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ERGODIC MEASURES WITH INFINITE ENTROPY

ELEONORA CATSIGERAS AND SERGE TROUBETZKOY

ABSTRACT. We construct ergodic, probability measures with infinite metric entropy for generic continuous maps and homeomorphisms on compact manifolds. We also construct sequences of such measures that converge to a zero-entropy measure.

1. INTRODUCTION

Let M be a C^1 compact manifold of finite dimension $m \geq 1$, equipped with a Riemannian metric dist . The manifold M may have or have not boundary. Let $C^0(M)$ be the space of continuous maps $f : M \rightarrow M$ with the uniform norm:

$$\|f - g\|_{C^0} := \max_{x \in M} \text{dist}(f(x), g(x)), \quad \forall f, g \in C^0(M).$$

We denote by $\text{Hom}(M)$ the space of homeomorphisms $f : M \rightarrow M$ with the uniform norm:

$$\|f - g\|_{\text{Hom}} := \max \left\{ \|f - g\|_{C^0}, \|f^{-1} - g^{-1}\|_{C^0} \right\} \quad \forall f, g \in \text{Hom}(M).$$

Since the metric spaces $C^0(M)$ and $\text{Hom}(M)$ are complete they are Baire spaces. A subset $\mathcal{S} \subset C^0(M)$ (or $\mathcal{S} \subset \text{Hom}(M)$) is called a G_δ -set if it is the countable intersection of open subsets of $C^0(M)$ (resp. $\text{Hom}(M)$). We say that a property P of the maps $f \in C^0(M)$ (or $f \in \text{Hom}(M)$) is *generic*, or that generic maps satisfy P , if the set of maps that satisfy P contains a *dense* G_δ -set in $C^0(M)$ (resp. $\text{Hom}(M)$).

The main result of this article is the following theorem.

Theorem 1. *The generic map $f \in C^0(M)$ has an ergodic Borel probability measure μ such that $h_\mu(f) = +\infty$, furthermore there exists $p \geq 1$ such that μ is mixing for the map f^p .*

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Remark. In the case that M is a compact interval, Theorem 1 was proved in [CT, Theorem 3.9 and p.33, para 2]. So, in this paper we will prove it only for m -dimensional manifolds where $m \geq 2$.

Yano proved that generic continuous maps of compact manifolds with or without boundary have infinite topological entropy [Ya]. Therefore, from the variational principle, there exists invariant measures with metric entropies as large as required. Nevertheless, this property alone does not imply the existence of invariant measures with infinite metric entropy. In fact, it is well known that the metric entropy function $\mu \rightarrow h_\mu(f)$ is not upper semi-continuous for C^0 -generic systems. Moreover, we prove that it is *strongly* non upper semi-continuous in the following sense:

Theorem 2. *For a generic map $f \in C^0(M)$ there exists a sequence of ergodic measures μ_n such that for all $n \geq 1$ we have $h_{\mu_n} = +\infty$ and*

$$\lim_{n \rightarrow +\infty}^* \mu_n = \mu \text{ with } h_\mu = 0,$$

where \lim^* denotes the limit in the space of probability measures endowed with the weak* topology.

Even if we had a priori some f -invariant measure μ with infinite metric entropy, we do not know if this property alone implies the existence of ergodic measures with infinite metric entropy as Theorems 1 and 2 state. Actually, if μ had infinitely many ergodic components, the proof that the metric entropy of at least one of those ergodic components must be larger or equal than the entropy of μ , uses the upper semi-continuity of the metric entropy function (see for instance [Ke, Theorem 4.3.7, p. 75]).

Yano also proved that generic homeomorphisms on manifolds of dimension 2 or larger, have infinite topological entropy [Ya]. Thus one wonders if Theorems 1 and 2 hold also for homeomorphisms. We give a positive answer to this question.

Theorem 3. *If $\dim(M) \geq 2$, then the generic homeomorphism $f \in \text{Hom}(M)$ has an ergodic Borel probability measure μ satisfying $h_\mu(f) = +\infty$, furthermore there exists $p \geq 1$ such that μ is mixing for the map f^p .*

Theorem 4. *If $\dim(M) \geq 2$, then for a generic homeomorphism $f \in \text{Hom}(M)$ there exists a sequence of ergodic measures μ_n such that for all $n \geq 1$ we have $h_{\mu_n} = +\infty$ and*

$$\lim_{n \rightarrow +\infty}^* \mu_n = \mu \text{ with } h_\mu = 0.$$

To prove the above theorems in dimension two or larger we construct a family \mathcal{H} , called models, of continuous maps in the cube $[0, 1]^m$, including some homeomorphisms of the cube onto itself, which have a complicated behavior on a Cantor set (Definition 2.4). A periodic shrinking box is a compact set $K \subset M$ that is homeomorphic to the cube $[0, 1]^m$ and such that for some $p \geq 1$: $K, f(K), \dots, f^{p-1}(K)$ are pairwise disjoint and $f^p(K) \subset \text{int}(K)$ (Definition 4.1). The main steps of the proofs of Theorems 1 and 3 are the following results.

- (1) **Lemma 3.1** *Any model $\Phi \in \mathcal{H}$ in the cube $[0, 1]^m$ has an Φ -invariant mixing measure ν such that $h_\nu(\Phi) = +\infty$.*
- (2) **Lemmas 4.2 and 4.6** *Generic maps in $C^0(M)$, and generic homeomorphisms, have a periodic shrinking box.*
- (3) **Lemmas 4.8 and 4.9** *Generic maps $f \in C^0(M)$, and generic homeomorphisms for $m \geq 2$, have a periodic shrinking box K such that the return map $f^p|_K$ is topologically conjugated to a model $\Phi \in \mathcal{H}$.*

A *good sequence of periodic shrinking boxes* is a sequence $\{K_n\}_{n \geq 1}$ of periodic shrinking boxes such that accumulate (with the Hausdorff distance) on a periodic point x_0 , and moreover their iterates $f^j(K_n)$ also accumulate on the periodic orbit of x_0 , uniformly for $j \geq 0$ (see Definition 5.1). The main step in the proof of Theorems 2 and 4 is Lemma 3.1 together with

- (4) **Lemma 5.2** *Generic maps $f \in C^0(M)$, and if $m \geq 2$ also generic homeomorphisms, have a good sequence $\{K_n\}$ of boxes, such that the return maps $f^{p_n}|_{K_n}$ are topologically conjugated to models $\Phi_n \in \mathcal{H}$.*

2. CONSTRUCTION OF THE FAMILY OF MODELS.

We call a compact set $K \subset D^m := [0, 1]^m$ or more generally $K \subset M$, where M is an m -dimensional manifold with $m \geq 2$, a *box* if it is homeomorphic to D^m . Models are certain continuous maps of $D^m := [0, 1]^m$ that we will define in this section.

We denote by $\text{RHom}(D^m)$ the space of relative homeomorphisms $\Phi : D^m \rightarrow D^m$ (i.e., Φ is a homeomorphism onto its image included in D^m), with the topology induced by:

$$\|\Phi - \Psi\|_{\text{RHom}} := \max \left\{ \|\Phi - \Psi\|_{C^0(D^m)}, \|\Phi^{-1} - \Psi^{-1}\|_{C^0(\Phi(D^m) \cap \Psi(D^m))} \right\}$$

if $\Phi(D^m) \cap \Psi(D^m) \neq \emptyset$. Throughout the article we will consider the distance between two relative homeomorphisms Φ, Ψ only if $\Phi(D^m) \cap$

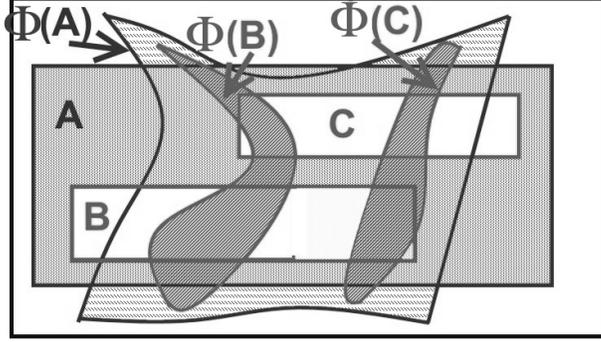


FIGURE 1. An atom A of generation 0 and two atoms B, C of generation 1 for a map Φ of D^2 .

$\Psi(D^m) \neq \emptyset$, and we mostly use the above distance in the case $\Phi(D^m) = \Psi(D^m)$.

Definition 2.1. (Φ -relation from a box to another).

Let $\Phi \in C^0(D^m)$. Let $B, C \subset \text{int}(D^m)$ be two boxes. We write

$$B \xrightarrow{\Phi} C \text{ if } \Phi(B) \cap \text{int}(C) \neq \emptyset.$$

Observe that this condition is open in $C^0(D^m)$ and also in $\text{RHom}(D^m)$.

Definition 2.2. (Atoms of generation 0 and 1) (See Figure 1)

We call a box $A \subset \text{int}(D^m)$ an *atom of generation 0* for Φ , if $A \xrightarrow{\Phi} A$.

We call two disjoint boxes $B_1, B_2 \subset \text{int}(A)$, *atoms of generation 1*, if

$$B_i \xrightarrow{\Phi} B_j, \quad \forall i, j \in \{1, 2\}.$$

If A is an atom of generation 0 and B_1, B_2 are the two atoms of generation 1, we denote $\mathcal{A}_0 := \{A\}$, $\mathcal{A}_1 := \{B_1, B_2\}$.

Definition 2.3. Atoms of generation $n \geq 2$ (See Figure 2)

Assume by induction that the finite families $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_{n-1}$ of atoms for $\Phi \in C^0(D^m)$ of generations up to $n-1$ are already defined, such that the atoms of the same generation $j = 1, \dots, n-1$ are pairwise disjoint, contained in the interior of the $(j-1)$ -atoms in such a way that all the $(j-1)$ -atoms contain the same number of j -atoms and

$$\#\mathcal{A}_j = 2^{j^2}, \quad \forall j = 0, 1, \dots, n-1.$$

Assume also that for all $j \in \{1, \dots, n-1\}$ and for all $B \in \mathcal{A}_j$:

$$\#\{C \in \mathcal{A}_j : B \xrightarrow{\Phi} C\} = 2^j, \quad \#\{D \in \mathcal{A}_j : D \xrightarrow{\Phi} B\} = 2^j.$$

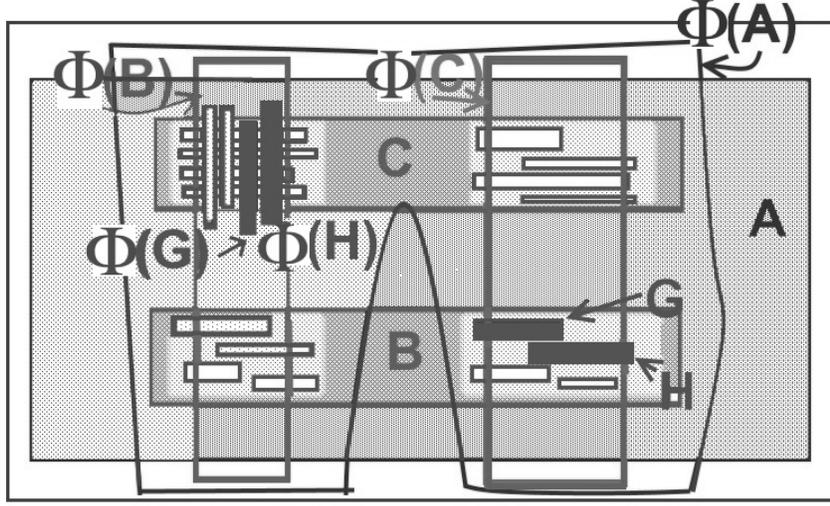


FIGURE 2. An atom A of generation 0, two atoms B, C of generation 1, and 16 atoms of generation 2. In particular the two atoms G, H of generation 2 satisfy $\Gamma_2(C, B, C) = \{G, H\}$.

Denote

$$\mathcal{A}_j^{2*} := \{(B, C) \in \mathcal{A}_j^2 : B \xrightarrow{\Phi} C\},$$

$$\mathcal{A}_j^{3*} := \{(D, B, C) \in \mathcal{A}_j^3 : D \xrightarrow{\Phi} B, B \xrightarrow{\Phi} C\}.$$

For fixed $(D, B, C) \in \mathcal{A}_{j-1}^{3*}$ denote

$$\Omega_j(B) := \{G \in \mathcal{A}_j : G \subset \text{int}(B)\},$$

$$\Omega_j(D, B) := \{G \in \Omega_j(B) : D \xrightarrow{\Phi} B\},$$

$$\Gamma_j(D, B, C) := \{G \in \Omega_j(D, B) : B \xrightarrow{\Phi} C\}.$$

We call the sets of a finite collection \mathcal{A}_n of pairwise disjoint boxes, *atoms of generation n* , or *n -atoms* if they satisfy the following conditions (see Figure 2):

a) Each atom of generation n is contained in the interior of an atom of generation $n - 1$, and the interior of each atom $B \in \mathcal{A}_{n-1}$ contains exactly $2^n \cdot 2^{n-1} = 2^{2n-1}$ pairwise disjoint n -atoms, which we call *the children of B* . In other words:

$$\#\Omega_n(B) = 2^{2n-1} \quad \forall B \in \mathcal{A}_{n-1}.$$

Therefore

$$\mathcal{A}_n = \bigcup_{B \in \mathcal{A}_{n-1}} \Omega_n(B),$$

where the families of atoms in the union are pairwise disjoint. Therefore

$$(1) \quad \#\mathcal{A}_n = (\#\mathcal{A}_{n-1})(\#\Omega_n(B)) = 2^{(n-1)^2} \cdot 2^{2n-1} = 2^{n^2}.$$

b) For each $B \in \mathcal{A}_{n-1}$, the collection of children of B is partitioned in the 2^{n-1} sub-collections $\Omega_n(D, B)$, where the atoms $D \in \mathcal{A}_{n-1}$ are such that $D \xrightarrow{\Phi} B$. Besides,

$$\#\Omega_n(D, B) = 2^n, \quad \forall (D, B) \in \mathcal{A}_{n-1}^{2*}.$$

In other words

$$\Omega_n(B) = \bigcup_{D: (D,B) \in \mathcal{A}_{n-1}^{2*}} \Omega_n(D, B), \quad \forall B \in \mathcal{A}_{n-1},$$

where the families of atoms in the above union are pairwise disjoint. Thus for any $B \in \mathcal{A}_{n-1}$ we have

$$\#\Omega_n(B) = (\#\{D \in \mathcal{A}_{n-1}: D \xrightarrow{\Phi} B\}) \cdot (\#\Omega_n(D, B)) = 2^{n-1} \cdot 2^n = 2^{2n-1}.$$

c) For each 2-tuple $(D, B) \in \mathcal{A}_{n-1}^{2*}$ the collection $\Omega_n(D, B)$ is partitioned in the 2^{n-1} sub-collections $\Gamma_n(D, B, C)$, where the atoms $C \in \mathcal{A}_{n-1}$ are such that $B \xrightarrow{\Phi} C$. Besides,

$$\#\Gamma_n(D, B, C) = 2 \quad \forall (D, B, C) \in \mathcal{A}_{n-1}^{3*}.$$

For example, in Figure 2 we have $\{F, G\} = \Gamma_2(C, B, C)$.

d) For each $(D, B, C) \in \mathcal{A}_n^{3*}$ and for each $G \in \Gamma_n(D, B, C)$

$$G \xrightarrow{\Phi} E, \quad \forall E \in \Omega_n(B, C),$$

$$\text{and } \Phi(G) \cap E' = \emptyset \quad \forall E' \in \mathcal{A}_n \setminus \Omega_n(B, C).$$

From the above conditions a) to d) we deduce:

$$(2) \quad \Omega_n(D, B) = \bigcup_{C: (B,C) \in \mathcal{A}_{n-1}^{2*}} \Gamma_n(D, B, C),$$

$$(3) \quad \mathcal{A}_n = \bigcup_{(D,B,C) \in \mathcal{A}_{n-1}^{3*}} \Gamma_n(D, B, C),$$

where the families of atoms in the unions are pairwise disjoint. Besides, for any pair $(G, E) \in \mathcal{A}_n^2$:

$$G \xrightarrow{\Phi} E \text{ if and only if } \exists (D, B, C) \in \mathcal{A}_{n-1}^{3*} \text{ such that}$$

$$G \in \Gamma_n(D, B, C), \quad E \in \Omega_n(B, C),$$

We also deduce the following properties for any atom $G \in \mathcal{A}_n$:

$$(4) \quad \#\{E \in \mathcal{A}_n: G \xrightarrow{\Phi} E\} = 2^n, \quad \#\{E \in \mathcal{A}_n: E \xrightarrow{\Phi} G\} = 2^n.$$

In fact, on the one hand, from condition d) we deduce that for each atom $G \in \mathcal{A}_n$, the number of atoms $E \in \mathcal{A}_n$ such that $G \xrightarrow{\Phi} E$ equals the number $\#\Omega_n(B, C) = 2^n$, where B and C are the unique $(n-1)$ -atoms such that $G \in \Gamma_n(D, B, C)$ for some $D \in \mathcal{A}_{n-1}$. On the other hand, the number of atoms $E \in \mathcal{A}_n$ such that $E \xrightarrow{\Phi} G$ equals

$$(\#\Gamma_n(D, B, C)) \cdot (\#\{D \in \mathcal{A}_{n-1} : D \xrightarrow{\Phi} B\}) = 2 \cdot 2^{n-1} = 2^n,$$

where B and C are the unique $(n-1)$ -atoms such that $G \in \Omega_n(B, C)$.

Since $2^n < 2^{n^2}$ if $n > 1$, we conclude that not all the pairs (G, E) of atoms of generation $n \geq 2$ satisfy $G \xrightarrow{\Phi} E$.

Definition 2.4. (Models)

We call $\Phi \in C^0(D^m)$ a *model* if $\Phi(D^m) \subset \text{int}(D^m)$, and there exists a sequence $\{\mathcal{A}_n\}_{n \geq 0}$ of finite families of pairwise disjoint boxes contained in $\text{int}(D^m)$ that are atoms of generation $0, 1, \dots, n, \dots$ respectively for Φ (according to Definitions 2.2, and 2.3) such that

$$(5) \quad \lim_{n \rightarrow +\infty} \max_{A \in \mathcal{A}_n} \text{diam} A = 0.$$

Denote by \mathcal{H} the family of all the models in $C^0(D^m)$.

Remark 2.5. For each fixed $n \geq 1$ the four conditions a) to d) of Definition 2.3, are open conditions. So, the family \mathcal{H}_n of maps that have atoms up to generation $n \geq 1$, is open in $C^0(D^m)$ and also in $\text{RHom}(D^m)$. Moreover, the conditions $\Phi(D^m) \subset \text{int}(D^m)$ and $\max_{A \in \mathcal{A}_n} \text{diam}(A) < \varepsilon_n$ are also open. Therefore, the intersection of the families satisfying the conditions for all $n \geq 1$, which is the family \mathcal{H} of model maps, is a G_δ -set in $C^0(D^m)$ and $\mathcal{H} \cap \text{RHom}(D^m)$ is a G_δ -set in $\text{RHom}(D^m)$.

Construction of models.

The rest of this section is dedicated to the proof of the following lemma.

Lemma 2.6. *The family \mathcal{H} of models is a nonempty G_δ -set in $C^0(D^m)$ and $\mathcal{H} \cap \text{RHom}(D^m)$ is a nonempty G_δ -set in $\text{RHom}(D^m)$.*

In Remark 2.5 we have noted that \mathcal{H} is a G_δ -set. The fact that it is nonempty is a consequence of the following result.

Lemma 2.7. *For all $f \in \text{RHom}(D^m)$ such that $f(D^m) \subset \text{int}(D^m)$, there exists $\psi \in \text{Hom}(D^m)$ and $\Phi \in \mathcal{H} \cap \text{RHom}(D^m)$ such that*

$$\psi|_{\partial D^m} = \text{id}|_{\partial D^m} \quad \text{and} \quad \psi \circ f = \Phi.$$

We begin by outline the strategy of the proof of Lemma 2.7. The homeomorphisms ψ and Φ are constructed as uniform limits of respective convergent sequences ψ_n and Φ_n of homeomorphisms, such that

$\psi_n \circ f = \Phi_n$ for all $n \geq 0$. The homeomorphism Φ_n has atoms up to generation n , and Φ_{n+1} coincides with Φ_n outside the interiors of its atoms of generation n . Therefore the collections of atoms of generation up to n for Φ_n is also a collection of atoms for Φ_{n+1} . To change Φ_n inside each atom A of generation n we change ψ_n only inside some adequately defined boxes $R \subset \text{int}(f(A))$, constructing $\psi_{n+1}|_R$ in such a way that $\psi_{n+1}|_{\partial R} = \psi_n|_{\partial R}$, and finally extending $\psi_{n+1}(x) := \psi_n(x)$ for all x in the complement of the union of all the boxes R . The definition of several boxes R inside each atom A of generation n , is useful to construct the atoms of generation $n+1$ for $\Phi_{n+1} = \psi_{n+1} \circ f$.

