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ERGODIC MEASURES WITH INFINITE ENTROPY

ELEONORA CATSIGERAS AND SERGE TROUBETZKOY

Abstract. We construct ergodic, probability measures with infinite metric entropy for generic continuous maps and homeomorphisms on compact manifolds. We also construct sequences of such measures that converge to a zero-entropy measure.

1. Introduction

Let M be a C^1 compact manifold of finite dimension $m \geq 1$, equipped with a Riemannian metric dist. The manifold M may or may not have boundary. Let $C^0(M)$ be the space of continuous maps $f : M \to M$ with the uniform norm:

$$\|f - g\|_{C^0} := \max_{x \in M} \operatorname{dist}(f(x), g(x)), \qquad \forall\ f, g \in C^0(M).$$

We denote by $\operatorname{Hom}(M)$ the space of homeomorphisms $f : M \to M$ with the uniform norm:

$$\|f - g\|_{\operatorname{Hom}} := \max \left\{ \|f - g\|_{C^0},\ \|f^{-1} - g^{-1}\|_{C^0} \right\} \quad \forall\ f, g \in \operatorname{Hom}(M).$$

Since the metric spaces $C^0(M)$ and $\operatorname{Hom}(M)$ are complete they are Baire spaces. A subset $\mathcal{S} \subset C^0(M)$ (or $\mathcal{S} \subset \operatorname{Hom}(M)$) is called *a G_δ-set* if it is the countable intersection of open subsets of $C^0(M)$ (resp. $\operatorname{Hom}(M)$). We say that a property P of the maps $f \in C^0(M)$ (or $f \in \operatorname{Hom}(M)$) is *generic,* or that generic maps satisfy P, if the set of maps that satisfy P contains a *dense G_δ-set* in $C^0(M)$ (resp. $\operatorname{Hom}(M)$).

The main result of this article is the following theorem.

Theorem 1. *The generic map $f \in C^0(M)$ has an ergodic Borel probability measure μ such that $h_\mu(f) = +\infty$, furthermore there exists $p \geq 1$ such that μ is mixing for the map f^p.*

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Remark. In the case that M is a compact interval, Theorem 1 was proved in [CT, Theorem 3.9 and p.33, para 2]. So, in this paper we will prove it only for m-dimensional manifolds where $m \geq 2$.

Yano proved that generic continuous maps of compact manifolds with or without boundary have infinite topological entropy [Ya]. Therefore, from the variational principle, there exists invariant measures with metric entropies as large as required. Nevertheless, this property alone does not imply the existence of invariant measures with infinite metric entropy. In fact, it is well known that the metric entropy function $\mu \to h_\mu(f)$ is not upper semi-continuous for C^0-generic systems. Moreover, we prove that it is *strongly* non upper semi-continuous in the following sense:

Theorem 2. *For a generic map $f \in C^0(M)$ there exists a sequence of ergodic measures μ_n such that for all $n \geq 1$ we have $h_{\mu_n}(f) = +\infty$ and*

$$\lim_{n\to+\infty}{}^{*} \mu_n = \mu \text{ with } h_\mu(f) = 0,$$

where $\lim^*$ denotes the limit in the space of probability measures endowed with the weak* topology.

Even if we had a priori some f-invariant measure μ with infinite metric entropy, we do not know if this property alone implies the existence of ergodic measures with infinite metric entropy as Theorems 1 and 2 state. Actually, if μ had infinitely many ergodic components, the proof that the metric entropy of at least one of those ergodic components must be larger or equal than the entropy of μ, uses the upper semi-continuity of the metric entropy function (see for instance [Ke, Theorem 4.3.7, p. 75]).

Yano also proved that generic homeomorphisms on manifolds of dimension 2 or larger, have infinite topological entropy [Ya]. Thus one wonders if Theorems 1 and 2 hold also for homeomorphisms. We give a positive answer to this question.

Theorem 3. *If $dim(M) \geq 2$, then the generic homeomorphism $f \in \mathrm{Hom}(M)$ has an ergodic Borel probability measure μ satisfying $h_\mu(f) = +\infty$, furthermore there exists $p \geq 1$ such that μ is mixing for the map f^p.*

Theorem 4. *If $dim(M) \geq 2$, then for a generic homeomorphism $f \in \mathrm{Hom}(M)$ there exists a sequence of ergodic measures μ_n such that for all $n \geq 1$ we have $h_{\mu_n}(f) = +\infty$ and*

$$\lim_{n\to+\infty}{}^{*} \mu_n = \mu \text{ with } h_\mu(f) = 0.$$

To prove the above theorems in dimension two or larger we construct a family $\mathcal{H}$, called models, of continuous maps in the cube $[0,1]^m$, including some homeomorphisms of the cube onto itself, which have a complicated behavior on a Cantor set (Definition 2.4). A periodic shrinking box is a compact set $K \subset M$ that is homeomorphic to the cube $[0,1]^m$ and such that for some $p \geq 1$: $K, f(K), \ldots, f^{p-1}(K)$ are pairwise disjoint and $f^p(K) \subset \mathrm{int}(K)$ (Definition 4.1). The main steps of the proofs of Theorems 1 and 3 are the following results.

Lemma 3.1: *Any model $\Phi \in \mathcal{H}$ in the cube $[0,1]^m$ has an Φ-invariant mixing measure ν such that $h_\nu(\Phi) = +\infty$.*

Lemmas 4.2 and 4.6: *Generic maps in $C^0(M)$, and generic homeormorphisms, have a periodic shrinking box.*

Lemmas 4.8 and 4.9: *Generic maps $f \in C^0(M)$, and generic homeomorphisms for $m \geq 2$, have a periodic shrinking box K such that the return map $f^p|_K$ is topologically conjugated to a model $\Phi \in \mathcal{H}$.*

A *good sequence of periodic shrinking boxes* is a sequence $\{K_n\}_{n\geq 1}$ of periodic shrinking boxes which accumulate (with the Hausdorff distance) on a periodic point x_0, and moreover their iterates $f^j(K_n)$ also accumulate on the periodic orbit of x_0, uniformly for $j \geq 0$ (see Definition 5.1). The main step in the proof of Theorems 2 and 4 is Lemma 3.1 together with

Lemma 5.7: *Generic maps $f \in C^0(M)$, and if $m \geq 2$ also generic homeomorphisms, have a good sequence $\{K_n\}$ of boxes, such that the return maps $f^{p_n}|_{K_n}$ are topologically conjugated to models $\Phi_n \in \mathcal{H}$.*

2. Construction of the family of models.

We call a compact set $K \subset D^m := [0,1]^m$ or more generally $K \subset M$, where M is an m-dimensional manifold with $m \geq 2$, *a box* if it is homeomorphic to D^m. Models are certain continuous maps of $D^m := [0,1]^m$ that we will define in this section.

We denote by $\mathrm{RHom}(D^m)$ the space of relative homeomorphisms $\Phi : D^m \to D^m$ (i.e., Φ is a homeomorphism onto its image included in D^m), with the topology induced by:

$$\|\Phi - \Psi\|_{\mathrm{RHom}} := \max\left\{\|\Phi - \Psi\|_{C^0(D^m)},\ \|\Phi^{-1} - \Psi^{-1}\|_{C^0(\Phi(D^m)\cap\Psi(D^m))}\right\}$$

if $\Phi(D^m) \cap \Psi(D^m)) \neq \emptyset$. Throughout the article we will consider the distance between two relative homeomorphisms Φ, Ψ only if $\Phi(D^m) \cap \Psi(D^m) \neq \emptyset$, and we mostly use the above topology in the case $\Phi(D^m) =$

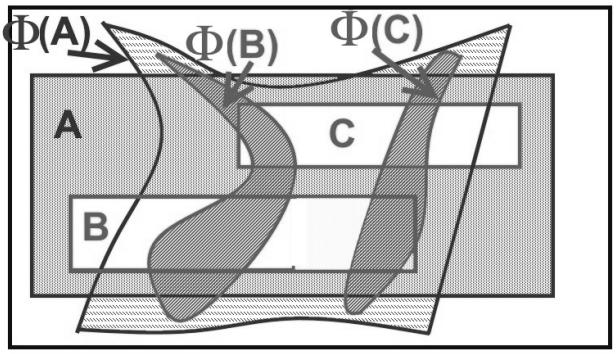

FIGURE 1. An atom A of generation 0 and two atoms B, C of generation 1 for a map Φ of D^2.

$\Psi(D^m)$. It is worth stating that $\| \cdot - \cdot \|_{\mathrm{RHom}}$ defines a topology and a uniformity, but it is not a metric.

Definition 2.1. (Φ-relation from a box to another).
Let $\Phi \in C^0(D^m)$. Let $B, C \subset \mathrm{int}(D^m)$ be two boxes. We write

$$B \xrightarrow{\Phi} C \text{ if } \Phi(B) \cap \mathrm{int}(C) \neq \emptyset.$$

Observe that this condition is open in $C^0(D^m)$ and also in $\mathrm{RHom}(D^m)$.

Definition 2.2. (Atoms of generation 0 and 1) (See Figure 1)
We call a box $A \subset \mathrm{int}(D^m)$ *an atom of generation 0* for Φ, if $A \xrightarrow{\Phi} A$. We call two disjoint boxes $B_1, B_2 \subset \mathrm{int}(A)$, *atoms of generation 1,* if

$$B_i \xrightarrow{\Phi} B_j, \qquad \forall\, i, j \in \{1, 2\}.$$

If A is an atom of generation 0 and B_1, B_2 are the two atoms of generation 1, we denote $\mathcal{A}_0 := \{A\}$, $\mathcal{A}_1 := \{B_1, B_2\}$.

Definition 2.3. Atoms of generation n $\geq$ 2 (See Figure 2)
Assume by induction that the finite families $\mathcal{A}_0, \mathcal{A}_1 \ldots, \mathcal{A}_{n-1}$ of atoms for $\Phi \in C^0(D^m)$ of generations up to $n-1$ are already defined, such that the atoms of the same generation $j = 1, \ldots, n-1$ are pairwise disjoint, contained in the interior of the $(j-1)$-atoms in such a way that all the $(j-1)$-atoms contain the same number of j-atoms and

$$\#\mathcal{A}_j = 2^{j^2}, \qquad \forall\, j = 0, 1, \ldots, n-1.$$

Assume also that for all $j \in \{1, \ldots, n-1\}$ and for all $B \in \mathcal{A}_j$:

$$\#\{C \in \mathcal{A}_j \colon B \xrightarrow{\Phi} C\} = 2^j, \qquad \#\{D \in \mathcal{A}_j \colon D \xrightarrow{\Phi} B\} = 2^j.$$

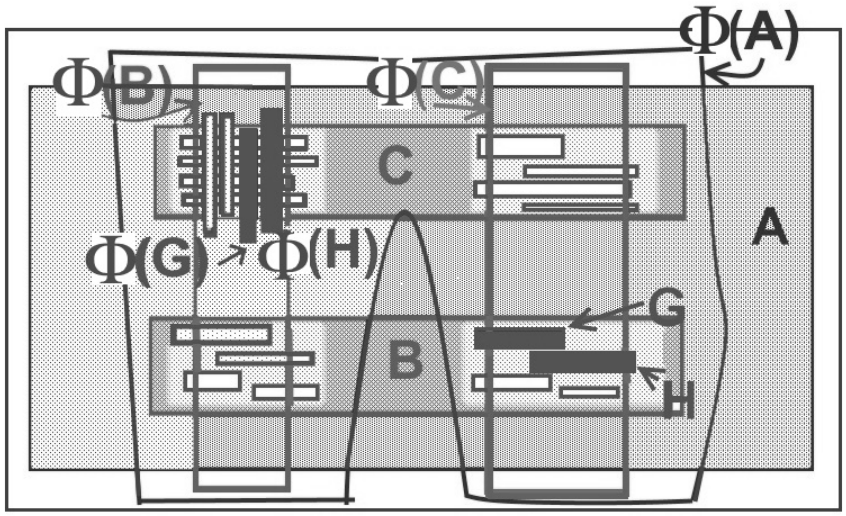

FIGURE 2. An atom A of generation 0, two atoms B, C of generation 1, and 16 atoms of generation 2. In particular the two atoms G, H of generation 2 satisfy $\Gamma_2(C, B, C) = \{G, H\}$.

Denote

$$\mathcal{A}_j^{2*} := \{(B, C) \in \mathcal{A}_j^2 : \quad B \xrightarrow{\Phi} C\},$$

$$\mathcal{A}_j^{3*} := \{(D, B, C) \in \mathcal{A}_j^3 : \quad D \xrightarrow{\Phi} B, \quad B \xrightarrow{\Phi} C\}.$$

For fixed $(D, B, C) \in \mathcal{A}_{j-1}^{3*}$ denote

$$\Omega_j(B) := \{G \in \mathcal{A}_j : G \subset \operatorname{int}(B)\},$$

$$\Omega_j(D, B) := \{G \in \Omega_j(B) : D \xrightarrow{\Phi} G\},$$

$$\Gamma_j(D, B, C) := \{G \in \Omega_j(D, B) : G \xrightarrow{\Phi} C\}.$$

We call the sets of a finite collection $\mathcal{A}_n$ of pairwise disjoint boxes, *atoms of generation n,* or *n-atoms* if they satisfy the following conditions (see Figure 2):

a) Each atom of generation n is contained in the interior of an atom of generation $n-1$, and the interior of each atom $B \in \mathcal{A}_{n-1}$ contains exactly $2^n \cdot 2^{n-1} = 2^{2n-1}$ pairwise disjoint n-atoms, which we call *the children of B.* In other words:

$$\#\Omega_n(B) = 2^{2n-1} \qquad \forall\ B \in \mathcal{A}_{n-1}.$$

Therefore

$$\mathcal{A}_n = \bigcup_{B \in \mathcal{A}_{n-1}} \Omega_n(B),$$

where the families of atoms in the union are pairwise disjoint. Therefore

$$\#\mathcal{A}_n = (\#\mathcal{A}_{n-1})(\#\Omega_n(B)) = 2^{(n-1)^2} \cdot 2^{2n-1} = 2^{n^2}. \tag{1}$$

b) For each $B \in \mathcal{A}_{n-1}$, the collection of children of B is partitioned into 2^{n-1} sub-collections $\Omega_n(D, B)$, where the atoms $D \in \mathcal{A}_{n-1}$ are such that $D \xrightarrow{\Phi} B$. Besides,

$$\#\Omega_n(D, B) = 2^n, \quad \forall (D, B) \in \mathcal{A}^{2*}_{n-1}.$$

In other words

$$\Omega_n(B) = \bigcup_{D\colon (D,B)\in\mathcal{A}^{2*}_{n-1}} \Omega_n(D, B), \qquad \forall\ B \in \mathcal{A}_{n-1},$$

where the families of atoms in the above union are pairwise disjoint. Thus for any $B \in \mathcal{A}_{n-1}$ we have

$$\#\Omega_n(B) = \big(\#\{D \in \mathcal{A}_{n-1}\colon D \xrightarrow{\Phi} B\}\big)\cdot\big(\#\Omega_n(D, B)\big) = 2^{n-1}\cdot 2^n = 2^{2n-1}.$$

c) For each 2-tuple $(D, B) \in \mathcal{A}^{2*}_{n-1}$ the collection $\Omega_n(D, B)$ is partitioned into 2^{n-1} sub-collections $\Gamma_n(D, B, C)$, where the atoms $C \in \mathcal{A}_{n-1}$ are such that $B \xrightarrow{\Phi} C$. Besides,

$$\#\Gamma_n(D, B, C) = 2 \ \ \forall\ (D, B, C) \in \mathcal{A}^{3*}_{n-1}.$$

For example, in Figure 2 we have $\{F, G\} = \Gamma_2(C, B, C)$.
d) For each $(D, B, C) \in \mathcal{A}^{3*}_n$ and for each $G \in \Gamma_n(D, B, C)$

$$G \xrightarrow{\Phi} E, \qquad \forall\ E \in \Omega_n(B, C),$$

$$\text{and} \ \ \Phi(G) \cap E' = \emptyset\ \forall\ E' \in \mathcal{A}_n \setminus \Omega_n(B, C).$$

From the above conditions a) to d) we deduce:

$$\Omega_n(D, B) = \bigcup_{C\colon (B,C)\in\mathcal{A}^{2*}_{n-1}} \Gamma_n(D, B, C), \tag{2}$$

$$\mathcal{A}_n = \bigcup_{(D,B,C)\in\mathcal{A}^{3*}_{n-1}} \Gamma_n(D, B, C), \tag{3}$$

where the families of atoms in the unions are pairwise disjoint. Besides, for any pair $(G, E) \in \mathcal{A}^2_n$:

$$G \xrightarrow{\Phi} E \text{ if and only if } \exists\ (D, B, C) \in \mathcal{A}^{3*}_{n-1} \text{ such that}$$

$$G \in \Gamma_n(D, B, C),\ E \in \Omega_n(B, C),$$

We also deduce the following properties for any atom $G \in \mathcal{A}_n$:

$$\#\{E \in \mathcal{A}_n\colon G \xrightarrow{\Phi} E\} = 2^n, \quad \#\{E \in \mathcal{A}_n\colon E \xrightarrow{\Phi} G\} = 2^n. \tag{4}$$

In fact, on the one hand, from condition d) we deduce that for each atom $G \in \mathcal{A}_n$, the number of atoms $E \in \mathcal{A}_n$ such that $G \xrightarrow{\Phi} E$ equals the number $\#\Omega_n(B,C) = 2^n$, where B and C are the unique $(n-1)$-atoms such that $G \in \Gamma_n(D,B,C)$ for some $D \in \mathcal{A}_{n-1}$. On the other hand, the number of atoms $E \in \mathcal{A}_n$ such that $E \xrightarrow{\Phi} G$ equals

$$(\#\Gamma_n(D,B,C)) \cdot (\#\{D \in \mathcal{A}_{n-1} : \ D \xrightarrow{\Phi} B\}) = 2 \cdot 2^{n-1} = 2^n,$$

where B and C are the unique $(n-1)$-atoms such that $G \in \Omega_n(B,C)$.

Since $2^n < 2^{n^2}$ if $n > 1$, we conclude that not all the pairs (G,E) of atoms of generation $n \geq 2$ satisfy $G \xrightarrow{\Phi} E$.

Definition 2.4. (Models)

We call $\Phi \in C^0(D^m)$ *a model* if $\Phi(D^m) \subset \mathrm{int}(D^m)$, and there exists a sequence $\{\mathcal{A}_n\}_{n\geq 0}$ of finite families of pairwise disjoint boxes contained in $\mathrm{int}(D^m)$ that are atoms of generation $0, 1, \ldots, n, \ldots$ respectively for Φ (according to Definitions 2.2, and 2.3) such that

$$\lim_{n\to+\infty} \max_{A\in\mathcal{A}_n} \mathrm{diam} A = 0. \tag{5}$$

Denote by $\mathcal{H}$ the family of all the models in $C^0(D^m)$.

Remark 2.5. For each fixed $n \geq 1$ the four conditions a) to d) of Definition 2.3, are open conditions. So, the family $\mathcal{H}_n$ of maps that have atoms up to generation $n \geq 1$, is open in $C^0(D^m)$ and also in $\mathrm{RHom}(D^m)$. Moreover, the conditions $\Phi(D^m) \subset \mathrm{int}(D^m)$ and $\max_{A\in\mathcal{A}_n} \mathrm{diam}(A) < \varepsilon_n$ are also open. Therefore, the intersection of the families satisfying the conditions for all $n \geq 1$, which is the family $\mathcal{H}$ of model maps, is a G_δ-set in $C^0(D^m)$ and $\mathcal{H} \cap \mathrm{RHom}(D^m)$ is a G_δ-set in $\mathrm{RHom}(D^m)$.

Construction of models.

The rest of this section is dedicated to the proof of the following lemma.

Lemma 2.6. *The family $\mathcal{H}$ of models is a nonempty G_δ-set in $C^0(D^m)$ and $\mathcal{H} \cap \mathrm{RHom}(D^m)$ is a nonempty G_δ-set in $\mathrm{RHom}(D^m)$.*

In Remark 2.5 we have noted that $\mathcal{H}$ is a G_δ-set. The fact that it is nonempty is a consequence of the following result.

Lemma 2.7. *For all $f \in \mathrm{RHom}(D^m)$ such that $f(D^m) \subset \mathrm{int}(D^m)$, there exists $\psi \in \mathrm{Hom}(D^m)$ and $\Phi \in \mathcal{H} \cap \mathrm{RHom}(D^m)$ such that*

$$\psi|_{\partial D^m} = \mathrm{id}|_{\partial D^m} \quad \textit{and} \quad \psi \circ f = \Phi.$$

We begin by outlining the strategy of the proof of Lemma 2.7. The homeomorphisms ψ and Φ are constructed as uniform limits of respective convergent sequences ψ_n and Φ_n of homeomorphisms, such that

$\psi_n \circ f = \Phi_n$ for all $n \geq 0$. The homeomorphism Φ_n has atoms up to generation n, and Φ_{n+1} coincides with Φ_n outside the interiors of its atoms of generation n. Therefore the collections of atoms of generation up to n for Φ_n is also a collection of atoms for Φ_{n+1}. To change Φ_n inside each atom A of generation n we change ψ_n only inside some adequately defined boxes $R \subset \mathrm{int}(f(A))$, constructing $\psi_{n+1}|_R$ in such a way that $\psi_{n+1}|_{\partial R} = \psi_n|_{\partial R}$, and finally extending $\psi_{n+1}(x) := \psi_n(x)$ for all x in the complement of the union of all the boxes R.

