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Nonstandard use of anti-windup loop for systems with input backlash

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Abstract

Control systems with backlash at the input are considered in this paper. The goal of this work is to characterize the attractor of such nonlinear dynamical systems, and to design anti-windup inspired loops such that the system is globally asymptotically stable with respect to this attractor. The anti-windup loops affect the dynamics of the controllers, and allow to increase the performance of the closed-loop systems. Different performance issues are considered throughout the paper such as robustness with respect to uncertainty in the backlash, and \mathcal{L}_2 gain when external disturbances affect the dynamics. Numerically tractable algorithms with feasibility guarantee are provided, as soon as the linear closed-loop system, obtained by neglecting the backlash effect, is asymptotically stable. The results are illustrated on an academic example and an open-loop unstable aircraft system.

Keywords: Stability analysis, stabilization, backlash, anti-windup loops.

1 Introduction

Backlash operators are nonlinearities present in many mechanical systems. They are involved in mechanical slacks, static friction, elastic displacements and ferromagnetism. Figure 1 shows some examples and [10] is a good introduction of this kind of nonlinearity. Since many smart actuators and materials contain a backlash, and since they are often used for precise control systems, such as position regulation, it is crucial to take the backlash operator into account in the design or the analysis of the closed loop. Neglecting them can reduce the performance or alleviate the stability objective.

There are many different models for backlash operators, see [10] for a survey of possible models. Here we will focus on a component-wise model considered in [10, 13, 19, 4, 12]. The undesirable effect of such a nonlinearity map has been already analyzed using circle criterion in [7], assuming that the control system is open-loop stable, an assumption that is not required in this paper. The present work is based on a Lyapunov method for control systems with backlash operators in the loop. The control objective is to reduce the undesirable

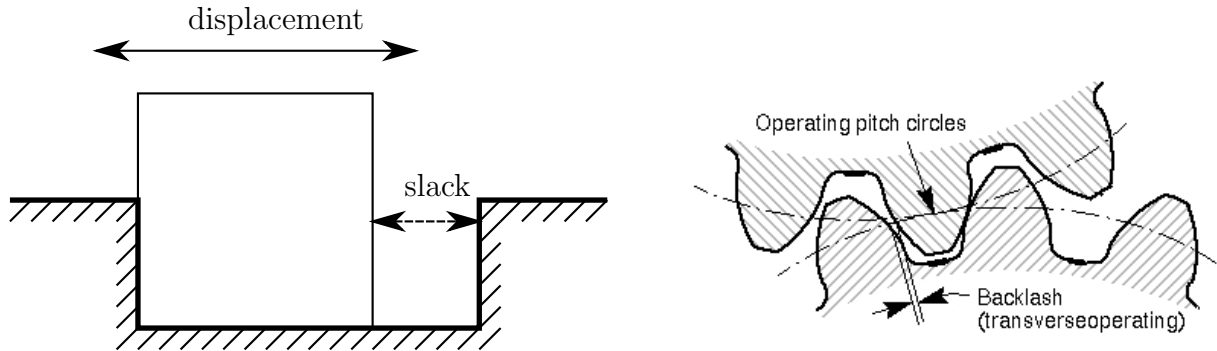


Figure 1: Left: mechanical slack with a longitudinal backlash. Right: rotational backlash.

effect of the nonlinearity by adding an extra dynamics. Such a strategy inspired by anti-windup techniques has been already employed for control systems with saturated inputs, see e.g. [16, 22], where it is shown how to design so called anti-windup gains to improve the performance of saturating closed-loop systems.

To be more specific, our contribution is as follows. We succeed to compute numerically tractable conditions for the design of nonstandard anti-windup (NSAW) gains that modify the controller dynamics and its output for different control objectives. In presence of backlash operators, the most natural control objective is probably the reduction of the attractor that is the reduction of the ultimate bounded set where all solutions converge. In an ideal case where the backlash is not present, the attractor is usually reduced to one point, thus our NSAW technique can be seen as a method to approach as much as possible the ideal case.