To prove Lemma 2.7, we need several technical lemmas and some more definitions. For each $(P, Q) \in \mathcal{A}_n^{2*}$ in the proof of Lemma 2.7 we will recursively choose a connected component $S(P, Q)$ of $\Phi_n(P) \cap Q$. For each $(D, B, C) \in \mathcal{A}_n^{3*}$ let $G_0(D, B, C)$, $G_1(D, B, C)$ be two disjoint boxes contained in $\text{int}(S(D, B)) \cap \Phi_n^{-1}S(B, C)$. Denote:

$$\mathcal{A}_{n+1} := \{G_j(D, B, C) : j \in \{0, 1\}, (D, B, C) \in \mathcal{A}_n^{3*}\};$$

$$\Omega_{n+1}(B) := \{G_j(D, B, C) : j \in \{0, 1\}, (D, B, C) \in \mathcal{A}_n^{3*}\}$$

for each fixed $B \in \mathcal{A}_n$;

$$\Omega_{n+1}(D, B) := \{G_j(D, B, C) : j \in \{0, 1\}, B \xrightarrow{\Phi_n} C\}$$

for each fixed $(D, B) \in \mathcal{A}_n^{2*}$;

$$\Gamma_{n+1}(D, B, C) := \{G_j(D, B, C) : j \in \{0, 1\}\}$$

for each fixed $(D, B, C) \in \mathcal{A}_n^{3*}$.

Lemma 2.8. *Fix $n \geq 0$. Let $\Phi_n \in R\text{Hom}(D^m)$. Assume that there exists families $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_n$ of atoms of generation up to n for Φ_n and that*

$$\text{int}(S(D, B)) \cap \Phi_n^{-1}S(B, C) \neq \emptyset \quad \forall (D, B, C) \in \mathcal{A}_n^{3*}.$$

Then

- a) $\#\mathcal{A}_n^{3*} = 2^{n^2+2n}$
- b) $\#\mathcal{A}_{n+1} = 2^{(n+1)^2}$ and $E \cap F = \emptyset$ for all $E, F \in \mathcal{A}_{n+1}$ such that $E \neq F$.
- c) The family \mathcal{A}_{n+1} is partitioned in the 2^{n^2} different subfamilies $\Omega_{n+1}(B)$ where $B \in \mathcal{A}_n$. Besides $\#\Omega_{n+1}(B) = 2^{2n+1} = 2^{n+1}2^n$ for all $B \in \mathcal{A}_n$
- d) For all $B \in \mathcal{A}_n$ the family of boxes $\Omega_{n+1}(B)$ is partitioned in the 2^n subfamilies $\Omega_{n+1}(D, B)$ where $D \in \mathcal{A}_n$ such that $D \xrightarrow{\Phi_n} B$.
- e) $\#\Omega_{n+1}(D, B) = 2^{n+1}$ for all $(D, B) \in \mathcal{A}_n^{2*}$.

- f) For all $(D, B) \in \mathcal{A}_n^{2*}$ the family of boxes $\Omega_{n+1}(D, B)$ is partitioned in the 2^n subfamilies $\Gamma_{n+1}(D, B, C)$ where $C \in \mathcal{A}_n$ such that $B \xrightarrow{\Phi_n} C$.
- g) $\#\Gamma_{n+1}(D, B, C) = 2$ for all $(D, B, C) \in \mathcal{A}_n^{3*}$.
- h) For all $(B, C) \in \mathcal{A}_n^{2*}$ and for all $E \in \mathcal{A}_{n+1}$, the intersection $E \cap B \cap \Phi_n^{-1}(C)$ is nonempty if and only if $E \in \Gamma_{n+1}(D, B, C)$ for some $D \in \mathcal{A}_n$ such that $D \xrightarrow{\Phi_n} B$.

Remark 2.9. Note that properties b) to g) in Lemma 2.8 imply that the family \mathcal{A}_{n+1} of boxes such constructed, satisfy conditions a), b) and c) of Definition 2.3 of atoms of generation $n + 1$ for Φ_n . Thus, to be a family of atoms of atoms of generation $n + 1$ it is only left to modify, if necessary, Φ_n in the interior of the boxes of \mathcal{A}_n in such a way that these boxes also satisfy condition d) of Definition 2.3.

Proof. of Lemma 2.8. a) All the n -tuples $(D, B, C) \in \mathcal{A}_n^{3*}$ can be constructed choosing freely $D \in \mathcal{A}_n$, for each D choosing $B \in \mathcal{A}_n$ such that $D \xrightarrow{\Phi_n} B$, and for each such B choosing $C \in \mathcal{A}_n$ such that $B \xrightarrow{\Phi_n} C$. Taking into account that by hypothesis \mathcal{A}_n is a family of atoms, equalities (1) and (4) of Definition 2.3 imply

$$\begin{aligned} \#\mathcal{A}_n^{3*} &= \#\{(D, B, C) \in \mathcal{A}_n^3 : D \xrightarrow{\Phi_n} B, B \xrightarrow{\Phi_n} C\} = \\ &(\#\mathcal{A}_n) \cdot (\#\{B \in \mathcal{A}_n : D \xrightarrow{\Phi_n} B\}) \cdot (\#\{C \in \mathcal{A}_n : B \xrightarrow{\Phi_n} C\}) = \\ &2^{n^2} 2^n 2^n = 2^{n^2+2n} \end{aligned}$$

g) By construction, $\Gamma_{n+1}(D, C, B) = \{G_0(B, C, D), G_1(B, C, D)\}$, where the two boxes inside the family are disjoint, hence different. Thus the cardinality of the family is 2.

b) By construction, $E = G_j(D, C, B)$, $F = G_{j'}(D', B', C')$. If they $E \neq F$ then, either $(D, C, B) = (D', C', B')$ and $i \neq i'$, or $(D, C, B) \neq (D', C', B')$. In the first case, by construction $G_0(D, C, B) \cap G_1(D, C, B) = \emptyset$, in other words $E \cap F = \emptyset$. In the second case, either $D \neq D'$ or $B \neq B'$ or $C \neq C'$. By construction $G_j(D, B, C) \subset \Phi_n(D) \cap B \cap \Phi_n^{-1}(C)$ and $G_{j'}(D', B', C') \subset \Phi_n(D') \cap B' \cap \Phi_n^{-1}(C')$. Since two different atoms of generation n are pairwise disjoint, and $\Phi_n \in RHom(D^m)$, we deduce that $G_j(D, B, C) \cap G_{j'}(D', B', C') = \emptyset$, hence $E \cap F = \emptyset$, as required.

By construction $\mathcal{A}_{n+1} = \bigcup_{(D,C,B) \in \mathcal{A}_n^{3*}} \Gamma_{n+1}(D, B, C)$, where the families in the union are pairwise disjoint and have each 2 different boxes of \mathcal{A}_{n+1} . Therefore, taking into account part a), we deduce that

$$\#\mathcal{A}_{n+1} = 2 \cdot (\#\mathcal{A}_n^{3*}) = 2 \cdot 2^{n^2+2n} = 2^{(n+1)^2}.$$

c) By construction of \mathcal{A}_{n+1} and $\Omega_{n+1}(B)$, we have

$$\mathcal{A}_{n+1} = \bigcup_{B \in \mathcal{A}_n} \Omega_{n+1}(B).$$

Besides $G \subset B$ for all $G \in \Omega_{n+1}(B)$, because by construction $G \subset S(D, B) \subset \Phi(D) \cap B$ for some $D \in \mathcal{A}_n$. Since two different atoms $B \neq B'$ of generation n are pairwise disjoint, we deduce that $\Omega_{n+1}(B) \cap \Omega_{n+1}(B') = \emptyset$ if $B \neq B'$. We conclude that the above union of different subfamilies $\Omega_{n+1}(B)$ is a partition of \mathcal{A}_{n+1} , as required.

Note that

$$\Omega_{n+1}(B) = \bigcup_{D \in \mathcal{A}_n, D \xrightarrow{\Phi_n} B} \bigcup_{C \in \mathcal{A}_n, B \xrightarrow{\Phi_n} C} \Gamma_{n+1}(D, B, C),$$

where the families in the union are pairwise disjoint and each of them has two different boxes. Therefore, taking into account that \mathcal{A}_n is a family of atoms (by hypothesis), equality (4) of Definition 2.3 implies:

$$\begin{aligned} \#\Omega_{n+1}(B) &= 2 \cdot (\#\{D \in \mathcal{A}_n, D \xrightarrow{\Phi_n} B\}) \cdot \#\{C \in \mathcal{A}_n, B \xrightarrow{\Phi_n} C\} = \\ &= 2 \cdot 2^n \cdot 2^n = 2^{2n+1}. \end{aligned}$$

d) By construction, $\Omega_{n+1}(B) = \bigcup_{D \in \mathcal{A}_n, D \xrightarrow{\Phi_n} B} \Omega_{n+1}(D, B)$. Besides, $\Omega_{n+1}(D, B) \cap \Omega_{n+1}(D', B) = \emptyset$ if $D \neq D'$ in \mathcal{A}_n , since different atoms of generation n are pairwise disjoint, and $G \in \Omega_{n+1}(D, B)$ implies $G \subset \Phi_n(D)$ which is disjoint with $\Phi_n(D')$ since Φ_n is a homeomorphism.

f) By construction, $\Omega_{n+1}(D, B) = \bigcup_{C \in \mathcal{A}_n, B \xrightarrow{\Phi_n} C} \Gamma_{n+1}(D, B, C)$. Besides, $\Gamma_{n+1}(D, B, C) \cap \Gamma_{n+1}(D, B, C') = \emptyset$ if $C \neq C'$ in \mathcal{A}_n , because two different atoms of generation n are pairwise disjoint and $G \in \Gamma_{n+1}(D, B, C)$ implies $G \subset \Phi_n^{-1}(C)$.

e) From Assertions f) and g) proved above, and from equality (4) of the definition of atoms of generation n , we deduce that

$$\#\Omega_{n+1}(D, B) = 2 \cdot (\#\{C \in \mathcal{A}_n : B \xrightarrow{\Phi_n} C\}) = 2 \cdot 2^n = 2^{n+1}.$$

h) By construction, $E = G_j(D, B, C)$ for some $j \in \{0, 1\}$ and some $(D, B, C) \in \mathcal{A}_n^{3*}$, i.e., $E \in \Gamma_{n+1}(D, B, C)$. Besides $E = G_j(D, B, C) \subset \text{int}(S(D, B) \cap \Phi_n^{-1}(S(B, C))) \subset \text{int}(\Phi_n(D) \cap B \cap \Phi_n^{-1}(C))$. But for $(B, C) \neq (B', C')$ in \mathcal{A}_n^{2*} the sets $B \cap \Phi_n^{-1}(C)$ and $B' \cap \Phi_n^{-1}(C')$ are disjoint, thus the box E belongs to $\Gamma_{n+1}(D, B, C)$ for some $D \in \mathcal{A}_n$, if and only if it intersects $B \cap \Phi_n^{-1}(C)$. \square

Lemma 2.10. *Assume the hypotheses of Lemma 2.8. Let $\tilde{L}_{n+1} \subset D^m$ be a finite set with cardinality $2^{(n+1)^2} 2^{n+1} s$, with a unique point $\tilde{e}_i(E) \in$*

\tilde{L}_{n+1} for each $(i, E) \in \{1, 2, \dots, 2^{n+1}\} \times \mathcal{A}_{n+1}$. Assume that

$$\tilde{e}_i(E) \in \text{int}(E) \quad \forall (i, E) \in \{1, 2, \dots, 2^{n+1}\} \times \mathcal{A}_{n+1}.$$

Then, there exists a permutation $\theta : \tilde{L}_{n+1} \mapsto \tilde{L}_{n+1}$ such that

a) For all $(i, E) \in \{1, 2, \dots, 2^{n+1}\} \times \Gamma_{n+1}(D, B, C)$

$$\theta(\tilde{e}_i(E)) = \tilde{e}_{i'}(E')$$

for a unique $i' \in \{1, 2, \dots, 2^{n+1}\}$ and a unique $E' \in \Omega_{n+1}(B, C)$.

b) For all $(D, B, C) \in \mathcal{A}_n^{3*}$, for all $E \in \Gamma_{n+1}(D, B, C)$ and for all $F \in \Omega_{n+1}(B, C)$ there exists unique $(i, i') \in \{1, 2, \dots, 2^{n+1}\}^2$ such that $\theta(\tilde{e}_i(E)) = \tilde{e}_{i'}(F)$.

c) For all $(B, C) \in \mathcal{A}_n^{2*}$

$$\theta\left(\left\{\tilde{e}_i(E) : E \in \bigcup_{D \in \mathcal{A}_n, (D, B) \in \mathcal{A}_n^{2*}} \Gamma(D, B, C), \quad i \in \{1, 2, \dots, 2^{n+1}\}\right\}\right) =$$

$$\left\{\tilde{e}_{i'}(F), F \in \Omega_{n+1}(B, C), i' \in \{1, 2, \dots, 2^{n+1}\}\right\} = \tilde{L}_{n+1} \cap S(B, C).$$

Proof. From the construction of the family \mathcal{A}_{n+1} (see Lemma 2.8), we deduce that for all $E \in \mathcal{A}_{n+1}$ there exists unique $j \in \{0, 1\}$ and unique $(D, B, C) \in \mathcal{A}_n^{3*}$ such that $E = G_j(D, B, C) \subset \Gamma_{n+1}(D, B, C)$. Therefore, for all $(i, E) \in \{1, 2, \dots, 2^{n+1}\} \times \mathcal{A}_{n+1}$, we have

$$\tilde{e}_i(E) = e_i(G_j(D, B, C))$$

By hypothesis \mathcal{A}_n is the family of atoms of generation n for Φ_n , thus we can apply Equalities (4) of Definition 2.3. So, for each $B \in \mathcal{A}_n$, we can index the different atoms $D \in \mathcal{A}_n$ such that $D \xrightarrow{\Phi_n} B$ as follows:

$$(6) \quad \{D \in \mathcal{A}_n : D \xrightarrow{\Phi_n} B\} = \{D_1^-(B), D_2^-(B), \dots, D_{2^n}^-(B)\},$$

where $D_{k_1}^-(B) \neq D_{k_2}^-(B)$ if $k_1 \neq k_2$ (actually, they are disjoint atoms of generation n).

Analogously

$$(7) \quad \{C \in \mathcal{A}_n : B \xrightarrow{\Phi_n} C\} = \{C_1^+(B), C_2^+(B), \dots, C_{2^n}^+(B)\},$$

where $C_{l_1}^+(B) \neq C_{l_2}^+(B)$ if $l_1 \neq l_2$.

Now, we index the distinct points of \tilde{L}_{n+1} as follows:

$$\hat{e}_{i,j}(k, B, l) := \tilde{e}_i(G_j(D, B, C)) = e_i(G_j(D_k^-(B), B, C_l^+(B))),$$

for all $(i, j, B, k, l) \in \{1, 2, \dots, 2^{n+1}\} \times \{0, 1\} \times \mathcal{A}_n \times \{1, 2, \dots, 2^n\}^2$.

Define the following correspondence $\theta : \tilde{L}_{n+1} \rightarrow \tilde{L}_{n+1}$:

$$\theta(\hat{e}_{i,j}(k, B, l)) = \hat{e}_{i',j'}(k', B', l'), \quad \text{where}$$

- $B' := C_l^+(B)$,

- k' is such that $B = D_{k'}^-(C)$ (such k' exists and is unique because $B \xrightarrow{\Phi_n} C$, using (6)),
- $l' = i \pmod{2^n}$,
- $j' = 0$ if $i \leq 2^n$ and $j' = 1$ if $i > 2^n$,
- $i' = k + j \cdot 2^n$.

Let us prove that θ is surjective; hence it is a permutation of the finite set \tilde{L}_{n+1} .

Let $e_{i',j'}(k', B', l') \in \tilde{L}_{n+1}$ be given, where

$$(i', j', B', k', l') \in \{1, 2, \dots, 2^{n+1}\} \times \{0, 1\} \times \mathcal{A}_n \times \{1, 2, \dots, 2^n\}^2.$$

Construct

- $i := l' + j' \cdot 2^n$. Then $l' = i \pmod{2^n}$, $j' = 0$ if $i \leq 2^n$ and $j' = 1$ if $i > 2^n$.
- $B := D_{k'}^-(B')$. Then $B \xrightarrow{\Phi_n} B'$. So, there exists l such that $B' = C_l^+(B)$.
- $k := i' \pmod{2^n}$, $j := 0$ if $i' \leq 2^n$ and $j := 1$ if $i' > 2^n$. Therefore $i' = k + 2^n j$.

By the above equalities we have constructed some θ^{-1} such that $\theta \circ \theta^{-1}$ is the identity map. So, θ is surjective, hence also one-to-one in the finite set \tilde{L}_{n+1} , as required.

Now, let us prove that θ satisfies assertions a), b) and c) of Lemma 2.10.

a) Fix $\tilde{e}_i(E) \in \tilde{L}_{n+1}$. By construction $\theta(\tilde{e}_i(E)) = \tilde{e}_{i'}(E') \subset \text{int}(E')$ for all (i, E) for some $(i, E) \in \{1, 2, \dots, 2^{n+1}\} \times \mathcal{A}_{n+1}$. Since two different boxes of \mathcal{A}_{n+1} are pairwise disjoint (recall Lemma 2.8-a)), the box E' is unique. Besides, by hypothesis, $\tilde{e}_{i'}(E') \neq \tilde{e}_{j'}(E')$ if $j \neq j'$. So, the index i' is also unique. Therefore, to finish the proof of a), it is enough to check that $E' \in \Omega_{n+1}(B, C)$ if $E \in \Gamma_{n+1}(D, B, C)$.

By the definition of the family $\Gamma_{n+1}(D, B, C)$ in Lemma 2.8, if $E \in \Gamma_{n+1}(D, B, C)$, there exists $j \in \{0, 1\}$ such that $E = G_j(D, B, C)$. Thus, using the notation at the beginning $\tilde{e}_i(E) = \tilde{e}_i(G_j(D, B, C)) = \hat{e}_{i,j}(k, B, l)$, where $D = D_k^-(B)$ and $C = C_l^+(B)$. Then, using the definition of the permutation θ , and the computation of its inverse θ^{-1} , we obtain $\tilde{e}_{i'}(E) = \theta(\tilde{e}_i(E)) = \hat{e}_{i',j'}(k', B', l')$, where

$$B' = C_l^+(B) = C, \quad D' = D_{k'}^-(B') = B.$$

We have proved that $\tilde{e}_{i'}(E') = \tilde{e}_{i'}(G_{j'}(B, C, C'))$. Finally, from the definition of the family $\Omega_{n+1}(B, C)$ in Lemma 2.8 we conclude that $E' \in \Omega_{n+1}(B, C)$ as asserted in part a).

b) Fix $(D, B, C) \in \mathcal{A}_n^{3*}$ and $E \subset \Gamma_{n+1}(D, B, C)$. Then, using the definition of the family $\Gamma_{n+1}(D, B, C)$ in Lemma 2.7, we have

unique $(j, k, l) \in \{0, 1\} \times \{1, 2, \dots, 2^n\}$ such that $E = G_j(D, B, C)$, $D = D_k^-(B)$, $C = C_l^+(B)$. Consider the finite set Z of 2^{n+1} distinct points $\tilde{e}_i(E) = \widehat{e}_{i,j}(k, B, l)$, with j, k, B, l fixed as above and $i \in \{1, 2, \dots, 2^{n+1}\}$. The image of each point in Z by the permutation θ is $\theta(\tilde{e}_i(E)) = \widehat{e}_{i',j'}(B, C, C')$ (here we use assertion a)), satisfying the equality $i' = k + 2^n j$. Since k, j are fixed, we deduce that there exists a unique i' such that all the points of $\theta(Z)$ are of the form $\tilde{e}_{i'}(F)$, $F = G_{j'}(B, C, C')$ with $j' \in \{0, 1\}$, $C' = C_{k'}^+(C)$, $k' \in \{1, 2, \dots, 2^{n+1}\}$. We have proved that the permutation $\theta|_Z$ is equivalent to

$$i \in \{1, 2, \dots, 2^{n+1}\} \rightarrow (j', k') \in \{0, 1\} \times \{1, 2, \dots, 2^n\}$$

such that $\theta(\widehat{e}_i(E)) = \widehat{e}_{i',j'}(B, C, C_{k'}^+(C))$ with i' fixed.