The collection of boxes R inside the atoms A of generation n will be used to construct the atoms of generation $n+1$ for $\Phi_{n+1} = \psi_{n+1} \circ f$, but they are not these atoms. The atoms of generation $n+1$ will be many little boxes in the interior of each box R, obtained after modifying the map only inside each R.

To prove Lemma 2.7, we need several technical lemmas and some more definitions. For each $(P, Q) \in \mathcal{A}_n^{2*}$ in the proof of Lemma 2.7 we will recursively choose a connected component $S(P, Q)$ of $\Phi_n(P) \cap Q$. For each $(D, B, C) \in \mathcal{A}_n^{3*}$ let $G_0(D, B, C)$, $G_1(D, B, C)$ be two disjoint boxes contained in $\mathrm{int}(S(D, B) \cap \Phi_n^{-1} S(B, C))$ (once the connected components are chosen the map Φ_n will be constructed such that the latter intersection is nonempty). Denote:

$$\mathcal{A}_{n+1} := \{G_j(D, B, C) : j \in \{0, 1\}, (D, B, C) \in \mathcal{A}_n^{3*}\};$$

$$\Omega_{n+1}(B) := \{G_j(D, B, C) : j \in \{0, 1\}, (D, B, C) \in \mathcal{A}_n^{3*}\}$$

for each fixed $B \in \mathcal{A}_n$;

$$\Omega_{n+1}(D, B) := \{G_j(D, B, C) : j \in \{0, 1\}, B \xrightarrow{\Phi_n} C\}$$

for each fixed $(D, B) \in \mathcal{A}_n^{2^*}$;

$$\Gamma_{n+1}(D, B, C) := \{G_j(D, B, C) : j \in \{0, 1\}\} \tag{6}$$

for each fixed $(D, B, C) \in \mathcal{A}_n^{3^*}$.

Lemma 2.8. *Fix $n \geq 0$. Let $\Phi_n \in \mathrm{RHom}(D^m)$. Assume that there exists families $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$ of atoms of generation up to n for Φ_n and that*

$$\mathrm{int}(S(D, B) \cap \Phi_n^{-1} S(B, C)) \neq \emptyset \quad \forall\ (D, B, C) \in \mathcal{A}_n^{3*}.$$

Then

a) $\#\mathcal{A}_n^{3^*} = 2^{n^2+2n}$

b) $\#\mathcal{A}_{n+1} = 2^{(n+1)^2}$ and $E \cap F = \emptyset$ for all $E, F \in \mathcal{A}_{n+1}$ such that $E \neq F$.

c) The family $\mathcal{A}_{n+1}$ is partitioned into 2^{n^2} different subfamilies $\Omega_{n+1}(B)$ where $B \in \mathcal{A}_n$. Besides $\#\Omega_{n+1}(B) = 2^{2n+1} = 2^{n+1}2^n$ for all $B \in \mathcal{A}_n$

d) For all $B \in \mathcal{A}_n$ the family of boxes $\Omega_{n+1}(B)$ is partitioned into 2^n subfamilies $\Omega_{n+1}(D,B)$ where $D \in \mathcal{A}_n$ such that $D \overset{\Phi_n}{\to} B$.

e) $\#\Omega_{n+1}(D,B) = 2^{n+1}$ for all $(D,B) \in \mathcal{A}_n^{2*}$.

f) For all $(D,B) \in \mathcal{A}_n^{2*}$ the family of boxes $\Omega_{n+1}(D,B)$ is partitioned into 2^n subfamilies $\Gamma_{n+1}(D,B,C)$ where $C \in \mathcal{A}_n$ such that $B \overset{\Phi_n}{\to} C$.

g) $\#\Gamma_{n+1}(D,B,C) = 2$ for all $(D,B,C) \in \mathcal{A}_n^{3*}$.

h) For all $(B,C) \in \mathcal{A}_n^{2*}$ and for all $E \in \mathcal{A}_{n+1}$, the intersection $E \cap B \cap \Phi_n^{-1}(C)$ is nonempty if and only if $E \in \Gamma_{n+1}(D,B,C)$ for some $D \in \mathcal{A}_n$ such that $D \overset{\Phi_n}{\to} B$.

Remark 2.9. Note that properties b) to g) in Lemma 2.8 imply that the family $\mathcal{A}_{n+1}$ of boxes such constructed, satisfy conditions a), b) and c) of Definition 2.3 of atoms of generation $n+1$ for Φ_n. Thus, to be a family of atoms of generation $n+1$ it is only left to modify, if necessary, Φ_n in the interior of the boxes of $\mathcal{A}_n$ in such a way that these boxes also satisfy condition d) of Definition 2.3.

Proof. of Lemma 2.8. a) All the n-tuples $(D,B,C) \in \mathcal{A}_n^{3*}$ can be constructed choosing freely $D \in \mathcal{A}_n$, for each D choosing $B \in \mathcal{A}_n$ such that $D \overset{\Phi_n}{\to} B$, and for each such B choosing $C \in \mathcal{A}_n$ such that $B \overset{\Phi_n}{\to} C$. Taking into account that by hypothesis $\mathcal{A}_n$ is a family of atoms, equalities (1) and (4) of Definition 2.3 imply

$$\#\mathcal{A}_n^{3*} = \#\{(D,B,C) \in \mathcal{A}_n^3 \colon D \overset{\Phi_n}{\to} B, B \overset{\Phi_n}{\to} C\} =$$

$$(\#\mathcal{A}_n) \cdot (\#\{B \in \mathcal{A}_n \colon D \overset{\Phi_n}{\to} B\}) \cdot (\#\{C \in \mathcal{A}_n \colon B \overset{\Phi_n}{\to} C\}) =$$

$$2^{n^2} 2^n 2^n = 2^{n^2+2n}$$

g) By construction, $\Gamma_{n+1}(D,C,B) = \{G_0(D,C,B),\ G_1(D,C,B)\}$, where the two boxes inside the family are disjoint, hence different. Thus the cardinality of the family is 2.

b) By construction, $E = G_j(D,C,B)$, $F = G_{j'}(D',B',C')$. If $E \neq F$ then, either $(D,C,B) = (D',C',B')$ and $j \neq j'$, or $(D,C,B) \neq (D',C',B')$. In the first case, by construction $G_0(D,C,B) \cap G_1(D,C,B) = \emptyset$, in other words $E \cap F = \emptyset$. In the second case, either $D \neq D'$ or $B \neq B'$ or $C \neq C'$. By construction $G_j(D,B,C) \subset \Phi_n(D) \cap B \cap \Phi_n^{-1}(C)$ and $G_{j'}(D',B',C') \subset \Phi_n(D') \cap B' \cap \Phi_n^{-1}(C')$. Since two different atoms of generation n are pairwise disjoint, and $\Phi_n \in \mathrm{RHom}(D^m)$, we deduce that $G_j(D,B,C) \cap G_{j'}(D',B',C') = \emptyset$, hence $E \cap F = \emptyset$, as required.

By construction $\mathcal{A}_{n+1} = \bigcup_{(D,C,B) \in \mathcal{A}_n^{3*}} \Gamma_{n+1}(D,B,C)$, where the families in the union are pairwise disjoint and have each 2 different boxes

of $\mathcal{A}_{n+1}$. Therefore, taking into account part a), we deduce that

$$\#\mathcal{A}_{n+1} = 2 \cdot (\#\mathcal{A}_n^{3*}) = 2 \cdot 2^{n^2+2n} = 2^{(n+1)^2}.$$

c) By construction of $\mathcal{A}_{n+1}$ and $\Omega_{n+1}(B)$, we have

$$\mathcal{A}_{n+1} = \bigcup_{B \in \mathcal{A}_n} \Omega_{n+1}(B).$$

Besides $G \subset B$ for all $G \in \Omega_{n+1}(B)$, because by construction $G \subset S(D,B) \subset \Phi(D) \cap B$ for some $D \in \mathcal{A}_n$. Since two different atoms $B \neq B'$ of generation n are pairwise disjoint, we deduce that $\Omega_{n+1}(B) \cap \Omega_{n+1}(B') = \emptyset$ if $B \neq B'$. We conclude that the above union of different subfamilies $\Omega_{n+1}(B)$ is a partition of $\mathcal{A}_{n+1}$, as required.

Note that

$$\Omega_{n+1}(B) = \bigcup_{D \in \mathcal{A}_n, D \overset{\Phi_n}{\to} B} \quad \bigcup_{C \in \mathcal{A}_n, B \overset{\Phi_n}{\to} C} \Gamma_{n+1}(D,B,C),$$

where the families in the union are pairwise disjoint and each of them has two different boxes. Therefore, taking into account that $\mathcal{A}_n$ is a family of atoms (by hypothesis), equality (4) of Definition 2.3 implies:

$$\#\Omega_{n+1}(B) = 2 \cdot (\#\{D \in \mathcal{A}_n, D \overset{\Phi_n}{\to} B\}) \cdot \#\{C \in \mathcal{A}_n, B \overset{\Phi_n}{\to} C\} =$$

$$2 \cdot 2^n \cdot 2^n = 2^{2n+1}.$$

d) By construction, $\Omega_{n+1}(B) = \bigcup_{D \in \mathcal{A}_n, D \overset{\Phi_n}{\to} B} \Omega_{n+1}(D,B)$. Besides, $\Omega_{n+1}(D,B) \cap \Omega_{n+1}(D',B) = \emptyset$ if $D \neq D'$ in $\mathcal{A}_n$, since different atoms of generation n are pairwise disjoint, and $G \in \Omega_{n+1}(D,B)$ implies $G \subset \Phi_n(D)$ which is disjoint with $\Phi_n(D')$ since Φ_n is a homeomorphism.

f) By construction, $\Omega_{n+1}(D,B) = \bigcup_{C \in \mathcal{A}_n, B \overset{\Phi_n}{\to} C} \Gamma_{n+1}(D,B,C)$. Besides, $\Gamma_{n+1}(D,B,C) \cap \Gamma_{n+1}(D,B,C') = \emptyset$ if $C \neq C'$ in $\mathcal{A}_n$, because two different atoms of generation n are pairwise disjoint and $G \in \Gamma_{n+1}(D,B,C)$ implies $G \subset \Phi_n^{-1}(C)$.

e) From Assertions f) and g) proved above, and from equality (4) of the definition of atoms of generation n, we deduce that

$$\#\Omega_{n+1}(D,B) = 2 \cdot (\#\{C \in \mathcal{A}_n \colon B \overset{\Phi_n}{\to} C\}) = 2 \cdot 2^n = 2^{n+1}.$$

h) By construction, $E = G_j(D,B,C)$ for some $j \in \{0,1\}$ and some $(D,B,C) \in \mathcal{A}_n^{3^*}$, i.e., $E \in \Gamma_{n+1}(D,B,C)$. Besides $E = G_j(D,B,C) \subset \operatorname{int}(S(D,B) \cap \Phi_n^{-1}(S(B,C))) \subset \operatorname{int}(\Phi_n(D) \cap B \cap \Phi_n^{-1}(C))$. But for $(B,C) \neq (B',C')$ in $\mathcal{A}_n^{2*}$ the sets $B \cap \Phi_n^{-1}(C)$ and $B' \cap \Phi_n^{-1}(C')$ are disjoint, thus the box E belongs to $\Gamma_{n+1}(D,B,C)$ for some $D \in \mathcal{A}_n$, if and only if it intersects $B \cap \Phi_n^{-1}(C)$. $\square$

Lemma 2.10. *Assume the hypotheses of Lemma 2.8. Let $\widetilde{L}_{n+1} \subset D^m$ be a finite set with cardinality $2^{(n+1)^2}2^{n+1}$, with a unique point $\widetilde{e}_i(E) \in \widetilde{L}_{n+1}$ for each $(i,E) \in \{1,2,\dots 2^{n+1}\} \times \mathcal{A}_{n+1}$. Assume that*

$$\widetilde{e}_i(E) \in \operatorname{int}(E) \ \ \forall\ (i,E) \in \{1,2,\dots 2^{n+1}\} \times \mathcal{A}_{n+1}.$$

Then, there exists a permutation $\theta : \widetilde{L}_{n+1} \to \widetilde{L}_{n+1}$ such that

a) For all $(i,E) \in \{1,2,\dots 2^{n+1}\} \times \Gamma_{n+1}(D,B,C)$

$$\theta(\widetilde{e}_i(E)) = \widetilde{e}_{i'}(E')$$

for a unique $i' \in \{1,2,\dots,2^{n+1}\}$ and a unique $E' \in \Omega_{n+1}(B,C)$.

b) For all $(D,B,C) \in \mathcal{A}_n^{3*}$, for all $E \in \Gamma_{n+1}(D,B,C)$ and for all $F \in \Omega_{n+1}(B,C)$ there exists unique $(i,i') \in \{1,2,\dots 2^{n+1}\}^2$ such that $\theta(\widetilde{e}_i(E)) = \widetilde{e}_{i'}(F)$.

c) For all $(B,C) \in \mathcal{A}_n^{2*}$

$$\theta\Big(\Big\{\widetilde{e}_i(E)\colon E \in \bigcup_{D\in\mathcal{A}_n,\ (D,B)\in\mathcal{A}_n^{2*}} \Gamma_{n+1}(D,B,C),\ \ i \in \{1,2,\dots 2^{n+1}\}\Big\}\Big) =$$

$$\Big\{\widetilde{e}_{i'}(F) : F \in \Omega_{n+1}(B,C), i' \in \{1,2,,\dots 2^{n+1}\}\Big\} = \widetilde{L}_{n+1} \cap S(B,C).$$

Proof. From the construction of the family $\mathcal{A}_{n+1}$ (see Lemma 2.8), we deduce that for all $E \in \mathcal{A}_{n+1}$ there exists unique $j \in \{0,1\}$ and unique $(D,B,C) \in \mathcal{A}_n^{3*}$ such that

$$E = G_j(D,B,C) \in \Gamma_{n+1}(D,B,C)$$

(recall Equality (6)) and thus we will write

$$\widetilde{e}_i(G_j(D,B,C)) = \widetilde{e}_i(E)$$

for all $(i,E) \in \{1,2,\dots,2^{n+1}\} \times \mathcal{A}_{n+1}$.

By hypothesis $\mathcal{A}_n$ is the family of atoms of generation n for Φ_n, thus we can apply Equalities (4) of Definition 2.3. So, for each $B \in \mathcal{A}_n$, we can index the different atoms $D \in \mathcal{A}_n$ such that $D \overset{\Phi_n}{\to} B$ as follows:

$$\{D \in \mathcal{A}_n\colon D \overset{\Phi_n}{\to} B\} = \{D_1^-(B), D_2^-(B), \dots D_{2^n}^-(B)\}, \tag{7}$$

where $D_{k_1}^-(B) \neq D_{k_2}^-(B)$ if $k_1 \neq k_2$ (actually, they are disjoint atoms of generation n).

Analogously

$$\{C \in \mathcal{A}_n\colon B \overset{\Phi_n}{\to} C\} = \{C_1^+(B), C_2^+(B), \dots C_{2^n}^+(B)\}, \tag{8}$$

where $C_{l_1}^+(B) \neq C_{l_2}^+(B)$ if $l_1 \neq l_2$.

Now, we index the distinct points of $\widetilde{L}_{n+1}$ as follows:

$$\widehat{e}_{i,j}(k,B,l) := \widetilde{e}_i(G_j(D,B,C)) = \widetilde{e}_i(G_j(D_k^-(B), B, C_l^+(B))),$$

for all $(i,j,B,k,l) \in \{1,2,\dots,2^{n+1}\} \times \{0,1\} \times \mathcal{A}_n \times \{1,2,\dots,2^n\}^2$.

Define the following correspondence $\theta : \widetilde{L}_{n+1} \to \widetilde{L}_{n+1}$:

$$\theta(\widehat{e}_{i,j}(k,B,l)) = \widehat{e}_{i',j'}(k',B',l'), \text{ where}$$

- $B' := C_l^+(B)$,
- k' is such that $B = D_{k'}^-(C)$ (such k' exists and is unique because $B \overset{\Phi_n}{\to} C$, using (7)),
- $l' = i \pmod{2^n}$,
- $j' = 0$ if $i \le 2^n$ and $j' = 1$ if $i > 2^n$,
- $i' = k + j \cdot 2^n$.

Let us prove that θ is surjective; hence it is a permutation of the finite set $\widetilde{L}_{n+1}$.

Let $\widehat{e}_{i',j'}(k',B',l') \in \widetilde{L}_{n+1}$ be given, where

$$(i',j',B',k',l') \in \{1,2,\dots,2^{n+1}\} \times \{0,1\} \times \mathcal{A}_n \times \{1,2,\dots,2^n\}^2.$$

Construct

- $i := l' + j' \cdot 2^n$. Then $l' = i \pmod{2^n}$, $j' = 0$ if $i \le 2^n$ and $j' = 1$ if $i > 2^n$.
- $B := D_{k'}^-(B')$. Then $B \overset{\Phi_n}{\to} B'$. So, there exists l such that $B' = C_l^+(B)$.
- $k := i' \pmod{2^n}$, $j := 0$ if $i' \le 2^n$ and $j := 1$ if $i' > 2^n$. Therefore $i' = k + 2^n j$.

By the above equalities we have constructed some θ^{-1} such that $\theta \circ \theta^{-1}$ is the identity map. So, θ is surjective, hence also one-to-one in the finite set $\widetilde{L}_{n+1}$, as required.

Now, let us prove that θ satisfies assertions a), b) and c) of Lemma 2.10.

a) Fix $\widetilde{e}_i(E) \in \widetilde{L}_{n+1}$. By construction $\theta(\widetilde{e}_i(E)) = \widetilde{e}_{i'}(E') \subset \mathrm{int}(E')$ for some $(i,E) \in \{1,2,\dots,2^{n+1}\} \times \mathcal{A}_{n+1}$. Since two different boxes of $\mathcal{A}_{n+1}$ are pairwise disjoint (recall Lemma 2.8-a)), the box E' is unique. Besides, by hypothesis, $\widetilde{e}_{i'}(E') \neq \widetilde{e}_{j'}(E')$ if $i' \neq j'$. So, the index i' is also unique. Therefore, to finish the proof of a), it is enough to check that $E' \in \Omega_{n+1}(B,C)$ if $E \in \Gamma_{n+1}(D,B,C)$.

By the definition of the family $\Gamma_{n+1}(D,B,C)$ in Lemma 2.8, if $E \in \Gamma_{n+1}(D,B,C)$, there exists $j \in \{0,1\}$ such that $E = G_j(D,B,C)$. Thus, using the notation at the beginning $\widetilde{e}_i(E) = \widetilde{e}_i(G_j(D,B,C)) = \widehat{e}_{i,j}(k,B,l)$, where $D = D_k^-(B)$ and $C = C_l^+(B)$. Then, using the definition of the permutation θ, and the computation of its inverse θ^{-1}, we obtain $\widetilde{e}_{i'}(E) = \theta(\widetilde{e}_i(E)) = \widehat{e}_{i',j'}(k',B',l')$, where

$$B' = C_l^+(B) = C, \quad D' = D_{k'}^-(B') = B.$$

We have proved that $\widetilde{e}_{i'}(E') = \widetilde{e}_{i'}(G_{j'}(B,C,C'))$. Finally, from the definition of the family $\Omega_{n+1}(B,C)$ in Lemma 2.8 we conclude that $E' \in \Omega_{n+1}(B,C)$ as asserted in part a).

b) Fix $(D,B,C) \in \mathcal{A}_n^{3*}$ and $E \subset \Gamma_{n+1}(D,B,C)$. Then, using the definition of the family $\Gamma_{n+1}(D,B,C)$ in Lemma 2.8, we have unique $(j,k,l) \in \{0,1\} \times \{1,2,\dots,2^n\}^2$ such that $E = G_j(D,B,C)$, $D = D_k^-(B)$, $C = C_l^+(B)$. Consider the finite set Z of 2^{n+1} distinct points $\widetilde{e}_i(E) = \widehat{e}_{i,j}(k,B,l)$, with j,k,B,l fixed as above and $i \in \{1,2,\dots,2^{n+1}\}$. Let $i' =: k+2^n j$, then the image of each point in Z by the permutation θ is $\theta(\widetilde{e}_i(E)) = \widetilde{e}_{i'}(G_{j'}(B,C,C')$ (here we use assertion a). Since k,j are fixed, we deduce that there exists a unique i' such that all the points of $\theta(Z)$ are of the form $\widetilde{e}_{i'}(F)$, $F = G_{j'}(B,C,C')$ with $j' \in \{0,1\}$, $C' = C_{k'}^+(C)$, $k' \in \{1,2,\dots,2^{n+1}\}$. We have proved that the permutation $\theta|_Z$ is equivalent to

$$i \in \{1,2,\dots,2^{n+1}\} \to (j',k') \in \{0,1\} \times \{1,2,\dots,2^n\}$$

such that $\theta(\widetilde{e}_i(E)) = \widetilde{e}_{i'}(G_{j'}(B,C,C_{k'}^+(C)))$ with i' fixed.