Another natural control objective is the robustness issue, that is to guarantee some stability properties in presence of external perturbations or disturbances in the loop and in the backlash operator. The Lyapunov method applies for such cases, and we show how to design NSAW gains guaranteeing the best performance in terms of robustness, with respect to uncertainty in the backlash and with respect to external disturbances. Again our design conditions are numerically tractable and apply as soon as the closed-loop system, without any backlash, is asymptotically stable. On the other hand, no restrictive hypothesis is necessary regarding the open-loop stability of the system. More precisely, our conditions are based on the solution to a convex problem written in terms of Linear Matrix Inequalities that are shown to have a solution allowing to explicitly compute NSAW gains.

The paper is organized as follows. The stability problem and the NSAW gain design problem are introduced in Section 2 for backlash control systems. The main results are given in Section 3 yielding a solution to the stability analysis and allowing to compute numerically tractable conditions for the design of NSAW gains. Some extensions are given in Section 4 when considering both external disturbances and backlash uncertainties. Illustrative examples are given in Section 5 and some concluding remarks are collected in Section 6.

Notation. For two vectors x, y of \mathbb{R}^n , the notation $x \succeq y$ means that $x_{(i)} - y_{(i)} \geq 0$, $\forall i = 1, \dots, n$. $\mathbf{1}$ and $\mathbf{0}$ denote the identity matrix and the null matrix of appropriate dimensions, respectively. $x \in \mathbb{R}_+^n$ means that $x \succeq 0$. The Euclidian norm is denoted $\|\cdot\|$. A' and $trace(A)$ denote the transpose and the trace of A , respectively. $He\{A\} = A + A'$. For two sym-

metric matrices, A and B , $A > B$ means that $A - B$ is positive definite. In partitioned symmetric matrices, the symbol \star stands for symmetric blocks. $\lambda_{max}(A)$ (respectively, $\lambda_{min}(A)$) denotes the maximal (respectively, minimal) eigenvalue of the matrix A .

2 Problem formulation

2.1 System description

The class of systems under consideration is described by:

$$\begin{aligned}\dot{x}_p &= A_p x_p + B_p u_p \\ y_p &= C_p x_p\end{aligned}\tag{1}$$

where $x_p \in \mathbb{R}^{n_p}$ is the state, $u_p \in \mathbb{R}^m$ is the input of the plant, $y_p \in \mathbb{R}^p$ is the measured output. A_p , B_p , C_p are matrices of appropriate dimensions. The pairs (A_p, B_p) and (C_p, A_p) are supposed to be controllable and observable, respectively. The function u_p is a backlash-like nonlinearity.

The connection between the plant and the controller through its output $y_c \in \mathbb{R}^m$ is realized as follows:

$$u_p = \Phi[y_c]\tag{2}$$

where Φ is a component-wise backlash operator (see, for example, [10, 13, 19, 4]). We denote the set of continuous, piecewise differentiable functions $f: [0, +\infty) \rightarrow \mathbb{R}^m$ by $\mathcal{C}_{pw}^1([0, +\infty); \mathbb{R}^m)$, that is the set of continuous functions f being, for some unbounded sequence $(t_j)_{j=0}^\infty$ in $[0, +\infty)$ with $t_0 = 0$, continuously differentiable on (t_{j-1}, t_j) for all $j \in \mathbb{N}$. Given the vector ρ in \mathbb{R}_+^m and $L = \text{diag}(\ell_{(i)})$, with positive values $\ell_{(i)}$, $i = 1, \dots, m$, the operator Φ is defined as follows, for all $f \in \mathcal{C}_{pw}^1([0, +\infty); \mathbb{R}^m)$, for all $j \in \mathbb{N}$, for all $t \in (t_{j-1}, t_j)$ and for all $i \in \{1, \dots, m\}$:

$$\widehat{(\Phi[f](t))}_{(i)} = \begin{cases} \ell_{(i)} \dot{f}_{(i)}(t) & \text{if } \dot{f}_{(i)}(t) \geq 0 \\ & \text{and } (\Phi[f](t))_{(i)} = \ell_{(i)}(f_{(i)}(t) - \rho_{(i)}) \\ \ell_{(i)} \dot{f}_{(i)}(t) & \text{if } \dot{f}_{(i)}(t) \leq 0 \\ & \text{and } (\Phi[f](t))_{(i)} = \ell_{(i)}(f_{(i)}(t) + \rho_{(i)}) \\ 0 & \text{otherwise} \end{cases}\tag{3}$$

where $0 = t_0 < t_1 < \dots$ is a partition of $[0, +\infty)$ such that f is continuously differentiable on each of the intervals (t_{j-1}, t_j) , $j \in \mathbb{N}$. Thus, Φ is a time-invariant nonlinearity with slope restriction, as in [14]. Note however that it is a memory-based operator, since to compute it, we need to have information about the past values of its input (this is not the case in [14]). Since it is possible to stack several backlash operators into one backlash operator (with the dimension equal to the sum of all dimensions), without loss of generality, the standard form defined here remains valid.