Since $\#\{1, 2, \dots, 2^{n+1}\} = \#\{0, 1\} \times \#\{1, 2, \dots, 2^n\}$, from the injectiveness of θ we deduce that $\theta(Z) = \{0, 1\} \times \{1, 2, \dots, 2^n\}$. In other words, for every $F \in \Omega(B, C)$ there exists unique i such that $\theta(\tilde{e}_i(E)) = \tilde{e}_{i'}(F)$ (where i' is uniquely defined given E). This ends the proof of assertion b).

c) For fixed $(B, C) \in \mathcal{A}_n^{2*}$, denote

$$P := \left\{ \tilde{e}_i(E) : E \in \bigcup_{D \in \mathcal{A}_n, D \stackrel{\Phi_n}{\rightarrow} B} \Gamma_{n+1}(D, B, C), i \in \{1, 2, \dots, 2^{n+1}\} \right\},$$

$$Q := \{ \tilde{e}_{i'}(F) : F \in \Omega_{n+1}(B, C), i' \in \{1, 2, \dots, 2^{n+1}\} \} \subset \tilde{L}_{n+1}.$$

Applying assertion a) we deduce that $\theta(P) \subset Q$. So, to prove that $\theta(P) = Q$ it is enough to prove that $\#P = \#Q$. In fact, applying parts e) and g) of Lemma 2.8 for the family of boxes \mathcal{A}_{n+1} , and equality (4) of Definition 2.2 for the family of atoms \mathcal{A}_n , we obtain

$$\#P = 2^{n+1} \cdot (\#\Gamma_{n+1}(D, B, C)) \cdot (\#\{D \in \mathcal{A}_n : D \stackrel{\Phi_n}{\rightarrow} B\}) = 2^{n+1} \cdot 2 \cdot 2^n$$

$$\#Q = 2^{n+1} \cdot (\#\Omega_{n+1})(B, C) = 2^{n+1} \cdot 2^{n+1},$$

which proves that $\#P = \#Q$ and thus that $\theta(P) = Q$.

Finally, let us prove that $Q = \tilde{L}_{n+1} \cap S(B, C)$.

On the one hand, if $F \in \Omega_{n+1}(B, C)$, then $F = G_j(B, C, C')$ for some (j, C') . Applying the construction of the boxes of \mathcal{A}_{n+1} in the hypothesis of Lemma 2.8, we obtain $F \subset S(B, C)$, hence $\tilde{e}_{i'}(F) \in \tilde{L}_{n+1} \cap \text{int}(F) \subset \tilde{L}_{n+1} \cap S(B, C)$. This proves that $Q \subset \tilde{L}_{n+1} \cap S(B, C)$.

On the other hand, if $\tilde{e}_{i'}(F) \in \tilde{L}_{n+1} \cup S(B, C)$, then $F \in \mathcal{A}_{n+1}$. Applying Lemma 2.8, we obtain $F = G_j(D', B', C') \subset S(D', B')$ for some $(D', B', C') \in \mathcal{A}_n^{3*}$. Since $S(D', B') \subset \Phi_n(D') \cap B'$ and $S(B, C) \subset \Phi_n(B) \cap C$, we obtain $S(D', B') \cap S(B, C) = \emptyset$ if $(D', B') \neq (B, C)$. But $\tilde{e}_{i'}(F) \in \text{int}(F) \cap S(B, C) \subset S(D', B') \cap S(B, C)$. We conclude

that $(D', B') = (B, C)$, thus $F = G_j(B, C, C') \in \Omega_{n+1}(B, C)$, hence $\tilde{e}_{i'}(F) \in Q$. We have proved that $\tilde{L}_{n+1} \cap S(B, C) \subset Q$. Therefore the proof of assertion c) is complete. \square

Lemma 2.11. *Assume the hypothesis of Lemmas 2.8 and 2.10. Let $\Phi_{n+1} \in RHom(D^m)$ be such that*

$$\Phi_{n+1}(x) = \Phi_n(x) \quad \forall x \notin \bigcup_{A \in \mathcal{A}_n} A, \quad \Phi_{n+1}(\tilde{e}) = \theta(\tilde{e}) \quad \forall \tilde{e} \in \tilde{L}_{n+1},$$

where θ is the permutation of \tilde{L}_{n+1} constructed in Lemma 2.10. Then,

- a) $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_{n+1}$ are collections of atoms up to generation $n+1$ for Φ_{n+1} .
- b) For each $(E, F) \in \mathcal{A}_{n+1}^2$ such that $E \xrightarrow{\Phi_{n+1}} F$, there exists exactly one point $\tilde{e}_i(E) \in \tilde{L}_{n+1} \cap \text{int}(E)$, and exactly one point $\tilde{e}_{i'}(F) \in \tilde{L}_{n+1} \cap \text{int}(F)$, such that

$$\tilde{\Phi}_{n+1}(\tilde{e}_i(E)) = \tilde{e}_{i'}(F).$$

Proof. a) Since $\Phi_{n+1}(x) = \Phi_n(x)$ for all $x \notin \bigcup_{B \in \mathcal{A}_n} B$, the same families $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_n$, of atoms up to generation n for Φ_n are families of atoms up to generation n for Φ_{n+1} . As observed in Remark 2.9, the family \mathcal{A}_{n+1} of pairwise disjoint boxes satisfy conditions a), b) and c) of Definition 2.3 with $n+1$ instead of n , for Φ_{n+1} . So, to prove that \mathcal{A}_{n+1} is a family of atoms of generation $n+1$ for Φ_{n+1} it is enough to prove that its boxes satisfy also condition d) of Definition 2.3.

Take $(D, B, C) \in \mathcal{A}_n^{3*}$, $E \in \Gamma_{n+1}(D, B, C)$ and $F \in \Omega_{n+1}(B, C)$. Applying Lemma 2.10-b), there exists (i, i') such that $\theta(\tilde{e}_i(E)) = \tilde{e}_{i'}(F)$. Therefore

$$\Phi_{n+1}(\tilde{e}_i(E)) = \tilde{e}_{i'}(F).$$

Since $\tilde{e}_i(E) \in \text{int}(E)$ and $\tilde{e}_{i'}(F) \in \text{int}(F)$, we conclude that $\Phi_{n+1}(E) \cap \text{int}(F) \neq \emptyset$, namely, $E \xrightarrow{\Phi_{n+1}} F$. We have proved that

$$E \xrightarrow{\Phi_{n+1}} F \quad \forall E \in \Gamma_{n+1}(D, B, C), \quad \forall F \in \Omega_{n+1}(B, C).$$

This proves the first half of condition d) of Definition 2.3.

Let us prove the second half. Let $E \in \Gamma_{n+1}(D, B, C)$ and $F \in \mathcal{A}_{n+1} \setminus \Omega_{n+1}(B, C)$. Then $F \in \Omega_{n+1}(B', C')$ with $(B', C') \neq (B, C)$ (recall that by Lemma 2.8, the union of all the families $\Omega_{n+1}(\cdot, \cdot)$ is \mathcal{A}_{n+1}). We have $F \in \Omega_{n+1}(B', C')$; therefore $F \subset \Phi_n(B') \cap C'$. Besides $\Phi_n(B') = \Phi_{n+1}(B')$ (recall that Φ_n and Φ_{n+1} are relative homeomorphisms that coincide outside the atoms of generation n ; so, by continuity, they

coincide in the borders of all those atoms, therefore the images by Φ_n and Φ_{n+1} of each atom of generation n coincide). We deduce that

$$(8) \quad F \subset \Phi_{n+1}(B') \cap C' \text{ with } (B'C') \neq (B, C).$$

Since $E \in \Gamma_{n+1}(D, B, C)$, we know that $E \subset B \cap \Phi_n^{-1}(C) = B \cap \Phi_{n+1}^{-1}(C)$ (recall the construction of the atoms of the family \mathcal{A}_{n+1} in the hypothesis of Lemma 2.8). Therefore

$$(9) \quad \Phi_{n+1}(E) \subset \Phi_{n+1}(B) \cap C$$

Since two different atoms of generation n are disjoint, and Φ_{n+1} is a relative homeomorphism, we have $(\Phi_{n+1}(B) \cap C) \cap (\Phi_{n+1}(B') \cap C') = \emptyset$ if $(B, C) \neq (B', C')$. We conclude that

$$\Phi_{n+1}(E) \cap F = \emptyset \quad \forall F \in \mathcal{A}_{n+1} \setminus \Omega_{n+1},$$

ending the proof of condition d) of Definition 2.3 for the family \mathcal{A}_{n+1} . The proof of assertion a) is complete.

b) Take $(E, F) \in \mathcal{A}_{n+1}^2$ such that $E \xrightarrow{\Phi_{n+1}^{-1}} F$. Then

$$(10) \quad \Phi_{n+1}(E) \cap F \neq \emptyset.$$

We have $E \in \mathcal{A}_{n+1}$. Therefore, by the construction of this family in Lemma 2.8, there exists $(D, B, C) \in \mathcal{A}_n^{3*}$ such that

$$E \in \Gamma_n(D, B, C).$$

Combining inequality (10) with property d) of Definition 2.3 (putting $n + 1$ instead of n), we obtain

$$F \in \Omega_{n+1}(B, C).$$

Applying Lemma 2.10-b) there exists a unique $(i, i') \in \{1, 2, \dots, 2^{n+1}\}^2$ such that $\Phi_{n+1}(\tilde{e}_i(E)) = \theta(\tilde{e}_i(E)) = \tilde{e}_{i'}(F)$. Since $\tilde{L}_{n+1} \cap (\text{int}(E)) = \{\tilde{e}_i(E) : i \in \{1, 2, \dots, 2^{n+1}\}\}$ and analogously for $\tilde{L}_{n+1} \cap (\text{int}(F))$, we conclude that there exists a unique point $\tilde{e}_i(E) \in \tilde{L}_{n+1} \cap (\text{int}(E))$ and a unique point $\tilde{e}_{i'}(F) \in \tilde{L}_{n+1} \cap (\text{int}(F))$ such that $\Phi_{n+1}(\tilde{e}_i(E)) = \tilde{e}_{i'}(F)$, as required. The proof of part b) is complete. \square

Lemma 2.12. *Let $\psi \in \text{Hom}(D^m)$, $P, Q \subset D^m$ be boxes such that $\psi(P) = Q$, $p_1, \dots, p_k \in \text{int}(P)$ be distinct points and $q_1, \dots, q_k \in \text{int}(Q)$ be also distinct points. Then, there exists $\psi^* \in \text{Hom}(D^m)$ such that*

$$\psi^*(x) = \psi(x) \quad \forall x \notin \text{int}(P) \text{ and } \psi^*(p_i) = q_i \quad \forall i \in \{1, \dots, k\}.$$

Proof. We argue by induction on $k \geq 1$.

For $k = 1$, it suffices to construct a homeomorphism $\chi : Q \mapsto Q$ such that $\chi|_{\partial Q}$ is the identity map, and $\chi(\psi(p_1)) = q_1$. Once χ is constructed, the homeomorphism defined by $\psi^*(x) := \psi(x)$ if $x \notin \text{int}(P)$ and $\psi^*(x) = \chi \circ \psi(x)$ if $x \in \text{int}(P)$, satisfies the required properties.

Since Q is a box we can consider a homeomorphism $\xi : Q \mapsto D^m$. Then, it is enough to construct $\chi' \in \text{Hom}(D^m)$ such that

$$(11) \quad \chi'(x) = x \quad \forall x \in \partial D^m \text{ and}$$

$$(12) \quad \chi'(\xi \circ \psi(p_1)) = \xi(q_1).$$

since $\chi := \xi^{-1} \circ \chi' \circ \xi$ satisfies the required properties.

Let $p'_1 := \xi(\psi(p_1)) \in \text{int}(D^m)$, $q'_1 = \xi(q_1) \in \text{int}(D^m)$. For each point $r \in \partial D_m$ denote by $S_p(r)$ the segment in D^m joining the point p'_1 to r , and by $S_q(r)$ the segment joining q'_1 with r . Let $\chi'|_{S_p(r)} : S_p(r) \mapsto S_q(r)$ be the affinity from one segment to the other, for each $r \in \partial D^m$, leaving r fixed. Define $\chi'(p'_1) := q'_1$ and, for all $x \in D^m \setminus \{p'_i\}$ define $\chi'(x) := \chi'|_{S_p(r_x)}(x)$, where r_x is the unique point in ∂D_m such that $x \in S_p(r(x))$. It is not hard to check that $\chi' : D_m \mapsto D_m$ is a homeomorphism. By construction χ' leaves fixed the points of ∂D_m and $\chi'(p'_1) = q'_1$ as required.

We turn to the inductive step, assume that the assertion of the lemma is true for $k - 1$, we add the subscript j to all the objects associated to step j (we will use $j = 1$, $j = k - 1$ and of course $j = k$ in the proof). Let us prove it for k . First, the induction hypothesis, yields homeomorphisms $\psi_{k-1}^* : D^m \mapsto D^m$ such that $\psi_{k-1}^*(x) = \psi(x)$ for all $x \notin \text{int}(P_j)$ and $\psi_{k-1}^*(p_i) = q_i$ for all $1 \leq i \leq k - 1$.

Since the points p_1, \dots, p_k are distinct we can choose a box $P_1 \subset \text{int}(P_k)$ such that $p_k \in \text{int}(P_1)$ and $p_1, \dots, p_{k-1} \notin P_1$. Consider the ball $Q_1 := \psi_1(P_1) \subset \psi_1(P_k) = \psi(P_k) = Q_k$ and choose a point $q'_k \in \text{int}(Q_1)$. Using that the lemma is true for $j = 1$ (as proved above), construct a homeomorphism $\psi_1^* : D^m \mapsto D^m$ such that $\psi_1^*(p_k) = q'_k$ and $\psi_1^*(x) = \psi_{k-1}^*(x)$ for all $x \notin P_1$ (hence $\psi_1^*(x) = \psi(x)$ for all $x \notin P$).

We claim that $q'_k, q_k \notin \{q_1, \dots, q_{k-1}\}$. By hypothesis q_1, q_2, \dots, q_k are all distinct points, on the other hand $q_i = \psi_{k-1}^*(p_i) = \psi_1^*(p_i)$ for all $i = 1, 2, \dots, k - 1$ and $q'_k = \psi_1(p_k)$. We have $q'_k \notin \{q_1, \dots, q_{k-1}\}$ since ψ_1 is a homeomorphism, and $p_1, p_2, \dots, p_{k-1}, p_k$ are all distinct points.

Now, consider a box $Q'_1 \subset \text{int}(Q_k)$ such that $q'_k, q_k \in \text{int}(Q'_1)$ and $q_1, \dots, q_{k-1} \notin Q'_1$. Since the lemma was already proved in the case $k = 1$, we can construct a homeomorphism $\xi : D_m \mapsto D_m$ such that $\xi(x) = x$ for all $x \notin \text{int}(Q'_1)$ and $\xi(q'_k) = q_k$. Define

$$\psi_k^* := \xi \circ \psi_1.$$

Then, since $\psi(x) \notin \text{int}(\psi(P)) = \text{int}(Q) \supset Q'_1$, we have

$$\psi_k^*(x) = \xi \circ \psi_1(x) = \xi \circ \psi(x) = \psi(x), \quad \text{for all } x \notin \text{int}(P).$$

Besides, since $q_i \notin Q'_1$ if $i \leq q-1$, we have

$$\psi(p_i) = \xi \circ \psi_1(p_i) = \xi(q_i) = q_i, \quad \forall i \in \{1, \dots, k-1\}.$$

Finally

$$\psi(p_k) = \xi \circ \psi_1(p_k) = \xi(q'_k) = q_k,$$

ending the proof of Lemma 2.12 \square

Lemma 2.13. *Let $\psi \in \text{Hom}(D^m)$, $r \geq 1$, $P_1, P_2, \dots, P_r \subset D^m$ be pairwise disjoint boxes, and $Q_j := \psi(P_j)$ for all $j \in \{1, 2, \dots, r\}$. For $k \geq 1$ and $j \in \{1, 2, \dots, r\}$, let $p_{1,j}, \dots, p_{k,j} \in \text{int}(P_j)$ be distinct points and $q_{1,j}, \dots, q_{k,j} \in \text{int}(Q_j)$ also be distinct points. Then, there exists a $\psi^* \in \text{Hom}(D^m)$ such that*

$$\psi^*(x) = \psi(x) \quad \forall x \notin \bigcup_{j=1}^r \text{int}(P_j) \text{ and}$$

$$\psi^*(p_{i,j}) = q_{i,j} \quad \forall (i, j) \in \{1, \dots, k\} \times \{1, 2, \dots, r\}.$$

Proof. Applying Lemma 2.12, for each $j \in \{1, 2, \dots, r\}$, yields homeomorphisms $\psi_j : D^m \mapsto D^m$ such that

$$\psi_j(x) = \psi(x) \quad \forall x \in D^m \setminus \text{int}(P_j) \quad \text{and} \quad \psi_j|_{\text{int}(P_j)}(p_{i,j}) = q_{i,j}.$$

Construct the homeomorphism

$$\psi^*(x) := \psi(x) \text{ if } x \notin \bigcup_{j=1}^r \text{int}(P_j),$$

and for each $j \in \{1, 2, \dots, r\}$:

$$\psi^*(x) := \psi_j(x) \text{ if } x \in \text{int}(P_j).$$

It is immediate to check that $\psi^* : D^m \mapsto D^m$ is a homeomorphism that satisfies the required properties. \square

Proof. of Lemma 2.7 We divide the construction of ψ and $\Phi \in \mathcal{H}$ into several steps:

Step 1. Construction of the atom of generation 0. Since $f(D^m) \subset \text{int}(D^m)$, there exists a box $A_0 \subset \text{int}(D^m)$ such that $f(D^m) \subset \text{int}(A_0)$. The box A_0 is the *atom of generation 0* for the relative homeomorphism $\Phi_0 := f$ which satisfies $\Phi_0 = \psi_0 \circ f$ where ψ_0 is the identity map. Applying the Brouwer Fixed Point Theorem, there exists a point $e_0 \in \text{int}(\Phi_0(A_0))$ such that $\Phi_0(e_0) = e_0$. Define $S(A_0, A_0)$ to be the connected component of $A_0 \cap \Phi_0(A_0)$ containing e_0 .

Step 2. Construction of the atoms of generation $n+1$. Inductively assume that we have constructed families $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_n$ of atoms up to generation n for $\Phi_n = \psi_n \circ f$, where $\psi_n \in \text{Hom}(D^m)$ is a homeomorphism satisfying $\psi_n|_{\partial D^m}$ is the identity map, satisfying:

- I) $\max_{B \in \mathcal{A}_i} \max\{\text{diam}(B), \text{diam}(f(B))\} < \frac{1}{2^i} \quad \forall i \in \{0, 1, \dots, n\}$;
- II) $\Phi_i(x) = \Phi_{i-1}(x), \quad \forall x \in D^m \setminus \bigcup_{B \in \mathcal{A}_{i-1}} B, \quad \forall i \in \{1, \dots, n\}$;
- III) for all $(D, B, C) \in \mathcal{A}_n^{3*}$ there exists a point $e(D, B, C)$ such that

$$L_n := \{e(D, B, C) : (D, B, C) \in \mathcal{A}_n^{3*}\}$$

is Φ_n -invariant, and

$$(13) \quad e(D, B, C) \in \text{int}(S(D, B) \cap \Phi_n^{-1}(S(B, C))),$$

where $S(D, B)$ and $S(B, C)$ are previously defined connected components of $B \cap \Phi_n(D)$ and of $C \cap \Phi_n(B)$ respectively;

- IV) the sets $S(D, B)$ and $S(D', B')$ are disjoint if $(D, B) \neq (D', B')$.

Let us construct the family \mathcal{A}_{n+1} of atoms of generation $n+1$ and the homeomorphisms Φ_{n+1} and ψ_{n+1} . First, for each $(D, B) \in (\mathcal{A}_n)^{2*}$ we choose a box $R(D, B)$ such that

$$(14) \quad e(D, B, C) \in \text{int}(R(D, B)) \subset \text{int}(S(D, B))$$

$$\forall C \in \mathcal{A}_n \text{ such that } B \xrightarrow{\Phi_n} C.$$

By (IV)) such boxes $R(\cdot, \cdot)$ are pairwise disjoint.

Recall that $e(D, B, C) \in L_n$, the set L_n is Φ_n -invariant and moreover $e(D, B, C) \in \text{int}(\Phi_n^{-1}(S(B, C)))$. Thus

$$e(D, B, C) \in \text{int}\left(R(D, B) \cap \Phi_n^{-1}(R(B, C))\right) \neq \emptyset$$

because

$$\Phi_n(e(D, B, C)) = e(B, C, \cdot) \in \text{int}(R(B, C)).$$

Next, for each $(D, B, C) \in \mathcal{A}_n^{3*}$ we choose two pairwise disjoint boxes, $G_0(D, B, C)$ and $G_1(D, B, C)$, contained in the interior of $R(D, B) \cap \Phi_n^{-1}(R(B, C))$, satisfying

$$(15) \quad \max\{\text{diam}(G_i(D, B, C)), \text{diam}(f(G_i(D, B, C)))\} < \frac{1}{2^{n+1}}$$

for $i = 0, 1$. Now, we use Lemma 2.8 to define the family \mathcal{A}_{n+1} of all the boxes $G_i(D, B, C)$ and use its properties. The boxes of the family \mathcal{A}_{n+1} will be the $(n+1)$ -atoms of two new homeomorphisms $\tilde{\Phi}_{n+1}$ and Φ_{n+1} that we will construct as follows.