Since $\#\{1,2,\dots,2^{n+1}\} = \#(\{0,1\} \times \{1,2,\dots,2^n\})$, from the injectiveness of θ we deduce that $\theta(Z) = \{0,1\} \times \{1,2,\dots,2^n\}$. In other words, for every $F \in \Omega(B,C)$ there exists unique i such that $\theta(\widetilde{e}_i(E)) = \widetilde{e}_{i'}(F)$ (where i' is uniquely defined given E). This ends the proof of assertion b).

c) For fixed $(B,C) \in \mathcal{A}_n^{2*}$, denote

$$P := \Big\{\widetilde{e}_i(E)\colon \quad E \in \bigcup_{D \in \mathcal{A}_n, D \overset{\Phi_n}{\to} B} \Gamma_{n+1}(D,B,C), \quad i \in \{1,2,\dots,2^{n+1}\}\Big\},$$

$$Q := \{\widetilde{e}_{i'}(F)\colon \quad F \in \Omega_{n+1}(B,C), \quad i' \in \{1,2,\dots,2^{n+1}\}\} \subset \widetilde{L}_{n+1}.$$

Applying assertion a) we deduce that $\theta(P) \subset Q$. So, to prove that $\theta(P) = Q$ it is enough to prove that $\#P = \#Q$. In fact, applying parts e) and g) of Lemma 2.8 for the family of boxes $\mathcal{A}_{n+1}$, and equality (4) of Definition 2.2 for the family of atoms $\mathcal{A}_n$, we obtain

$$\#P = 2^{n+1} \cdot (\#\Gamma_{n+1}(D,B,C)) \cdot (\#\{D \in \mathcal{A}_n\colon D \overset{\Phi_n}{\to} B\}) = 2^{n+1} \cdot 2 \cdot 2^n$$
$$\#Q = 2^{n+1} \cdot (\#\Omega_{n+1})(B,C)) = 2^{n+1} \cdot 2^{n+1},$$

which proves that $\#P = \#Q$ and thus that $\theta(P) = Q$.

Finally, let us prove that $Q = \widetilde{L}_{n+1} \cap S(B,C)$.

On the one hand, if $F \in \Omega_{n+1}(B,C)$, then $F = G_j(B,C,C')$ for some (j,C'). Applying the construction of the boxes of $\mathcal{A}_{n+1}$ in the hypothesis of Lemma 2.8, we obtain $F \subset S(B,C)$, hence $\widetilde{e}_{i'}(F) \in \widetilde{L}_{n+1} \cap \mathrm{int}(F) \subset \widetilde{L}_{n+1} \cap S(B,C)$. This proves that $Q \subset \widetilde{L}_{n+1} \cap S(B,C)$.

On the other hand, if $\widetilde{e}_{i'}(F) \in \widetilde{L}_{n+1} \cup S(B,C)$, then $F \in \mathcal{A}_{n+1}$. Applying Lemma 2.8, we obtain $F = G_j(D',B',C') \subset S(D',B')$ for some $(D',B',C') \in \mathcal{A}_n^{3*}$. Since $S(D',B') \subset \Phi_n(D') \cap B'$ and $S(B,C) \subset \Phi_n(B) \cap C$, we obtain $S(D',B') \cap S(B,C) = \emptyset$ if $(D',B') \neq (B,C)$. But $\widetilde{e}_{i'}(F) \in \mathrm{int}(F) \cap S(B,C) \subset S(D',B') \cap S(B,C)$. We conclude that $(D',B') = (B,C)$, thus $F = G_j(B,C,C') \in \Omega_{n+1}(B,C)$, hence $\widetilde{e}_{i'}(F) \in Q$. We have proved that $\widetilde{L}_{n+1} \cap S(B,C) \subset Q$. Therefore the proof of assertion c) is complete. □

Lemma 2.11. *Assume the hypothesis of Lemmas 2.8 and 2.10. Let* $\Phi_{n+1} \in \mathrm{RHom}(D^m)$ *be such that*

$$\Phi_{n+1}(x) = \Phi_n(x) \ \ \forall\, x \notin \bigcup_{A \in \mathcal{A}_n} A, \quad \Phi_{n+1}(\widetilde{e}) = \theta(\widetilde{e}) \ \ \forall\, \widetilde{e} \in \widetilde{L}_{n+1},$$

where θ *is the permutation of* $\widetilde{L}_{n+1}$ *constructed in Lemma 2.10. Then,*

a) $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_{n+1}$ *are collections of atoms up to generation* $n+1$ *for* Φ_{n+1}.

b) For each $(E,F) \in \mathcal{A}_{n+1}^2$ *such that* $E \overset{\Phi_{n+1}}{\to} F$*, there exists exactly one point* $\widetilde{e}_i(E) \in \widetilde{L}_{n+1} \cap \mathrm{int}(E)$*, and exactly one point* $\widetilde{e}_{i'}(F) \in \widetilde{L}_{n+1} \cap \mathrm{int}(F)$*, such that*

$$\widetilde{\Phi}_{n+1}(\widetilde{e}_i(E)) = \widetilde{e}_{i'}(F).$$

Proof. a) Since $\Phi_{n+1}(x) = \Phi_n(x)$ for all $x \notin \bigcup_{B \in \mathcal{A}_n} B$, the same families $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_n$, of atoms up to generation n for Φ_n are families of atoms up to generation n for Φ_{n+1}. As observed in Remark 2.9, the family $\mathcal{A}_{n+1}$ of pairwise disjoint boxes satisfy conditions a), b) and c) of Definition 2.3 with $n+1$ instead of n, for Φ_{n+1}. So, to prove that $\mathcal{A}_{n+1}$ is a family of atoms of generation $n+1$ for Φ_{n+1} it is enough to prove that its boxes satisfy also condition d) of Definition 2.3.

Take $(D,B,C) \in \mathcal{A}_n^{3^*}$, $E \in \Gamma_{n+1}(D,B,C)$ and $F \in \Omega_{n+1}(B,C)$. Applying Lemma 2.10-b), there exists (i,i') such that $\theta(\widetilde{e}_i(E)) = \widetilde{e}_{i'}(F)$. Therefore

$$\Phi_{n+1}(\widetilde{e}_i(E))) = \widetilde{e}_{i'}(F).$$

Since $\widetilde{e}_i(E) \in \mathrm{int}(E)$ and $\widetilde{e}_{i'}(F) \in \mathrm{int}(F)$, we conclude that $\Phi_{n+1}(E) \cap \mathrm{int}(F) \neq \emptyset$, namely, $E \overset{\Phi_n}{\to} F$. We have proved that

$$E \overset{\Phi_n}{\to} F \ \ \forall E \in \Gamma_{n+1}(D,B,C), \ \forall F \in \Omega_{n+1}(B,C).$$

This proves the first half of condition d) of Definition 2.3.

Let us prove the second half. Let $E \in \Gamma_{n+1}(D,B,C)$ and $F \in \mathcal{A}_{n+1} \setminus \Omega_{n+1}(B,C)$. Then $F \in \Omega_{n+1}(B',C')$ with $(B',C') \neq (B,C)$ (recall that by Lemma 2.8, the union of all the families $\Omega_{n+1}(\cdot,\cdot)$ is $\mathcal{A}_{n+1}$). We

have $F \in \Omega_{n+1}(B', C')$; therefore $F \subset \Phi_n(B') \cap C'$. Besides $\Phi_n(B') = \Phi_{n+1}(B')$ (recall that Φ_n and Φ_{n+1} are relative homeomorphisms that coincide outside the atoms of generation n; so, by continuity, they coincide in the borders of all those atoms, therefore the images by Φ_n and Φ_{n+1} of each atom of generation n coincide). We deduce that

$$F \subset \Phi_{n+1}(B') \cap C' \text{ with } (B', C') \neq (B, C). \tag{9}$$

Since $E \in \Gamma_{n+1}(D, B, C)$, we know that $E \subset B \cap \Phi_n^{-1}(C) = B \cap \Phi_{n+1}^{-1}(C)$ (recall the construction of the atoms of the family $\mathcal{A}_{n+1}$ in the hypothesis of Lemma 2.8). Therefore

$$\Phi_{n+1}(E) \subset \Phi_{n+1}(B) \cap C \tag{10}$$

Since two different atoms of generation n are disjoint, and Φ_{n+1} is a relative homeomorphism, we have $(\Phi_{n+1}(B) \cap C) \cap (\Phi_{n+1}(B') \cap C') = \emptyset$ if $(B, C) \neq (B', C')$. We conclude that

$$\Phi_{n+1}(E) \cap F = \emptyset \quad \forall\ F \in \mathcal{A}_{n+1} \setminus \Omega_{n+1}(B, C),$$

ending the proof of condition d) of Definition 2.3 for the family $\mathcal{A}_{n+1}$. The proof of assertion a) is complete.

b) Take $(E, F) \in \mathcal{A}_{n+1}^2$ such that $E \overset{\Phi_{n+1}}{\to} F$. Then

$$\Phi_{n+1}(E) \cap F \neq \emptyset. \tag{11}$$

We have $E \in \mathcal{A}_{n+1}$. Therefore, by the construction of this family in Lemma 2.8, there exists $(D, B, C) \in \mathcal{A}_n^{3*}$ such that

$$E \in \Gamma_{n+1}(D, B, C).$$

Combining property (11) with property d) of Definition 2.3 (putting $n+1$ instead of n), we obtain

$$F \in \Omega_{n+1}(B, C).$$

Applying Lemma 2.10-b) there exists a unique $(i, i') \in \{1, 2, \ldots, 2^{n+1}\}^2$ such that $\Phi_{n+1}(\widetilde{e}_i(E)) = \theta(\widetilde{e}_i(E)) = \widetilde{e}_{i'}(F)$. The proof of part b) is complete. □

Lemma 2.12. *Let $\psi \in \mathrm{RHom}(D^m)$, $P, Q \subset D^m$ be boxes such that $\psi(P) = Q$, $p_1, \ldots, p_k \in \mathrm{int}(P)$ be distinct points and $q_1, \ldots, q_k \in \mathrm{int}(Q)$ be also distinct points. Then, there exists $\psi^* \in \mathrm{RHom}(D^m)$ such that*

$$\psi^*(x) = \psi(x) \quad \forall\ x \notin \mathrm{int}(P) \text{ and } \psi^*(p_i) = q_i \quad \forall\ i \in \{1, \ldots, k\}.$$

Proof. It is straightforward. □

Lemma 2.13. *Let $\psi \in \mathrm{RHom}(D^m)$, $r \geq 1$, $P_1, P_2, \ldots P_r \subset D^m$ be pairwise disjoint boxes, and $Q_j := \psi(P_j)$ for all $j \in \{1, 2, \ldots, r\}$. For $k \geq 1$ and $j \in \{1, 2, \ldots, r\}$, let $p_{1,j}, \ldots, p_{k,j} \in \mathrm{int}(P_j)$ be distinct points*

and $q_{1,j}, \dots, q_{k,j} \in \mathrm{int}(Q_j)$ *also be distinct points. Then, there exists a* $\psi^* \in \mathrm{RHom}(D^m)$ *such that*

$$\psi^*(x) = \psi(x) \ \ \forall\ x \notin \bigcup_{j=1}^{r} \mathrm{int}(P_j) \ \textit{and}$$

$$\psi^*(p_{i,j}) = q_{i,j} \ \ \forall\ (i,j) \in \{1,\dots,k\} \times \{1,2,\dots,r\}.$$

Proof. It is straightforward. □

Proof. of Lemma 2.7 We divide the construction of ψ and $\Phi \in \mathcal{H}$ into several steps:

Step 1. Construction of the atom of generation 0. Since $f(D^m) \subset \mathrm{int}(D^m)$, there exists a box $A_0 \subset \mathrm{int}(D^m)$ such that $f(D^m) \subset \mathrm{int}(A_0)$. The box A_0 is the *atom of generation 0* for the relative homeomorphism $\Phi_0 := f$ which satisfies $\Phi_0 = \psi_0 \circ f$ where ψ_0 is the identity map. Applying the Brower Fixed Point Theorem, there exists a point $e_0 \in \mathrm{int}(\Phi_0(A_0))$ such that $\Phi_0(e_0) = e_0$. Define $S(A_0, A_0)$ to be the connected component of $A_0 \cap \Phi_0(A_0)$ containing e_0.

Step 2. Construction of the atoms of generation *n+1*. Inductively assume that we have constructed families $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_n$ of atoms up to generation n for $\Phi_n = \psi_n \circ f$, where $\psi_n \in \mathrm{Hom}(D^m)$ is a homeomorphism satisfying:

I) $\psi_n|_{\partial D^m}$ is the identity map,
II) $\max_{B \in \mathcal{A}_i} \max\{\mathrm{diam}(B), \mathrm{diam}(f(B))\} < \frac{1}{2^i} \ \ \forall i \in \ \{0,1,\dots n\}$;
III) $\Phi_i(x) = \Phi_{i-1}(x), \qquad \forall\ x \in D^m \setminus \bigcup_{B \in \mathcal{A}_{i-1}} B,\ \forall i \in \{1,\dots,n\}$;
IV) for all $(D,B,C) \in \mathcal{A}_n^{3*}$ there exists a point $e(D,B,C)$ such that

$$L_n := \{e(D,B,C) \colon (D,B,C) \in \mathcal{A}_n^{3*}\}$$

is Φ_n-invariant, and

$$e(D,B,C) \in \mathrm{int}\big(S(D,B) \cap \Phi_n^{-1}(S(B,C))\big), \tag{12}$$

where $S(D,B)$ and $S(B,C)$ are previously defined connected components of $B \cap \Phi_n(D)$ and of $C \cap \Phi_n(B)$ respectively;
V) the sets $S(D,B)$ and $S(D',B')$ are disjoint if $(D,B) \neq (D',B')$.

Let us construct the family $\mathcal{A}_{n+1}$ of atoms of generation $n+1$ and the homeomorphisms Φ_{n+1} and ψ_{n+1}. First, for each $(D,B) \in \mathcal{A}_n^{2^*}$ we choose a box $R(D,B)$ such that

$$e(D,B,C) \in \mathrm{int}(R(D,B)) \subset \mathrm{int}\big(S(D,B)\big) \tag{13}$$

$$\forall\ C \in \mathcal{A}_n \text{ such that } B \overset{\Phi_n}{\to} C.$$

By (V) such boxes $R(\cdot,\cdot)$ are pairwise disjoint.

Recall that $e(D,B,C) \in L_n$, the set L_n is Φ_n-invariant and moreover $e(D,B,C) \in \operatorname{int}(\Phi_n^{-1}(S(B,C)))$. Thus

$$e(D,B,C) \in \operatorname{int}\Big(R(D,B) \cap \Phi_n^{-1}(R(B,C))\Big) \neq \emptyset$$

because

$$\Phi_n(e(D,B,C)) = e(B,C,\cdot) \in \operatorname{int}(R(B,C)).$$

Next, for each $(D,B,C) \in \mathcal{A}_n^{3*}$ we choose two pairwise disjoint boxes, $G_0(D,B,C)$ and $G_1(D,B,C)$, contained in the interior of $R(D,B) \cap \Phi_n^{-1}(R(B,C))$, satisfying

$$\max\{\operatorname{diam}(G_i(D,B,C)), \operatorname{diam}(f(G_i(D,B,C)))\} < \frac{1}{2^{n+1}} \tag{14}$$

for $i = 0,1$ Now, we use Lemma 2.8 to define the family $\mathcal{A}_{n+1}$ of all the boxes $G_i(D,B,C)$ and use its properties. The boxes of the family $\mathcal{A}_{n+1}$ will be the $(n+1)$-atoms of two new homeomorphisms $\widetilde{\Phi}_{n+1}$ and Φ_{n+1} that we will construct as follows.

First, in the interior of each box $E \in \mathcal{A}_{n+1}$ we choose 2^{n+1} distinct points $\widetilde{e}_i(E), i = 1, 2 \ldots, 2^{n+1}$, and denote

$$\widetilde{L}_{n+1} := \{\widetilde{e}_i(E) \colon E \in \mathcal{A}_{n+1},\ 1 \leq i \leq 2^{n+1}\}.$$

Second, we build a permutation $\widetilde{\theta}$ of $\widetilde{L}_{n+1}$ satisfying the properties of Lemma 2.10.

Third, applying Lemma 2.13, we construct $\widetilde{\psi}_{n+1} \in \operatorname{Hom}(D^m)$ defined satisfying the following constraints.

(a)

$$\widetilde{\psi}_{n+1}|_{\psi_n^{-1}R(B,C)} : \psi_n^{-1}(R(B,C)) \to R(B,C) \quad \forall (B,C) \in \mathcal{A}_n^{2^*},$$

(b)

$$\widetilde{\psi}_{n+1}(x) = \psi_n(x),$$
$$\forall x \notin \bigcup_{(B,C)\in\mathcal{A}_n^{2*}} \psi_n^{-1}(R(B,C)),$$

(c)

$$\widetilde{\psi}_{n+1}(f(\widetilde{e})) = \widetilde{\theta}(\widetilde{e}), \qquad \forall\ \widetilde{e} \in \widetilde{L}_{n+1}.$$

To prove the existence of such a homeomorphism $\widetilde{\psi}_{n+1}$ we must verify the hypotheses of Lemma 2.13. On the one hand, the boxes $R(B,C)$ where $(B,C) \in \mathcal{A}_n^{2*}$ are pairwise disjoint. So their preimages by the homeomorphism ψ_n also are pairwise disjoint. On the other hand, the finite set

$$\{f(\widetilde{e}) : \quad \widetilde{e} \in \widetilde{L}_{n+1} \cap \operatorname{int}(\Phi_n^{-1}(R(B,C)))\}$$

is contained in the interior of $f \circ \Phi_n^{-1}(R(B,C)) = \psi_n^{-1}(R(B,C))$. Besides, it coincides with

$$\{f(\widetilde{e}_i(E))\colon E \in \Gamma_{n+1}(D,B,C) \text{ for some } D \in \mathcal{A}_n,\ i = 1,\dots,2^{n+1}\}$$

(recall that $\widetilde{e}_i(E) \in E \cap B \cap \Phi_n^{-1}(C)$ and apply Lemma 2.8-h)). So, its image by the permutation $\widetilde{\theta}$ is the finite set

$$\{\widetilde{\theta}(\widetilde{e}_i(E))\colon E \in \Gamma_{n+1}(D,B,C) \quad \text{for some } D \in \mathcal{A}_n,\ i = 1,2,\dots 2^{n+1}\}$$

Applying Lemma 2.10-c), the latter set is

$$\{e_k(F)\colon F \in \Omega_{n+1}(B,C), k = 1,2,\dots 2^{n+1}\} = \widetilde{L}_{n+1} \cap R(B,C),$$

which is contained in the interior of $R(B,C)$. The hypothesis of Lemma 2.13 is satisfied.

We construct

$$\widetilde{\Phi}_{n+1} := \widetilde{\psi}_{n+1} \circ f.$$

Since $\widetilde{\Phi}_{n+1}(x) = \Phi_n(x),\ \forall x \notin \cup_{A \in \mathcal{A}_n} A$, the same atoms up to generation n for Φ_n are still atoms up to generation n for $\widetilde{\Phi}_{n+1}$. But moreover, applying Lemma 2.11-a), the boxes of the new family $\mathcal{A}_{n+1}$ are now $(n+1)$-atoms for $\widetilde{\Phi}_{n+1}$.

Step 3. Construction of Φ_{n+1} and ψ_{n+1}. To argue by induction, we will not use the homeomorphisms $\widetilde{\Phi}_{n+1}$ and $\widetilde{\psi}_{n+1}$, rather we need to modify them to obtain new homeomorphisms Φ_{n+1} and ψ_{n+1} such that Assertion (12) also holds for $n+1$ instead of n. Precisely, modifying $\widetilde{\psi}_{n+1}$ only in the interiors of the boxes $f(G)$ for all the atoms $G \in \mathcal{A}_{n+1}$, we construct a new homeomorphism ψ_{n+1} such that $\Phi_{n+1} := \psi_{n+1} \circ f$ has the same atoms up to generation $n+1$ as $\widetilde{\Phi}_{n+1}$.

From the above construction of $\widetilde{\psi}_{n+1}$ and $\widetilde{\Phi}_{n+1}$, and from Lemma 2.11-b), we know that for each $(G,E) \in \mathcal{A}_{n+1}^{2*}$ there exists a unique point $\widetilde{e}_i(G) \in \mathrm{int}(G)$, and a unique point $\widetilde{e}_k(E)$, such that

$$\widetilde{\Phi}_{n+1}(\widetilde{e}_i(G)) = \widetilde{\psi}_{n+1} \circ f(\widetilde{e}_i(G)) = \widetilde{e}_k(E) \in \mathrm{int}(E).$$

Therefore

$$\widetilde{e}_k(E) \in \mathrm{int}(E \cap \widetilde{\Phi}_{n+1}(G)).$$

Denote by

$$S(G,E)$$

the connected component of $E \cap \widetilde{\Phi}_{n+1}(G)$ that contains the point $\widetilde{e}_k(E)$. Choose 2^{n+1} distinct points

$$e_i(G,E) \in \mathrm{int}(S(G,E)), \quad i = 1,\dots,2^{n+1}$$

and a permutation θ of the finite set

$$L_{n+1} := \{e_i(G,E) : \quad (G,E) \in \mathcal{A}_{n+1}^{2*}, \quad i = 1,\dots,2^{n+1}\} \tag{15}$$

such that for each fixed $(G,E,F)\in\mathcal{A}^{3*}_{n+1}$ there exists a unique point $e_i(G,E)$, and a unique point $e_k(E,F)$, satisfying

$$\theta(e_i(G,E))=e_k(E,F).$$

The proof of the existence of such permutation is similar to the proof of Lemma 2.10.