The plant is controlled by the following output dynamical controller

$$\begin{aligned}\dot{x}_c &= A_c x_c + B_c y_p + \theta_1 \\ y_c &= C_c x_c + D_c y_p + \theta_2\end{aligned}\tag{4}$$

where $x_c \in \mathbb{R}^{n_c}$ is the state, $y_p \in \mathbb{R}^{n_p}$ is the output of the plant, $y_c \in \mathbb{R}^m$ is the output of the controller. $\theta_1 \in \mathbb{R}^{n_c}$ and $\theta_2 \in \mathbb{R}^m$ are input signals to be designed to perform a suitable correction inspired by anti-windup action (see, for example, [16], [22] in the context of saturation nonlinearities) for mitigating the undesired effects of the backlash. The principle of the nonstandard anti-windup (NSAW) loop considered consists in picking the difference between the output of the nonlinear actuator ($\Phi[y_c]$) and the output of the linearized one (Ly_c), which are available signals for building θ_1 and θ_2 :

$$\begin{aligned}\theta_1 &= E_c(\Phi[y_c] - Ly_c) \\ \theta_2 &= F_c(\Phi[y_c] - Ly_c)\end{aligned}\tag{5}$$

with E_c and F_c matrices of appropriate dimensions.

Remark 2.1 *The design of the controller (4) may be performed with classical techniques to stabilize the plant, disregarding the effects of the backlash nonlinearity (with $\theta_1 = 0$ and $\theta_2 = 0$). In other words, it is assumed that the controller (4) stabilizes the plant (1) through the linear interconnection $u_p = Ly_c$ (which corresponds to take $\Phi[y_c] = Ly_c$) and therefore the matrix:*

$$A_0 = \begin{bmatrix} A_p + B_pLD_cC_p & B_pLC_c \\ B_cC_p & A_c \end{bmatrix}\tag{6}$$

is Hurwitz.

2.2 Problem formulation

The closed-loop system issued from (1), (2), (3), (4) and (5) reads as follows

$$\begin{aligned}\dot{x}_p &= A_px_p + B_p\Phi[y_c] \\ \dot{x}_c &= A_cx_c + B_cy_p + E_c(\Phi[y_c] - Ly_c) \\ y_p &= C_px_p \\ y_c &= C_cx_c + D_cy_p + F_c(\Phi[y_c] - Ly_c)\end{aligned}\tag{7}$$

Let us note that according to (3), one gets $\Phi[y_c](t) \in I_\Phi$ with

$$I_\Phi = \{\Phi[y_c] \in \mathbb{R}^m; L(y_c + \rho) \succeq \Phi[y_c] \succeq L(y_c - \rho)\}\tag{8}$$

One can observe that, by definition, from any initial condition in I_Φ , the solution $\Phi[y_c](t)$ remains confined in I_Φ , $\forall t \geq 0$. According to [11], [5], that means that the nonlinearity Φ is active.

The presence of the backlash operator Φ may induce the existence of multiple equilibrium points or a limit cycle around the origin. Furthermore, in a neighborhood of the origin, system (1) operates in open loop. Hence, we are concerned with the asymptotic behavior of the state $x = [x'_p \ x'_c] \in \mathbb{R}^n$, $n = n_p + n_c$ but not of the operator Φ . Therefore, we want to study the stability properties of the following attractor:

$$\mathcal{A} = \mathcal{S}_0 \subseteq \mathbb{R}^n\tag{9}$$

It is important to emphasize that the proposed technique does not require for the open-loop system to be stable, contrary to [7], [17].

Then the problem we intend to solve can be summarized as follows.

