$	47.26%	46.42%	47189.9
Medium ($p = 20\%$)	(0)	1.18% / 1 078 824 (0)	$p_0 = 0.05$	5.06%	3.88%	5179.8
			$p_0 = 0.15$	13.15%	11.97%	13182.3
			$p_0 = 0.3$	25.35%	24.17%	25290.8
			$p_0 = 0.5$	41.3%	40.12%	41233.5
Hard ($p = 42\%$)	(0)	1.48% / 1 056 459 (0)	$p_0 = 0.05$	4.28%	2.8%	4253.6
			$p_0 = 0.15$	10.13%	8.65%	10036
			$p_0 = 0.3$	18.76%	17.28%	18769.1
			$p_0 = 0.5$	30.21%	28.73%	30260.3

Table 2

Estimations of differences of *economic capital* (in terms of surrender rate) and account value with different correlations in various scenarios.

depending on the product. To fix ideas, we shall assume that the so-called Account Value (AV) is adjusted with withdrawals so that the funds returns are re-balanced. A quite classical way for insurers to manage surrenders is to set base surrenders and then to adjust it dynamically. The point is that this adjustment is still based on independent behaviors of policyholders, which is precisely what we suggest to avoid in this paper. Consider the simplified but realistic model of adjusted AV:

$$AV_{adj}(t) = AV_{adj}(t-1) \times [1 + \% \text{ periodical benefit} \times (1 - \text{surrender rate})]$$

It is now possible to estimate the sensibility of the Account Value in terms of behavior risk by using Monte Carlo simulations of various scenarios, corresponding to surrender shocks. This is precisely the recommendations of Solvency II which stimulates the development of internal risk models and the use of multiple scenarios to assess robust risk margins, the so-called *economic capital* (EC), which is the difference between the *VaR* and the *best estimate* (BE) (the BE is our expectation). The bi-modal distribution of surrender rate implies significant changes in these estimations and so in reserving: first the account value is estimated with best estimate assumptions and then with the $VaR_{99.5\%}$ in the case of Normal distribution (independent behaviors) and Bi-modal distribution (correlated behaviors). The difference of estimated economic capitals represents the gap of reserves for the insurer in terms of surrender rate.

Numerical application The model used to introduce dependence between policyholders' behaviors is described in first section. Hereafter, assume that:

Portfolio size	BE	EC / AV ($VaR_{99.5\%}^{Normal}$)	Correlation	EC ($VaR_{99.5\%}^{Bi-modal}$)	ΔEC	ΔAV
Little:			$p_0 = 0.05$	6.26%	4.5%	6280
5000	8.1%	1.76% / 1 090 140	$p_0 = 0.2$	20.42%	18.66%	20440
policyholders	($p_0 = 0$)	($p_0 = 0$)	$p_0 = 0.5$	48.54%	46.78%	48560
Medium:			$p_0 = 0.05$	5.1%	4.59%	5102
50 000	8.08%	0.51% / 1 091 408	$p_0 = 0.2$	19.01%	18.5%	19012
policyholders	($p_0 = 0$)	($p_0 = 0$)	$p_0 = 0.5$	46.63%	46.12%	46634
Big:			$p_0 = 0.05$	4.73%	4.59%	4731.2
500 000	8.08%	0.1426% / 1 091 777	$p_0 = 0.2$	18.56%	18.41%	18565
policyholders	($p_0 = 0$)	($p_0 = 0$)	$p_0 = 0.5$	46.16%	46.01%	46158.6

Table 3

Impact of portfolio size on account value in a **soft** context (100 000 simulations).

- the risk term is one year ;
- the economical and financial conditions are getting worse ;
- today, the account value equals 1 000 000 US\$;
- the yearly benefit is 10% ;
- the VaR , EC and AV are computed for the coming year.

Those parameters were chosen to reflect reality without betraying confidentiality. The portfolio size is still 17657 and the number of simulations is set to 100 000. Table 2 highlights the potential impact of surrender modeling on the estimations of reserves on the consider products. It summarizes the differences of reserves in \$ for different economical contexts and levels of correlation. The results tell us that in a soft context, considering independent behaviors instead of 50% correlated behaviors can cause the insurer to experiment an unexpected loss up to 47 190\$ of the account value at the end of the year (4.72% of the initial wealth)! The absolute difference between the mean expected surrender rate and the stressed one is 47.26%!

Besides the initial account value has been set to 1 000 000\$, which is very low as compared to usual equity capitals of insurance companies. Guess what would be the actual loss of the insurer in such a casual situation...

The size of the company is also a key-factor: let us extract the part of Table 2 concerning the “soft” context and study the impact of the number of policyholders in the portfolio. Initially, there were 17657 policyholders. Some huge insurance companies may think that their size prevents them from experimenting such scenario because of mutualization. The analysis of Table 3 demonstrates that the number of policyholders does not have a strong influence on the computation of the risk margin. To reserve enough money to cover the correlation risk of surrender behaviors is very important in predicting capital needs: underestimating this risk could make shareholders dissatisfied because of unexpected important margin calls. Because undiversifiable risks are present, increasing the size of the portfolio would not reduce the impact of a correlation crisis on the considered quantities.

7 Conclusion

We have shown in this article that the impact of the surrender rate distribution is major on both predicting the economic capital needs and the reserves of the company. This distribution results from a different point of view on behaviors' modeling: from conditionally independent behaviors, we move to correlated policyholders' decisions that better reflect the reality in our opinion. For further research, it would be essential to explore the path-dependent aspect or memory properties of the surrender rate trajectory (the present value depends on the previous ones), which should be meaningful to forecast it and understand its dynamics. Of course it requires to have empirical data, which is not always the case in practice.

To look into the question of the feedback impact of massive surrenders on interest rates and inflation rates could also be interesting as it could create some kind of vicious circle. Another source of risk for the insurer are options embedded in the contracts that enable to change the underlying investment type: for example the policyholder can move his money from Unit-Link to Euro products (Loisel (2010)) lately.

To sum up, the surrender risk is extremely complicated to study because it depends on many factors: policyholders' characteristics, personal desires and needs, contract features and time elapsed, economical and financial context (liquidity issues), social and cultural aspects, but also regulator decisions. A potential intervention of the regulator might require the company to suspend surrender payments until things calm down a bit in very adverse scenarios. Finally the insurer cannot control all this but her target is to settle the best Asset and Liabilities Management that she can for a robust and sustainable solvency; and this can start by trying to manage behavior risk at the creation of the contract!

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