First, in the interior of each box $E \in \mathcal{A}_{n+1}$ we choose 2^{n+1} distinct points $\tilde{e}_i(E), i = 1, 2, \dots, 2^{n+1}$, and denote

$$\tilde{L}_{n+1} := \{\tilde{e}_i(E) : E \in \mathcal{A}_{n+1}, 1 \leq i \leq 2^{n+1}\}.$$

Second, we build a permutation $\tilde{\theta}$ of \tilde{L}_{n+1} satisfying the properties of Lemma 2.10.

Third, applying Lemma 2.13, we construct $\tilde{\psi}_{n+1} \in \text{Hom}(D^m)$ defined satisfying the following constraints.

(a)

$$\tilde{\psi}_{n+1}|_{\psi_n^{-1}R(B,C)} : \psi_n^{-1}(R(B,C)) \rightarrow R(B,C) \quad \forall (B,C) \in \mathcal{A}_n^{2*},$$

(b)

$$\begin{aligned} \tilde{\psi}_{n+1}(x) &= \psi_n(x), \\ \forall x \notin \bigcup_{(B,C) \in \mathcal{A}_n^{2*}} \psi_n^{-1}(R(B,C)) &= \bigcup_{(B,C) \in \mathcal{A}_n^{2*}} f \circ \Phi_n^{-1}(R(B,C)), \end{aligned}$$

(c)

$$\tilde{\psi}_{n+1}(f(\tilde{e})) = \tilde{\theta}(\tilde{e}), \quad \forall \tilde{e} \in \tilde{L}_{n+1}.$$

To prove the existence of such a homeomorphism $\tilde{\psi}_{n+1}$ we must verify the hypotheses of Lemma 2.13. On the one hand, the boxes $R(B,C)$ where $B, C \in \mathcal{A}_n^{2*}$ are pairwise disjoint. So their preimages by the homeomorphism ψ_n also are pairwise disjoint. On the other hand, the finite set

$$\{f(\tilde{e}) : \tilde{e} \in \tilde{L}_{n+1} \cap \text{int}(\Phi_n^{-1}(R(B,C)))\}$$

is contained in the interior of $f \circ \Phi_n^{-1}(R(B,C)) = \psi_n^{-1}(R(B,C))$. Besides, it coincides with

$$\{f(\tilde{e}_i(E)) : E \in \Gamma_{n+1}(D, B, C) \text{ for some } D \in \mathcal{A}_n, i = 1, \dots, 2^{n+1}\}$$

(recall that $\tilde{e} \in E \cap B \cap \Phi_n^{-1}(C)$ and apply Lemma 2.8-h)). So, its image by the permutation $\tilde{\theta}$ is the finite set

$$\{\tilde{\theta}(\tilde{e}_i(G)) : G \in \Gamma_{n+1}(D, B, C) \text{ for some } D \in \mathcal{A}_n, i = 1, 2, \dots, 2^{n+1}\}$$

Applying Lemma 2.10-c), the latter set is

$$\{e_k(F) : F \in \Omega_{n+1}(B, C), k = 1, 2, \dots, 2^{n+1}\} = \tilde{L}_{n+1} \cap R(B, C),$$

which is contained in the interior of $R(B, C)$. The hypothesis of Lemma 2.13 is satisfied.

We construct

$$\tilde{\Phi}_{n+1} := \tilde{\psi}_{n+1} \circ f.$$

Since $\tilde{\Phi}_{n+1}(x) = \Phi_n(x)$, $\forall x \notin \cup_{A \in \mathcal{A}_n} A$, the same atoms up to generation n for Φ_n are still atoms up to generation n for $\tilde{\Phi}_{n+1}$. But moreover, applying Lemma 2.11-a), the boxes of the new family \mathcal{A}_{n+1} are now $(n+1)$ -atoms for $\tilde{\Phi}_{n+1}$.

Step 3. Construction of Φ_{n+1} and Ψ_{n+1} . To argue by induction, we will not use the homeomorphisms $\tilde{\Phi}_{n+1}$ and $\tilde{\psi}_{n+1}$, rather we need to

modify them to obtain new homeomorphisms Φ_{n+1} and ψ_{n+1} such that Assertion (13) also holds for $n+1$ instead of n . Precisely, modifying $\tilde{\psi}_{n+1}$ only in the interiors of the boxes $f(G)$ for all the atoms $G \in \mathcal{A}_{n+1}$, we construct a new homeomorphism ψ_{n+1} such that $\Phi_{n+1} := \psi_{n+1} \circ f$ has the same atoms up to generation $n+1$ as $\tilde{\Phi}_{n+1}$.

From the above construction of $\tilde{\psi}_{n+1}$ and $\tilde{\Phi}_{n+1}$, and from Lemma 2.11-b), we know that for each $(G, E) \in \mathcal{A}_{n+1}^{2*}$ there exists a unique point $\tilde{e}_i(G) \in \text{int}(G)$, and a unique point $\tilde{e}_k(E)$, such that

$$\tilde{\Phi}_{n+1}(\tilde{e}_i(G)) = \tilde{\psi}_{n+1} \circ f(\tilde{e}_i(G)) = \tilde{e}_k(E) \in \text{int}(E).$$

Therefore

$$\tilde{e}_k(E) \in \text{int}(E \cap \tilde{\Phi}_{n+1}(G)).$$

Denote by

$$S(G, E)$$

the connected component of $E \cap \tilde{\Phi}_{n+1}(G)$ that contains the point $\tilde{e}_k(E)$. Choose 2^{n+1} distinct points

$$e_i(G, E) \in \text{int}(S(G, E)), \quad i = 1, \dots, 2^{n+1}$$

and a permutation θ of the finite set

$$(16) \quad L_{n+1} := \{e_i(G, E) : (G, E) \in \mathcal{A}_{n+1}^{2*}, \quad i = 1, \dots, 2^{n+1}\}$$

such that for each fixed $(G, E, F) \in (\mathcal{A}_{n+1})^{3*}$ there exists a unique point $e_i(G, E)$, and a unique point $e_k(E, F)$, satisfying

$$\theta(e_i(G, E)) = e_k(E, F).$$

The proof of the existence of such permutation is similar to the proof of Lemma 2.10.)

Applying Lemma 2.13, construct a homeomorphism

$$\psi_{n+1} \in \text{Hom}(D^m)$$

such that

$$\psi_{n+1}|_{f(G)} : f(G) \rightarrow \tilde{\psi}_{n+1}(f(G)) = \tilde{\Phi}_{n+1}(G) \quad \forall G \in \mathcal{A}_{n+1},$$

$$\psi_{n+1}(x) = \tilde{\psi}_{n+1}(x) \quad \forall x \notin \bigcup_{G \in \mathcal{A}_{n+1}} f(G)$$

$$\psi_{n+1}(f(e_i(G, E))) = \theta(e_i(G, E))$$

$$\forall (E, G) \in \mathcal{A}_{n+1}^2 \text{ such that } G \xrightarrow{\Phi} E, \quad \forall i = 1, \dots, 2^{n+1}$$

and extend ψ_{n+1} to the whole box D^m by defining $\psi_{n+1}(x) = \tilde{\psi}_{n+1}(x)$, $\forall x \in D^m \setminus \bigcup_{G \in \mathcal{A}_{n+1}} f(G)$. In particular

$$\psi_{n+1}|_{\partial D^m} = \tilde{\psi}_{n+1}|_{\partial D^m} = \text{id}|_{\partial D^m}.$$

Define

$$\Phi_{n+1} := \psi_{n+1} \circ f.$$

As said above, the property that Φ_{n+1} coincides with $\tilde{\Phi}_{n+1}$ outside all the atoms of \mathcal{A}_{n+1} implies that the boxes of the families $\mathcal{A}_0, \dots, \mathcal{A}_{n+1}$ are also atoms up to generation $n+1$ for Φ_{n+1} . But now, they have the following additional property: there exists a one-to-one correspondence between the 3-tuples $(G, E, F) \in (\mathcal{A}_{n+1})^{3*}$ and the points of the set L_{n+1} of Equality (16), such that

$$(17) \quad e(G, E, F) := e_i(G, E) \in \text{int}(S(G, E) \cap \Phi_{n+1}^{-1}(S(E, F))),$$

where $S(G, E)$ and $S(E, F)$ are the previously chosen connected components of $E \cap \Phi_n(G)$ and of $F \cap \Phi_n(E)$ respectively. Besides, by construction the finite set L_{n+1} is Φ_{n+1} -invariant. In fact $\Phi_{n+1}(L_n) = \psi_{n+1}(f(L_{n+1})) = L_{n+1}$. Therefore, Assertion (13) holds for $n+1$ and the inductive construction is complete.

Step 4. The limit homeomorphisms. From the above construction we have:

$$\psi_{n+1}(x) = \tilde{\psi}_{n+1}(x) = \psi_n(x) \text{ if } x \notin \bigcup_{B,C} \psi_n^{-1}(R(B, C)) \subset \bigcup_B f(B)$$

$$\begin{aligned} \psi_{n+1} \circ \psi_n^{-1}(R(B, C)) &= \tilde{\psi}_{n+1} \circ \psi_n^{-1}(R(B, C)) = \\ &= \psi_n \circ \psi_n^{-1}(R(B, C)) = R(B, C) \subset C. \end{aligned}$$

Therefore,

$$\text{dist}(\psi_{n+1}^{-1}(x), \psi_n^{-1}(x)) \leq \max_{B \in \mathcal{A}_n} \text{diam}(f(B)) < \frac{1}{2^n}, \quad \forall x \in D^m;$$

$$\text{dist}(\psi_{n+1}(x), \psi_n(x)) \leq \max_{C \in \mathcal{A}_n} \text{diam}(C) < \frac{1}{2^n}, \quad \forall x \in D^m,$$

$$(18) \quad \|\psi_{n+1} - \psi_n\|_{\text{Hom}} < \frac{1}{2^n}.$$

From Inequality (18) we deduce that the sequence ψ_n is Cauchy in $\text{Hom}(D^m)$. Therefore, it converges to a homeomorphism ψ . Moreover, by construction $\psi_n|_{\partial D^m} = \text{id}|_{\partial D^m}$ for all $n \geq 1$. Then $\psi|_{\partial D^m} = \text{id}|_{\partial D^m}$.

The convergence of ψ_n to ψ in $\text{Hom}(D^m)$ implies that $h_n = \psi_n \circ f \in \text{RHom}(D^m)$ converges to $\Phi = \psi \circ f \in \text{RHom}(D^m)$ as $n \rightarrow +\infty$. Since $f(D^m) \subset \text{int}(D^m)$ and $\psi \in \text{Hom}(D^m)$, we deduce that $\Phi(D^m) \subset \text{int}(D^m)$. Moreover, by construction $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_n$ are families of atoms up to generation n for Φ_n , and $\Phi_m(x) = \Phi_n(x)$ for all $x \in D^m \setminus \bigcup_{B \in \mathcal{A}_n} B$ and for all $m \geq n$. Since $\lim_m \Phi_m = \Phi$, the boxes

of the family \mathcal{A}_n are n -atoms for Φ for all $n \geq 0$. Finally, from Inequality (I), the diameters of the n -atoms converge uniformly to zero as $n \rightarrow +\infty$. Thus Φ is a model according to Definition 2.4. \square

3. INFINITE METRIC ENTROPY AND MIXING PROPERTY OF THE MODELS.

The purpose of this section is to prove the following Lemma.

Lemma 3.1. (Main Lemma) *Let $\mathcal{H} \subset C^0(D^m)$ be the family of models with $m \geq 2$ (Definition 2.4). Then, for each $\Phi \in \mathcal{H}$ there exists a Φ -invariant mixing (hence ergodic) measure ν supported on an Φ -invariant Cantor set $\Lambda \subset D^m$ such that $h_\nu(\Phi) = +\infty$.*

Remark 3.2. Lemma 3.1 holds, in particular, for $\mathcal{H} \cap \text{RHom}(D^m)$.

To prove Lemma 3.1 we need to define the paths of atoms and to discuss their properties. We also need to define the invariant Cantor set Λ that will support the measure ν and prove some of its topological dynamical properties.

Definition 3.3. (Paths of atoms)

Let $\Phi \in \mathcal{H} \subset C^0(D^m)$, $l \geq 2$ and let (A_1, A_2, \dots, A_l) be a l -tuple of atoms for Φ of the same generation n , such that

$$A_i \xrightarrow{\Phi} A_{i+1}, \quad \forall i \in \{1, 2, \dots, l-1\}.$$

We call (A_1, A_2, \dots, A_l) an l -path of n -atoms from A_1 to A_l . Let \mathcal{A}_n^* denote the family of all the l -paths of atoms of generation l .

Lemma 3.4. *For all $n \geq 1$, for all $l \geq 2n$, and for all $A_1, A_2 \in \mathcal{A}_n$ there exists an l -path of n -atoms from A_1 to A_2 .*

Proof. For $n = 1$, the result is trivial for all $l \geq 2$ (see Definition 2.2). Let us assume by induction that the result holds for some $n - 1 \geq 1$ and let us prove it for n .

Let $E, F \in \mathcal{A}_n$. From equality (3) of Definition 2.3, there exists unique atoms $B_{-1}, B_0, B_1 \in \mathcal{A}_{n-1}$ such that $E \in \Gamma_n(B_{-1}, B_0, B_1)$. Then $B_{-1} \xrightarrow{\Phi} B_0$, $E \subset B_0$ and, by condition d) of Definition 2.3:

$$(19) \quad E \xrightarrow{\Phi} E_1, \quad \forall E_1 \in \Omega_n(B_0, B_1).$$

Analogously, there exists unique atoms $B_*, B_{*+1} \in \mathcal{A}_{n-1}$ such that $F \in \Omega_n(B_*, B_{*+1})$. Then $B_* \xrightarrow{\Phi} B_{*+1}$, $F \subset B_{*+1}$ and

$$(20) \quad E_* \xrightarrow{\Phi} F, \quad \forall E_* \in \bigcup_{\substack{B_{*-1} \in \mathcal{A}_{n-1}: \\ B_{*-1} \xrightarrow{\Phi} B_*}} \Gamma_n(B_{*-1}, B_*, B_{*+1})$$

Since $B_1, B_* \in \mathcal{A}_{n-1}$ the induction hypothesis ensures that for all $l \geq 2n-2$ there exists an l -path (B_1, B_2, \dots, B_l) from B_1 to $B_l = B_*$. We write $B_{*-1} = B_{l-1}$, $B_* = B_l$, $B_{*+1} = B_{l+1}$. So Assertion (20) becomes

$$(21) \quad E_l \xrightarrow{\Phi} F, \quad \forall E_l \in \Gamma_n(B_{l-1}, B_l, B_{l+1})$$

Taking into account that $B_{i-1} \xrightarrow{\Phi} B_i$ for $1 < i \leq l$, and applying condition d) of Definition 2.3, we deduce that, if $E_{i-1} \in \Gamma_n(B_{i-2}, B_{i-1}, B_i) \subset \mathcal{A}_n$, then

$$(22) \quad E_{i-1} \xrightarrow{\Phi} E_i, \quad \forall E_i \in \Omega_n(B_{i-1}, B_i), \quad \forall 1 < i \leq l.$$

Combining (19), (21) and (22) yields an $(l+2)$ -path (E, E_1, \dots, E_l, F) of atoms of generation n from E to F , as required. \square

Lemma 3.5. *Let $n, l \geq 2$. For each l -path (B_1, \dots, B_l) of $(n-1)$ -atoms there exists an l -path (E_1, E_2, \dots, E_l) of n -atoms such that $E_i \subset \text{int}(B_i)$ for all $i = 1, 2, \dots, l$.*

Proof. In the proof of Lemma 3.4 for each l -path (B_1, B_2, \dots, B_l) of $(n-1)$ -atoms we have constructed the l -path (E_1, E_2, \dots, E_l) of n -atoms as required. \square

Definition 3.6. (The Λ -set) Let $\Phi \in \mathcal{H} \subset C^0(D^m)$ be a model map. Let $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_n, \dots$ be its sequence of families of atoms. The subset

$$\Lambda := \bigcap_{n \geq 0} \bigcup_{A \in \mathcal{A}_n} A$$

of $\text{int}(D^m)$ is called *the Λ -set* of the map Φ .

From Definitions 2.2 and 2.3, we know that, for each fixed $n \geq 0$, the set $\Lambda_n := \bigcup_{A \in \mathcal{A}_n} A$, is nonempty, compact, and $\text{int}(\Lambda_n) \supset \Lambda_{n+1}$. Therefore, Λ is also nonempty and compact. Moreover, Λ_n is composed of a finite number of connected components $A \in \mathcal{A}_n$, which by Definition 2.4, satisfy $\lim_{n \rightarrow +\infty} \max_{A \in \mathcal{A}_n} \text{diam} A = 0$. Since $\Lambda := \bigcap_{n \geq 0} \Lambda_n$, we deduce that the Λ -set is a *Cantor set* contained in $\text{int}(D^m)$.

Lemma 3.7. *Let $n, l \geq 1$ and $A_1, A_2 \in \mathcal{A}_n$. If there exists an $l+1$ -path from A_1 to A_2 , then $\Phi^l(A_1 \cap \Lambda) \cap (A_2 \cap \Lambda) \neq \emptyset$.*

Proof. Assume that there exists an $(l+1)$ -path from A_1 to A_2 . So, from Lemma 3.5, for all $j \geq n_0$ there exists atoms $B_{j,1}, B_{j,2} \in \mathcal{A}_j$ and an $(l+1)$ -path from $B_{j,1}$ to $B_{j,2}$ (with constant length $l+1$) such that

$$B_{n,i} = A_i, \quad B_{j+1,i} \subset B_{j,i}, \quad \forall j \geq n_0, \quad i = 1, 2.$$

Construct the following two points x_1 and x_2 :]

$$\{x_i\} = \bigcap_{j \geq n_0} B_{j,i}, \quad i = 1, 2.$$

By Definition 3.6, $x_i \in A_i \cap \Lambda$. So, to finish the proof of Lemma 3.7 it is enough to prove that $\Phi^l(x_1) = x_2$.

Recall that l is fixed. Since Φ is uniformly continuous, for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $(y_0, y_1, \dots, y_l) \in (D^m)^l$ satisfies $d(\Phi(y_i), y_{i+1}) < \delta$ for $0 \leq i \leq l-1$, then the points y_0 and y_l satisfy $d(\Phi^l(y_0), y_l) < \varepsilon$. We choose δ small enough such that additional $d(\Phi^l(x), \Phi^l(y)) < \varepsilon$ if $d(x, y) < \delta$.

From (5), there exists $j \geq n_0$ such that $\text{diam}(B_{j,i}) < \delta$. Since there exists an $(l+1)$ -path from $B_{j,1}$ to $B_{j,2}$, there exists a (y_0, \dots, y_l) as in the previous paragraph with $y_0 \in B_{j,1}$ and $y_l \in B_{j,2}$. Thus

$$\begin{aligned} d(\Phi^l(x_1), x_2) &\leq d(\Phi^l(x_1), \Phi^l(y_0)) + d(\Phi^l(y_0), y_l) + d(y_l, x_2) \\ &< \text{diam}(\Phi^l(B_{j,1})) + \varepsilon + \text{diam}(B_{j,2}) < 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\Phi^l(x_1) = x_2$, as required. \square

Lemma 3.8. (Topological dynamical properties of Λ)

- a) The Λ -set of a model map $\Phi \in \mathcal{H}$ is Φ -invariant, i.e., $\Phi(\Lambda) = \Lambda$.
- b) The map Φ restricted to the Λ -set is topologically mixing.
- c) In particular, $\Phi^l(A_1 \cap \Lambda) \cap (A_2 \cap \Lambda) \neq \emptyset$, for all $n \geq 1$, for any two atoms $A_1, A_2 \in \mathcal{A}_n$ and for all $l \geq 2n - 1$.