Problem 2.1 *Characterize the region \mathcal{S}_0 of the state space, containing the origin, and design the NSAW gains E_c and F_c such that system (7) is globally asymptotically stable with respect to \mathcal{S}_0 , when initialized as in (8). In other words, \mathcal{S}_0 is a global asymptotic attractor for the closed-loop dynamics (7), for any initial value of Φ in I_Φ .*

2.3 Closed-loop description and well-posedness

For conciseness, throughout the paper, we denote $\dot{\Phi}$ instead of $\widehat{\Phi[y_c]}$, and Φ instead of $\Phi[y_c]$. Let us define the nonlinearity Ψ :

$$\Psi = \Phi[y_c] - Ly_c \quad (10)$$

Hence, with the augmented state $x = [x'_p \ x'_c]^\top \in \mathbb{R}^n$, the closed-loop system reads:

$$\begin{aligned} \dot{x} &= A_0x + (B + RE_c + BLF_c)\Psi \\ y_c &= Kx + F_c\Psi \end{aligned} \quad (11)$$

with A_0 defined in (6) and

$$K = [D_c C_p \ C_c]; B = \begin{bmatrix} B_p \\ \mathbf{0} \end{bmatrix}; R = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$$

When $F_c = \mathbf{0}$, system (11) does not contain any algebraic loop. However, it can be interesting to consider $F_c \neq 0$ in order to have more degrees of freedom to characterize the attractor set \mathcal{S}_0 . A particular attention should be paid to the well-posedness of the implicit relation defining y_c in (11). Indeed, the algebraic loop $y_c = Kx + F_c\Psi$ is said to be well-posed if there exists a unique solution y_c of the second line of (11) for each $Kx + F_c\Psi$. We have the following well-posedness result:

Proposition 2.1 *Under the assumption that $\mathbf{1} + F_cL$ is nonsingular, the algebraic loop in (11) is well defined, and the system (11) is well-posed.*

To prove the proposition, note that from the definition of Φ , one can prove that the implicit function $g(y_c) = y_c - Kx + F_c\Psi = \mathbf{0}$ has a solution, and therefore, provided that $\mathbf{1} + F_cL$ is nonsingular, Proposition 2.1 holds. See, for example, [22] (Chapter 2, page 38) for more details.

3 Main Results

3.1 Theoretical conditions

The following result provides a solution to Problem 2.1, by adapting Lemma 1 in [18] to system (11).

Theorem 3.1 *Suppose there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, two diagonal positive definite matrices $S_2 \in \mathbb{R}^{m \times m}$, $T_3 \in \mathbb{R}^{m \times m}$, two matrices $E_c \in \mathbb{R}^{n_c \times m}$, $F_c \in \mathbb{R}^{m \times m}$ and a positive scalar τ satisfying the following conditions*

$$M_1 < \mathbf{0} \quad (12)$$

$$\rho'LT_3L\rho - \tau \leq 0 \quad (13)$$

with

$$M_1 = \begin{bmatrix} He\{A_0W\} + \tau W & \star & \star \\ (B + RE_c + BLF_c)' & -T_3 & \star \\ -LKA_0W & -LK(B + RE_c + BLF_c) & -He\{(\mathbf{1} + LF_c)S_2\} \end{bmatrix} \quad (14)$$

Then, for any admissible initial conditions $(x(0), \Psi(0))$, the closed-loop system (11) is globally asymptotically stable with respect to the set \mathcal{S}_0 defined as follows:

$$\mathcal{S}_0 = \{x \in \mathbb{R}^n; x'W^{-1}x \leq 1\} \quad (15)$$

In other words, E_c , F_c and \mathcal{S}_0 are solution to Problem 2.1.

Proof. First note that the satisfaction of relation (12) guarantees that the matrix $\mathbf{1} + LF_c$ is Hurwitz and therefore nonsingular, which implies that Proposition 2.1 holds.