Applying Lemma 2.13, construct a homeomorphism

$$\psi_{n+1}\in\mathrm{Hom}(D^m)$$

such that

$$\psi_{n+1}|_{f(G)}:f(G)\to\widetilde{\psi}_{n+1}(f(G))=\widetilde{\Phi}_{n+1}(G)\ \ \forall\ G\in\mathcal{A}_{n+1},$$

$$\psi_{n+1}(x)=\widetilde{\psi}_{n+1}(x)\ \ \forall\ x\notin\bigcup_{G\in\mathcal{A}_{n+1}}f(G)$$

$$\psi_{n+1}(f(e_i(G,E))=\theta(e_i(G,E))$$

$$\forall\ (E,G)\in\mathcal{A}^2_{n+1}\text{ such that }G\xrightarrow{\Phi}E,\qquad\forall\ i=1,\dots,2^{n+1}$$

and extend ψ_{n+1} to the whole box D^m by defining $\psi_{n+1}(x)=\widetilde{\psi}_{n+1}(x)$, $\forall\ x\in D^m\setminus\bigcup_{G\in\mathcal{A}_{n+1}}f(G)$. In particular

$$\psi_{n+1}|_{\partial D^m}=\widetilde{\psi}_{n+1}|_{\partial D^m}=\mathrm{id}|_{\partial D^m}.$$

Define

$$\Phi_{n+1}:=\psi_{n+1}\circ f.$$

As said above, the property that Φ_{n+1} coincides with $\widetilde{\Phi}_{n+1}$ outside all the atoms of $\mathcal{A}_{n+1}$ implies that the boxes of the families $\mathcal{A}_0,\dots,\mathcal{A}_{n+1}$ are also atoms up to generation $n+1$ for Φ_{n+1}. But now, they have the following additional property: there exists a one-to-one correspondence between the 3-tuples $(G,E,F)\in\mathcal{A}^{3*}_{n+1}$ and the points of the set L_{n+1} of Equality (15), such that

$$e(G,E,F):=e_i(G,E)\in\mathrm{int}\big(S(G,E)\cap\Phi^{-1}_{n+1}(S(E,F))\big),\tag{16}$$

where $S(G,E)$ and $S(E,F)$ are the previously chosen connected components of $E\cap\Phi_n(G)$ and of $F\cap\Phi_n(E)$ respectively. Besides, by construction the finite set L_{n+1} is Φ_{n+1}-invariant. In fact $\Phi_{n+1}(L_{n+1})=\psi_{n+1}(f(L_{n+1}))=L_{n+1}$. Therefore, Assertion (12) holds for $n+1$ and the inductive construction is complete.

Step 4. The limit homeomorphisms. From the above construction we have:

$$\psi_{n+1}(x)=\widetilde{\psi}_{n+1}(x)=\psi_n(x)\text{ if }x\notin\bigcup_{B,C}\psi_n^{-1}(R(B,C))\subset\bigcup_{B}f(B)$$

$$\psi_{n+1}\circ\psi_n^{-1}(R(B,C))=\widetilde{\psi}_{n+1}\circ\psi_n^{-1}(R(B,C))=$$

$$\psi_n \circ \psi_n^{-1}(R(B,C)) = R(B,C) \subset C.$$

Therefore,

$$\mathrm{dist}(\psi_{n+1}^{-1}(x), \psi_n^{-1}(x)) \leq \max_{B\in\mathcal{A}_n} \mathrm{diam}(f(B)) < \frac{1}{2^n}, \qquad \forall\, x \in D^m;$$

$$\mathrm{dist}(\psi_{n+1}(x), \psi_n(x)) \leq \max_{C\in\mathcal{A}_n} \mathrm{diam}(C) < \frac{1}{2^n}, \qquad \forall\, x \in D^m,$$

$$\|\psi_{n+1} - \psi_n\|_{\mathrm{Hom}} < \frac{1}{2^n}. \tag{17}$$

From Inequality (17) we deduce that the sequence ψ_n is Cauchy in $\mathrm{Hom}(D^m)$. Therefore, it converges to a homeomorphism ψ. Moreover, by construction $\psi_n|_{\partial D^m} = \mathrm{id}|_{\partial D^m}$ for all $n \geq 1$. Then $\psi|_{\partial D^m} = \mathrm{id}|_{\partial D^m}$.

The convergence of ψ_n to ψ in $\mathrm{Hom}(D^m)$ implies that $\Phi_n = \psi_n \circ f \in \mathrm{RHom}(D^m)$ converges to $\Phi = \psi \circ f \in \mathrm{RHom}(D^m)$ as $n \to +\infty$. Since $f(D^m) \subset \mathrm{int}(D^m)$ and $\psi \in \mathrm{Hom}(D^m)$, we deduce that $\Phi(D^m) \subset \mathrm{int}(D^m)$. Moreover, by construction $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$ are families of atoms up to generation n for Φ_n, and $\Phi_j(x) = \Phi_n(x)$ for all $x \in D^m \setminus \bigcup_{B\in\mathcal{A}_n} B$ and for all $j \geq n$. Since $\lim_j \Phi_j = \Phi$, the boxes of the family $\mathcal{A}_n$ are n-atoms for Φ for all $n \geq 0$. Finally, from Inequality (II), the diameters of the n-atoms converge uniformly to zero as $n \to +\infty$. Thus Φ is a model according to Definition 2.4. □

3. Infinite metric entropy and mixing property of the models.

The purpose of this section is to prove the following Lemma.

Lemma 3.1. (Main Lemma) *Let $\mathcal{H} \subset C^0(D^m)$ be the family of models with $m \geq 2$ (Definition 2.4). Then, for each $\Phi \in \mathcal{H}$ there exists a Φ-invariant mixing (hence ergodic) measure ν supported on an Φ-invariant Cantor set $\Lambda \subset D^m$ such that $h_\nu(\Phi) = +\infty$.*

Remark 3.2. Lemma 3.1 holds, in particular, for $\mathcal{H} \cap \mathrm{RHom}(D^m)$.

To prove Lemma 3.1 we need to define the paths of atoms and to discuss their properties. We also need to define the invariant Cantor set Λ that will support the measure ν and prove some of its topological dynamical properties.

Definition 3.3. (Paths of atoms)
Let $\Phi \in \mathcal{H} \subset C^0(D^m)$, $l \geq 2$ and let $(A_1, A_2, \ldots, A_l)$ be a l-tuple of atoms for Φ of the same generation n, such that

$$A_i \xrightarrow{\Phi} A_{i+1}, \qquad \forall\, i \in \{1, 2, \ldots, l-1\}.$$

We call $(A_1, A_2, \dots, A_l)$ an *l-path of n-atoms from A_1 to A_l*. Let $\mathcal{A}_n^{l*}$ denote the family of all the l-paths of atoms of generation l.

Lemma 3.4. *For all $n \geq 1$, for all $l \geq 2n$, and for all $A_1, A_2 \in \mathcal{A}_n$ there exists an l-path of n-atoms from A_1 to A_2.*

Proof. For $n = 1$, the result is trivial for all $l \geq 2$ (see Definition 2.2). Let us assume by induction that the result holds for some $n - 1 \geq 1$ and let us prove it for n.

Let $E, F \in \mathcal{A}_n$. From equality (3) of Definition 2.3, there exists unique atoms $B_{-1}, B_0, B_1 \in \mathcal{A}_{n-1}$ such that $E \in \Gamma_n(B_{-1}, B_0, B_1)$. Then $B_{-1} \xrightarrow{\Phi} B_0$, $E \subset B_0$ and, by condition d) of Definition 2.3:

$$E \xrightarrow{\Phi} E_1, \qquad \forall\ E_1 \in \Omega_n(B_0, B_1). \tag{18}$$

Analogously, there exists unique atoms $B_*, B_{*+1} \in \mathcal{A}_{n-1}$ such that $F \in \Omega_n(B_*, B_{*+1})$. Then $B_* \xrightarrow{\Phi} B_{*+1}$, $F \subset B_{*+1}$ and

$$E_* \xrightarrow{\Phi} F, \qquad \forall\ E_* \in \bigcup_{\substack{B_{*-1} \in \mathcal{A}_{n-1}: \\ B_{*-1} \xrightarrow{\Phi} B_*}} \Gamma_n(B_{*-1}, B_*, B_{*+1}) \tag{19}$$

Since $B_1, B_* \in \mathcal{A}_{n-1}$ the induction hypothesis ensures that that for all $l \geq 2n-2$ there exists an l-path $(B_1, B_2, \dots, B_l)$ from B_1 to $B_l = B_*$. We write $B_{*-1} = B_{l-1}$, $B_* = B_l$, $B_{*+1} = B_{l+1}$. So Assertion (19) becomes

$$E_l \xrightarrow{\Phi} F, \qquad \forall\ E_l \in \Gamma_n(B_{l-1}, B_l, B_{l+1}) \tag{20}$$

Taking into account that $B_{i-1} \xrightarrow{\Phi} B_i$ for $1 < i \leq l$, and applying condition d) of Definition 2.3, we deduce that, if $E_{i-1} \in \Gamma_n(B_{i-2}, B_{i-1}, B_i) \subset \mathcal{A}_n$, then

$$E_{i-1} \xrightarrow{\Phi} E_i, \qquad \forall\ E_i \in \Omega_n(B_{i-1}, B_i), \qquad \forall\ 1 < i \leq l. \tag{21}$$

Combining (18), (20) and (21) yields an $(l+2)$-path $(E, E_1, \dots, E_l, F)$ of atoms of generation n from E to F, as required. □

Lemma 3.5. *Let $n, l \geq 2$. For each l-path $(B_1, \dots, B_l)$ of $(n-1)$-atoms there exists an l- path $(E_1, E_2, \dots, E_l)$ of n-atoms such that $E_i \subset \operatorname{int}(B_i)$ for all $i = 1, 2, \dots, l$.*

Proof. In the proof of Lemma 3.4 for each l-path $(B_1, B_2, \dots, B_l)$ of $(n-1)$-atoms we have constructed the l-path $(E_1, E_2, \dots, E_l)$ of n-atoms as required. □

Definition 3.6. (The Λ-set) Let $\Phi \in \mathcal{H} \subset C^0(D^m)$ be a model map. Let $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n, \ldots$ be its sequence of families of atoms. The subset

$$\Lambda := \bigcap_{n\geq 0} \bigcup_{A\in\mathcal{A}_n} A$$

of $\mathrm{int}(D^m)$ is called *the* Λ*-set* of the map Φ.

From Definitions 2.2 and 2.3, we know that, for each fixed $n \geq 0$, the set $\Lambda_n := \bigcup_{A\in\mathcal{A}_n} A$, is nonempty, compact, and $\mathrm{int}(\Lambda_n) \supset \Lambda_{n+1}$. Therefore, Λ is also nonempty and compact. Moreover, Λ_n is composed of a finite number of connected components $A \in \mathcal{A}_n$, which by Definition 2.4, satisfy $\lim_{n\to+\infty} \max_{A\in\mathcal{A}_n} \mathrm{diam} A = 0$. Since $\Lambda := \bigcap_{n\geq 0} \Lambda_n$, we deduce that the Λ-set is *a Cantor set* contained in $\mathrm{int}(D^m)$.

Lemma 3.7. *Let $n, l \geq 1$ and $A_1, A_2 \in \mathcal{A}_n$. If there exists an $l+1$-path from A_1 to A_2, then $\Phi^l(A_1 \cap \Lambda) \cap (A_2 \cap \Lambda) \neq \emptyset$.*

Proof. Assume that there exists an $(l+1)$-path from A_1 to A_2. So, from Lemma 3.5, for all $j \geq n$ there exists atoms $B_{j,1}, B_{j,2} \in \mathcal{A}_j$ and an $(l+1)$-path from $B_{j,1}$ to $B_{j,2}$ (with constant length $l+1$) such that

$$B_{n,i} = A_i, \quad B_{j+1,i} \subset B_{j,i}, \quad \forall\, j \geq n, \quad i = 1, 2.$$

Construct the following two points x_1 and x_2:

$$\{x_i\} = \bigcap_{j\geq n_0} B_{j,i}, \qquad i = 1, 2.$$

By Definition 3.6, $x_i \in A_i \cap \Lambda$. So, to finish the proof of Lemma 3.7 it is enough to prove that $\Phi^l(x_1) = x_2$.

Recall that l is fixed. Since Φ is uniformly continuous, for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $(y_0, y_1, \ldots, y_l) \in (D^m)^l$ satisfies $d(\Phi(y_i), y_{i+1}) < \delta$ for $0 \leq i \leq l-1$, then the points y_0 and y_l satisfy $d(\Phi^l(y_0), y_l) < \varepsilon$. We choose δ small enough such that additional $d(\Phi^l(x), \Phi^l(y)) < \varepsilon$ if $d(x, y) < \delta$.

From (5), there exists $j \geq n$ such that $\mathrm{diam}(B_{j,i}) < \delta$. Since there exists an $(l+1)$-path from $B_{j,1}$ to $B_{j,2}$, there exists a $(y_0, \ldots, y_l)$ as in the previous paragraph with $y_0 \in B_{j,1}$ and $y_l \in B_{j,2}$. Thus

$$\begin{aligned} d(\Phi^l(x_1), x_2) &\leq d(\Phi^l(x_1), \Phi^l(y_0)) + d(\Phi^l(y_0), y_l) + d(y_l, x_1) \\ &< \mathrm{diam}(\Phi^l(B_{j,1})) + \varepsilon + \mathrm{diam}(B_{j,2}) < 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\Phi^l(x_1) = x_2$, as required. □

Lemma 3.8. (Topological dynamical properties of Λ)

a) The Λ-set of a model map $\Phi \in \mathcal{H}$ is Φ-invariant, i.e., $\Phi(\Lambda) = \Lambda$.
b) The map Φ restricted to the Λ-set is topologically mixing.

c) In particular, $\Phi^l(A_1 \cap \Lambda) \cap (A_2 \cap \Lambda) \neq \emptyset$, *for all* $n \geq 1$, *for any two atoms* $A_1, A_2 \in \mathcal{A}_n$ *and for all* $l \geq 2n-1$.

Proof. a) Let $x \in \Lambda$ and let $\{A_n(x)\}_{n\geq 0}$ the unique sequence of atoms such that $x \in A_n(x)$ and $A_n(x) \in \mathcal{A}_n$ for all $n \geq 0$. Then, $\Phi(x) \in \Phi(A_n(x))$ for all $n \geq 0$. From Definition 2.3, for all $n \geq 0$ there exists an atom $B_n \in \mathcal{A}_n$ such that $A_n(x) \xrightarrow{\Phi} B_n$. Therefore $\Phi(A_n(x)) \cap B_n \neq \emptyset$. Let d denote the Hausdorff distance between subsets of D^m, we deduce

$$d(\Phi(x), B_n) \leq \operatorname{diam}\big(\Phi(A_n(x))\big) + \operatorname{diam}\big(B_n\big).$$

Moreover, Equality (5) and the continuity of Φ imply

$$\lim_{n\to+\infty} \max\big\{\operatorname{diam}\big(\Phi(A_n(x))\big), \operatorname{diam}\big(B_n\big)\big\} = 0.$$

Then, for all $\varepsilon > 0$ there exists $n_0 \geq 0$ such that $d(\Phi(x), B_n) < \varepsilon$ for some atom $B_n \in \mathcal{A}_n$ for all $n \geq n_0$. Since any atom of any generation intersects Λ, we deduce that $d(\Phi(x), \Lambda) < \varepsilon$ for each $\varepsilon > 0$. Since Λ is compact, this implies $\Phi(x) \in \Lambda$. We have proved that $\Phi(\Lambda) \subset \Lambda$.

Now, let us prove the other inclusion. Let $y \in \Lambda$ and let $\{B_n(y)\}_{n\geq 0}$ the unique sequence of atoms such that $y \in B_n(y)$ and $B_n(y) \in \mathcal{A}_n$ for all $n \geq 0$. From Definition 2.3, for all $n \geq 0$ there exists an atom $A_n \in \mathcal{A}_n$ such that $A_n \xrightarrow{\Phi} B_n(y)$. Therefore $\Phi(A_n) \cap B_n(y) \neq \emptyset$. We deduce that, for all $n \geq 0$, there exists a point $x_n \in A_n \in \mathcal{A}_n$ such that $\Phi(x_n) \in B_n(y)$. Since any atom A_n contains points of Λ, we obtain

$$d(x_n, \Lambda) \leq \operatorname{diam}(A_n) \text{ and } d(\Phi(x_n), y) \leq \operatorname{diam}(B_n(y)), \qquad \forall\, n \geq 0.$$

Let x be the limit of a convergent subsequence of $\{x_n\}_{n\geq 0}$. Applying Equality (5) and the continuity of Φ, we deduce that $d(x, \Lambda) = 0$ and $d(\Phi(x),\, y) = 0$. This means that $y = \Phi(x)$ and $x \in \Lambda$. We have proved that $y \in \Phi(\Lambda)$ for all $y \in \Lambda$; namely $\Lambda = \Phi(\Lambda)$, as required.

c) We will prove a stronger assertion: for any two atoms, even of different generation, there exists $l_0 \geq 1$ such that

$$\Phi^l(A_1 \cap \Lambda) \cap (A_2 \cap \Lambda) \neq \emptyset \quad \forall\, l \geq l_0. \tag{22}$$

It is not restrictive to assume that A_1 and A_2 are atoms of the same generation n_0 (if not, take n_0 equal to the largest of both generation and substitute A_i by an atom of generation n_0 contained in A_i). Applying Lemma 3.4, for all $l \geq 2n_0 - 1$ there exists an $(l+1)$-path from A_1 to A_2. So, from Lemma 3.7 $\Phi^l(A_1 \cap \Lambda) \cap (A_2 \cap \Lambda) \neq \emptyset$, as required.

b) The intersection of Λ with the atoms of all the generations generates its topology, thus Equation (22) implies that Λ is topologically mixing. □

For fixed $(A_0, A_l) \in \mathcal{A}_n^2$ we set

$$\mathcal{A}_n^{l+1\,*}(A_0, A_l) := \{(A_0, A_1, \dots, A_{l-1}, A_l) \in \mathcal{A}_n^{l+1\,*}\}.$$

Lemma 3.9. *Let $l, n \geq 1$. Then*

a) $\#\mathcal{A}_n^{l+1\,*} = 2^{nl} \cdot (\#\mathcal{A}_n)$.

b) $\#\mathcal{A}_n^{l+1\,*}(A_0, A_l) = \frac{2^{nl}}{\#\mathcal{A}_n} \;\; \forall\, (A_0, A_l) \in \mathcal{A}_n^2$, *for all* $l \geq 2n-1$.

Proof. a) Each $(l+1)$-path $(A_0, A_1, \dots, A_l)$ of n-atoms is determined by a free choice of the atom $A_0 \in \mathcal{A}_n$, followed by the choice of the atoms $A_j \in \mathcal{A}_n$ such that $A_j \xrightarrow{\Phi} A_{j-1}$ for all $j = 1, \dots, l$. From equality (4) of Definition 2.3, we know that for any fixed $A \in \mathcal{A}_n$ the number of atoms $B \in \mathcal{A}_n$ such that $B \xrightarrow{\Phi} A$ is 2^n. This implies a), as required.

b) We argue by induction on n. Fix $n = 1$ and $l \geq 1$. Since any two atoms $A_j, A_{j+1} \in \mathcal{A}_1$ satisfies $A_j \xrightarrow{\Phi} A_{j+1}$, the number of $(l+1)$-paths

$$(A_0, A_1, \dots, A_j, A_{j+1}, \dots A_{l-1}, A_l)$$

of 1-atoms with (A_0, A_l) fixed, equals $\#(\mathcal{A}_1)^{l-1} = 2^{l-1} = 2^l/2 = 2^{nl}/(\#\mathcal{A}_n)$ with $n = 1$.