Proof. a) Let $x \in \Lambda$ and let $\{A_n(x)\}_{n \geq 0}$ the unique sequence of atoms such that $x \in A_n(x)$ and $A_n(x) \in \mathcal{A}_n$ for all $n \geq 0$. Then, $\Phi(x) \in \Phi(A_n(x))$ for all $n \geq 0$. From Definition 2.3, for all $n \geq 0$ there exists an atom $B_n \in \mathcal{A}_n$ such that $A_n(x) \xrightarrow{\Phi} B_n$. Therefore $\Phi(A_n(x)) \cap B_n \neq \emptyset$. Let d denote the Hausdorff distance between subsets of D^m , we deduce

$$d(\Phi(x), B_n) \leq \text{diam}(\Phi(A_n(x))) + \text{diam}(B_n).$$

Moreover, Equality (5) and the continuity of Φ imply

$$\lim_{n \rightarrow +\infty} \max \{ \text{diam}(\Phi(A_n(x))), \text{diam}(B_n) \} = 0.$$

Then, for all $\varepsilon > 0$ there exists $n_0 \geq 0$ such that $d(\Phi(x), B_n) < \varepsilon$ for some atom $B_n \in \mathcal{A}_n$ for all $n \geq n_0$. Since any atom of any generation intersects Λ , we deduce that $d(\Phi(x), \Lambda) < \varepsilon$ for each $\varepsilon > 0$. Since Λ is compact, this implies $\Phi(x) \in \Lambda$. We have proved that $\Phi(\Lambda) \subset \Lambda$.

Now, let us prove the other inclusion. Let $y \in \Lambda$ and let $\{B_n(y)\}_{n \geq 0}$ the unique sequence of atoms such that $y = \Phi(x) \in B_n(y)$ and $B_n(y) \in \mathcal{A}_n$ for all $n \geq 0$. From Definition 2.3, for all $n \geq 0$ there exists an atom

$A_n \in \mathcal{A}_n$ such that $A_n \xrightarrow{\Phi} B_n(y)$. Therefore $\Phi(A_n) \cap B_n(y) \neq \emptyset$. We deduce that, for all $n \geq 0$, there exists a point $x_n \in A_n \in \mathcal{A}_n$ such that $\Phi(x_n) \in B_n(y)$. Since any atom A_n contains points of Λ , we obtain

$$d(x_n, \Lambda) \leq \text{diam}(A_n) \text{ and } d(\Phi(x_n), y) \leq \text{diam}(B_n(y)), \quad \forall n \geq 0.$$

Let x be the limit of a convergent subsequence of $\{x_n\}_{n \geq 0}$, applying Equality (5) and the continuity of Φ , we deduce that $d(x, \Lambda) = 0$ and $d(\Phi(x), y) = 0$. This means that $y = \Phi(x)$ and $x \in \Lambda$. We have proved that $y \in \Phi(\Lambda)$ for all $y \in \Lambda$; namely $\Lambda = \Phi(\Lambda)$, as required.

c) We will prove a stronger assertion: for any two atoms, even of different generation, there exists $l_0 \geq 1$ such that

$$(23) \quad \Phi^l(A_1 \cap \Lambda) \cap (A_2 \cap \Lambda) \neq \emptyset \quad \forall l \geq l_0.$$

It is not restrictive to assume that A_1 and A_2 are atoms of the same generation n_0 (if not, take n_0 equal to the largest of both generation and substitute A_i by an atom of generation n_0 contained in A_i). Applying Lemma 3.4, for all $l \geq 2n_0 - 1$ there exists an $(l + 1)$ -path from A_1 to A_2 . So, from Lemma 3.7 $\Phi^l(A_1 \cap \Lambda) \cap (A_2 \cap \Lambda) \neq \emptyset$, as required.

b) The intersection of Λ with the atoms of all the generations generates its topology, thus Equation (23) implies that Λ is topologically mixing. \square

For fixed $(A_0, A_l) \in \mathcal{A}_n^2$ we set

$$\mathcal{A}_n^{l+1*}(A_0, A_l) := \{(A_0, A_1, \dots, A_{l-1}, A_l) \in \mathcal{A}_n^{l+1*}\}.$$

Lemma 3.9. *Let $l, n \geq 1$. Then*

- a) $\mathcal{A}_n^{l+1*} = 2^{nl} \cdot (\#\mathcal{A}_n)$.
- b) $\mathcal{A}_n^{l+1*}(A_0, A_l) = \frac{2^{nl}}{\#\mathcal{A}_n} \quad \forall (A_0, A_l) \in \mathcal{A}_n^2$, for all $l \geq 2n - 1$.

Proof. a) Each $(l + 1)$ -path (A_0, A_1, \dots, A_l) of n -atoms is determined by a free choice of the atom $A_0 \in \mathcal{A}_n$, followed by the choice of the atoms $A_j \in \mathcal{A}_n$ such that $A_j \xrightarrow{\Phi} A_{j-1}$ for all $j = 1, \dots, l$. From equality (4) of Definition 2.3, we know that for any fixed $A \in \mathcal{A}_n$ the number of atoms $B \in \mathcal{A}_n$ such that $B \xrightarrow{\Phi} A$ is 2^n . This implies a), as required.

b) We argue by induction on n . Fix $n = 1$ and $l \geq 1$. Since any two atoms $A_j, A_{j+1} \in \mathcal{A}_1$ satisfies $A_j \xrightarrow{\Phi} A_{j+1}$, the number of $(l + 1)$ -paths

$$(A_0, A_1, \dots, A_j, A_{j+1}, \dots, A_{l-1}, A_l)$$

of 1-atoms with (A_0, A_l) fixed, equals $\#(\mathcal{A}_1)^{l-1} = 2^{l-1} = 2^l/2 = 2^{nl}/(\#\mathcal{A}_n)$ with $n = 1$.

Now, let us assume that assertion b) holds for some $n \geq 1$ and let us prove it for $n + 1$. Let $l \geq 2(n + 1) - 1 = 2n + 1 \geq 3$ and let $(B_0, B_l) \in$

\mathcal{A}_{n+1}^2 . From equality (3) and conditions a) and b) of Definition 2.3, there exists unique $(A_{-1}, A_0, A_1) \in \mathcal{A}_n^{3*}$ and unique $(A_{l-1}, A_l) \in \mathcal{A}_n^{2*}$ such that

$$B_0 \in \Gamma_{n+1}(A_{-1}, A_0, A_1), \quad B_l \in \Omega_{n+1}(A_{l-1}, A_l).$$

As $(A_1, A_{l-1}) \in \mathcal{A}_n^2$ and $l-2 \geq 2n-1$, the induction hypothesis ensures that the number of $(l-1)$ -paths $(A_1, A_2, \dots, A_{l-1})$ from A_1 to A_{l-1} is

$$(24) \quad \#\mathcal{A}_n^{l-1*}(A_1, A_{l-1}) = \frac{2^{n(l-2)}}{\#\mathcal{A}_n} = \frac{2^{n(l-2)}}{2^{n^2}} = 2^{nl-2n-n^2}.$$

Let $\mathcal{C}(B_0, B_l)$ be the set

$$\bigcup_{(A_1, \dots, A_{l-1}) \in \mathcal{A}_n^{l-1*}(A_1, A_{l-1})} \{(B_0, B_1, \dots, B_l) \in \mathcal{A}_{n+1}^{l+1*} : B_j \in \Gamma_{n+1}(A_{j-1}, A_j, A_{j+1}) \forall j\},$$

where the families in the above union are pairwise disjoint. It is standard to check that the families in the union $\mathcal{C}(B_0, B_l)$ are pairwise disjoint, because for $A \neq \tilde{A}$ in \mathcal{A}_n , the families $\Gamma_{n+1}(\cdot, A, \cdot)$ and $\Gamma_{n+1}(\cdot, \tilde{A}, \cdot)$ are disjoint. We assert that

$$(25) \quad \mathcal{A}_{n+1}^{l+1*}(B_0, B_l) = \mathcal{C}(B_0, B_l)$$

First, let us prove that $\mathcal{A}_{n+1}^{l+1*}(B_0, B_l) \subset \mathcal{C}(B_0, B_l)$.

In fact, if $(B_0, B_1, \dots, B_{l-1}, B_l) \in \mathcal{A}_{n+1}^{l+1*}(B_0, B_l)$, there exists unique $A_j \in \mathcal{A}_n$ such that $B_j \subset A_j$ for all $j \in \{0, 1, \dots, l\}$. Since $B_{j-1} \xrightarrow{\Phi} B_j \xrightarrow{\Phi} B_{j+1}$ for all $j \in \{1, \dots, l-1\}$, we deduce that $A_j \xrightarrow{\Phi} B_{j+1} \xrightarrow{\Phi} A_{j+1}$; hence, by definition of the families Γ_{n+1} we have

$$B_j \in \Gamma_{n+1}(A_{j-1}, A_j, A_{j+1}) \quad \forall j \in \{1, \dots, l-1\}.$$

Thus, $(B_0, B_1, \dots, B_{l-1}, B_l) \in \mathcal{C}(B_0, B_l)$, as required.

Now, let us prove that $\mathcal{A}_{n+1}^{l+1*}(B_0, B_l) \supset \mathcal{C}(B_0, B_l)$.

If $(B_0, B_1, \dots, B_{l-1}, B_l) \in \mathcal{C}(B_0, B_l)$, then there exists a $(l-1)$ chain A_1, \dots, A_{l-1} of n -atoms from A_1 to A_{l-1} such that

$$B_j \in \Gamma_{n+1}(A_{j-1}, A_j, A_{j+1}) \subset \Omega_{n+1}(A_{j-1}, A_j)$$

for all $j = 1, 2, \dots, l-1$. By construction we have

$$B_0 \in \Gamma_{n+1}(A_{-1}, A_0, A_1) \quad \text{and} \quad B_l \in \Omega_{n+1}(A_{l-1}, A_l).$$

Therefore, applying condition d) of Definition 2.3, we deduce that

$$B_j \xrightarrow{\Phi} B_{j+1} \quad \forall j = 0, 1, \dots, l-1.$$

In other words, $(B_0, B_1, \dots, B_{l-1}, B_l) \in \mathcal{A}_{n+1}^{l+1*}(B_0, B_l)$, ending the proof of Equality (25).

Now, applying (24) and (25), we obtain $\#\mathcal{A}_{n+1}^{l+1*}(B_0, B_l) =$

$$\begin{aligned}
 & \sum_{(A_1, \dots, A_{l-1}) \in \mathcal{A}_n^{l-1*}(A_1, A_{l-1})} \#\{(B_0, B_1, \dots, B_l) \in \mathcal{A}_{n+1}^{l+1} : B_j \in \Gamma_{n+1}(A_{j-1}, A_j, A_{j+1}) \forall j\} \\
 &= \sum_{(A_1, \dots, A_{l-1}) \in \mathcal{A}_n^{l-1*}(A_1, A_{l-1})} \prod_{j=1}^{l-1} \#\Gamma_{n+1}(A_{j-1}, A_j, A_{j+1}) \\
 &= (\#\mathcal{A}_n^{l-1*}(A_1, A_{l-1})) \cdot 2^{l-1} = 2^{nl-2n-n^2+l-1} = 2^{(n+1)l-(n+1)^2} = \frac{2^{(n+1)l}}{\#\mathcal{A}_{n+1}}
 \end{aligned}$$

as required. \square

Lemma 3.10. (Intersection of Λ with l -paths) Fix $l, n \geq 1$. Then

a) For any $G \in \mathcal{A}_{n+l}$, there exists a unique $(l+1)$ -path (A_0, A_1, \dots, A_l) of n -atoms such that $G \cap \Lambda \subset \bigcap_{j=0}^l \Phi^{-j}(A_j)$.

b) For any atoms $G \in \mathcal{A}_{n+l}$, $A \in \mathcal{A}_n$ and $j \in \{0, 1, \dots, l\}$:

$$(G \cap \Lambda) \cap \Phi^{-j}(A) \neq \emptyset \Leftrightarrow G \cap \Lambda \subset \Phi^{-j}(A).$$

c) For any $(l+1)$ -path $\vec{A}_n^l := (A_0, A_1, \dots, A_l)$ of n -atoms,

$$(26) \quad \Lambda \cap \bigcap_{j=0}^l \Phi^{-j}(A_j) = \bigcup_{G \in \mathcal{F}_{n,l}(\vec{A}_n^l)} G \cap \Lambda,$$

$$\text{where } \mathcal{F}_{n,l}(\vec{A}_n^l) := \left\{ G \in \mathcal{A}_{n+l} : G \cap \Lambda \subset \bigcap_{j=0}^l \Phi^{-j}(A_j) \right\}.$$

d) For any atom $G \in \mathcal{A}_{n+l}$ and any path $\vec{A}_n^l \in \mathcal{A}_n^{l+1*}$:

$G \in \mathcal{F}_{n,l}(\vec{A}_n^l)$ if and only if there exists $(G_0, G_1, \dots, G_l) \in \mathcal{A}_{n+1}^{l+1*}$ such that $G_0 = G$ and $G_j \subset A_j$ for all $j = 0, 1, \dots, l$.

e) For any $(l+1)$ -path (A_0, A_1, \dots, A_l) of n -atoms,

$$\#\mathcal{F}_{n,l}(\vec{A}_n^l) = \frac{1}{2^{nl}} \cdot \frac{\#\mathcal{A}_{n+l}}{\#\mathcal{A}_n}.$$

Proof. a) From equalities (2) and (3) of Definition 2.3, for any atom G of generation $n+l$ there exist two unique atoms B, C of generation $n+l-1$ such that $B \xrightarrow{\Phi} C$, $G \subset B$ and $G \xrightarrow{\Phi} E$ for all $E \in \Omega_{n+l}(B, C)$. Moreover, from condition d) of Definition 2.3, we have

$$(27) \quad \Phi(G) \cap (F) \neq \emptyset \text{ if and only if } F \in \Omega_{n+l}(B, C).$$

We claim that

$$(28) \quad \Phi(G \cap \Lambda) \subset \text{int}(C).$$

Since Λ is Φ -invariant, for any $x \in G \cap \Lambda$, we have $\Phi(x) \in \Phi(G) \cap \Lambda$. Therefore $\Phi(x)$ is in the interior of some atom $E(x)$ of generation $n+l$

(see Definition 3.6). From (27), $E(x) \in \Omega_{n+l}(B, C)$. Thus $E(x) \subset \text{int}(C)$ and $\Phi(x) \in \text{int}(C)$ for all $x \in G \cap \Lambda$ proving (28).

So, there exists $C_1 \in \mathcal{A}_{n+l-1}$ such that $\Phi(G \cap \Lambda) \subset \text{int}(C_1) \cap \Lambda$. Applying the same assertion to C_1 instead of G , we deduce that there exists $C_2 \in \mathcal{A}_{n+l-2}$ such that $\Phi(C_1 \cap \Lambda) \subset \text{int}(C_2) \cap \Lambda$. So, by induction, we construct atoms

$$C_1, C_2, \dots, C_l \quad \text{such that} \quad C_j \in \mathcal{A}_{n+l-j} \quad \text{and}$$

$$\Phi^j(G \cap \Lambda) \subset \text{int}(C_j) \cap \Lambda, \quad \forall j = 1, \dots, l.$$

Since any atom of generation larger than n is contained in a unique atom of generation n , there exists $A_0, A_1, \dots, A_l \in \mathcal{A}_n$ such that $A_0 \supset G$ and $A_i \supset C_i$, $\forall i = 1, \dots, l$. We obtain

$$\Phi^j(G \cap \Lambda) \subset \text{int}(A_j), \quad \forall j = 0, 1, \dots, l.$$

Besides, (A_0, A_1, \dots, A_l) is an $(l+1)$ -path since $\emptyset \neq \Phi^j(G \cap \Lambda) \subset \Phi(A_{j-1}) \cap \text{int}(A_j)$; hence $A_{j-1} \xrightarrow{\Phi} A_j$ for all $j = 1, \dots, l$. Then, $G \cap \Lambda \subset \Phi^{-j}(A_j)$ for all $j = 0, 1, \dots, l$; proving the existence statement in a).

To prove uniqueness assume that (A_0, A_1, \dots, A_l) and $(A'_0, A'_1, \dots, A'_l)$ are paths of n -atoms such that

$$G \cap \Lambda \subset \Phi^{-j}(A_j) \cap \Phi^{-j}(A'_j) \quad \forall j \in \{0, 1, \dots, l\}.$$

Then $A_j \cap A'_j \neq \emptyset$ for all $j \in \{0, 1, \dots, l\}$. Since two different atoms of the same generation are pairwise disjoint, we deduce that $A_j = A'_j$ for all $j \in \{0, 1, \dots, l\}$ as required.

Trivially, if $\Lambda \cap G \subset \Phi^{-j}(A)$ then $\Lambda \cap G \subset \Phi^{-j}(A) \neq \emptyset$. Let us prove the converse assertion. Let $G \in \mathcal{A}_{n+1}$ such that $\Lambda \cap G \subset \Phi^{-j_0}(A) \neq \emptyset$ for some $j_0 \in \{0, 1, \dots, l\}$. Applying part a) there exists an $(l+1)$ -path $(\tilde{A}_0, \dots, \tilde{A}_l)$ of n -atoms such that $G \cap \Lambda \subset \Phi^{-j}(\tilde{A}_j)$ for all $j \in \{0, 1, \dots, l\}$. So, $\Phi^{-j_0}(A_{j_0}) \cap \Phi^{-j_0}(\tilde{A}_{j_0}) \neq \emptyset$. But, if $\tilde{A}_{j_0} \neq A_{j_0}$, then since they are atoms of the same generation they would be disjoint, and the above intersection would be empty. We deduce that $A_{j_0} = \tilde{A}_{j_0}$, hence $G \cap \Lambda \subset \Phi^{-j_0}(A_{j_0})$, as required.

b) Trivially, if $G \cap \Lambda \subset \Phi^{-j}(A)$, then $(G \cap \Lambda) \cap \Phi^{-j}(A) \neq \emptyset$. Now, let us prove the converse assertion. Fix $G \in \mathcal{A}_{n+l}$ and $A \in \mathcal{A}_n$ satisfying $(G \cap \Lambda) \cap \Phi^{-j}(A) \neq \emptyset$. Applying part a) there exists $\tilde{A} \in \mathcal{A}_n$ such that $G \cap \Lambda \subset \Phi^{-j}(\tilde{A})$. Therefore $G \cap \Lambda \cap \Phi^{-j}(A) \subset \Phi^{-j}(\tilde{A} \cap A) \neq \emptyset$. Since A and \tilde{A} are atoms of generation n , and two different atoms of the same generation are disjoint, we conclude that $\tilde{A} = A$, hence $G \cap \Lambda \subset \Phi^{-j}(A)$, as required.

c) For the $(l + 1)$ -path $\vec{A}_n^l = (A_0, A_1, \dots, A_l)$ of n -atoms, construct

$$(29) \quad \tilde{\mathcal{F}}_{n,l}(\vec{A}_n^l) := \{G \in \mathcal{A}_{n+l} : G \cap \Lambda \cap \Phi^{-j}(A_j) \neq \emptyset \quad \forall j \in \{0, 1, \dots, l\}\}.$$

From the definitions of the the families $\mathcal{F}_{n,l}$ and $\tilde{\mathcal{F}}_{n,l}$, and taking into account that Λ is contained in the union of $(n + 1)$ -atoms, we obtain:

$$\bigcup_{G \in \mathcal{F}_{n,l}(\vec{A}_n^l)} G \cap \Lambda \subset \Lambda \cap \left(\bigcap_{j=0}^l \Phi^{-j}(A_j) \right) \subset \bigcup_{G \in \tilde{\mathcal{F}}_{n,l}(\vec{A}_n^l)} G \cap \Lambda.$$

Therefore, to prove Equality (26) it is enough to show that

$$(30) \quad \tilde{\mathcal{F}}_{n,l}(\vec{A}_n^l) = \mathcal{F}_{n,l}(\vec{A}_n^l),$$

but this equality immediately follows from the construction of the families $\mathcal{F}_{n,l}(\vec{A}_n^l)$ and $\tilde{\mathcal{F}}_{n,l}(\vec{A}_n^l)$ by assertion b).

d) For each $(l + 1)$ -path $\vec{A}_n^l = (A_0, A_1, \dots, A_l)$ of n -atoms construct the family $\mathcal{G}_n^l(\vec{A}_n^l) :=$

$$\{G_0 \in \mathcal{A}_{n+1} : \exists (G_0, G_1, \dots, G_l) \in \mathcal{A}_{n+1}^{l+1*} \text{ such that } G_j \subset A_j \quad \forall j\}$$

We will first prove that $\mathcal{G}_n^l(\vec{A}_n^l) \supset \mathcal{F}_n^l(\vec{A}_n^l)$. In fact, take $G \in \mathcal{F}_n^l(\vec{A}_n^l)$, and take any point $x \in G \cap \Lambda$. We have $\Phi^j(x) \in A_j \cap \Lambda$ for all $j \in \{0, 1, \dots, l\}$ (recall that Λ is Φ -invariant). Since any point in Λ is contained in the interior of some atom of any generation, there exists an atom G_j of generation $n+l$ such that $\Phi_j(x) \in \text{int}(G_j)$. Recall that each atom of generation $n+l$ is contained in a unique atom of generation n . As $\Phi_j(x) \in G_j \cap A_j \neq \emptyset$, and different atoms of the same are disjoint, we conclude that $G_j \subset A_j$. Besides $G_0 = G$ because $x \in G \cap G_0$. Finally (G_0, G_1, \dots, G_l) is a $(l+1)$ -path because $\Phi_{j+1}(x) = \Phi(\Phi_j(x)) \in \Phi(G_j) \cap \text{int}(G_{j+1})$ for all $j \in \{0, 1, \dots, l-1\}$; namely $G_j \xrightarrow{\Phi} G_{j+1}$. We have proved that $G \in \mathcal{G}_n^l(\vec{A}_n^l)$, as required.