To prove Theorem 3.1, we consider a quadratic Lyapunov function candidate V defined by $V(x) = x'Px$, $P = P' > \mathbf{0}$, for all x in \mathbb{R}^n . We want to verify that there exists a class \mathcal{K} function α such that $\dot{V}(x) \leq -\alpha(V(x))$, for all x such that $x'Px \geq 1$ (i.e. for any $x \in \mathbb{R}^n \setminus \mathcal{S}_0$), and for all nonlinearities Ψ satisfying Lemma 1 in [18]. By using the S-procedure, it is sufficient to check that $\mathcal{L} < 0$, where

$$\begin{aligned} \mathcal{L} = & \dot{V}(x) - \tau(1 - x'Px) \\ & - \Psi'T_3\Psi + \rho'LT_3L\rho - 2(\dot{\Psi} + L\dot{y}_c)'N_1\Psi \\ & - 2(\dot{\Psi} + L\dot{y}_c)'N_2(\dot{\Psi} + (\mathbf{1} - N_3)L\dot{y}_c) \end{aligned} \quad (16)$$

with τ a positive scalar and T_3 a positive diagonal matrix. Choosing $N_1 = \mathbf{0}^1$ and $N_3 = \mathbf{1}$, from the definition of Ψ in (10), noting that $\dot{V}(x) = x'(A_0'P + PA_0)x + 2x'P(B + RE_c + BLF_c)\Psi$, it follows that $\mathcal{L} = \mathcal{L}_0 + \rho'LT_3L\rho - \tau$ with

$$\mathcal{L}_0 = \begin{bmatrix} x \\ \Psi \\ \dot{\Psi} \end{bmatrix}' M_2 \begin{bmatrix} x \\ \Psi \\ \dot{\Psi} \end{bmatrix}$$

with M_2 defined as follows

$$M_2 = \begin{bmatrix} He\{PA_0\} + \tau P & \star & \star \\ (B + RE_c + BLF_c)'P & -T_3 & \star \\ -N_2LKA_0 & -N_2LK(B + RE_c + BLF_c) & -He\{N_2(\mathbf{1} + LF_c)\} \end{bmatrix} \quad (17)$$

The matrix M_1 of relation (12) is directly obtained by pre- and post-multiplying M_2 by $\begin{bmatrix} W & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_2 \end{bmatrix}$, where $W = P^{-1}$ and $S_2 = N_2^{-1}$.

The satisfaction of relations (12) and (13) implies both $\mathcal{L}_0 < 0$ and $\rho'LT_3L\rho - \tau \leq 0$, and then $\mathcal{L} < 0$, for all $(x, \Psi, \dot{\Psi}) \neq \mathbf{0}$, and for any $x \in \mathbb{R}^n \setminus \mathcal{S}_0$.

¹The positive definiteness constraint of N_1 of Lemma 1 in [18] can be relaxed in a positive semi-definiteness constraint without loss of generality. It is the reason why in the current paper we fix $N_1 = 0$.

Therefore, the satisfaction of relation (12) ensures that there exists $\varepsilon > 0$, such that $\mathcal{L} \leq -\varepsilon \| [x' \ \Psi' \ \dot{\Psi}']' \|^2 \leq -\varepsilon x'x$. Hence, since by definition one gets $\dot{V}(x) \leq \dot{V}(x) - \tau(1 - x'Px) \leq \mathcal{L}$, one can also verify

$$\dot{V}(x) \leq -\varepsilon x'x, \quad \forall x \text{ such that } x'Px \geq 1 \quad (18)$$

Furthermore, from (18), there exists a time $T \geq t_0 + (x(t_0)'Px(t_0) - 1)\lambda_{max}(P)/\varepsilon$ such that $x(t) \in \mathcal{S}_0, \forall t \geq T$, and therefore, \mathcal{S}_0 is an invariant set for the trajectories of system (11). Hence, in accordance with [8], it concludes the proof of Theorem 3.1. ■

Let us comment the feasibility of conditions of Theorem 3.1.

Proposition 3.1 *Theorem 3.1 enjoys the following properties:*

1. Given $E_c = \mathbf{0}$ and $F_c = \mathbf{0}$, condition (12) is feasible if and only if matrix A_0 is Hurwitz.
2. There always exist E_c and F_c non null such that condition (12) holds.

Proof. Let us first define matrix M_0 corresponding to M_1 in the case $E_c = \mathbf{0}$ and $F_c = \mathbf{0}$:

$$M_0 = \begin{bmatrix} He\{A_0W\} + \tau W & \star & \star \\ B' & -T_3 & \star \\ -LKA_0W & -LKB & -2S_2 \end{bmatrix} \quad (19)$$

Then, in this case, condition (12) corresponds to $M_0 < \mathbf{0}$. Recall that matrix A_0 is Hurwitz by construction (see Remark 2.1). Then one can show that the inequality $M_0 < \mathbf{0}$ is always feasible by using

- 1) the fact that the stability property of A_0 implies the existence of a matrix $W = W' > \mathbf{0}$ such that $He\{A_0W\} + \tau W < \mathbf{0}$;
- 2) the Schur complement to show that all the minors of matrix $-M_0$ are positive for large enough values of T_3 and S_2 .