Now, let us assume that assertion b) holds for some $n \geq 1$ and let us prove it for $n+1$. Let $l \geq 2(n+1) - 1 = 2n+1 \geq 3$ and let $(B_0, B_l) \in \mathcal{A}_{n+1}^2$. From equality (3) and conditions a) and b) of Definition 2.3, there exists unique $(A_{-1}, A_0, A_1) \in \mathcal{A}_n^{3*}$ and unique $(A_{l-1}, A_l) \in \mathcal{A}_n^{2*}$ such that

$$B_0 \in \Gamma_{n+1}(A_{-1}, A_0, A_1), \qquad B_l \in \Omega_{n+1}(A_{l-1}, A_l).$$

As $(A_1, A_{l-1}) \in \mathcal{A}_n^2$ and $l-2 \geq 2n-1$, the induction hypothesis ensures that the number of $(l-1)$-paths $(A_1, A_2, \dots, A_{l-1})$ from A_1 to A_{l-1} is

$$\#\mathcal{A}_n^{l-1\,*}(A_1, A_{l-1}) = \frac{2^{n(l-2)}}{\#\mathcal{A}_n} = \frac{2^{n(l-2)}}{2^{n^2}} = 2^{nl-2n-n^2}. \tag{23}$$

Let $\mathcal{C}(B_0, B_l)$ be the set

$$\bigcup_{(A_1,\dots,A_{l-1}) \in \mathcal{A}_n^{l-1\,*}(A_1, A_{l-1})} \big\{(B_0, B_1, \dots, B_l) \in \mathcal{A}_{n+1}^{l+1} \colon B_j \in \Gamma_{n+1}(A_{j-1}, A_j, A_{j+1})\ \forall j\big\},$$

where the families in the above union are pairwise disjoint. It is standard to check that the families in the union $\mathcal{C}(B_0, B_l)$ are pairwise disjoint, because for $A \neq \widetilde{A}$ in $\mathcal{A}_n$, the families $\Gamma_{n+1}(\cdot, A, \cdot)$ and $\Gamma_{n+1}(\cdot, \widetilde{A}, \cdot)$ are disjoint.

A straightforward verification shows that

$$\mathcal{A}_{n+1}^{l+1\,*}(B_0, B_l) = \mathcal{C}(B_0, B_l). \tag{24}$$

Now, applying (23) and (24), we obtain

$$\begin{aligned}
&\#\mathcal{A}_{n+1}^{l+1\,*}(B_0,B_l) = \\
&= \sum_{(A_1,\dots,A_{l-1})\in\mathcal{A}_n^{l-1\,*}(A_1,A_{l-1})} \#\big\{(B_0,B_1,\dots,B_l)\in\mathcal{A}_{n+1}^{l+1}\colon B_j\in\Gamma_{n+1}(A_{j-1},A_j,A_{j+1})\ \forall j\big\} \\
&= \sum_{(A_1,\dots,A_{l-1})\in\mathcal{A}_n^{l-1\,*}(A_1,A_{l-1})} \prod_{j=1}^{l-1}\#\Gamma_{n+1}(A_{j-1},A_j,A_{j+1}) \\
&= \ (\#\mathcal{A}_n^{l-1\,*}(A_1,A_{l-1}))\cdot 2^{l-1} = 2^{nl-2n-n^2+l-1} = 2^{(n+1)l-(n+1)^2} = \frac{2^{(n+1)l}}{\#\mathcal{A}_{n+1}}
\end{aligned}$$

as required. □

Let $\vec{A}_n^l := (A_0,A_1,\dots,A_l)$ be an $(l+1)$-path of n-atoms, and $\mathcal{F}_{n,l}(\vec{A}_n^l) := \Big\{G\in\mathcal{A}_{n+l}\colon G\cap\Lambda\subset\bigcap_{j=0}^l\Phi^{-j}(A_j)\Big\}$.

Lemma 3.10. (Intersection of Λ with l-paths) *Fix $l,n\geq 1$. Then*

a) For any $G\in\mathcal{A}_{n+l}$, there exists a unique $(l+1)$-path $(A_0,A_1,\dots,A_l)$ of n-atoms such that $G\cap\Lambda\subset\bigcap_{j=0}^l\Phi^{-j}(A_j)$.

b) For any atoms $G\in\mathcal{A}_{n+l}$, $A\in\mathcal{A}_n$ and $j\in\{0,1,\dots,l\}$:

$$(G\cap\Lambda)\cap\Phi^{-j}(A)\neq\emptyset \quad\Leftrightarrow\quad G\cap\Lambda\subset\Phi^{-j}(A).$$

c) For any $(l+1)$-path $\vec{A}_n^l=(A_0,A_1,\dots,A_l)$ of n-atoms,

$$\Lambda\cap\bigcap_{j=0}^l\Phi^{-j}(A_j) = \bigcup_{G\in\mathcal{F}_{n,l}(\vec{A}_n^l)} G\cap\Lambda, \tag{25}$$

d) For any atom $G\in\mathcal{A}_{n+l}$ and any path $\vec{A}_n^l\in\mathcal{A}_n^{l+1\,*}$: $G\in\mathcal{F}_{n,l}(\vec{A}_n^l)$ if and only if there exists $(G_0,G_1,\dots,G_l)\in\mathcal{A}_{n+l}^{l+1\,*}$ such that $G_0=G$ and $G_j\subset A_j$ for all $j=0,1,\dots,l$.

e) For any $(l+1)$-path $(A_0,A_1,\dots,A_l)$ of n-atoms,

$$\#\mathcal{F}_{n,l}(\vec{A}_n^l) = \frac{1}{2^{nl}}\cdot\frac{\#\mathcal{A}_{n+l}}{\#\mathcal{A}_n}.$$

Proof. a) From equalities (2) and (3) of Definition 2.3, for any atom G of generation $n+l$ there exist two unique atoms B,C of generation $n+l-1$ such that $B\xrightarrow{\Phi}C$, $G\subset B$ and $G\xrightarrow{\Phi}E$ for all $E\in\Omega_{n+l}(B,C)$. Moreover, from condition d) of Definition 2.3, we have

$$\Phi(G)\cap F\neq\emptyset \text{ if and only if } F\in\Omega_{n+l}(B,C). \tag{26}$$

We claim that

$$\Phi(G \cap \Lambda) \subset \mathrm{int}(C). \tag{27}$$

Since Λ is Φ-invariant, for any $x \in G \cap \Lambda$, we have $\Phi(x) \in \Phi(G) \cap \Lambda$. Therefore $\Phi(x)$ is in the interior of some atom $E(x)$ of generation $n+l$ (see Definition 3.6). From (26), $E(x) \in \Omega_{n+l}(B, C)$. Thus $E(x) \subset \mathrm{int}(C)$ and $\Phi(x) \in \mathrm{int}(C)$ for all $x \in G \cap \Lambda$ proving (27).

So, there exists $C_1 \in \mathcal{A}_{n+l-1}$ such that $\Phi(G \cap \Lambda) \subset \mathrm{int}(C_1) \cap \Lambda$. Applying the same assertion to C_1 instead of G, we deduce that there exists $C_2 \in \mathcal{A}_{n+l-2}$ such that $\Phi(C_1 \cap \Lambda) \subset \mathrm{int}(C_2) \cap \Lambda$. So, by induction, we construct atoms

$$C_1, C_2, \dots, C_l \quad \text{such that} \quad C_j \in \mathcal{A}_{n+l-j} \quad \text{and}$$

$$\Phi^j(G \cap \Lambda) \subset \mathrm{int}(C_j) \cap \Lambda, \qquad \forall\ j = 1, \dots, l.$$

Since any atom of generation larger than n is contained in a unique atom of generation n, there exists $A_0, A_1, \dots, A_l \in \mathcal{A}_n$ such that $A_0 \supset G$ and $A_i \supset C_i, \qquad \forall\ i = 1, \dots, l$. We obtain

$$\Phi^j(G \cap \Lambda) \subset \mathrm{int}(A_j), \qquad \forall\ j = 0, 1, \dots, l.$$

Besides, $(A_0, A_1, \dots, A_l)$ is an $(l+1)$-path since $\emptyset \neq \Phi^j(G \cap \Lambda) \subset \Phi(A_{j-1}) \cap \mathrm{int}(A_j)$; hence $A_{j-1} \xrightarrow{\Phi} A_j$ for all $j = 1, \dots, l$. Then, $G \cap \Lambda \subset \Phi^{-j}(A_j)$ for all $j = 0, 1, \dots, l$; proving the existence statement in a).

To prove uniqueness assume that $(A_0, A_1, \dots, A_l)$ and $(A'_0, A'_1, \dots, A'_l)$ are paths of n-atoms such that

$$G \cap \Lambda \subset \Phi^{-j}(A_j) \cap \Phi^{-j}(A'_j) \ \ \forall\ j \in \{0, 1, \dots, l\}.$$

Then $A_j \cap A'_j \neq \emptyset$ for all $j \in \{0, 1, \dots, l\}$. Since two different atoms of the same generation are pairwise disjoint, we deduce that $A_j = A'_j$ for all $j \in \{0, 1, \dots, l\}$ as required.

b) Trivially, if $G \cap \Lambda \subset \Phi^{-j}(A)$, then $(G \cap \Lambda) \cap \Phi^{-j}(A) \neq \emptyset$. Now, let us prove the converse assertion. Fix $G \in \mathcal{A}_{n+l}$ and $A \in \mathcal{A}_n$ satisfying $(G \cap \Lambda) \cap \Phi^{-j}(A) \neq \emptyset$. Applying part a) there exists $\widetilde{A} \in \mathcal{A}_n$ such that $G \cap \Lambda \subset \Phi^{-j}(\widetilde{A})$. Therefore $G \cap \Lambda \cap \Phi^{-j}(A) \subset \Phi^{-j}(\widetilde{A} \cap A) \neq \emptyset$. Since A and $\widetilde{A}$ are atoms of generation n, and two different atoms of the same generation are disjoint, we conclude that $\widetilde{A} = A$, hence $G \cap \Lambda \subset \Phi^{-j}(A)$, as required.

c) For the $(l+1)$-path $\vec{A}^l_n = (A_0, A_1, \dots, A_l)$ of n-atoms, construct

$$\widetilde{\mathcal{F}}_{n,l}(\vec{A}^l_n) := \big\{G \in \mathcal{A}_{n+l} \colon G \cap \Lambda \cap \Phi^{-j}(A_j) \neq \emptyset \ \ \forall j \in \{0, 1, \dots, l\}\big\}. \tag{28}$$

From the definitions of the families $\mathcal{F}_{n,l}$ and $\widetilde{\mathcal{F}}_{n,l}$, and taking into account that Λ is contained in the union of $(n+l)$-atoms, we obtain:

$$\bigcup_{G\in\mathcal{F}_{n,l}(\vec{A}_n^l)} G\cap\Lambda \;\subset\; \Lambda\cap\big(\bigcap_{j=0}^{l}\Phi^{-j}(A_j)\big) \;\subset\; \bigcup_{G\in\widetilde{\mathcal{F}}_{n,l}(\vec{A}_n^l)} G\cap\Lambda.$$

Therefore, to prove Equality (25) it is enough to show that

$$\widetilde{\mathcal{F}}_{n,l}(\vec{A}_n^l)=\mathcal{F}_{n,l}(\vec{A}_n^l), \tag{29}$$

but this equality immediately follows from the construction of the families $\mathcal{F}_{n,l}(\vec{A}_n^l)$ and $\widetilde{\mathcal{F}}_{n,l}(\vec{A}_n^l)$ by assertion b).

d) For each $(l+1)$-path $\vec{A}_n^l=(A_0,A_1\ldots,A_l)$ of n-atoms construct the family $\mathcal{G}_{n,l}(\vec{A}_n^l):=$

$$\big\{G_0\in\mathcal{A}_{n+l}\colon\ \exists(G_0,G_1,\ldots,G_l)\in\mathcal{A}_{n+l}^{l+1\,*}\text{ such that }G_j\subset A_j\ \forall j\big\}$$

We will first prove that $\mathcal{G}_{n,l}(\vec{A}_n^l)\supset\mathcal{F}_{n,l}(\vec{A}_n^l)$. In fact, take $G\in\mathcal{F}_{n,l}(\vec{A}_n^l)$, and take any point $x\in G\cap\Lambda$. We have $\Phi^j(x)\in A_j\cap\Lambda$ for all $j\in\{0,1,\ldots,l\}$ (recall that Λ is Φ-invariant). Since any point in Λ is contained in the interior of some atom of any generation, there exists an atom G_j of generation $n+l$ such that $\Phi^j(x)\in\operatorname{int}(G_j)$. Recall that each atom of generation $n+l$ is contained in a unique atom of generation n. As $\Phi^j(x)\in G_j\cap A_j\neq\emptyset$, and different atoms of the same generation are disjoint, we conclude that $G_j\subset A_j$. Besides $G_0=G$ because $x\in G\cap G_0$. Finally $(G_0,G_1,\ldots,G_l)$ is a $(l+1)$-path because $\Phi^{j+1}(x)=\Phi(\Phi^j(x))\in\Phi(G_j)\cap\operatorname{int}(G_{j+1})$ for all $j\in\{0,1,\ldots,l-1\}$; namely $G_j\overset{\Phi}{\to}G_{j+1}$. We have proved that $G\in\mathcal{G}_{n,l}(\vec{A}_n^l)$, as required.

Now, let us prove that $\mathcal{G}_{n,l}(\vec{A}_n^l)\subset\mathcal{F}_{n,l}(\vec{A}_n^l)$. Assume that $G_0\in\mathcal{A}_{n+l}$ and $(G_0,G_1,\ldots,G_l)\in\mathcal{A}_{n+l}^{l+1\,*}$ satisfies $G_j\subset A_j$ for all $j\in\{0,1,\ldots,l\}$. Therefore $(G_0,G_1,\ldots,G_j)$ is a $(j+1)$-path of $(n+l)$-atoms for all $j\in\{1,2,\ldots,l\}$. Applying Lemma 3.7, we obtain $G_0\cap\Lambda\cap\Phi^{-j}(G_j)\neq\emptyset$. Therefore, taking into account that $G_j\subset A_j$, we deduce that

$$G_0\cap\Lambda\cap\Phi^{-j}(A_j)\neq\emptyset\ \ \forall\ j\in\{0,1,\ldots,l\}.$$

Therefore $G_0\in\widetilde{\mathcal{F}}_{n,l}(\vec{A}_n^l)=\mathcal{F}_{n,l}(\vec{A}_n^l)$ (recall (28) and (29)). This holds for any $G_0\in\mathcal{G}_{n,l}(\vec{A}_n^l)$, thus $\mathcal{G}_{n,l}(\vec{A}_n^l)\subset\mathcal{F}_{n,l}(\vec{A}_n^l)$, as required.

e) From Assertion a) we obtain:

$$\mathcal{A}_{n+l}=\bigcup_{\vec{A}_n^l\in\mathcal{A}_n^{l+1\,*}}\mathcal{F}_{n,l}(\vec{A}_n^l), \tag{30}$$

where the families in the above union are pairwise disjoint, due to the uniqueness property of assertion a).

Recall the characterization of the family $\mathcal{F}_{n,1}(\vec{A}_n^l)$ given by Assertion d). From Definition 2.3 (condition a) and equality (4), the number of atoms of each generation larger than n that are contained in each $A_j \in \mathcal{A}_n$, and also the number of atoms $G_j \in \mathcal{A}_{n+1}$ such that $G_j \overset{\Phi}{\to} G_{j+1}$, are constants that depend only on the generations but not on the chosen atom. Therefore, there exists a constant $k_{n,l}$ such that $\#\mathcal{F}_{n,l}(\vec{A}_n^l) = \#\mathcal{G}_{n,l}(\vec{A}_n^l) = k_{n,l}$ for all the $(l+1)$-paths of n-atoms. So, from Equality (30) we obtain:

$$\#\mathcal{A}_{n+l} = (\#\mathcal{A}_n^{l+1\,*}) \cdot (\#\mathcal{F}_{n,l}(\{A_j\})),$$

and applying Lemma 3.9, we conclude

$$\#\mathcal{A}_{n+l} = 2^{nl} \cdot (\#\mathcal{A}_n) \cdot (\#\mathcal{F}_{n,l}(\{A_j\})),$$

as required. □

We turn to the proof of Lemma 3.1. We will first construct the measure ν and then prove that it has the required properties.

We start by defining an additive pre-measure on the Λ-set of Φ by

$$\nu^*(A \cap \Lambda) := \frac{1}{\#\mathcal{A}_n}, \qquad \forall\ A \in \mathcal{A}_n, \qquad \forall\ n \geq 0.$$

Since ν^* is a pre-measure defined in a family of sets that generates the Borel σ-algebra of Λ, there exists a unique Borel probability measure ν supported on Λ such that

$$\nu(A \cap \Lambda) := \frac{1}{\#\mathcal{A}_n}, \qquad \forall\ A \in \mathcal{A}_n, \qquad \forall\ n \geq 0. \tag{31}$$

In the following lemmas we will prove that ν is Φ-invariant, mixing, and that the metric entropy $h_\nu(\Phi)$ is infinite.

Lemma 3.11. *ν is invariant by Φ.*

Proof. Since the atoms of all generation intersected with Λ generates the Borel σ-algebra of Λ, it is enough to prove that

$$\nu(C \cap \Lambda) = \nu(\Phi^{-1}(C \cap \Lambda)), \qquad \forall\ C \in \mathcal{A}_n, \qquad \forall\ n \geq 0. \tag{32}$$

From (3), taking into account that Λ is invariant and that any point in Λ belongs to an atom of generation $n+1$, we obtain:

$$\Phi^{-1}(C \cap \Lambda) = \bigcup_{\substack{B \in \mathcal{A}_n \\ B \overset{\Phi}{\to} C}} \ \bigcup_{\substack{D \in \mathcal{A}_n \\ D \overset{\Phi}{\to} B}} \ \bigcup_{G \in \Gamma_{n+1}(D,B,C)} (G \cap \Lambda),$$

where both unions are of pairwise disjoint sets. Using (4) we obtain

$$\nu(\Phi^{-1}(C\cap\Lambda)) = \sum_{\substack{B\in\mathcal{A}_n\\ B\xrightarrow{\Phi}C}}\ \sum_{\substack{D\in\mathcal{A}_n\\ D\xrightarrow{\Phi}B}}\ \sum_{G\in\Gamma_{n+1}(B,C,D)} \nu(G\cap\Lambda)$$

$$= N_C\cdot N_B\cdot(\#\Gamma_{n+1}(B,C,D))\cdot\frac{1}{\#\mathcal{A}_{n+1}}, \tag{33}$$

where $N_X := \#\{Y\in\mathcal{A}_n\colon Y\xrightarrow{\Phi}X\}) = 2^n$ for all $X\in\mathcal{A}_n$. Since $\#\Gamma_{n+1}(B,C,D)) = 2$ (this is part c) of Definition 2.3) and $\#\mathcal{A}_{n+1} = 2^{(n+1)^2}$, we conclude

$$\nu(\Phi^{-1}(C\cap\Lambda)) = 2^n\cdot 2^n\cdot 2\cdot\frac{1}{2^{(n+1)^2}} = \frac{1}{2^{n^2}} = \frac{1}{\#\mathcal{A}_n} = \nu(C\cap\Lambda),$$

proving Equality (32) as required. □

Lemma 3.12. *ν is mixing.*

Proof. The family of atoms of all generations intersected with Λ generates the Borel σ-algebra of Λ, thus it is enough to prove that for any pair (C_0, D_0) of atoms (of equal or different generations) there exists $l_0\geq 1$ such that

$$\nu(\Phi^{-l}(D_0\cap\Lambda)\cap(C_0\cap\Lambda)) = \nu(C_0\cap\Lambda)\cdot\nu(D_0\cap\Lambda)\ \ \forall\, l\geq l_0. \tag{34}$$

Let us first prove this in the case that C_0 and D_0 are atoms of the same generation n. Take $l\geq 2n-1$. Applying Lemma 3.8-c), we have $\Phi^{-l}(D_0\cap\Lambda)\cap(C_0\cap\Lambda)\neq\emptyset\ \ \forall\, l\geq 2n-1$.

Fix $l\geq 2n-1$. We will use the notation

$$\vec{A}_n^l := (C_0, A_1, \ldots, A_{l-1}, D_0)\in\mathcal{A}_n^{l+1\,*}(C_0, D_0)$$

to denote any one of the $2^{nl}/(\#\mathcal{A}_n)$ different $l+1$-paths of n-atoms from C_0 to D_0 (see Lemma 3.9-b)).

We assert that

$$\Phi^{-l}(D_0\cap\Lambda)\cap(C_0\cap\Lambda) = T := \bigcup_{\vec{A}_n^l\in\mathcal{A}_n^{l+1\,*}(C_0,D_0)}\ \bigcup_{B\in\mathcal{F}_{n,l}(\vec{A}_n^l)}(B\cap\Lambda), \tag{35}$$

where the family $\mathcal{F}_{n,l}(\vec{A}_n^l)$ of $(n+l)$-atoms is defined in Lemma 3.10-c).

First, let us prove that $\Phi^{-l}(D_0\cap\Lambda)\cap(C_0\cap\Lambda)\subset T$. Fix $x\in(D_0\cap\Lambda)\cap(C_0\cap\Lambda)$. Then C_0, D_0 are the unique atoms of generation n that contain x and $\Phi^l(x)\in\Phi^l(\Lambda)=\Lambda$ respectively. Since $x\in\Lambda$, there exists a unique atom B of generation $n+l$ that contains x. Applying Lemma 3.10-a) there exists a unique $(A_0, A_1, \ldots, A_l)\in\mathcal{A}_n^{l+1\,*}$ such that $B\cap\Lambda\subset\Phi^{-j}(A_j)$ for all $j\in\{0,1,\ldots,l\}$. Since the n-atom that contains x is C_0, and two different n-atoms are disjoint, we deduce that

$A_0 = C_0$. Analogously, since the n-atom that contains $\Phi^l(x)$ is D_0 and the preimages of two different n-atoms are disjoint, we deduce that $A_l = D_0$. Therefore we have found $\vec{A}_n^l = (C_0, A_1, \dots, A_{l-1}, D_0)$ and $B \in \mathcal{F}_{n,l}(\vec{A}_n^l)$ such that $x \in B \cap \Lambda$. In other words, $x \in T$, as required.