Now, let us prove that $\mathcal{G}_n^l(\vec{A}_n^l) \subset \mathcal{F}_n^l(\vec{A}_n^l)$. Assume that $G_0 \in \mathcal{A}_{n+1}$ and $(G_0, G_1, \dots, G_l) \in \mathcal{A}_{n+1}^{l+1*}$ satisfies $G_j \subset A_j$ for all $j \in \{0, 1, \dots, l\}$. Therefore (G_0, G_1, \dots, G_j) is a $(j+1)$ -path of $(n+1)$ -atoms for all $j \in \{1, 2, \dots, l\}$. Applying Lemma 3.7, we obtain $G_0 \cap \Lambda \cap \Phi^{-j}(G_j) \neq \emptyset$. Therefore, taking into account that $G_j \subset A_j$, we deduce that

$$G_0 \cap \Lambda \cap \Phi^{-j}(A_j) \neq \emptyset \quad \forall j \in \{0, 1, \dots, l\}.$$

Therefore $G_0 \in \tilde{\mathcal{F}}_n^l(\vec{A}_n^l) = \mathcal{F}_n^l(\vec{A}_n^l)$ (recall (29) and (30)). This holds for any $G_0 \in \mathcal{G}_n^l(\vec{A}_n^l)$, thus $\mathcal{G}_n^l(\vec{A}_n^l) \subset \mathcal{F}_n^l(\vec{A}_n^l)$, as required.

e) From Assertion a) we obtain:

$$(31) \quad \mathcal{A}_{n+l} = \bigcup_{\vec{A}_n^l \in \mathcal{A}_n^{l+1*}} \mathcal{F}_{n,1}(\vec{A}_n^l),$$

where the families in the above union are pairwise disjoint, due to the uniqueness property of assertion a).

Recall the characterization of the family $\mathcal{F}_{n,1}(\vec{A}_n^l)$ given by Assertion d). From Definition 2.3 (condition a) and equality (4), the number of atoms of each generation larger than n that are contained in each $A_j \in \mathcal{A}_n$, and also the number of atoms $G_j \in \mathcal{A}_{n+1}$ such that $G_j \xrightarrow{\Phi} G_{j+1}$, are constants that depend only on the generations but not on the chosen atom. Therefore, there exists a constant $k_{n,l}$ such that $\#\mathcal{F}_{n,l}(\vec{A}_n^l) = \#\mathcal{G}_{n,l}(\vec{A}_n^l) = k_{n,l}$ for all the $(l+1)$ -paths of n -atoms. So, from Equality (31) we obtain:

$$\#\mathcal{A}_{n+l} = (\#\mathcal{A}_n^{l+1*}) \cdot (\#\mathcal{F}_{n,l}(\{A_j\})),$$

and applying Lemma 3.9, we conclude

$$\#\mathcal{A}_{n+l} = 2^{nl} \cdot (\#\mathcal{A}_n) \cdot (\#\mathcal{F}_{n,l}(\{A_j\})),$$

as required. \square

We turn to the proof of Lemma 3.1. We will first construct the measure ν and then prove that it has the required properties.

We start by defining an additive pre-measure on the Λ -set of Φ by

$$\nu^*(A \cap \Lambda) := \frac{1}{\#\mathcal{A}_n}, \quad \forall A \in \mathcal{A}_n, \quad \forall n \geq 0.$$

Since ν^* is a pre-measure defined in a family of sets that generates the Borel σ -algebra of Λ , there exists a unique Borel probability measure ν supported on Λ such that

$$(32) \quad \nu(A \cap \Lambda) := \frac{1}{\#\mathcal{A}_n}, \quad \forall A \in \mathcal{A}_n, \quad \forall n \geq 0.$$

In the following lemmas we will prove that ν is Φ -invariant, mixing, and that the metric entropy $h_\nu(\Phi)$ is infinite.

Lemma 3.11. *ν is invariant by Φ .*

Proof. Since the atoms of all generation intersected with Λ generates the Borel σ -algebra of Λ , it is enough to prove that

$$(33) \quad \nu(C \cap \Lambda) = \nu(\Phi^{-1}(C \cap \Lambda)), \quad \forall C \in \mathcal{A}_n, \quad \forall n \geq 0.$$

From (3), taking into account that Λ is invariant and that any point in Λ belongs to an atom of generation $n+1$, we obtain:

$$\Phi^{-1}(C \cap \Lambda) = \bigcup_{\substack{(D,B) \in \mathcal{A}_n^{2*} \\ B \xrightarrow{\Phi} C}} \bigcup_{G \in \Gamma_{n+1}(D,B,C)} (G \cap \Lambda),$$

where both unions are of pairwise disjoint sets. Using (4) we obtain

$$\begin{aligned} \nu(\Phi^{-1}(C \cap \Lambda)) &= \sum_{\substack{B \in \mathcal{A}_n \\ B \xrightarrow{\Phi} C}} \sum_{\substack{D \in \mathcal{A}_n \\ D \xrightarrow{\Phi} B}} \sum_{G \in \Gamma_{n+1}(B,C,D)} \nu(G \cap \Lambda) \\ (34) \qquad \qquad \qquad &= N_C \cdot N_B \cdot (\#\Gamma_{n+1}(B, C, D)) \cdot \frac{1}{\#\mathcal{A}_{n+1}}, \end{aligned}$$

where $N_X := \#\{Y \in \mathcal{A}_n : Y \xrightarrow{\Phi} X\} = 2^n$ for all $X \in \mathcal{A}_n$. Since $\#\Gamma_{n+1}(B, C, D) = 2$ (this is part c) of Definition 2.3) and $\#\mathcal{A}_{n+1} = 2^{(n+1)^2}$, we conclude

$$\nu(\Phi^{-1}(C \cap \Lambda)) = 2^n \cdot 2^n \cdot 2 \cdot \frac{1}{2^{(n+1)^2}} = \frac{1}{2^{n^2}} = \frac{1}{\#\mathcal{A}_n} = \nu(C \cap \Lambda),$$

proving Equality (33) as required. \square

Lemma 3.12. *ν is mixing.*

Proof. The family of atoms of all generations intersected with Λ generates the Borel σ -algebra of Λ , thus it is enough to prove that for any pair (C_0, D_0) of atoms (of equal or different generations) there exists $l_0 \geq 1$ such that

$$(35) \quad \nu(\Phi^{-l}(D_0 \cap \Lambda) \cap (C_0 \cap \Lambda)) = \nu(C_0 \cap \Lambda) \cdot \nu(D_0 \cap \Lambda) \quad \forall l \geq l_0.$$

Let us first prove this in the case that C_0 and D_0 are atoms of the same generation n . Take $l \geq 2n - 1$. Applying Lemma 3.8-c), we have $\Phi^{-l}(D_0 \cap \Lambda) \cap (C_0 \cap \Lambda) \neq \emptyset \quad \forall l \geq 2n - 1$.

Fix $l \geq 2n - 1$. Let

$$\vec{A}_n^l := (C_0, A_1, \dots, A_{l-1}, D_0) \in \mathcal{A}_n^{l+1*}(C_0, D_0)$$

denote one of the $2^{nl}/(\#\mathcal{A}_n)$ different $l + 1$ -paths of n -atoms from C_0 to D_0 (see Lemma 3.9-b)).

We assert that

$$(36) \quad \Phi^{-l}(D_0 \cap \Lambda) \cap (C_0 \cap \Lambda) = T := \bigcup_{\vec{A}_n^l \in \mathcal{A}_n^{l+1*}(C_0, D_0)} \bigcup_{B \in \mathcal{F}_{n,l}(\vec{A}_n^l)} (B \cap \Lambda),$$

where the family $\mathcal{F}_{n,l}(\vec{A}_n^l)$ of $(n+l)$ -atoms is defined in Lemma 3.10-c).

First, let us prove that $\Phi^{-l}(D_0 \cap \Lambda) \cap (C_0 \cap \Lambda) \subset T$. Fix $x \in (D_0 \cap \Lambda) \cap (C_0 \cap \Lambda)$. Then C_0, D_0 are the unique atoms of generation n that contain x and $\Phi^l(x) \in \Phi^l(\Lambda) = \Lambda$ respectively. Since $x \in \Lambda$, there

exists a unique atom B of generation $n + l$ that contains x . Applying Lemma 3.10-a) there exists a unique $(A_0, A_1, \dots, A_l) \in \mathcal{A}_n^{l+1*}$ such that $B \cap \Lambda \subset \Phi^{-j}(A_j)$ for all $j \in \{0, 1, \dots, l\}$. Since the n -atom that contains x is C_0 , and two different n -atoms are disjoint, we deduce that $A_0 = C_0$. Analogously, since the n -atom that contains $\Phi^l(x)$ is D_0 and the preimages of two different n -atoms are disjoint, we deduce that $A_l = D_0$. Therefore we have found $\vec{A}_n^l = (C_0, A_1, \dots, A_{l-1}, D_0)$ and $B \in \mathcal{F}_{n,l}(\vec{A}_n^l)$ such that $x \in B \cap \Lambda$. In other words, $x \in T$, as required.

Next, let us prove that $\Phi^{-l}(D_0 \cap \Lambda) \cap (C_0 \cap \Lambda) \supset T$. Take $B \in \mathcal{F}_{n,l}(\vec{A}_n^l)$ for some $\vec{A}_n^l = (C_0, A_1, \dots, A_{l-1}, D_0)$. From the definition of the family $\mathcal{F}_{n,l}(\vec{A}_n^l)$ in Lemma 3.10-c), we have $B \cap \Lambda \subset (C_0 \cap \Lambda) \cap \Phi^{-l}(D_0)$. Besides $B \cap \Lambda \in \Phi^l(\Lambda)$ because $\Phi^l(\Lambda) = \Lambda$. We conclude that $B \cap \Lambda \subset (C_0 \cap \Lambda) \cap \Phi^{-l}(D_0 \cap \Lambda)$, proving that $T \subset \Phi^{-l}(D_0 \cap \Lambda) \cap (C_0 \cap \Lambda)$, as required. This ends the proof of equality (36).

n -atoms are pairwise disjoint, thus the sets in the union constructing T are pairwise disjoint. Therefore, from (36), and applying Lemma 3.9-b) and Lemma 3.10-e), we deduce

$$\begin{aligned} \nu((C_0 \cap \Lambda) \cap \Phi^{-l}(D_0 \cap \Lambda)) &= \sum_{\vec{A}_n^l \in \mathcal{A}_n^{l+1*}(C_0, D_0)} \sum_{B \in \mathcal{F}_{n,l}(\vec{A}_n^l)} \nu(B \cap \Lambda) \\ &= (\#\mathcal{A}_n^{l+1*}(C_0, D_0)) \cdot (\#\mathcal{F}_{n,l}(\vec{A}_n^l)) \cdot \frac{1}{\#\mathcal{A}_{n+l}} \\ &= \frac{2^{nl}}{\#\mathcal{A}_n} \cdot \frac{1}{2^{nl}} \cdot \frac{\#\mathcal{A}_{n+l}}{\#\mathcal{A}_n} \cdot \frac{1}{\#\mathcal{A}_{n+l}} = \frac{1}{\#\mathcal{A}_n} \cdot \frac{1}{\#\mathcal{A}_n} \\ &= \nu(C_0 \cap \Lambda) \cdot \nu(D_0 \cap \Lambda). \end{aligned}$$

This ends the proof of equality (35) in the case that C_0 and D_0 are atoms of the same generation n , taking $l_0 = 2n - 1$.

Now, let us prove equality (35) when C_0 and D_0 are atoms of different generations. Let n equal the maximum of both generations. Take $l \geq 2n - 1$. Since Λ is contained in the union of the atoms of any generation, we have

$$C_0 \cap \Lambda = \bigcup_{C \in \mathcal{A}_n, C \subset C_0} C \cap \Lambda,$$

where the sets in the union are pairwise disjoint. Analogously

$$\Phi^{-l}(D_0 \cap \Lambda) = \bigcup_{D \in \mathcal{A}_n, D \subset D_0} \Phi^{-l}(D \cap \Lambda),$$

where also the sets in this union are pairwise disjoint. So,

$$(C_0 \cap \Lambda) \cap \Phi^{-l}(D_0 \cap \Lambda) = \bigcup_{C \in \mathcal{A}_n, C \subset C_0} \bigcup_{D \in \mathcal{A}_n, C \subset D_0} (C \cap \Lambda) \cap \Phi^{-l}(D \cap \Lambda).$$

Since the sets in the union are pairwise disjoint, we deduce

$$\nu((C_0 \cap \Lambda) \cap \Phi^{-l}(D_0 \cap \Lambda)) = \sum_{C \in \mathcal{A}_n, C \subset C_0} \sum_{D \in \mathcal{A}_n, C \subset D_0} \nu((C \cap \Lambda) \cap \Phi^{-l}(D \cap \Lambda)).$$

As C, D are atoms of the same generation n , and $l \geq 2n - 1$, we can apply the first case proved above, to deduce that

$$\nu((C_0 \cap \Lambda) \cap \Phi^{-l}(D_0 \cap \Lambda)) =$$

$$(37) \quad \#\{C \in \mathcal{A}_n, C \subset C_0\} \cdot \#\{D \in \mathcal{A}_n, C \subset D_0\} \cdot \frac{1}{(\#\mathcal{A}_n)^2}.$$

The number of atoms of generation n contained in an atom C_0 of generation n_1 larger or equal than n , does not depend of the chosen atom C_0 . Therefore,

$$\#\{C \in \mathcal{A}_n, C \subset C_0\} = \frac{\#\mathcal{A}_n}{\#\mathcal{A}_{n_1}} = (\#\mathcal{A}_n) \cdot \nu(C_0 \cap \Lambda).$$

Analogously

$$\#\{D \in \mathcal{A}_n, D \subset D_0\} = (\#\mathcal{A}_n) \cdot \nu(D_0 \cap \Lambda).$$

Finally, substituting in equality (37) we conclude that

$$\nu(\Phi^{-l}(D_0 \cap \Lambda) \cap (C_0 \cap \Lambda)) = \nu(C_0 \cap \Lambda) \cdot \nu(D_0 \cap \Lambda) \quad \forall l \geq 2n - 1,$$

ending the proof. \square

Lemma 3.13. $h_\nu(\Phi) = +\infty$.

Proof. For $n \geq 1$ we consider the partition \mathcal{A}_n of Λ consisting of all the n -atoms intersected with Λ . By the definition of metric entropy

$$(38) \quad h_\nu(\Phi) := \sup_{\mathcal{P}} h(\mathcal{P}, \nu) \geq h(\mathcal{A}_n, \nu), \text{ where}$$

$$(39) \quad h(\mathcal{A}_n, \nu) := \lim_{l \rightarrow +\infty} \frac{1}{l} H\left(\bigvee_{j=0}^l (\Phi^{-j} \mathcal{A}_n), \nu\right),$$

$$\mathcal{Q}_l := \bigvee_{j=0}^l \Phi^{-j} \mathcal{A}_n := \left\{ \bigcap_{j=0}^l \Phi^{-j} A_j \cap \Lambda \neq \emptyset : A_j \in \mathcal{A}_n \right\},$$

$$(40) \quad H(\mathcal{Q}_l, \nu) := - \sum_{X \in \mathcal{Q}_l} \nu(X) \log \nu(X).$$

For any nonempty $X := \Lambda \cap \left(\bigcap_{j=0}^l \Phi^{-j} A_j \right) \in \mathcal{Q}_l$, Lemma 3.10-c) yields

$$\nu(X) = \nu \left(\bigcap_{j=0}^l \Phi^{-j} A_j \cap \Lambda \right) = \sum_{G \in \mathcal{F}_{n,l}(\{\bar{A}_j^l\})} \nu(G \cap \Lambda).$$

Since G is an atom of generation $n+l$, we have $\nu(G \cap \Lambda) = 1/(\#\mathcal{A}_{n+l})$, thus applying Lemma 3.10-e), yields

$$\nu(X) = \frac{\#\mathcal{F}_{n,l}(\{A_j\})}{\#\mathcal{A}_{n+l}} = \frac{1}{2^{nl} \cdot \#\mathcal{A}_n}.$$

Combining this with (40) yields $H(\mathcal{Q}_l) = \log(\#\mathcal{A}_n) + nl \cdot \log 2$. Finally, substituting in Equality (39), we conclude

$$h(\mathcal{A}_n, \nu) := \lim_{l \rightarrow +\infty} \frac{1}{l} H(\mathcal{Q}_l, \nu) = n \log 2.$$

Combining with (38) yields $h_\nu(\Phi) \geq n \log 2$, for all $n \geq 1$; hence $h_\nu(\Phi) = +\infty$. \square

Proof of Lemma 3.1. As proved in Lemmas 3.11, 3.12 and 3.13, the probability measure ν constructed by equality (32) is Φ -invariant, mixing and has infinite metric entropy, as required. \square

4. PERIODIC SHRINKING BOXES

In this section we will prove Theorems 1 and 3. The proofs are based on the properties of the models proved in the previous sections, and on the existence of the periodic shrinking boxes which we construct here.

Definition 4.1. (Periodic shrinking box) Let $f \in C^0(M)$ and $K \subset M$ be a box. We call K *periodic shrinking with period* $p \geq 1$ for f , if $K, f(K), f^2(K), \dots, f^{p-1}(K)$ are pairwise disjoint, and $f^p(K) \subset \text{int}(K)$. If so, we call $f^p|_K : K \rightarrow \text{int}(K)$ the *return map*.

Lemma 4.2. *For any $\delta > 0$, there exists an open and dense set of maps $f \in C^0(M)$ that have a periodic shrinking box K with $\text{diam}(K) < \delta$. For a dense set of $f \in C^0(M)$ the return map to K is a homeomorphism onto its image.*

The proof of this lemma uses the following definition and technical result.

Definition 4.3. Let $H \subset K$ be two boxes in the manifold M . We say that H is *homothetic* to K if there exists a homeomorphism $\phi : K \mapsto [0, 1]^m$ such that $\phi(H) = [a, b]^m \subset [0, 1]^m$

Lemma 4.4. *Let $f \in C^0(M)$ and $x_0 \in M$. For all $\varepsilon > 0$, there exists $g \in C^0(M)$ and a neighborhood H of x_0 such that $\|g - f\|_{C^0} < \varepsilon$ and a homeomorphism $g|_H$ onto its image. Moreover, g can be constructed to coincide with f except in an arbitrarily small neighborhood of x_0 .*

Proof. Fix $0 < \delta < \varepsilon/2$ small enough such that if $\text{dist}(x, y) < \delta$, then $\text{dist}(f(x), f(y)) < \varepsilon/2$. Construct boxes H and K such that $\text{diam}(K) < \delta$ and H is homothetic to K and $x_0 \in \text{int}(H) \subset H \subset \text{int}(K)$. Let $\phi : K \mapsto [0, 1]^m$ be the homeomorphism of Definition 4.3. Consider the compact neighborhood $K' \subset M$ of radius $\varepsilon/2$ and center $f(x_0)$, and a homeomorphism $\phi' : K' \mapsto [0, 1]^m$ (if necessary reduce ε from the very beginning so the compact balls of radius $\varepsilon/2$ in the manifold are homeomorphic to $[0, 1]^m$). Consider a box $H' \subset \text{int}(K')$.

Fix a homeomorphism $\chi : H \mapsto H'$, Define $g \in C^0(M)$ by

$$g(x) := f(x) \text{ if } x \notin \text{int}(K)$$

$$g(x) := \chi(x) \text{ if } x \in H$$

$$g(x) := \phi'^{-1} \circ \xi \circ \phi(x) \text{ if } x \in \text{int}(K) \setminus H,$$

where $\xi : [0, 1]^m \setminus [a, b]^m \rightarrow [0, 1]^m$ is constructed as follows. Fix $z_0 \in \text{int}([a, b]^m)$. For each point $z \in [0, 1]^m \setminus [a, b]^m$, consider the half-line starting at z_0 containing z with z_0 . Consider the segment $S(y) = [s_1(y), s_2(y)]$ contained in this half-line, where $s_1(y)$ is the unique point of the half-line in $\partial[0, 1]^m$, and $s_2(y)$ is the unique point in $\partial[a, b]^m$. Now define ξ such that $\xi|_{S(y)} : S(y) \rightarrow [\phi' \circ f \circ \phi^{-1}(y), \phi' \circ \chi \circ \phi^{-1}(y)]$ is the affinity mapping $s_1(y)$ to $\phi' \circ f \circ \phi^{-1}(y)$ and $s_2(y)$ to $\phi' \circ \chi \circ \phi^{-1}(y)$.