Let us now consider the case where E_c and F_c are non null. In this case, condition (12) can be written as

$$M_1 = M_0 + He\left\{ \begin{bmatrix} R \\ \mathbf{0} \\ -LKR \end{bmatrix} E_c \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix} \right\} + He\left\{ \begin{bmatrix} BL & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ -LKBL & -L \end{bmatrix} (\mathbf{1} \otimes F_c) \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_2 \end{bmatrix} \right\} \quad (20)$$

Hence, if $M_0 < \mathbf{0}$ is feasible, it is always possible to find some value for E_c, F_c such that relation (20) is feasible, or equivalently such that $M_1 < \mathbf{0}$ is feasible ■

Remark 3.1 *Conditions of Theorem 2 in [18] can be deduced from that one of Theorem 3.1. Indeed, as pointed out previously, the part due to N_1 appearing in [18] is not useful to solve Problem 2.1 and may be removed. Hence, by setting $E_c = \mathbf{0}$ and $F_c = \mathbf{0}$ in the conditions of Theorem 3.1 one retrieves the conditions of Theorem 2 in [18] with $N_1 = \mathbf{0}$ and $P = W^{-1}$.*

3.2 Computational issues

Theorem 3.1 may be used to solve both the NSAW design Problem 2.1 and the analysis problem (E_c and F_c given). It may then be noted that there are two sources of nonlinearities in the conditions of Theorem 3.1. The first one is issued from the use of the S-procedure, with the product τW , and is the unique source of nonlinearity in the stability analysis problem and in the NSAW design problem of E_c only ($F_c = \mathbf{0}$ or given). This nonlinearity is easily managed as τ is a simple parameter without much influence on the solution. It may be selected arbitrarily by try-and-error until a feasible condition is found, or thanks to a grid search, or encapsulated in an optimization problem with the Matlab function *fminsearch*. On the other hand, when F_c is a decision variable, another nonlinearity occurs due to the product $F_c S_2$. An iterative procedure may then be used, where S_2 and F_c are alternatively a decision variable, the other one being fixed to the solution to the previous step. The initialization of the iterative process may be done with S_2 solution to the analysis problem. Finally optimization problems may be solved to evaluate the smallest set \mathcal{S}_0 , typically described by its volume, proportional to $\sqrt{\det(W)}$ [3], and related to the trace of matrix W .

The following algorithm can then be considered for the analysis problem or to design the gain E_c (F_c given, typically equal to $\mathbf{0}$) such that Theorem 3.1 holds.

Algorithm 3.1

$$\min_{\tau} \min_{W, S_2, T_3, E_c} \text{trace}(W) \quad (21)$$

subject to conditions (12), (13)

Similarly, the following algorithm can be considered to design the gain F_c (and eventually E_c) such that Theorem 3.1 holds.

Algorithm 3.2

- *Step 1. Initialization. Given $E_c = \mathbf{0}$ and $F_c = \mathbf{0}$, solve the optimization problem (21).*
- *Step 2. Design. Keep the value of S_2 and solve the following*

$$\min_{\tau} \min_{W, T_3, E_c, F_c} \text{trace}(W) \quad (22)$$

subject to conditions (12), (13)

- *Step 3. Analysis. Keep the values of E_c and F_c and solve (21).*
- *Step 4. Iterate between Step 2 and Step 3 until $\text{trace}(W)$ does not decrease more than a given tolerance $\epsilon > 0$.*

In both algorithms, the Matlab function *fminsearch* is used to find the minimal value of τ allowing to obtain a solution to optimization problems (21) and (22).

4 Extensions

4.1 Existence conditions

Note that Theorem 3.1 implies that system (7) is asymptotically stable with respect to the set $\mathcal{S}_0 \times I_\Phi$, as soon as Φ is initialized in I_Φ . Then, let us state an existence condition regarding the robustness of the solution to Problem 2.1. To do that, denote respectively the (set-valued) right-hand side of the first line of (11), the closed unit ball and the closed convex hull of a set by F , \mathbb{B} and \overline{co} .