Next, let us prove that $\Phi^{-l}(D_0 \cap \Lambda) \cap (C_0 \cap \Lambda) \supset T$. Take $B \in \mathcal{F}_{n,l}(\vec{A}_n^l)$ for some $\vec{A}_n^l = (C_0, A_1, \dots, A_{l-1}, D_0)$. From the definition of the family $\mathcal{F}_{n,l}(\vec{A}_n^l)$ in Lemma 3.10-c), we have $B \cap \Lambda \subset (C_0 \cap \Lambda) \cap \Phi^{-l}(D_0)$. Besides $B \cap \Lambda \in \Phi^l(\Lambda)$ because $\Phi^l(\Lambda) = \Lambda$. We conclude that $B \cap \Lambda \subset (C_0 \cap \Lambda) \cap \Phi^{-l}(D_0 \cap \Lambda)$, proving that $T \subset \Phi^{-l}(D_0 \cap \Lambda) \cap (C_0 \cap \Lambda)$, as required. This ends the proof of equality (35).

By definition n-atoms are pairwise disjoint, thus the sets in the union constructing T are pairwise disjoint. Therefore, from (35), and applying Lemma 3.9-b) and Lemma 3.10-e), we deduce

$$\begin{aligned}
\nu((C_0 \cap \Lambda) \cap \Phi^{-l}(D_0 \cap \Lambda)) &= \sum_{\vec{A}_n^l \in \mathcal{A}_n^{l+1\,*}(C_0, D_0)} \sum_{B \in \mathcal{F}_{n,l}(\vec{A}_n^l)} \nu(B \cap \Lambda) \\
&= (\#\mathcal{A}_n^{l+1\,*}(C_0, D_0)) \cdot (\#\mathcal{F}_{n,l}(\vec{A}_n^l)) \cdot \frac{1}{\#\mathcal{A}_{n+l}} \\
&= \frac{2^{nl}}{\#\mathcal{A}_n} \cdot \frac{1}{2^{nl}} \cdot \frac{\#\mathcal{A}_{n+l}}{\#\mathcal{A}_n} \cdot \frac{1}{\#\mathcal{A}_{n+l}} = \frac{1}{\#\mathcal{A}_n} \cdot \frac{1}{\#\mathcal{A}_n} \\
&= \nu(C_0 \cap \Lambda) \cdot \nu(D_0 \cap \Lambda).
\end{aligned}$$

This ends the proof of equality (34) in the case that C_0 and D_0 are atoms of the same generation n, taking $l_0 = 2n - 1$.

Now, let us prove equality (34) when C_0 and D_0 are atoms of different generations. Let n equal the maximum of both generations. Take $l \geq 2n - 1$. Since Λ is contained in the union of the atoms of any generation, we have

$$C_0 \cap \Lambda = \bigcup_{C \in \mathcal{A}_n, C \subset C_0} C \cap \Lambda,$$

where the sets in the union are pairwise disjoint. Analogously

$$\Phi^{-l}(D_0 \cap \Lambda) = \bigcup_{D \in \mathcal{A}_n, D \subset D_0} \Phi^{-l}(D \cap \Lambda),$$

where also the sets in this union are pairwise disjoint. So,

$$(C_0 \cap \Lambda) \cap \Phi^{-l}(D_0 \cap \Lambda) = \bigcup_{C \in \mathcal{A}_n, C \subset C_0} \bigcup_{D \in \mathcal{A}_n, C \subset D_0} (C \cap \Lambda) \cap \Phi^{-l}(D \cap \Lambda).$$

Since the sets in the union are pairwise disjoint, we deduce

$$\nu((C_0\cap\Lambda)\cap\Phi^{-l}(D_0\cap\Lambda)) = \sum_{C\in\mathcal{A}_n, C\subset C_0}\ \sum_{D\in\mathcal{A}_n, C\subset D_0} \nu((C\cap\Lambda)\cap\Phi^{-l}(D\cap\Lambda)).$$

As C, D are atoms of the same generation n, and $l \geq 2n-1$, we can apply the first case proved above, to deduce that

$$\nu((C_0\cap\Lambda)\cap\Phi^{-l}(D_0\cap\Lambda)) =$$

$$\#\{C\in\mathcal{A}_n, C\subset C_0\}\cdot\#\{D\in\mathcal{A}_n, C\subset D_0\}\cdot\frac{1}{(\#\mathcal{A}_n)^2}. \tag{36}$$

The number of atoms of generation n contained in an atom C_0 of generation n_1 larger or equal than n, does not depend of the chosen atom C_0. Therefore,

$$\#\{C\in\mathcal{A}_n, C\subset C_0\} = \frac{\#\mathcal{A}_n}{\#\mathcal{A}_{n_1}} = (\#\mathcal{A}_n)\cdot\nu(C_0\cap\Lambda).$$

Analogously

$$\#\{D\in\mathcal{A}_n, D\subset D_0\} = (\#\mathcal{A}_n)\cdot\nu(D_0\cap\Lambda).$$

Finally, substituting in equality (36) we conclude that

$$\nu(\Phi^{-l}(D_0\cap\Lambda)\cap(C_0\cap\Lambda)) = \nu(C_0\cap\Lambda)\cdot\nu(D_0\cap\Lambda)\ \ \forall\ l\geq 2n-1,$$

ending the proof. □

Lemma 3.13. $h_\nu(\Phi) = +\infty$.

Proof. For $n\geq 1$ we consider the partition $\mathcal{A}_n$ of Λ consisting of all the n-atoms intersected with Λ. By the definition of metric entropy

$$h_\nu(\Phi) := \sup_{\mathcal{P}} h(\mathcal{P},\nu) \geq h(\mathcal{A}_n,\nu),\ \text{where} \tag{37}$$

$$h(\mathcal{A}_n,\nu) := \lim_{l\to+\infty}\frac{1}{l}H\Big(\bigvee_{j=0}^{l}(\Phi^{-j}\mathcal{A}_n),\nu\Big), \tag{38}$$

$$\mathcal{Q}_l := \bigvee_{j=0}^{l}\Phi^{-j}\mathcal{A}_n := \Big\{\bigcap_{j=0}^{l}\Phi^{-j}A_j\cap\Lambda\neq\emptyset:\quad A_j\in\mathcal{A}_n\Big\},$$

$$H(\mathcal{Q}_l,\nu) := -\sum_{X\in\mathcal{Q}_l}\nu(X)\log\nu(X). \tag{39}$$

For any nonempty $X := \Lambda \cap \Big(\bigcap_{j=0}^{l} \Phi^{-j} A_j \Big) \in \mathcal{Q}_l$, Lemma 3.10-c) yields

$$\nu(X) = \nu\Big(\bigcap_{j=0}^{l} \Phi^{-j} A_j \cap \Lambda \Big) = \sum_{G \in \mathcal{F}_{n,l}(\vec{A}_j^l)} \nu(G \cap \Lambda).$$

Since G is an atom of generation $n+l$, we have $\nu(G \cap \Lambda) = 1/(\#\mathcal{A}_{n+l})$, thus applying Lemma 3.10-e), yields

$$\nu(X) = \frac{\#\mathcal{F}_{n,l}(\{A_j\})}{\#\mathcal{A}_{n+l}} = \frac{1}{2^{nl} \cdot \#\mathcal{A}_n}.$$

Combining this with (39) yields $H(\mathcal{Q}_l) = \log(\#\mathcal{A}_n) + nl \cdot \log 2$. Finally, substituting in Equality (38), we conclude

$$h(\mathcal{A}_n, \nu) := \lim_{l \to +\infty} \frac{1}{l} H\Big(\mathcal{Q}_l, \nu \Big) = n \log 2.$$

Combining with (37) yields $h_\nu(\Phi) \geq n \log 2$, for all $n \geq 1$; hence $h_\nu(\Phi) = +\infty$. □

Proof of Lemma 3.1. As proved in Lemmas 3.11, 3.12 and 3.13, the probability measure ν constructed by equality (31) is Φ-invariant, mixing and has infinite metric entropy, as required. □

4. Periodic Shrinking Boxes

In this section we will prove Theorems 1 and 3. The proofs are based on the properties of the models proved in the previous sections, and on the existence of the periodic shrinking boxes which we construct here.

Definition 4.1. (Periodic shrinking box) Let $f \in C^0(M)$ and $K \subset M$ be a box. We call K *periodic shrinking with period* $p \geq 1$ for f, if $K,\ f(K),\ f^2(K),\ \ldots, f^{p-1}(K)$ are pairwise disjoint, and $f^p(K) \subset \mathrm{int}(K)$. If so, we call $f^p|_K : K \to \mathrm{int}(K)$ the *return map.*

Recall that the manifold M is compact. This assumption is important to obtain the following Lemmas 4.2 and 4.4. We will construct periodic shrinking boxes whose return maps are homeomorphisms onto their images. Although this latter condition is unnecessary for the construction of the periodic shrinking boxes, it will be used later in the proofs of Lemmas 4.8 and 4.9 where the return maps must be topologically conjugated to model maps.

Lemma 4.2. *For any $\delta > 0$, there exists an open and dense set of maps $f \in C^0(M)$ that have a periodic shrinking box K with* $\mathrm{diam}(K) < \delta$.

For a dense set of $f \in C^0(M)$ the return map to K is a homeomorphism onto its image.

The proof of this lemma uses the following definition and technical result.

Definition 4.3. Let $H \subset K$ be two boxes in the manifold M. We say that H is *homothetic* to K if there exists a homeomorphism $\phi : K \to [0,1]^m$ such that $\phi(H) = [a,b]^m \subset [0,1]^m$

Lemma 4.4. *Let $f \in C^0(M)$ and $x_0 \in M$. For all $\varepsilon > 0$, there exists $g \in C^0(M)$ and a neighborhood H of x_0 such that $\|g - f\|_{C^0} < \varepsilon$ and $g|_H$ is a homeomorphism onto its image. Moreover, g can be constructed to coincide with f except in an arbitrarily small neighborhood of x_0.*

Proof. Fix $0 < \delta < \varepsilon/2$ small enough such that if $\mathrm{dist}(x,y) < \delta$, then $\mathrm{dist}(f(x), f(y)) < \varepsilon/2$. Construct boxes H and K such that $\mathrm{diam}(K) < \delta$ and H is homothetic to K and $x_0 \in \mathrm{int}(H) \subset H \subset \mathrm{int}(K)$. Let $\phi : K \to [0,1]^m$ be the homeomorphism of Definition 4.3. Consider the compact neighborhood $K' \subset M$ of radius $\varepsilon/2$ and center $f(x_0)$, and a homeomorphism $\phi' : K' \to [0,1]^m$ (if necessary reduce ε from the very beginning so the compact balls of radius $\varepsilon/2$ in the manifold are homeomorphic to $[0,1]^m$). Consider a box $H' \subset \mathrm{int}(K')$.

Fix a homeomorphism $\chi : H \to H'$, Define $g \in C^0(M)$ by

$$g(x) := f(x) \text{ if } x \notin \mathrm{int}(K)$$

$$g(x) := \chi(x) \text{ if } x \in H$$

$$g(x) := {\phi'}^{-1} \circ \xi \circ \phi(x) \text{ if } x \in \mathrm{int}(K) \setminus H,$$

where $\xi : [0,1]^m \setminus [a,b]^m \to [0,1]^m$ is constructed as follows. Fix $z_0 \in \mathrm{int}([a,b]^m)$. For each point $z \in [0,1]^m \setminus [a,b]^m$, consider the half-line starting a z_0 containing z. Consider the segment $S(y) = [s_1(y), s_2(y)]$ contained in this half-line, where $s_1(y)$ is the unique point of the half-line in the boundary of $[0,1]^m$, and $s_2(y)$ is the unique point in the boundary of $[a,b]^m$. Now define ξ such that $\xi|_{S(y)} : S(y) \to [\phi' \circ f \circ \phi^{-1}(y),\ \ \phi' \circ \chi \circ \phi^{-1}(y)]$ is the affinity mapping $s_1(y)$ to $\phi' \circ f \circ \phi^{-1}(y)$ and $s_2(y)$ to $\phi' \circ \chi \circ \phi^{-1}(y)$.

It is standard to check that ξ is continuous and that ${\phi'}^{-1} \circ \xi \circ \phi|_{\partial K} = f$. Therefore $g \in C^0(M)$. By construction $g|_H = \chi : H \to H'$ is a homeomorphism. Besides, $g(x)$ may differ from $f(x)$ only in the points x of K; but both images are inside K' which is a ball of radius $\varepsilon/2$ in M. Therefore $\|g - f\|_{C^0} < \varepsilon$, as required. The final statement holds since H can be chosen arbitrarily small.

□

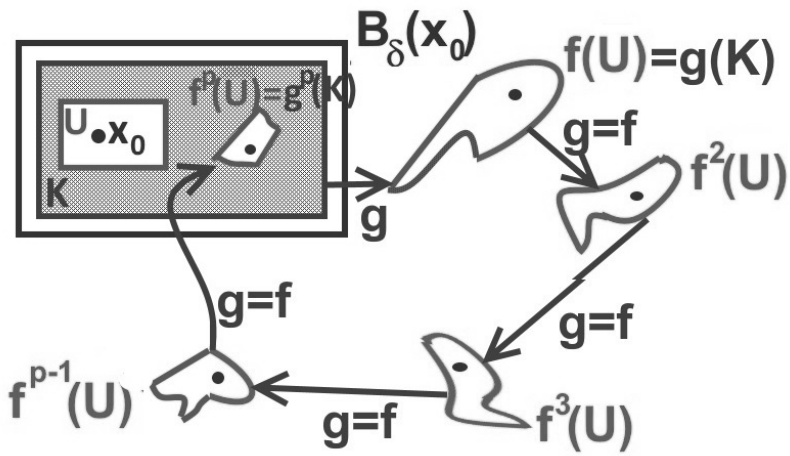

FIGURE 3. Construction of g near f with a periodic shrinking box K for g.

Proof of Lemma 4.2. According to Definition 4.1, the same periodic shrinking box K for f is also a periodic shrinking box with the same period for all $g \in C^0(M)$ near enough f, proving the openness assertion.

We turn to the denseness assertion. Let $f \in C^0(M)$ and $\varepsilon > 0$. We will construct $g \in C^0(M)$ and a periodic shrinking box K for g with $\mathrm{diam}(K) < \delta$, such that $\|g - f\|_{C^0} < \varepsilon$. We suppose $\delta > 0$ is to be smaller than the ε-modulus of continuity of f.

By the Krylov-Bogolyubov theorem invariant measures exist (recall that the manifold M is compact), and thus by the Poincaré Lemma, there exists a recurrent point $x_0 \in M$ for f. First assume that $x_0 \notin \partial M$. So, there exists a box $B \subset M$ with $\mathrm{diam}(B) < \delta$ such that $x_0 \in \mathrm{int}(B)$. Since x_0 is a recurrent point, there exists a smallest $p \in \mathbb{N}$ such that $f^p(x_0) \in \mathrm{int}(B)$. Taking B slightly smaller if necessary, we can assume that $f^j(x_0) \notin B$ for all $j = 1, 2, \ldots, p-1$. So, there exists a small compact box $U \subset \mathrm{int}(B)$ as in Figure 3, such that $x_0 \in \mathrm{int}(U)$, the sets $U, f(U), \ldots, f^{p-1}(U)$ are pairwise disjoint, and $f^p(U) \subset \mathrm{int}(B)$.

We can choose U homothetic to B. Since $U, f^p(U) \subset \mathrm{int}(B)$, there exists a box K such that $U, f^p(U) \subset \mathrm{int}(K) \subset K \subset \mathrm{int}(B)$. We also choose K homothetic to B. Therefore, there exists a homeomorphism $\psi : B \to B$ such that $\psi(x) = x$ for all $x \in \partial B$, and $\psi(K) = U$.

Finally, we construct $g \in C^0(M)$ as follows:

$$g(x) := f(x), \qquad \forall\, x \notin B, \qquad g(x) = f \circ \psi(x), \qquad \forall\, x \in B.$$

By construction, K is a periodic shrinking box of g; by the choice of δ we have $\|g - f\| < \varepsilon$.

Now, let us study the case for which M is a compact manifold with boundary and all the recurrent points of f belong to ∂M. Choose one such recurrent point $x_0 \in \partial M$. For any $\delta > 0$, there exists a compact box $B \subset M$, with $\operatorname{diam}(B) \leq \delta$ such that $x_0 \in \partial M \cap B$. Since x_0 is recurrent, there exists a smallest natural number $p \geq 1$ such that $f^p(x_0) \in B$. But $f^p(x_0)$ is also recurrent. So, $f^p(x_0) \in \partial M \cap B$. The previous proof does not work as is. To overcome the problem, we choose a new point $\widetilde{x}_0 \neq x_0$, near enough x_0, such that $\widetilde{x}_0 \in \operatorname{int}(B) \setminus \partial M$ and $f^p(\widetilde{x}_0) \in B$. By applying Lemma 4.4 and slightly perturbing f, if necessary, we can assume that the restriction of f to a small neighborhood of $\widetilde{x}_0$ is a local homeomorphism onto its image. Hence, $f^p(\widetilde{x}_0) \in \operatorname{int}(B) \setminus \partial M$. (If f were not a local homeomorphism onto its image, then $f^p(B)$ could be contained in ∂M.) To conclude, we repeat the construction of g and K above replacing the recurrent point x_0 by $\widetilde{x}_0$.

Now, let us show that we can construct densely in $C^0(M)$ a periodic shrinking box K such that the return map $f^p|_K$ is a homeomorphism onto its image. We repeat the beginning of the proof, up to the construction of the points $x_0, f(x_0), \ldots, f^p(x_0)$ such that $x_0, f^p(x_0) \in \operatorname{int}(B)$ and $f^j(x_0) \notin B$. Apply Lemma 4.4, slightly perturb f, if necessary, inside small open neighborhoods $W_0, W_1, \ldots, W_{p-1}$ of the points $x_0, f(x_0), \ldots, f^{p-1}(x_0)$ respectively, so that $f|_{\overline{W}_i}$ is a homeomorphism onto its image for all $i = 0, 1, \ldots, p-1$. Finally, construct the box U (Figure 3), but small enough so $f^j(U) \subset W_j$ for all $j = 0, 1, \ldots, p-1$, and repeat the construction of K and g as above. $\square$

Remark 4.5. Note that to obtain the dense property in the proof of Lemma 4.2, we only need to perturb the map f in the interior of the initial box B with diameter smaller than δ.

The following lemma is the homeomorphism version of Lemma 4.2.

Lemma 4.6. *For any $\delta > 0$, there exists an open and dense set of maps $f \in \operatorname{Hom}(M)$ that have a periodic shrinking box K with $\operatorname{diam}(K) < \delta$.*

Proof. The proof of Lemma 4.2 also works in the case that $f \in \operatorname{Hom}(M)$: in fact, the ε-perturbed map g constructed there is a homeomorphism, and to obtain $\|g - f\|_{\operatorname{Hom}(M)} < \varepsilon$ it is enough to reduce $\delta > 0$ to be smaller than the ε-continuity modulus of f and f^{-1}. $\square$

Remark 4.7. In the proof of Lemmas 4.2 and 4.6, if the starting recurrent point x_0 were a periodic point of period p, then the periodic shrinking box K so constructed would contain x_0 in its interior and have the same period p.

Lemma 4.8. *Let $\delta > 0$. A generic map $f \in C^0(M)$ has a periodic shrinking box K with* $\operatorname{diam}(K) < \delta$ *such that the return map $f^p|_K$ is topologically conjugated to a model map $\Phi \in \mathcal{H}$* (recall Definition 2.4).

Proof. Let $K \subset M$ be a periodic shrinking box for f. Fix a homeomorphism $\phi : K \to D^m$.

To prove the G_δ property, assume that $f \in C^0(M)$ has a periodic shrinking box K with $\operatorname{diam}(K) < \delta$, such that $\phi \circ f^p|_K \circ \phi^{-1} = \Phi \in \mathcal{H}$. From Definition 4.1, the same box K is also periodic shrinking with period p for all $g \in \mathcal{N}$, where $\mathcal{N} \subset C^0(M)$ is an open neighborhood of f. From Lemma 2.6, $\mathcal{H}$ is a G_δ-set in $C^0(D^m)$, i.e., it is the countable intersection of open families $\mathcal{H}_n \subset C^0(D^m)$. We define

$$\mathcal{V}_n := \{g \in \mathcal{N} \colon \phi \circ g^p|_K \circ \phi^{-1} \in \mathcal{H}_n\}.$$

Since the restriction to K of a continuous map g, and the composition of continuous maps, are continuous operations in $C^0(M)$, we deduce that $\mathcal{V}_n$ is an open family in $C^0(M)$. Besides

$$\phi \circ g^p|_K \circ \phi^{-1} \in \mathcal{H} = \bigcap_{n\geq 1} \mathcal{H}_n \quad \text{if} \quad g \in \bigcap_{n\geq 1} \mathcal{V}_n \subset C^0(M). \tag{40}$$

In other words, the set of maps $g \in C^0(M)$ that have periodic shrinking box K with $\operatorname{diam}(K) < \delta$, such that the return map $g^p|_K$ coincides, up to a conjugation, with a model map Φ, is a G_δ-set in $C^0(M)$.