It is standard to check that ξ is continuous and that $\phi'^{-1} \circ \xi \circ \phi|_{\partial K} = f$. Therefore $g \in C^0(M)$. By construction $g|_H = \chi : H \mapsto H'$ is a homeomorphism. Besides, $g(x)$ may differ from $f(x)$ only in the points x of K ; but both images are inside K' which is a ball of radius $\varepsilon/2$ in M . Therefore $\|g - f\|_{C^0} < \varepsilon$, as required. The final statement holds since H can be chosen arbitrarily small. □

Proof of Lemma 4.2. According to Definition 4.1, the same periodic shrinking box K for f is also a periodic shrinking box with the same period for all $g \in C^0(M)$ near enough f , proving the openness assertion.

We turn to the denseness assertion. Let $f \in C^0(M)$ and $\varepsilon > 0$. We will construct $g \in C^0(M)$ and a periodic shrinking box K for g with

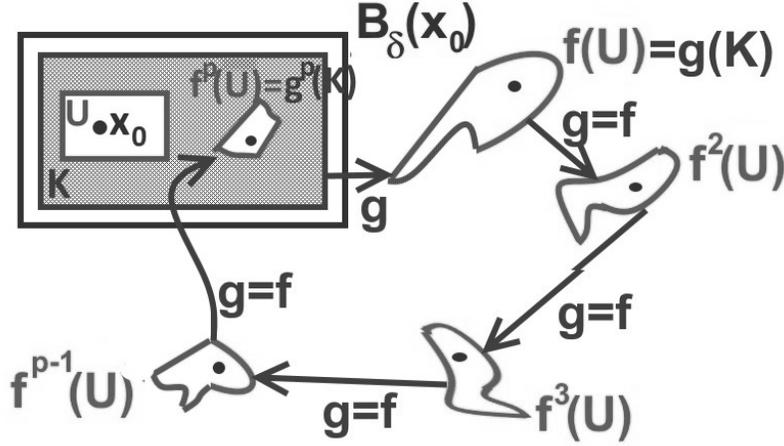


FIGURE 3. Construction of g near f with a periodic shrinking box K for g .

$\text{diam}(K) < \delta$, such that $\|g - f\|_{C^0} < \varepsilon$. We suppose $\delta > 0$ is to be smaller than the ε -modulus of continuity of f .

By the Krylov-Bogolyubov theorem invariant measures exist, and thus by the Poincaré Lemma, there exists a recurrent point $x_0 \in M$ for f . First assume that $x_0 \notin \partial M$. So, there exists a box $B \subset M$ with $\text{diam}(B) < \delta$ such that $x_0 \in \text{int}(B)$. Since x_0 is a recurrent point, there exists a smallest $p \in \mathbb{N}$ such that $f^p(x_0) \in \text{int}(B)$. Taking B slightly smaller if necessary, we can assume that $f^j(x_0) \notin B$ for all $j = 1, 2, \dots, p-1$. So, there exists a small compact box $U \subset \text{int}(B)$ as in Figure 3, such that $x_0 \in \text{int}(U)$, the sets $U, f(U), \dots, f^{p-1}(U)$ are pairwise disjoint, and $f^p(U) \subset \text{int}(B)$.

We can choose U homothetic to B . Since $U, f^p(U) \subset \text{int}(B)$, there exists a box K such that $U, f^p(U) \subset \text{int}(K) \subset K \subset \text{int}(B)$. We also choose K homothetic to B . Therefore, there exists a homeomorphism $\psi : B \rightarrow B$ such that $\psi(x) = x$ for all $x \in \partial B$, and $\psi(K) = U$.

Finally, we construct $g \in C^0(M)$ as follows:

$$g(x) := f(x), \quad \forall x \notin B, \quad g(x) = f \circ \psi(x), \quad \forall x \in B.$$

By construction, K is a periodic shrinking box of g ; by the choice of δ we have $\|g - f\| < \varepsilon$.

Now, let us study the case for which M is a manifold with boundary and all the recurrent points of f belong to ∂M . Choose one such recurrent point $x_0 \in \partial M$. For any $\delta > 0$, there exists a compact box $B \subset M$, with $\text{diam}(B) \leq \delta$ such that $x_0 \in \partial M \cap B$. Since x_0 is recurrent, there exists a smallest natural number $p \geq 1$ such that

$f^p(x_0) \in B$. But $f^p(x_0)$ is also recurrent. So, $f^p(x_0) \in \partial M \cap B$. The previous proof does not work as is. To overcome the problem, we choose a new point $\tilde{x}_0 \neq x_0$, near enough x_0 , such that $\tilde{x}_0 \in \text{int}(B) \setminus \partial M$ and $f^p(\tilde{x}_0) \in B$. By applying Lemma 4.4 and slightly perturbing f , if necessary, we can assume that the restriction of f to a small neighborhood of \tilde{x}_0 is a local homeomorphism onto its image. Hence, $f^p(\tilde{x}_0) \in \text{int}(B) \setminus \partial M$. To conclude, we repeat the construction of g and K above replacing the recurrent point x_0 by \tilde{x}_0 .

Now, let us show that we can construct densely in $C^0(M)$ a periodic shrinking box K such that the return map $f^p|_K$ is a homeomorphism onto its image. We repeat the beginning of the proof, up to the construction of the points $x_0, f(x_0), \dots, f^p(x_0)$ such that $x_0, f^p(x_0) \in \text{int}(B)$ and $f^j(x_0) \notin B$. Apply Lemma 4.4, slightly perturb f , if necessary, inside small open neighborhoods W_0, W_1, \dots, W_{p-1} of the points $x_0, f(x_0), \dots, f^{p-1}(x_0)$ respectively, so that $f|_{W_i}$ is a homeomorphism onto its image for all $i = 0, 1, \dots, p-1$. Finally, construct the box U (Figure 3), but small enough so $f^j(U) \subset W_j$ for all $j = 0, 1, \dots, p-1$, and repeat the construction of K and g as above. \square

Remark 4.5. Note that to obtain the dense property in the proof of Lemma 4.2, we only need to perturb the map f in the interior of the initial box B with diameter smaller than δ .

The following lemma is the homeomorphism version of Lemma 4.2.

Lemma 4.6. *For any $\delta > 0$, there exists an open and dense set of maps $f \in \text{Hom}(M)$ that have a periodic shrinking box K with $\text{diam}(K) < \delta$.*

Proof. The proof of Lemma 4.2 also works in the case that $f \in \text{Hom}(M)$: in fact, the ε -perturbed map g constructed there is a homeomorphism, and to obtain $\|g - f\|_{\text{Hom}(M)} < \varepsilon$ it is enough to reduce $\delta > 0$ to be smaller than the ε -continuity modulus of f and f^{-1} . \square

Remark 4.7. In the proof of Lemmas 4.2 and 4.6, if the starting recurrent point x_0 were a periodic point of period p , then the periodic shrinking box K so constructed would contain x_0 in its interior and have the same period p .

Lemma 4.8. *Let $\delta > 0$. A generic map $f \in C^0(M)$ has a periodic shrinking box K with $\text{diam}(K) < \delta$ such that the return map $f^p|_K$ is topologically conjugated to a model map $\Phi \in \mathcal{H}$ (recall Definition 2.4).*

Proof. Let $K \subset M$ be a periodic shrinking box for f . Fix a homeomorphism $\phi : K \rightarrow D^m$.

To prove the G_δ property, assume that $f \in C^0(M)$ has a periodic shrinking box K with $\text{diam}(K) < \delta$, such that $\phi \circ f^p|_K \circ \phi^{-1} = \Phi \in \mathcal{H}$.

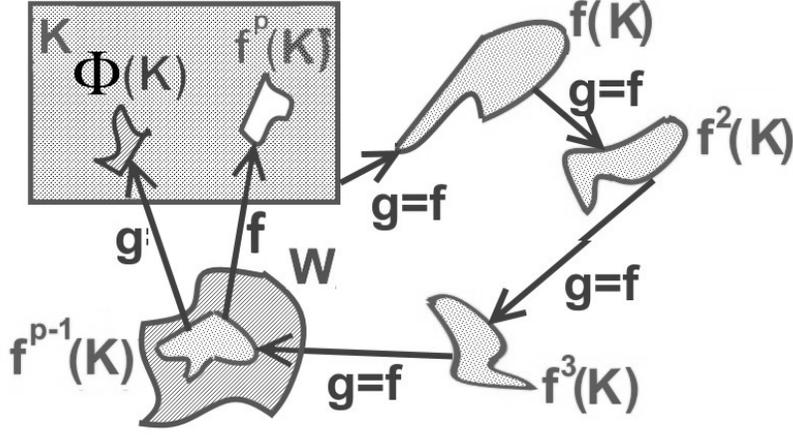


FIGURE 4. Perturbation g of f such that $g^p|_K = \Phi$.

From Definition 4.1, the same box K is also periodic shrinking with period p for all $g \in \mathcal{N}$, where $\mathcal{N} \subset C^0(M)$ is an open neighborhood of f . From Lemma 2.6, \mathcal{H} is a G_δ -set in $C^0(D^m)$, i.e., it is the countable intersection of open families $\mathcal{H}_n \subset C^0(D^m)$. We define

$$\mathcal{V}_n := \{f \in \mathcal{N} : \phi \circ f^p|_K \circ \phi^{-1} \in \mathcal{H}_n\}.$$

Since the restriction to K of a continuous map f , and the composition of continuous maps, are continuous operations in $C^0(M)$, we deduce that \mathcal{V}_n is an open family in $C^0(M)$. Besides

$$(41) \quad \phi \circ g^p|_K \circ \phi^{-1} \in \mathcal{H} = \bigcap_{n \geq 1} \mathcal{H}_n \quad \text{if} \quad g \in \bigcap_{n \geq 1} \mathcal{V}_n \subset C^0(M).$$

In other words, the set of maps $g \in C^0(M)$ that have periodic shrinking box K with $\text{diam}(K) < \delta$, such that the return map $g^p|_K$ coincides, up to a conjugation, with a model map Φ , is a G_δ -set in $C^0(M)$.

To show the denseness fix $f \in C^0(M)$ and $\varepsilon > 0$. Applying Lemma 4.2, it is not restrictive to assume that f has a periodic shrinking box K with $\text{diam}(K) < \min\{\delta, \varepsilon\}$, such that $f^p|_K$ is a homeomorphism onto its image. We will construct $g \in C^0(M)$ to be ε -near f and such that $\phi \circ g^p|_K \circ \phi^{-1} \in \mathcal{H}$.

Choose a box W such that $f^{p-1}(K) \subset \text{int}(W)$. If $p \geq 2$, take W disjoint with $f^j(K)$ for all $j \in \{0, 1, \dots, p-2\}$ (Figure 4). Reducing δ if necessary, we can take W with an arbitrarily small diameter.

To construct $g \in C^0(M)$ (see Figure 4) take $\Phi \in \mathcal{H}$ and let $g(x) := f(x)$ if $x \notin W$ and

$$g(x) := \phi^{-1} \circ \Phi \circ \phi \circ (f^p|_K)^{-1} \circ f(x), \quad \forall x \in f^{p-1}(K).$$

This defines a continuous map $g : f^{p-1}(K) \cup (M \setminus W) \rightarrow M$ such that $|g(x) - f(x)| < \text{diam}(K) < \varepsilon$ for all $x \in f^{p-1}(K) \subset W$ and $g(x) = f(x)$ for all $x \in M \setminus W$. Applying the Tietze Extension Theorem, there exists a continuous extension of g to the whole compact box W , hence to M , such that $\|g - f\|_{C^0} < \varepsilon$. Finally, by construction we obtain

$$g^p|_K = g|_{f^{p-1}(K)} \circ f^{p-1}|_K = \phi^{-1} \circ \Phi \circ \phi \circ (f^p|_K)^{-1} \circ f \circ f^{p-1}|_K = \phi^{-1} \circ \Phi \circ \phi,$$

ending the proof of Lemma 4.8. \square

Lemma 4.9. *Let $\delta > 0$. A generic homeomorphism $f \in \text{Hom}(M)$ has a periodic shrinking box K with $\text{diam}(K) < \delta$, such that the return map $f^p|_K$ is topologically conjugated to a model homeomorphism $\Phi \in \mathcal{H} \cap \text{RHom}(D^m)$.*

Proof. We repeat the proof of the G_δ -set property of Lemma 4.8, using $\mathcal{H} \cap \text{RHom}(D^m)$ instead of \mathcal{H} , and $\text{Hom}(M)$ instead of $C^0(M)$ (notice that taking the inverse is also a continuous operation in $\text{Hom}(M)$).

To show the denseness fix $f \in \text{Hom}(M)$ and $\varepsilon > 0$. Let $\delta \in (0, \varepsilon)$ be smaller than the ε -modulus of continuity of f and f^{-1} , and consider a periodic shrinking box K with $\text{diam}(K) < \delta$ (Lemma 4.6). Fix a homeomorphism $\phi : K \rightarrow D^m$. We will construct $g \in \text{Hom}(M)$ to be ε -near f in $\text{Hom}(M)$, with $\phi \circ g^p|_K \circ \phi^{-1} = \Phi \in \mathcal{H} \cap \text{RHom}(D^m)$.

From Definition 4.1 we know that the boxes $K, f(K), \dots, f^{p-1}(K)$ are pairwise disjoint and that $f^p(K) \subset \text{int}(K)$. Denote $W := f^{-1}(K)$. Since f is a homeomorphism, we deduce that W is a box as in Figure 4, such that $W \cap f^j(K) = \emptyset$ for all $j = 0, 1, \dots, p-2$ if $p \geq 2$, and $f^{p-1}(K) \subset \text{int}(W)$. Since $\text{diam}(K) < \delta$ we have $\text{diam}(W) < \varepsilon$.

Consider $\phi \circ f^p|_K \circ \phi^{-1} \in \text{RHom}(D^m)$. Applying part b) of Lemma 2.6, there exists a homeomorphism $\psi : D^m \rightarrow D^m$ such that

$$\psi|_{\partial D^m} = \text{id}|_{\partial D^m}, \quad \psi \circ \phi \circ f^p|_K \circ \phi^{-1} = \Phi \in \mathcal{H} \cap \text{RHom}(D^m).$$

So, we can construct $g \in \text{Hom}(M)$ such that $g(x) := f(x)$ for all $x \notin W$, and $g(x) := \phi^{-1} \circ \psi \circ \phi \circ f(x)$ for all $x \in W$. Since $\psi|_{\partial D^m}$ is the identity map, we obtain $g|_{\partial W} = f|_{\partial W}$. Thus, the above equalities define a continuous map $g : M \rightarrow M$. Moreover g is invertible because $g|_W : W \rightarrow K$ is a composition of homeomorphisms, and $g|_{M \setminus W} = f|_{M \setminus W} : M \setminus W \rightarrow M \setminus K$ is also a homeomorphism. So, $g \in \text{Hom}(M)$. Moreover, by construction we have $|g(x) - f(x)| < \text{diam}(K) < \varepsilon$ for all $x \in W$, and $g(x) = f(x)$ for all $x \notin W$. Also the inverse maps satisfy

$|g^{-1}(x) - f^{-1}(x)| < \text{diam}(f^{-1}(K)) = \text{diam}(W) < \varepsilon$ for all $x \in K$, and $g^{-1}(x) = f^{-1}(x)$ for all $x \notin K$. Therefore $\|g - f\|_{\text{Hom}} < \varepsilon$.

Finally, let us check that $g^p|_K$ is topologically conjugated to Φ :

$$\begin{aligned} g^p|_K &= g|_{f^{p-1}(K)} \circ f^{p-1}|_K = g|_W \circ f^{p-1}|_K = \\ &\quad \phi^{-1} \circ \psi \circ \phi \circ f \circ f^{p-1}|_K = \\ &\quad \phi^{-1} \circ (\psi \circ \phi \circ f^p|_K \circ \phi^{-1}) \circ \phi = \phi^{-1} \circ \Phi \circ \phi, \end{aligned}$$

ending the proof of Lemma 4.9. \square

Remark 4.10. In the proof of the dense property in Lemmas 4.8 and 4.9, once a periodic shrinking box K is constructed with period $p \geq 1$, we only need to perturb the map f inside $W \cup \bigcup_{j=0}^{p-1} f^j(K)$, where $W = f^{-1}(K)$ if f is a homeomorphism, and $\text{int}(W) \supset f^{p-1}(W)$ otherwise. In both cases, by reducing $\delta > 0$ from the very beginning, if necessary, we can construct W such that $\text{diam}(W) < \varepsilon$ for a previously specified small $\varepsilon > 0$.

Proof of Theorems 1 and 3. From Lemmas 4.8 and 4.9, a generic map $f \in C^0(M)$ and also a generic $f \in \text{Hom}(M)$, has a periodic shrinking box K such that the return map $f^p|_K : K \rightarrow \text{int}(K)$ is conjugated to a model map $\Phi \in \mathcal{H}$. We consider the homeomorphism $\phi^{-1} : K \rightarrow D^m$ such that $\phi^{-1} \circ f^p \circ \phi = \Phi \in \mathcal{H}$. Lemma 3.1 states that every map $\Phi \in \mathcal{H}$ has an Φ -invariant mixing measure ν with infinite metric entropy for Φ . Consider the push-forward measure $\phi_*\nu$, defined by $(\phi_*\nu)(B) := \nu(\phi^{-1}(B \cap K))$ for all the Borel sets $B \subset M$. By construction, $\phi_*\nu$ is supported on $K \subset M$. Since ϕ is a conjugation between Φ and $f^p|_K$, the push-forward measure $\phi_*\nu$ is f^p -invariant and mixing for f^p and moreover $h_{\phi_*\nu}(f^p) = +\infty$.

From $\phi_*\nu$, we will construct an f -invariant and f -ergodic measure μ supported on $\bigcup_{j=0}^{p-1} f^j(K)$, with infinite metric entropy for f . Precisely, for each Borel set $B \subset M$, define

$$(42) \quad \mu(B) := \frac{1}{p} \sum_{j=0}^{p-1} (f^j)_*(\phi_*\nu)(B \cap f^j(K)).$$

Applying Equality (42), and the fact that $\phi_*\nu$ is f^p -invariant and f^p -mixing, it is standard to check that μ is f -invariant and f -ergodic. From the convexity of the metric entropy function, we deduce that

$$h_\mu(f^p) = \frac{1}{p} \sum_{j=0}^{p-1} h_{(f^j)_*(\phi_*\nu)}(f^p) = +\infty.$$

Finally, recalling that $h_\mu(f^p) \leq p h_\mu(f)$ for any f -invariant measure μ and any natural number $p \geq 1$, we conclude that $h_\mu(f) = +\infty$. \square

5. GOOD SEQUENCES OF PERIODIC SHRINKING BOXES

The purpose of this section is to prove Theorems 2 and 4.

Definition 5.1. Let $f \in C^0(M)$ and let $K_1, K_2, \dots, K_n, \dots$ be a sequence of periodic shrinking boxes for f . We call $\{K_n\}_{n \geq 1}$ *good* if it has the following properties (see Figure 5):

- $\{K_n\}_{n \geq 1}$ is composed of pairwise disjoint boxes.
- There exists a natural number $p \geq 1$, independent of n , such that K_n is a periodic shrinking box for f with a period p_n , a multiple of p .
- There exists a sequence $\{H_n\}_{n \geq 0}$ of periodic shrinking boxes, all with period p , such that $K_n \cup H_n \subset H_{n-1}$, $K_n \cap H_n = \emptyset$ for all $n \geq 1$, and $\text{diam}(H_n) \rightarrow 0$ as $n \rightarrow +\infty$.

Remark. Definition 5.1 implies that $\bigcap_{n \geq 1} H_n = \{x_0\}$, where x_0 is periodic with period p . Furthermore, for any $j \geq 0$ we have

$$d(f^j(K_n), f^j(x_0)) \leq \text{diam}(f^j(H_{n-1})) \leq \max_{0 \leq k \leq p-1} \text{diam}(f^k(H_{n-1})) \xrightarrow{n \rightarrow \infty} 0,$$

and thus

$$(43) \quad \lim_{n \rightarrow +\infty} \sup_{j \geq 0} d(f^j(K_n), f^j(x_0)) = 0.$$

Lemma 5.2. *Generic maps $f \in C^0(M)$, and generic $f \in \text{Hom}(M)$ if $m \geq 2$, have good sequences $\{K_n\}_{n \geq 1}$ of periodic shrinking boxes such that the return maps $f^{p_n}|_{K_n}$ are topologically conjugated to model maps.*

Proof. To see the G_δ property assume that f has a good sequence $\{K_n\}_n$ of periodic shrinking boxes. For each fixed n , the boxes K_n and H_n are also periodic shrinking with periods p_n and p respectively, for all g in an open set in $C^0(M)$ or in $\text{Hom}(M)$ (see Definition 4.1). Taking the intersection of such open sets for all $n \geq 1$, we deduce that the same sequence $\{K_n\}$ is also a good sequence of periodic shrinking boxes for all g in a G_δ -set. Now, also assume that $f^{p_n}|_{K_n}$ is a model map for all $n \geq 1$. From Lemmas 4.8 and 4.9, for each fixed $n \geq 1$, the family of continuous maps g such that the return map $g^{p_n}|_{K_n}$ is topologically conjugated to a model, is a G_δ -set in $C^0(M)$ or in $\text{Hom}(M)$. The (countable) intersection of these G_δ -sets, produces a G_δ -set, as required.