Proposition 4.1 *Under the hypotheses of Theorem 3.1, there exists a continuous function $\delta : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ which is positive outside \mathcal{S}_0 such that the differential inclusion*

$$\dot{x} \in \overline{co}F(x + \delta(x)\mathbb{B}) + \delta(x)\mathbb{B} \quad (23)$$

is asymptotically stable with respect to the set $\mathcal{S}_0 \times I_\Phi$, for any admissible initial condition, i.e. Φ is initialized in I_Φ . It means that it exists a class \mathcal{KL} -function β such that, all solutions to (23), with Φ initialized in (8), satisfy

$$\|x(t)\|_{\mathcal{S}_0} \leq \beta(\|x(0)\|_{\mathcal{S}_0}, t), \forall t \geq 0.$$

Proof. To prove this result, we first note that we may include all the dynamics in the two first lines of (7) together with (3) in terms of a differential inclusion of the state x :

$$\dot{x} \in F(x) \quad (24)$$

where F is a nonempty, compact, convex and locally Lipschitz set-valued function, by considering the Filippov regularization of (7). Moreover the attractor \mathcal{S}_0 is compact and, with Theorem 3.1, the system (24) is globally asymptotically stable with the attractor \mathcal{S}_0 . Therefore, according to [20, Theorem 1], system (7) is robustly globally asymptotically stable with the attractor \mathcal{S}_0 . Moreover the distance of any point x to the set \mathcal{S}_0 is $\|x\|_{\mathcal{S}_0}$. This implies, with [20, Definition 8], the existence of a δ function and of a class \mathcal{KL} -function β as considered in the statement of Proposition 4.1. ■

Remark 4.1 *The notion of global ultimate boundedness with respect to a compact set \mathcal{S}_0 (see, for example, [8]) could be used, but the use of such a notion does not allow to ensure Lyapunov stability of the considered compact set \mathcal{S}_0 . That means that it may exist some trajectories starting close to \mathcal{S}_0 , which may not converge to it. However, the property of global asymptotic stability of a compact set \mathcal{S}_0 , as guaranteed in Theorem 3.1, allows to inherit robustness property with respect to small perturbations, as developed in Proposition 4.1. Similar facts appear in the context of quantized systems as discussed in [6].*

4.2 Uncertainty on the backlash

In this section, we address the case where the backlash operator is uncertain, in order to study the impact of the uncertain backlash operator on the asymptotic stability of the closed loop as done for example in [2] for the Coulomb friction. To be more specific, the uncertain

parameter of backlash operator is ρ , that is $\rho = \rho_N + \Delta\rho$, where ρ_N is the nominal part and $\Delta\rho$ the uncertain part with:

$$-\mu \preceq \Delta\rho \preceq \mu \quad (25)$$

for some constant vector μ in \mathbb{R}^m with positives entries.

Remark 4.2 *Taking into account explicitly the backlash uncertainty makes sense when associated to an NSAW strategy. Actually, in the case without NSAW, the smallest set \mathcal{S}_0 in which the trajectories of the closed-loop system are uniformly ultimately bounded is forced by the worst case $\rho = \rho_N + \mu$. On the other hand, one cannot expect to access the true value of the backlash in the NSAW scheme, and it has to be built by considering the a priori nominal value of the uncertain backlash dead-zone ρ .*

Then, in the control scheme (4), the NSAW signals are modified as follows:

$$\begin{aligned} \theta_1 &= E_c(\Phi_N[y_c] - Ly_c) \\ \theta_2 &= F_c(\Phi_N[y_c] - Ly_c) \end{aligned} \quad (26)$$

where Φ_N (which corresponds to ρ_N) represents the nominal backlash. Then, the closed-loop system is modified as follows:

$$\begin{aligned} \dot{x} &= A_0x + (B + RE_c + BLF_c)\Psi_N + B(\Psi - \Psi_N) \\ y_c &= Kx + F_c\Psi_N \\ z &= Cx \end{aligned} \quad (27)$$

with $\Psi_N = \Phi_N - Ly_c$ (associated to ρ_N), $\Psi = \Phi - Ly_c$ (associated to ρ). Note also that:

$$-2L\rho - L\mu \leq \Psi - \Psi_N \leq 2L\rho + L\mu.$$

Problem 2.1 is unchanged, excepted that it now concerns system (27) and a solution to this uncertain problem is given by the following theorem.