To show the denseness fix $f \in C^0(M)$ and $\varepsilon > 0$. Applying Lemma 4.2, it is not restrictive to assume that f has a periodic shrinking box K with $\operatorname{diam}(K) < \min\{\delta, \varepsilon\}$, such that $f^p|_K$ is a homeomorphism onto its image. We will construct $g \in C^0(M)$ to be ε-near f and such that $\phi \circ g^p|_K \circ \phi^{-1} \in \mathcal{H}$.

Choose a box W such that $f^{p-1}(K) \subset \operatorname{int}(W)$. If $p \geq 2$, take W disjoint with $f^j(K)$ for all $j \in \{0, 1, \ldots, p-2\}$ (Figure 4). Let us see that we can assume that W has an arbitrarily small diameter. It is enough to prove that f can be chosen such that $f^{p-1}(K)$ has an arbitrarily small diameter. In fact, in the construction of f in the proof of Lemma 4.2, we can choose the box U (see Figure 3), after choosing K, as small as needed. So, we choose U small enough such that the $(p-1)$-th. image of U by the map before the perturbation, has a small diameter. (Note that we do not change p). After that, we construct the perturbed map, which we are calling f here, as in the proof of Lemma 4.2: the image $f^{p-1}(K)$ of the new map f coincides with the $(p-1)$-th. image of U by the map before the perturbation (Figure 3). So, it has an arbitrarily small diameter, as required.

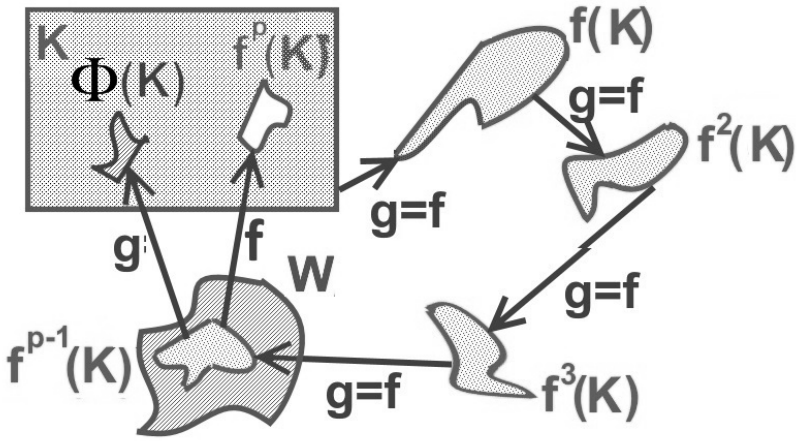

FIGURE 4. Perturbation g of f such that $g^p|_K = \Phi$.

To construct $g \in C^0(M)$ (see Figure 4) take $\Phi \in \mathcal{H}$ and let $g(x) := f(x)$ if $x \notin W$ and

$$g(x) := \phi^{-1} \circ \Phi \circ \phi \circ (f^p|_K)^{-1} \circ f(x), \qquad \forall\, x \in f^{p-1}(K).$$

This defines a continuous map $g : f^{p-1}(K) \cup (M \setminus W) \to M$ such that $|g(x) - f(x)| < \operatorname{diam}(K) < \varepsilon$ for all $x \in f^{p-1}(K) \subset W$ and $g(x) = f(x)$ for all $x \in M \setminus W$. Applying the Tietze Extension Theorem, there exists a continuous extension of g to the whole compact box W, hence to M, such that $\|g - f\|_{C^0} < \varepsilon$. Finally, by construction we obtain

$$g^p|_K = g|_{f^{p-1}(K)} \circ f^{p-1}|_K = \phi^{-1} \circ \Phi \circ \phi \circ (f^p|_K)^{-1} \circ f \circ f^{p-1}|_K = \phi^{-1} \circ \Phi \circ \phi,$$

ending the proof of Lemma 4.8. □

Lemma 4.9. *Let $\delta > 0$. A generic homeomorphism $f \in \operatorname{Hom}(M)$ has a periodic shrinking box K with $\operatorname{diam}(K) < \delta$, such that the return map $f^p|_K$ is topologically conjugated to a model homeomorphism $\Phi \in \mathcal{H} \cap \operatorname{RHom}(D^m)$.*

Proof. We repeat the proof of the G_δ-set property of Lemma 4.8, using $\mathcal{H} \cap \operatorname{RHom}(D^m)$ instead of $\mathcal{H}$, and $\operatorname{Hom}(M)$ instead of $C^0(M)$ (notice that taking the inverse is also a continuous operation in $\operatorname{Hom}(M)$).

To show the denseness fix $f \in \operatorname{Hom}(M)$ and $\varepsilon > 0$. Applying Lemma 4.2, it is not retrictive to assume that f has periodic shrinking boxes of arbitrarily small diameters. Let $\delta \in (0, \varepsilon)$ be smaller the the ε-modulus of continuity of f and f^{-1}. Consider a periodic shrinking box K with $\operatorname{diam}(K) < \delta$ (Lemma 4.6). Fix a homeomorphism $\phi : K \to$

D^m. We will construct $g \in \mathrm{Hom}(M)$ to be ε-near f in $\mathrm{Hom}(M)$, with $\phi \circ g^p|_K \circ \phi^{-1} = \Phi \in \mathcal{H} \cap \mathrm{RHom}(D^m)$.

From Definition 4.1 we know that the boxes $K, f(K), \dots, f^{p-1}(K)$ are pairwise disjoint and that $f^p(K) \subset \mathrm{int}(K)$. Denote $W := f^{-1}(K)$. Since f is a homeomorphism, we deduce that W is a box as in Figure 4, such that $W \cap f^j(K) = \emptyset$ for all $j = 0, 1, \dots, p-2$ if $p \geq 2$, and $f^{p-1}(K) \subset \mathrm{int}(W)$. Since $\mathrm{diam}(K) < \delta$ we have $\mathrm{diam}(W) < \varepsilon$.

Consider $\phi \circ f^p|_K \circ \phi^{-1} \in \mathrm{RHom}(D^m)$. Applying part b) of Lemma 2.6, there exists a homeomorphism $\psi : D^m \to D^m$ such that

$$\psi|_{\partial D^m} = \mathrm{id}|_{\partial D^m}, \qquad \psi \circ \phi \circ f^p|_K \circ \phi^{-1} = \Phi \in \mathcal{H} \cap \mathrm{RHom}(D^m).$$

So, we can construct $g \in \mathrm{Hom}(M)$ such that $g(x) := f(x)$ for all $x \notin W$, and $g(x) := \phi^{-1} \circ \psi \circ \phi \circ f(x)$ for all $x \in W$. Since $\psi|_{\partial D^m}$ is the identity map, we obtain $g|_{\partial W} = f|_{\partial W}$. Thus, the above equalities define a continuous map $g : M \to M$. Moreover g is invertible because $g|_W : W \to K$ is a composition of homeomorphisms, and $g|_{M \setminus W} = f|_{M \setminus W} : M \setminus W \to M \setminus K$ is also a homeomorphism. So, $g \in \mathrm{Hom}(M)$. Moreover, by construction we have $|g(x) - f(x)| < \mathrm{diam}(K) < \varepsilon$ for all $x \in W$, and $g(x) = f(x)$ for all $x \notin W$. Also the inverse maps satisfy $|g^{-1}(x) - f^{-1}(x)| < \mathrm{diam}(f^{-1}(K)) = \mathrm{diam}(W) < \varepsilon$ for all $x \in K$, and $g^{-1}(x) = f^{-1}(x)$ for all $x \notin K$. Therefore $\|g - f\|_{\mathrm{Hom}} < \varepsilon$.

Finally, let us check that $g^p|_K$ is topologically conjugated to Φ:

$$g^p|_K = g|_{f^{p-1}(K)} \circ f^{p-1}|_K = g|_W \circ f^{p-1}|_K =$$
$$\phi^{-1} \circ \psi \circ \phi \circ f \circ f^{p-1}|_K =$$
$$\phi^{-1} \circ (\psi \circ \phi \circ f^p|_K \circ \phi^{-1}) \circ \phi = \phi^{-1} \circ \Phi \circ \phi,$$

ending the proof of Lemma 4.9. □

Remark 4.10. In the proof of the dense property in Lemmas 4.8 and 4.9, once a periodic shrinking box K is constructed with period $p \geq 1$, we only need to perturb the map f inside $W \cup \bigcup_{j=0}^{p-1} f^j(K)$, where $W = f^{-1}(K)$ if f is a homeomorphism, and $\mathrm{int}(W) \supset f^{p-1}(K)$ otherwise. In both cases, by reducing the set U of Figure 3 from the very beginning, we can construct W such that $\mathrm{diam}(W) < \varepsilon$ for a previously specified small $\varepsilon > 0$.

Proof of Theorems 1 and 3. From Lemmas 4.8 and 4.9, a generic map $f \in C^0(M)$ and also a generic $f \in \mathrm{Hom}(M)$, has a periodic shrinking box K such that the return map $f^p|_K : K \to \mathrm{int}(K)$ is conjugated to a model map $\Phi \in \mathcal{H}$. We consider the homeomorphism $\phi^{-1} : K \to D^m$ such that $\phi^{-1} \circ f^p \circ \phi = \Phi \in \mathcal{H}$. Lemma 3.1 states that every map $\Phi \in \mathcal{H}$ has an Φ-invariant mixing measure ν with infinite metric entropy for Φ. Consider the push-forward measure $\phi_* \nu$, defined by $(\phi_* \nu)(B) :=$

$\nu(\phi^{-1}(B \cap K))$ for all the Borel sets $B \subset M$. By construction, $\phi_*\nu$ is supported on $K \subset M$. Since ϕ is a conjugation between Φ and $f^p|_K$, the push-forward measure $\phi_*\nu$ is f^p-invariant and mixing for f^p and moreover $h_{\phi_*\nu}(f^p) = +\infty$.

From $\phi_*\nu$, we will construct an f-invariant and f-ergodic measure μ supported on $\bigcup_{j=0}^{p-1} f^j(K)$, with infinite metric entropy for f. Precisely, for each Borel set $B \subset M$, define

$$\mu(B) := \frac{1}{p}\sum_{j=0}^{p-1}(f^j)_*(\phi_*\nu)(B \cap f^j(K)). \tag{41}$$

Applying Equality (41), and the fact that $\phi_*\nu$ is f^p-invariant and f^p-mixing, it is standard to check that μ is f-invariant and f-ergodic. From the convexity of the metric entropy function, we deduce that

$$h_\mu(f^p) = \frac{1}{p}\sum_{j=0}^{p-1} h_{(f^j)_*(\phi_*\nu)}(f^p) = +\infty.$$

Finally, recalling that $h_\mu(f^p) \leq p\, h_\mu(f)$ for any f-invariant measure μ and any natural number $p \geq 1$, we conclude that $h_\mu(f) = +\infty$. □

5. Good sequences of periodic shrinking boxes

In this section we prove Theorems 2 and 4. Throughout this section we assume that $\dim(M) \geq 2$. In the case that M is a compact interval, Theorem 2 can be proved analogously to the 2-dimensional case, but substituting Definition 2.4 of model maps in D^2, by the interval maps with measures with infinite entropy constructed in [CT].

Definition 5.1. Let $f \in C^0(M)$ and let $K_1, K_2, \ldots, K_n, \ldots$ be a sequence of periodic shrinking boxes for f. We call $\{K_n\}_{n\geq 1}$ *good* if it has the following properties (see Figure 5):
- $\{K_n\}_{n\geq 1}$ is composed of pairwise disjoint boxes.
- There exists a natural number $p \geq 1$, independent of n, such that K_n is a periodic shrinking box for f whose period p_n is a multiple of p.
- There exists a sequence $\{H_n\}_{n\geq 0}$ of periodic shrinking boxes, all with period p, such that $K_n \cup H_n \subset H_{n-1}$, $K_n \cap H_n = \emptyset$ for all $n \geq 1$, and $\operatorname{diam}(H_n) \to 0$ as $n \to +\infty$.

Remark. Definition 5.1 implies that $\bigcap_{n\geq 1} H_n = \{x_0\}$, where x_0 is periodic with period p. Furthermore, for any $j \geq 0$ we have

$$d(f^j(K_n), f^j(x_0)) \leq \operatorname{diam}(f^j(H_{n-1})) \leq \max_{0\leq k\leq p-1} \operatorname{diam}(f^k(H_{n-1})) \overset{n\to\infty}{\to} 0,$$

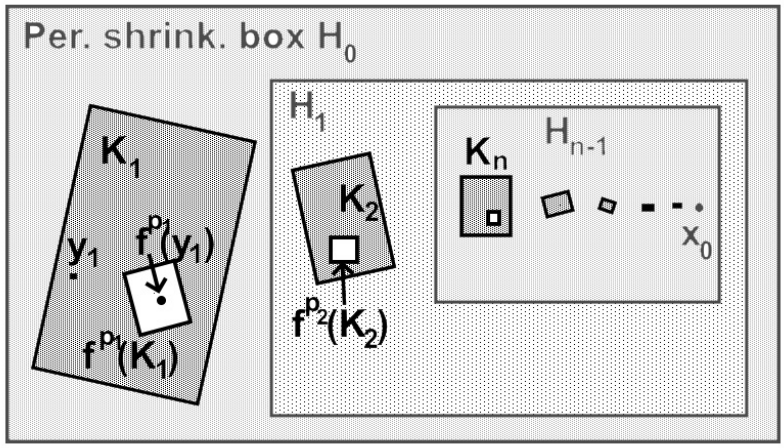

FIGURE 5. Construction of a good sequence of periodic shrinking boxes.

and thus

$$\lim_{n\to+\infty} \sup_{j\geq 0} d(f^j(K_n), f^j(x_0)) = 0. \tag{42}$$

We will construct a good sequence of periodic shrinking boxes for maps that are arbitrarily near a given f. We start by constructing the zeroth level boxes as follows:

Lemma 5.2. *Let $f \in C^0(M)$ (resp. $f \in \mathrm{Hom}(M)$) and $\varepsilon, \delta > 0$. Then, there exists $g_1 \in C^0(M)$ (resp. $g_1 \in \mathrm{Hom}(M)$), periodic shrinking boxes H_0 and K_1 for g_1 with periods p and p_1 respectively, where p_1 is multiple of p, and a periodic point $x_0 \in \mathrm{int}(H_0)$ for g_1 such that $K_1 \subset H_0 \setminus \{x_0\}$,*

$$g_1^{p_1}|_{K_1} \text{ is topologically conjugated to } \Phi_1 \in \mathcal{H} \quad \text{and}$$

$$\mathrm{diam}(H_0) < \delta, \quad \|g_1 - f\| < \frac{\varepsilon}{2}.$$

Proof. A generic map $\widetilde{f} \in C^0(M)$ (resp. $\widetilde{f} \in \mathrm{Hom}(M)$) has a periodic shrinking box H_0 with period $p \geq 1$, such that $\mathrm{diam}(H_0) < \delta$ and $\widetilde{f}^p|_{H_0}$ is conjugate to a model map $\Phi \in \mathcal{H}$ (Lemmas 4.8, resp. 4.9). Fix such an $\widetilde{f}$ in the $(\varepsilon/6)$-neighborhood of f. The same box H_0 will be a shrinking periodic box for the map g_1 to be constructed.

Since $\widetilde{f}^p : H_0 \to \mathrm{int}(H_0) \subset H_0$ is continuous, by the Brouwer Fixed Point Theorem there exists a periodic point $x_0 \in \mathrm{int}(H_0)$ of period p. Lemma 3.1 and the argument at the end of the proofs of Theorems 1 and 3, show that the map $\widetilde{f}$ has an ergodic measure μ supported on

$\bigcup_{j=0}^{p-1} \widetilde{f}^j(H_0)$ such that $h_\mu(\widetilde{f}) = +\infty$. Therefore, by Poincaré Recurrence Lemma, there exists some recurrent point $y_1 \in \mathrm{int}(H_0)$ for $\widetilde{f}$. We can choose such recurrent point $y_1 \neq x_0$ (see Figure 5) because μ is not supported on the orbit of the periodic point x_0 (recall that μ has infinite entropy and by construction its support is a perfect set).

Choose $\delta_1 > 0$ small enough and construct a box B_1 such that $y_1 \in \mathrm{int}(B_1)$, $\mathrm{diam}(B_1) < \delta_1$, the $\widetilde{f}$-orbit of x_0 (which is finite) does not intersect the finite piece of the $\widetilde{f}$-orbit of B_1 (until the first iterate of y_1 is in H_0) and $B_1 \subset \mathrm{int}(H_0)$. We repeat the proofs of the dense property of Lemmas 4.2 and 4.6, using the recurrent point y_1 instead of x_0, and the box B_1 instead of B (see Figure 3). We deduce that there exists an $(\varepsilon/6)$-perturbation $\hat{f}$ of $\widetilde{f}$, and a periodic shrinking box $K_1 \subset B_1$ for $\hat{f}$, with some period $p_1 \geq p$ (see Figure 5). Moreover, $\hat{f}$ coincides with $\widetilde{f}$ in $M \setminus \mathrm{int}(B_1)$ (recall Remark 4.5). Therefore, the same periodic point x_0 of $\widetilde{f}$ survives for $\hat{f}$. Besides, by the openness of the existence of the periodic shrinking box H_0, the same initial box H_0 is still periodic shrinking with period p for $\hat{f}$, provided that $\hat{f}$ is near enough $\widetilde{f}$. So, the compact sets of the family $\{\hat{f}^j(H_0)\}_{j=0,1,\ldots,p-1}$ are pairwise disjoint, and $\hat{f}^p(H_0) \subset \mathrm{int}(H_0)$. This implies that the period p_1 of the new periodic shrinking box K_1 for $\hat{f}$ is a multiple of p.

Now, we apply the proofs of the dense property of Lemmas 4.8 and 4.9, using the shrinking box K_1 instead of K (see Figure 4). We deduce that there exists an $(\varepsilon/6)$-perturbation g_1 of $\hat{f}$, such that K_1 is still a periodic shrinking box for g_1 with the same period p_1, but moreover, the return map $g_1^{p_1}|_{K_1}$ is now topologically conjugated to $\Phi_1 \in \mathcal{H}$.

Consider a box W_1 satisfying $\hat{f}^{p_1-1}(K_1) \subset W_1 \subset K_1$, small enough so its $\hat{f}$-orbit is disjoint from the $\hat{f}$-orbit of the periodic point x_0. Taking into account Remark 4.10, we can construct g_1 to coincide with $\hat{f}$ in the complement of $W_1 \bigcup \left(\bigcup_{j=0}^{p_1-1} \hat{f}^j(K_1) \right)$. Then, if g_1 is sufficiently near $\hat{f}$, the point x_0 is still periodic of period p for g_1, and besides H_0 is still a periodic shrinking box of period p for g_1 (recall that such property is open). Finally,

$$\|g_1 - f\| < \|g_1 - \hat{f}\| + \|\hat{f} - \widetilde{f}\| + \|\widetilde{f} - f\| < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2},$$

ending the proof. □

Now, assume that we have constructed the j-th level of periodic shrinking boxes for all $j \geq 0$ up to $n-1$ of a good sequence. We will construct the periodic shrinking boxes of the n-th level by perturbing

the given map once more. Let us first define the following family of maps:

Definition 5.3. Fix $\delta > 0$, and let p, n be natural numbers such that $p, n \geq 1$. We denote by $\mathcal{G}_{p,n,\delta} \subset C^0(M)$ the family of all the maps $g \in C^0(M)$ such that there exists n boxes $K_1, \ldots, K_n$ satisfying the following properties:
• $\{K_j\}_{1 \leq j \leq n}$ is composed of pairwise disjoint boxes.
• For all $1 \leq j \leq n$ the box K_j is a periodic shrinking for g with period p_j that is a multiple of p, and

$$g^{p_j}|_{K_j} \text{ is topologically conjugated to } \Phi_j \in \mathcal{H}.$$

• There exists a sequence $\{H_j\}_{0 \leq j \leq n-1}$ of periodic shrinking boxes for g, all of period p, and a periodic point $x_{n-1} \in \text{int}(H_{n-1})$ of period p, such that $K_j \cup H_j \subset H_{j-1}$, $K_j \cap H_j = \emptyset$ for all $1 \leq j \leq n-1$, $K_n \subset H_{n-1} \setminus x_{n-1}$ and $\text{diam}(H_j) < \delta/2^j$ for all $0 \leq j \leq n-1$ (see Figure 5).

Lemma 5.4. *Fix $\varepsilon > 0$, $\delta > 0$ and the natural numbers $n, p \geq 1$. Assume that $g_n \in \mathcal{G}_{p,n,\delta}$ or $g_n \in \mathcal{G}_{p,n,\delta} \cap \text{Hom}(M)$.*

Then, there exists an $\varepsilon/2^{n+1}$-perturbation g_{n+1} of g_n, such that $g_{n+1} \in \mathcal{G}_{p,n+1,\delta}$ or $g_{n+1} \in \mathcal{G}_{p,n+1,\delta} \cap \text{Hom}(M)$, respectively.