To prove denseness fix $f \in C^0(M)$ or $f \in \text{Hom}(M)$, and $\varepsilon > 0$. We will construct g in the ε -neighborhood of f and a good sequence of periodic shrinking boxes K_n for g such that, $g^{p_n}|_{K_n} =_\phi \Phi_n \in \mathcal{H}$ for all $n \geq 1$.

Generic maps $\tilde{f} \in C^0(M)$ and generic $\tilde{f} \in \text{Hom}(M)$ have a periodic shrinking box H_0 with period $p \geq 1$, such that $\tilde{f}^p|_{H_0}$ is conjugate to a model map $\Phi \in \mathcal{H}$ (Lemmas 4.8 and 4.9). Take such \tilde{f} in the

$(\varepsilon/6)$ -neighborhood of f . Since $\tilde{f}^p : H_0 \rightarrow \text{int}(H_0) \subset H_0$ is continuous, by the Brouwer Fixed Point Theorem there exists a periodic point $x_0 \in \text{int}(H_0)$ of period p . Lemma 3.1 and the argument at the end of the proofs of Theorems 1 and 3, show that the map \tilde{f} has an ergodic measure μ supported on $\bigcup_{j=0}^{p-1} \tilde{f}^j(H_0)$ such that $h_\mu(\tilde{f}) = +\infty$. Therefore, by Poincaré Recurrence Lemma, there exists some recurrent point $y_1 \in \text{int}(H_0)$ for \tilde{f} such that $y_1 \neq x_0$ (see Figure 5).

Choose $\delta_1 > 0$ small enough and construct a box B_1 such that $y_1 \in \text{int}(B_1)$, $\text{diam}(B_1) < \delta_1$, $x_0 \notin B_1$ and $B_1 \subset \text{int}(H_0)$. We repeat the proofs of the dense property of Lemmas 4.2 and 4.6, using the recurrent point y_1 instead of x_0 , and the box B_1 instead of B (see Figure 3). So, we deduce that there exists an $(\varepsilon/6)$ -perturbation f^* of \tilde{f} , and a periodic shrinking box $K_1 \subset B_1$ for f^* , with some period $p_1 \geq p$ (see Figure 5). Moreover, f^* coincides with \tilde{f} in $H_0 \setminus \text{int}(B_1)$ (recall Remark 4.5). Therefore, the same periodic point x_0 of \tilde{f} survives for f^* , and the same initial box H_0 is still periodic shrinking with period p for f^* . So, the compact sets of the family $\{f^{*j}(H_0)\}_{j=0,1,\dots,p-1}$ are pairwise disjoint, and $f^{*p}(H_0) \subset \text{int}(H_0)$. This implies that the period p_1 of the new periodic shrinking box K_1 for f^* , is a multiple of p .

Now, we apply the proofs of the dense property of Lemmas 4.8 and 4.9, using the shrinking box K_1 instead of K (see Figure 4). We deduce that there exists an $(\varepsilon/6)$ -perturbation g_1 of f^* , such that K_1 is still a periodic shrinking box for g_1 with the same period p_1 , but moreover, the return map is now $g_1^{p_1}|_{K_1} = \Phi_1 \in \mathcal{H}$. Taking into account Remark 4.10, we can construct g_1 to coincide with f^* in the complement of $W_1 \cup \left(\bigcup_{j=0}^{p_1-1} f^{*j}(K_1) \right)$, where $W_1 \supset f^{p-1}(K_1)$ is a box, small enough not to contain the periodic point x_0 , and to be contained in the interior of the shrinking box H_0 . Therefore, x_0 and H_0 are still periodic with period p for g_1 .

To summarize, we have built the periodic shrinking boxes H_0 and K_1 for a continuous map or homeomorphism g_1 , with periods p and p_1 respectively, where p_1 is multiple of p , and a periodic point $x_0 \in \text{int}(H_0)$ (see Figure 5), such that:

$$K_1 \subset H_0 \setminus \{x_0\}, \quad g_1^{p_1}|_{K_1} =_\phi \Phi_1 \in \mathcal{H} \quad \text{and}$$

$$\|g_1 - f\| < \|g_1 - f^*\| + \|f^* - \tilde{f}\| + \|\tilde{f} - f\| < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}.$$

We proceed by induction on $n \geq 1$, assume that H_0, \dots, H_{n-1} and K_1, \dots, K_n are periodic shrinking boxes (see Figure 5) of $g_n \in C^0(M)$ or $g_n \in \text{Hom}(M)$, with periods p and p_1, \dots, p_n respectively, where p_i is multiple of p , and that $x_{n-1} \in \text{int}(H_{n-1})$ is a periodic point of period

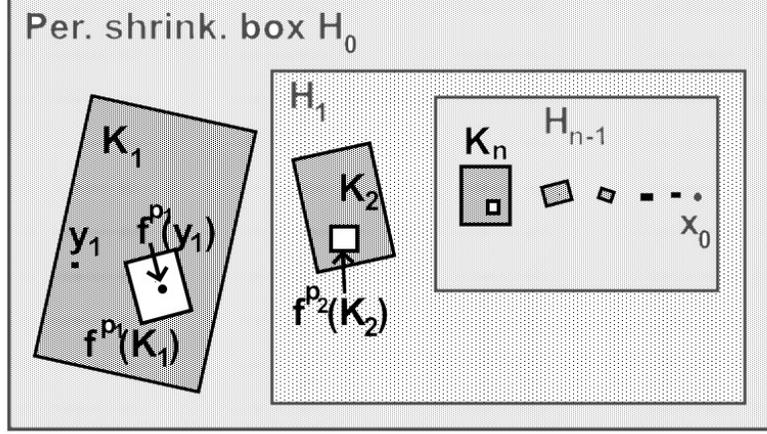


FIGURE 5. Construction of a good sequence of periodic shrinking boxes.

p for g_n . Assume also that $K_n \subset H_{n-1} \setminus \{x_{n-1}\}$, that for $1 \leq j \leq n-1$

$$H_j, K_j \subset H_{j-1}; \quad H_j \cap K_j = \emptyset; \quad \text{diam}(H_j) < \frac{\varepsilon}{2^j};$$

$$(44) \quad g_n^{p_j}|_{K_j} \text{ is topologically conjugated to } \Phi_j \in \mathcal{H},$$

and that we have a finite number of the previously constructed maps g_1, \dots, g_n such that

$$(45) \quad \|g_1 - f\| < \frac{\varepsilon}{2}, \quad \|g_j - g_{j-1}\| < \frac{\varepsilon}{2^j}, \quad \forall j = 2, \dots, n.$$

We will construct g_{n+1} and the boxes H_n and K_{n+1} that satisfy the above properties for $n+1$ instead of n , and such that for all $j = 1, \dots, n$, the boxes H_{j-1} and K_j are still periodic shrinking for g_{n+1} with the same periods p, p_j .

From the inductive hypothesis, g_n has a periodic shrinking box H_{n-1} of period p , a periodic point $x_{n-1} \in \text{int}(H_{n-1})$ of period p , and a periodic shrinking box $K_n \subset H_{n-1} \setminus \{x_{n-1}\}$ of period p_n , a multiple of p . We choose $0 < \tilde{\delta}_n < \varepsilon/2^n$ small enough, and construct a box $\tilde{B}_n \subset H_{n-1}$ containing the periodic point x_{n-1} in its interior, disjoint from K_n , and such that $\text{diam}(\tilde{B}_n) < \tilde{\delta}_n$. Repeating the proof of the density properties in Lemmas 4.2 and 4.6 (putting x_{n-1} instead of x_0 , and $\tilde{\delta}_n > 0$ small enough), we construct an $\varepsilon/(3 \cdot 2^{n+1})$ -perturbation \tilde{g}_n of g_n and a periodic shrinking box $H_n \subset \text{int}(\tilde{B}_n)$ for \tilde{g}_n . Moreover, since x_{n-1} is a periodic point with period p , the period of H_n can be made equal

to p (see Remark 4.7). By construction $H_n \subset \tilde{B}_n \subset H_{n-1}$ is disjoint from K_n and from ∂H_{n-1} . To construct \tilde{g}_n we only need to modify g_n inside \tilde{B}_n (recall Remark 4.5). Therefore the same periodic shrinking boxes H_0, H_1, \dots, H_{n-1} and K_1, K_2, \dots, K_n of g_n , are preserved for \tilde{g}_n , with the same periods.

Now, as in the proof of Lemmas 4.8 and 4.9, we construct a new $\varepsilon/(3 \cdot 2^{n+1})$ -perturbation g_n^* of \tilde{g}_n , such that $g_n^{*p}|_{H_n}$ is conjugated to a map in \mathcal{H} . To construct g_n^* we only need to modify \tilde{g}_n in $\tilde{W}_n \cup \bigcup_{j=0}^{p-1} g_n^j(H_n)$, where \tilde{W}_n is a neighborhood of $\tilde{g}_n^{p-1}(H_n)$ that can be taken arbitrarily small by choosing H_n small enough from the beginning (see Remark 4.10). Therefore we do not need to modify \tilde{g}_n or g_n outside H_{n-1} or inside K_n . We conclude that the same shrinking boxes $K_1, \dots, K_n; H_0, \dots, H_{n-1}$ for \tilde{g}_n and g_n , are still periodic shrinking for g_n^* , with the same periods and that $g_n^{*p_j}|_{K_j} = \tilde{g}_n^{p_j}|_{K_j}$ which is conjugated to $\Phi_j \in \mathcal{H}$ for all $j = 1, \dots, n$.

When modifying g_n inside $H_{n-1} \setminus K_n$ to obtain \tilde{g}_n and g_n^* , the periodic point x_{n-1} of period p for g_n , may not be preserved. But since $H_n \subset H_{n-1} \setminus K_n$ is a periodic shrinking box with period p for g_n^* , by the Brouwer Fixed Point Theorem, there exists a periodic point $x_n \in \text{int}(H_n) \setminus K_n$ for g_n^* , with the same period p .

Since the return map $g_n^{*p}|_{H_n}$ is conjugated to a model, there exists an ergodic measure with infinite entropy (see Lemma 3.1), supported on the g_n^* -orbit of H_n . Therefore, there exists a recurrent point $y_n \in \text{int}(H_n)$ such that $y_n \neq x_n$. We choose $\delta_n > 0$ small enough, and a compact box $B_n \subset \text{int}(H_n) \setminus \{x_n\}$ such that $y_n \in \text{int}(B_n)$ and $\text{diam}(B_n) < \delta_n$. Repeating the above arguments, and if δ_n is small enough, we construct a new $\varepsilon/(3 \cdot 2^{n+1})$ -perturbation g_{n+1} of g_n^* and a box $K_{n+1} \subset \text{int}(B_n)$ that is periodic with some period p_{n+1} for g_{n+1} , and such that $g_{n+1}^{p_{n+1}}|_{K_{n+1}}$ is conjugate to a map in \mathcal{H} .

To construct such a perturbation g_{n+1} of g_n^* , we only need to modify g_n^* in $\text{int}(B_n)$, and in a box W_{n+1} containing $g_n^{*p_{n+1}-1}(K_{n+1})$ in its interior (recall Remarks 4.5 and 4.10). Recall that W_{n+1} is a small set, provided that $\delta_n > 0$ is small enough. Therefore, g_{n+1} can be constructed so the point x_n is still periodic with period p for g_{n+1} , and the same boxes $H_0, H_1, \dots, H_n, K_1, \dots, K_n$ are still shrinking periodic for g_{n+1} , with the same periods. Moreover, $g_{n+1}^{p_j}|_{K_j} = g_j^{p_j}|_{K_j}$ is conjugated to $\Phi_j \in \mathcal{H}$ for all $j = 1, \dots, n$.

In particular H_n is periodic shrinking with period p for g_{n+1} , and it contains K_{n+1} by construction. This implies that the period p_{n+1} of

K_{n+1} is a multiple of p . By construction we have,

$$\|g_{n+1} - g_n\| \leq \|g_{n+1} - g_n^*\| + \|g_n^* - \tilde{g}_n\| + \|\tilde{g}_n - g_n\| < 3 \cdot \frac{\varepsilon}{3 \cdot 2^{n+1}} = \frac{\varepsilon}{2^{n+1}}.$$

Moreover $\text{diam}(H_n) < \tilde{\delta}_n < \varepsilon/2^n$. We have constructed g_{n+1} and the boxes H_n and K_{n+1} that satisfy the inductive properties for $n + 1$ instead of n , as required.

We have defined a sequence $\{g_n\}_{n \geq 1}$ of continuous maps or homeomorphisms on M , and sequences $\{H_n\}_{n \geq 1}$, $\{K_n\}$ of compact boxes such that the Properties (44) and (45) are satisfied for all $n \geq 1$. Since $\|g_{n+1} - g_n\| \leq \varepsilon/2^{n+1}$ for all $n \geq 1$ the sequence $\{g_n\}_{n \geq 1}$ is Cauchy. So, there exists a limit map g . Since g_n is an ε -perturbation of f for all $n \geq 1$, the limit map g also is. Finally, by construction we have $g_k(x) = g_n(x)$ for all $x \in \bigcup_{j=0}^{p_n} g_n^j(K_n)$, $g_k^{p_n}|_{K_n}$ is topologically conjugated to $\Phi_n \in \mathcal{H}$ for all $n \geq 1$ and for all $k \geq n$. Thus $\{K_n\}_{n \geq 1}$ is a good sequence of periodic shrinking boxes for g , as required. \square

Remark 5.3. As a consequence of Lemmas 5.2 and 3.1 (after applying the same arguments at the end of the proof of Theorems 1 and 3), generic continuous maps and homeomorphisms f have a sequence of ergodic measures μ_n , each one supported on the f -orbit of a box K_n of a good sequence $\{K_n\}_{n \geq 1}$ of periodic shrinking boxes for f , satisfying $h_{\mu_n}(f) = +\infty$ for all $n \geq 1$.

Let \mathcal{M} denote the metrizable space of Borel probability measures on a compact metric space M , endowed with the weak* topology. Fix a metric dist^* in \mathcal{M} .

Lemma 5.4. *For all $\varepsilon > 0$ there exists $\delta > 0$ that satisfies the following property: if $\mu, \nu \in \mathcal{M}$ and $\{B_1, B_2, \dots, B_r\}$ is a finite family of pairwise disjoint compact balls $B_i \subset M$, and if $\text{supp}(\mu) \cup \text{supp}(\nu) \subset \bigcup_{i=1}^r B_i$, and $\mu(B_i) = \nu(B_i)$, $\text{diam}(B_i) < \delta$ for all $i = 1, 2, \dots, r$, then $\text{dist}^*(\mu, \nu) < \varepsilon$.*

Proof. If $M = [0, 1]$ the proof can be found for instance in [CT, Lemma 4]. If M is any other compact manifold of finite dimension $m \geq 1$, with or without boundary, just copy the proof of [CT, Lemma 4] by substituting the pairwise disjoint compact intervals $I_1, I_2, \dots, I_r \subset [0, 1]$ in that proof, by the family of pairwise disjoint compact boxes $B_1, B_2, \dots, B_r \subset M$. \square

Proofs of Theorems 2 and 4. For any $\varepsilon > 0$, take $\delta > 0$ as in Lemma 5.4. Applying Lemma 5.2, generic continuous maps or homeomorphisms f have a good sequence of periodic shrinking boxes $\{K_n\}_{n \geq 1}$, and a sequence $\{\mu_n\}$ of ergodic f -invariant measures such that $h_{\mu_n}(f) = +\infty$ (see Remark 5.3) and such that $\text{supp}(\mu_n) \subset \bigcup_{j=0}^{p_n-1} f^j(K_n)$, where

$p_n = l_n \cdot p$, multiple of p , is the period of the shrinking box K_n . Taking into account that $\{f^j(K_n)\}_{0 \leq j \leq p_n-1}$ is a family of pairwise disjoint compact sets, and $f^{p_n}(K_n) \subset \text{int}(K_n)$, we obtain for each $j \in \{0, 1, \dots, p_n\}$

$$\begin{aligned} \mu_n(f^j(K_n)) &= \mu_n(f^{-j}(f^j(K_n))) = \mu_n(f^{-j}(f^j(K_n)) \cap \text{supp}(\mu_n)) = \\ &= \mu_n(K_n). \end{aligned}$$

Since $1 = \sum_{j=0}^{p_n-1} \mu_n(f^j(K_n)) = p_n \cdot \mu_n(K_n)$; we obtain

$$\mu_n(f^j(K_n)) = \mu_n(K_n) = \frac{1}{p_n} = \frac{1}{l_n p}, \quad \forall j = 0, 1, \dots, p_n.$$

From Definition 5.1, there exists a periodic point x_0 of period p such that $\lim_{n \rightarrow +\infty} \sup_{j \geq 0} \text{Hdist}(f^j(K_n), f^j(x_0)) = 0$, where Hdist denotes the Hausdorff distance. Therefore, there exists $n_0 \geq 1$ such that $d(f^j(K_n), f^j(x_0)) < \delta'$ for all $j \geq 0$ and for all $n \geq n_0$, where $\delta' < \delta/2$ is chosen such that the family of compact balls B_0, B_1, \dots, B_{p-1} , centered at the points $f^j(x_0)$ and with radius δ' , are pairwise disjoint. We obtain $f^j(K_n) \subset B_{j(\text{mod. } p)}$ for all $j \geq 0$ and for all $n \geq 0$. Therefore,

$$\mu_n(B_j) = \frac{1}{p}, \quad \forall j = 0, 1, \dots, p-1, \quad \forall n \geq n_0.$$

Finally, applying Lemma 5.4, we conclude $\text{dist}^*(\mu_n, \mu_0) < \varepsilon$ for all $n \geq n_0$, where $\mu_0 := (1/p) \sum_{j=0}^{p-1} \delta_{f^j(p)}$ is the f -invariant probability measure supported on the periodic orbit of x_0 , which has zero entropy. \square

6. OPEN QUESTIONS

If f is Lipschitz then no invariant measure has infinite entropy, since its topological entropy is finite. The following question arises: do Theorems 1 and 3 hold also for maps with more regularity than continuity but lower regularity than Lipschitz? For instance, do they hold for Hölder-continuous maps?

A-priori there is a chance to answer positively this question for one-dimensional Hölder continuous endomorphisms, because in such a case, the topological entropy is generically infinite. This is a simple corollary of the arguments in [FHT1]. Also for bi-Hölder homeomorphisms on manifolds of dimension 2 or larger, there is a chance to answer positively the above question, because their topological entropy is also generically infinite [FHT], [FHT1]. In this article we focus only on the C^0 -case, and leave for further research the eventual adaptation of our proofs, if this adaptation is possible, to C^α -maps or homeomorphisms with $0 < \alpha < 1$.

The hypothesis of Theorems 1 and 3 states that M is a compact manifold. It arises the following question: do some of the results also hold

in other compact metric spaces that are not manifolds? For instance, do they hold if the space is a Cantor set K ?

If the aim were just to construct $f \in \text{Hom}(K)$ with ergodic measures with infinite metric entropy, the answer would be positive. But if the purpose were to prove that such homeomorphisms are generic in $\text{Hom}(K)$, the answer would be negative.

In fact, Theorem 3 holds in particular for the 2-dimensional square $D^2 := [0, 1]^2$. One of the steps of the proofs consists in constructing some fixed Cantor set $\Lambda \subset D^2$, and a homeomorphism Φ on M that leaves Λ invariant, and possesses an Φ -invariant ergodic measure supported on Λ with infinite metric entropy (see Lemma 3.1 and Remark 3.2). Since any pair of Cantor sets K and Λ are homeomorphic, we deduce that any Cantor set K supports a homeomorphism f and an f -ergodic measure with infinite metric entropy.

Nevertheless, the above phenomenon is not generic on a Cantor set K . On the one hand, there also exists homeomorphisms on K with finite, and even zero, topological entropy. (Take for instance $f \in \text{Hom}(K)$ conjugated to the homeomorphism on the attractor of a Smale horseshoe, or to the attractor of the C^1 -Denjoy example on the circle.) On the other hand, it is known that each homeomorphism on a Cantor set K is topologically locally unique; i.e., it is conjugated to any of its small perturbations [AGW]. Therefore, the topological entropy is locally constant in $\text{Hom}(K)$. We conclude that the homeomorphisms on the Cantor set K with infinite metric entropy, that do exist, are not dense in $\text{Hom}(K)$; hence they are not generic.

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