Moreover, for all $j = 1, \ldots, n$ the same boxes $K_1, K_2, \ldots, K_n$ and $H_0, H_1, \ldots, H_{n-1}$ are shrinking periodic for the new map g_{n+1} and for the given map g_n, with the same periods, and

$$g_n^i|_{K_j} = g_{n+1}^i|_{K_j} \quad \forall\ i = 1, \ldots, p_j.$$

Proof. All the perturbations of g_n that we will construct are sufficiently close to g_n so that the same boxes $H_0, H_1, \ldots, H_{n-1}$ and $K_1, K_2, \ldots, K_n$ that are periodic shrinking for g_n are still periodic shrinking with the same periods for the perturbed maps. This is possible because the periodic shrinking property of a box and its period, are open conditions. Besides, we will only consider perturbations of g_n that coincide with g_n except in the interior of a finite number of boxes B, W, etc, whose g_n-iterates, up to the $(\max_{1 \leq j \leq n} p_j)$-th iterate, are disjoint with all the boxes of the family $\{g_n^i(K_j) : 1 \leq j \leq n,\ 0 \leq i \leq p_j - 1\}$. Therefore, if such a perturbation $\widetilde{g}$ of g_n is near enough g_n, then the iterates by $\widetilde{g}$ of the boxes B, W, etc (where $\widetilde{g}$ differs from g_n) are still disjoint with the g_n-iterates of K_j. This implies that for all $1 \leq j \leq n$,

$$g_n^i|_{K_j} = \widetilde{g}^i|_{K_j} \quad \forall\ i = 1, \ldots, p_j \quad \text{and then}$$

$$\widetilde{g}^{p_j}|_{K_j} = g_n^{p_j}|_{K_j} \text{ is topologically conjugated to } \Phi_j \in \mathcal{H}.$$

Now let us perturb g_n as above, in several steps, to construct the boxes H_n and K_{n+1}.

By hypothesis, g_n has a periodic shrinking box H_{n-1} of period p, a periodic point $x_{n-1} \in \mathrm{int}(H_{n-1})$ of period p, and a periodic shrinking box $K_n \subset H_{n-1} \setminus \{x_{n-1}\}$ of period p_n, multiple of p. It also has periodic shrinking boxes $K_1, \dots, K_{n-1}, K_n$ whose g_n-orbits are disjoint with the periodic orbit of x_{n+1}. So, we can construct a box $\widetilde{B}_n \subset H_{n-1}$ containing the periodic point x_{n-1} in its interior, whose g_n-orbit up to the $(\max_{1\le j\le n} p_j)$-th iterate is disjoint from all the sets of the family $\{f^i(K_j) :\ 1 \le j \le n,\ 0 \le i \le p_j - 1\}$. Besides, we construct the box $\widetilde{B}_n$ such that $\mathrm{diam}(\widetilde{B}_n) < \widetilde{\delta}/2^n$. Repeating the proof of the density properties in Lemmas 4.2 and 4.6 (putting x_{n-1} instead of x_0), we construct an $\varepsilon/(3 \cdot 2^{n+1})$-perturbation $\widetilde{g}_n$ of g_n, near enough g_n and a periodic shrinking box $H_n \subset \mathrm{int}(\widetilde{B}_n)$ for $\widetilde{g}_n$. Moreover, since x_{n-1} is a periodic point with period p for g_n, the period of H_n for $\widetilde{g}_n$ can be made equal to p (see Remark 4.7). By construction $H_n \subset \widetilde{B}_n \subset H_{n-1}$ is disjoint from K_n. To construct $\widetilde{g}_n$ we only needed to modify g_n inside $\widetilde{B}_n$ (recall Remark 4.5). Therefore, if $\widetilde{g}_n$ is near enough g_n, as observed at the beginning, the same periodic shrinking boxes $H_0, H_1, \dots, H_{n-1}$ and $K_1, K_2, \dots, K_n$ of g_n, are preserved for $\widetilde{g}_n$ with the same periods, and $\widetilde{g}_n$ coincides with g_n on the g_n-orbit of the boxes $K_1, \dots, K_n$.

Now, as in the proof of Lemmas 4.8 and 4.9, we construct a new $\varepsilon/(3{\cdot}2^{n+1})$-perturbation $\hat{g}_n$ of $\widetilde{g}_n$, such that $\hat{g}_n^p|_{H_n}$ is conjugated to a map in $\mathcal{H}$. To construct $\hat{g}_n$ we only need to modify $\widetilde{g}_n$ in $\widetilde{W}_n \cup \bigcup_{j=0}^{p-1} \widetilde{g}_n^j(H_n)$, where $\widetilde{W}_n$ is a small neighborhood of $\widetilde{g}_n^{p-1}(H_n)$ (see Remark 4.10). Since the $\widetilde{g}_n$-orbit of H_n is disjoint from the $\widetilde{g}_n$ orbits of K_j for all $1 \le j \le n$ (because H_n and K_j are disjoint periodic shrinking boxes for $\widetilde{g}_j$, we can choose W_n near enough $\widetilde{g}_n^{p-1}(H_n)$ and $\hat{g}_n$ near enough $\widetilde{g}_n$ so $\hat{g}_n$ coincides with $\widetilde{g}_n$ on the orbit of the boxes K_j, as observed at the beginning.

We conclude that the same shrinking boxes $K_1, \dots, K_n; H_0, \dots, H_{n-1}$ for $\widetilde{g}_n$ and g_n, are still periodic shrinking for $\hat{g}_n$, with the same periods and that $\hat{g}_n^{p_j}|_{K_j} = \widetilde{g}_n^{p_j}|_{K_j}$ which is conjugated to $\Phi_j \in \mathcal{H}$ for all $j = 1, \dots, n$.

When modifying g_n to obtain $\widetilde{g}_n$ and $\hat{g}_n$, the periodic point $x_{n-1} \in \mathrm{int}(H_{n-1})$ of period p for g_n, may not be preserved as periodic for $\hat{g}_n$. But since $H_n \subset H_{n-1} \setminus K_n$ is a periodic shrinking box with period p for $\hat{g}_n$, by the Brouwer Fixed Point Theorem, there exists a periodic point $x_n \in \mathrm{int}(H_n) \setminus K_n$ for $\hat{g}_n$, with the same period p.

Since the return map $\hat{g}_n^p|_{H_n}$ is conjugated to a model map, there exists an ergodic measure μ with infinite entropy for $\hat{g}_n$ (see Lemma 3.1), supported on the $\hat{g}_n$-orbit of H_n. Therefore, there exists a recurrent

point $y_n \in \mathrm{int}(H_n)$. We can choose such recurrent point $y_n \neq x_n$, because μ is not supported on the periodic orbit of x_n (in fact, μ has infinite entropy).

We now argue as in the proof of Lemma 5.2, (using $\hat{g}_n$, H_n and x_n in the role of $\tilde{f}$, H_0 and x_0) to construct an $\varepsilon/(3 \cdot 2^n)$- perturbation g_{n+1} of $\hat{g}_n$, and a box $K_{n+1} \subset H_n \setminus \{x_n\}$ that is periodic shrinking for g_{n+1} of period p_{n+1} which is a multiple of p, and such that $g_{n+1}^{p_{n+1}}|_{K_{n+1}}$ is topologically conjugated to a model map.

As observed at the beginning, if choosing g_{n+1} near enough $\hat{g}_n$, the boxes $H_0, \ldots, H_n$ and $K_1, \ldots, K_n$ are still periodic shrinking for g_{n+1} with the same periods, and

$$g_{n+1}^{p_j}|_{K_j} = \hat{g}_n^{p_j}|_{K_j} = g_n^{p_j}|_{K_j}$$

is topologically conjugated to a model map, for all $1 \leq j \leq n$.

By construction we have $g_{n+1} \in \mathcal{G}_{p,n,\delta}$ and

$$\|g_{n+1} - g_n\| \leq \|g_{n+1} - \hat{g}_n\| + \|\hat{g}_n - \widetilde{g}_n\| + \|\widetilde{g}_n - g_n\| < 3 \cdot \frac{\varepsilon}{3 \cdot 2^{n+1}} = \frac{\varepsilon}{2^{n+1}},$$

as required. □

Definition 5.5. Fix $\delta > 0$. We denote by $\mathcal{G}_\delta \subset C^0(M)$ the family of all the maps $g \in \bigcup_{p \geq 1} \bigcap_{n \geq 1} \mathcal{G}_{p,n,\delta}$ such that for all $n \geq 1$, the boxes $H_0, \ldots, H_{n-1}$ and $K_1, \ldots, K_n$ of Definition 5.3 for g as belonging to $\mathcal{G}_{p,n,\delta}$ coincide with the boxes for g as belonging to $\mathcal{G}_{p,n+1,\delta}$.

Lemma 5.6. *Fix $\delta > 0$. The family $\mathcal{G}_\delta$ is dense in $C^0(M)$ and its intersection with* $\mathrm{Hom}(M)$ *is dense in* $\mathrm{Hom}(M)$.

Proof. Let $f \in C^0(M)$ or $f \in \mathrm{Hom}(M)$, and $\varepsilon > 0$. We will construct $g \in \mathcal{G}_\delta$ such that $\mathrm{dist}(g, f) \leq \varepsilon$.

Applying Lemma 5.2, there exists $p \geq 1$ and $g_1 \in \mathcal{G}_{p,1,\delta}$ such that $\mathrm{dist}(g_1, f) \leq \varepsilon/2$. Denote by $H_0, K_1 \subset H_0$ the periodic shrinking boxes for g_1 as a map of $\mathcal{G}_{p,1,\delta}$ (recall Definition 5.3 for $n = 1$). By continuity, there exists $0 < \varepsilon_1 < \varepsilon$ such that for all g in the ε_1-neighborhood of g_1, H_0 is still a periodic shrinking box of period p for g.

By induction on $n \geq 1$, (Lemma 5.4 provides the inductive step), there exists a sequence of maps $g_1, g_2, \ldots, g_n, \ldots$ and a strictly decreasing sequence of positive real numbers $\varepsilon > \varepsilon_1 > \varepsilon_2 > \ldots > \varepsilon_n > \ldots$ such that, for all $n \geq 1$, $g_n \in \mathcal{G}_{p,n,\delta}$, $\mathrm{dist}(g_{n+1}, g_n) \leq \varepsilon_n/2^n$, the boxes $H_0, H_1, \ldots, H_{n-1}$ and $K_1, K_2, \ldots, K_n$ are still periodic shrinking for g_{n+1} with the same periods $p, p_1, p_2, \ldots, p_n$ as for g_n, and $g_{n+1} = g_n$ when restricted to the g_n-orbits of the boxes K_j for $j = 1, \ldots, n$. Besides, for all g in the ε_n-neighborhood of g_n, H_{n-1} is still a periodic shrinking box of period p for g.

Since $\|g_{n+1} - g_n\| \leq \varepsilon/2^{n+1}$ for all $n \geq 1$ the sequence $\{g_n\}_{n\geq 1}$ is Cauchy in $C^0(M)$ or $\mathrm{Hom}(M)$. So, there exists a limit map g. Since g_n is an ε-perturbation of f for all $n \geq 1$, the limit map g satisfies $\mathrm{dist}(g, f) \leq \varepsilon$.

Besides, by construction $g_k(x) = g_n(x)$ for all $x \in \bigcup_{j=0}^{p_n} g_n^j(K_n)$, for all $k \geq n \geq 1$. So $g_k^{p_n}|_{K_n} = g_n^{p_n}|_{K_n}$ is topologically conjugated to $\Phi_n \in \mathcal{H}$ for all $n \geq 1$ and for all $k \geq n$ (recall that $g_n \in \mathcal{G}_{p,n,\delta}$ and Definition 5.3). Thus K_n is still a periodic shrinking box for g of period p_n, and $g^{p_n}|_{K_n} = g_n^{p_n}|_{K_n}$ is topologically conjugated to a model map for all $n \geq 1$. Finally, for all $k > n \geq 1$ we have, by construction, $\mathrm{dist}(g_k, g_n) < \varepsilon_n(1/2^{n+1} + 1/2^{n+2} + \cdots + 1/2^k) \leq \varepsilon_n$. So, taking limit as $k \to +\infty$, we obtain $\mathrm{dist}(g, g_n) \leq \varepsilon_n$. This implies that H_{n-1} is still a periodic shrinking box of period p for g as it was for g_n. We have proved that $g \in \mathcal{G}_\delta$, as required. □

Lemma 5.7. *Generic maps $f \in C^0(M)$, and generic $f \in \mathrm{Hom}(M)$ have good sequences $\{K_n\}_{n\geq 1}$ of periodic shrinking boxes such that the return maps $f^{p_n}|_{K_n}$ are topologically conjugated to model maps.*

Proof. To see the G_δ property assume that f has a good sequence $\{K_n\}_n$ of periodic shrinking boxes. For each fixed n, the boxes K_n and H_n are also periodic shrinking with periods p_n and p respectively, for all g in an open set in $C^0(M)$ or in $\mathrm{Hom}(M)$ (see Definition 4.1). Taking the intersection of such open sets for all $n \geq 1$, we deduce that the same sequence $\{K_n\}$ is also a good sequence of periodic shrinking boxes for all g in a G_δ-set. Now, assume that besides $f^{p_n}|_{K_n}$ is topologically conjugated to a model map for all $n \geq 1$. From Lemmas 4.8 and 4.9, for each fixed $n \geq 1$, the family of continuous maps g such that the return map $g^{p_n}|_{K_n}$ is topologically conjugated to a model, is a G_δ-set in $C^0(M)$ or in $\mathrm{Hom}(M)$. The (countable) intersection of these G_δ-sets, produces a G_δ-set, as required.

To prove denseness, recall Definitions 5.3 and 5.5. Observe that the family of continuous maps or homeomorphisms that have a good sequence $\{K_n\}_{n\geq 1}$ of periodic shrinking boxes such that the return map to each K_n is topologically conjugated to a model map, contains the family $\mathcal{G}_\delta$ (or the intersection of this family with $\mathrm{Hom}(M)$), for any value of $\delta > 0$. Applying Lemma 5.6, this latter family is dense, ending the proof. □

Remark 5.8. As a consequence of Lemmas 5.7 and 3.1 (after applying the same arguments at the end of the proof of Theorems 1 and 3), generic continuous maps and homeomorphisms f have a sequence of ergodic measures μ_n, each one supported on the f-orbit of a box K_n of

a good sequence $\{K_n\}_{n\geq 1}$ of periodic shrinking boxes for f, satisfying $h_{\mu_n}(f) = +\infty$ for all $n \geq 1$.

Let $\mathcal{M}$ denote the metrizable space of Borel probability measures on a compact metric space M, endowed with the weak* topology. Fix a metric dist* in $\mathcal{M}$.

Lemma 5.9. *For all $\varepsilon > 0$ there exists $\delta > 0$ that satisfies the following property: if $\mu, \nu \in \mathcal{M}$ and $\{B_1, B_2, \ldots, B_r\}$ is a finite family of pairwise disjoint compact balls $B_i \subset M$, and if* $\mathrm{supp}(\mu) \cup \mathrm{supp}(\nu) \subset \bigcup_{i=1}^r B_i$, and $\mu(B_i) = \nu(B_i)$, $\mathrm{diam}(B_i) < \delta$ for all $i = 1, 2, \ldots, r$, *then* $\mathrm{dist}^*(\mu, \nu) < \varepsilon$.

Proof. If $M = [0, 1]$ the proof can be found for instance in [CT, Lemma 4]. If M is any other compact manifold of finite dimension $m \geq 1$, with or without boundary, just copy the proof of [CT, Lemma 4] by substituting the pairwise disjoint compact intervals $I_1, I_2, \ldots, I_r \subset [0, 1]$ in that proof, by the family of pairwise disjoint compact boxes $B_1, B_2, \ldots, B_r \subset M$. □

Proofs of Theorems 2 and 4. For any $\varepsilon > 0$, take $\delta > 0$ as in Lemma 5.9. Applying Lemma 5.7, generic continuous maps or homeomorphisms f have a good sequence of periodic shrinking boxes $\{K_n\}_{n\geq 1}$, and a sequence $\{\mu_n\}$ of ergodic f-invariant measures such that $h_{\mu_n}(f) = +\infty$ (see Remark 5.8) and such that $\mathrm{supp}(\mu_n) \subset \bigcup_{j=0}^{p_n-1} f^j(K_n)$, where $p_n = l_n \cdot p$, multiple of p, is the period of the shrinking box K_n. Taking into account that $\{f^j(K_n)\}_{0\leq j\leq p_n-1}$ is a family of pairwise disjoint compact sets, and $f^{p_n}(K_n) \subset \mathrm{int}(K_n)$, we obtain for each $j \in \{0, 1, \ldots, p_n\}$

$$\mu_n(f^j(K_n)) = \mu_n(f^{-j}(f^j(K_n))) = \mu_n(f^{-j}(f^j(K_n)) \cap \mathrm{supp}(\mu_n)) = \mu_n(K_n).$$

Since $1 = \sum_{j=0}^{p_n-1} \mu_n(f^j(K_n)) = p_n \cdot \mu_n(K_n)$; we obtain

$$\mu_n(f^j(K_n)) = \mu_n(K_n) = \frac{1}{p_n} = \frac{1}{l_n\, p}, \qquad \forall\ j = 0, 1, \ldots, p_n.$$

From Definition 5.1, there exists a periodic point x_0 of period p such that $\lim_{n\to+\infty} \sup_{j\geq 0} \mathrm{Hdist}(f^j(K_n, f^j(x_0))) = 0$, where Hdist denotes the Hausdorff distance. Therefore, there exists $n_0 \geq 1$ such that $d\left(f^j\left(K_n, f^j(x_0)\right)\right) < \delta'$ for all $j \geq 0$ and for all $n \geq n_0$, where $\delta' < \delta/2$ is chosen such that the family of compact balls B_0, B_1, …, B_{p-1}, centered at the points $f^j(x_0)$ and with radius δ', are pairwise disjoint. We obtain $f^j(K_n) \subset B_{j \pmod p}$ for all $j \geq 0$ and for all $n \geq 0$. Therefore,

$$\mu_n(B_j) = \frac{1}{p}, \qquad \forall\ j = 0, 1, \ldots, p-1, \qquad \forall\ n \geq n_0.$$

Finally, applying Lemma 5.9, we conclude $\mathrm{dist}^*(\mu_n, \mu_0) < \varepsilon$ for all $n \geq n_0$, where $\mu_0 := (1/p)\sum_{j=0}^{p-1} \delta_{f^j(p)}$ is the f-invariant probability measure supported on the periodic orbit of x_0, which has zero entropy. $\square$

6. Open questions

If f is Lipschitz then no invariant measure has infinite entropy, since its topological entropy is finite. The following question arises: do Theorems 1 and 3 hold also for maps with more regularity than continuity but lower regularity than Lipschitz? For instance, do they hold for Hölder-continuous maps?

A-priori there is a chance to answer positively this question for one-dimensional Hölder continuous endomorphisms, because in such a case, the topological entropy is generically infinite. This is a simple corollary of the arguments in [FHT1]. Also for bi-Hölder homeomorphisms on manifolds of dimension 2 or larger, there is a chance to answer positively the above question, because their topological entropy is also generically infinite [FHT], [FHT1]. In this article we focus only on the C^0-case, and leave for further research the eventual adaptation of our proofs, if this adaptation is possible, to C^α-maps or homeomorphisms with $0 < \alpha < 1$.

The hypothesis of Theorems 1 and 3 states that M is a compact manifold. It arises the following question: do some of the results also hold in other compact metric spaces that are not manifolds? For instance, do they hold if the space is a Cantor set K?

If the aim were just to construct $f \in \mathrm{Hom}(K)$ with ergodic measures with infinite metric entropy, the answer would be positive. But if the purpose were to prove that such homeomorphisms are generic in $\mathrm{Hom}(K)$, the answer would be negative.

In fact, Theorem 3 holds in particular for the 2-dimensional square $D^2 := [0,1]^2$. One of the steps of the proofs consists in constructing some fixed Cantor set $\Lambda \subset D^2$, and a homeomorphism Φ on M that leaves Λ invariant, and possesses an Φ-invariant ergodic measure supported on Λ with infinite metric entropy (see Lemma 3.1 and Remark 3.2). Since any pair of Cantor sets K and Λ are homeomorphic, we deduce that any Cantor set K supports a homeomorphism f and an f-ergodic measure with infinite metric entropy.

Nevertheless, the above phenomenon is not generic on a Cantor set K. On the one hand, there also exists homeomorphisms on K with finite, and even zero, topological entropy. (Take for instance $f \in \mathrm{Hom}(K)$ conjugated to the homeomorphism on the attractor of a Smale horseshoe, or to the attractor of the C^1- Denjoy example on

the circle.) On the other hand, it is known that each homeomorphism on a Cantor set K is topologically locally unique; i.e., it is conjugated to any of its small perturbations [AGW]. Therefore, the topological entropy is locally constant in $\mathrm{Hom}(K)$. We conclude that the homeomorphisms on the Cantor set K with infinite metric entropy, that do exist, are not dense in $\mathrm{Hom}(K)$; hence they are not generic.

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