Theorem 4.1 *Suppose there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, three diagonal positive definite matrices $S_2 \in \mathbb{R}^{m \times m}$, $T_3 \in \mathbb{R}^{m \times m}$, $T_4 \in \mathbb{R}^{m \times m}$, two matrices $E_c \in \mathbb{R}^{n_c \times m}$, $F_c \in \mathbb{R}^{m \times m}$ and a positive scalar τ satisfying*

$$\begin{bmatrix} M_1 & \star \\ B' [\mathbf{1} \ \mathbf{0} \ \mathbf{0}] & -T_4 \end{bmatrix} < \mathbf{0} \quad (28)$$

$$\rho' LT_3 L\rho - \tau + (2\rho' + \mu') LT_4 L(2\rho + \mu) \leq 0 \quad (29)$$

with M_1 defined in (14). Then, for any admissible initial conditions $(x(0), \Psi(0))$, the closed-loop system (27) is well posed and globally asymptotically stable with respect to the set \mathcal{S}_0 defined in (15) and E_c , F_c and \mathcal{S}_0 are solution to the uncertain problem 2.1.

Algorithms 3.1 and 3.2 are simply modified by updating the conditions with (28) and (29) instead of (12) and (13).

4.3 External stability

Let us consider that plant (1) is affected by an additive disturbance as follows:

$$\begin{aligned}\dot{x}_p &= A_p x_p + B_p u_p + B_{pw} w \\ y_p &= C_p x_p \\ z &= C_{pz} x_p\end{aligned}\tag{30}$$

where $w \in \mathbb{R}^q$ is the additive perturbation and $z \in \mathbb{R}^{n_z}$ is the performance output to be attenuated. The exogenous signal w is supposed to be limited in energy:

$$\int_0^\infty w(t)' w(t) dt \leq \delta^{-1}\tag{31}$$

with a positive finite scalar δ .

In that case, the plant (30) in closed loop with the controller (4) reads:

$$\begin{aligned}\dot{x} &= A_0 x + (B + RE_c + BLF_c)\Psi + B_w w \\ y_c &= Kx + F_c \Psi \\ z &= Cx\end{aligned}\tag{32}$$

with

$$B_w = \begin{bmatrix} B_{pw} \\ \mathbf{0} \end{bmatrix}; C = \begin{bmatrix} C_{pz} & \mathbf{0} \end{bmatrix}$$

Problem 2.1 is then modified as follows.

Problem 4.1 *Characterize the regions \mathcal{S}_0 and \mathcal{S}_∞ of the state space, containing the origin, and design the NSAW gains E_c and F_c such that:*

1. *When $w = \mathbf{0}$, the system (32) is globally asymptotically stable with respect to \mathcal{S}_0 , when initialized as in (8). In other words, \mathcal{S}_0 is a global asymptotic attractor for the closed-loop dynamics (32), for any initial value of Φ in I_Φ ;*
2. *When $w \neq \mathbf{0}$, the closed-loop trajectories of system (32) remains bounded in the set \mathcal{S}_∞ .*
3. *When $w \neq \mathbf{0}$, characterize the \mathcal{L}_2 -gain of the map from w to z . Then in this case we are interested to prove that the map from w to z is finite \mathcal{L}_2 -gain stable with*

$$\int_0^T z(t)' z(t) dt \leq \gamma^2 \int_0^T w(t)' w(t) dt + g(x(0)), \quad \forall T \geq 0\tag{33}$$

where $g(x(0))$ is a bias related to the initial condition $x(0) = [x_p(0)' \quad x_c(0)']' \in \mathbb{R}^n$.

Theorem 3.1 is modified to deal with Problem 4.1 as follows.

Theorem 4.2 Suppose there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, two diagonal positive definite matrices $S_2 \in \mathbb{R}^{m \times m}$, $T_3 \in \mathbb{R}^{m \times m}$, two matrices $E_c \in \mathbb{R}^{n_c \times m}$, $F_c \in \mathbb{R}^{m \times m}$ and two positive scalars τ and γ satisfying (13) and the following condition

$$\begin{bmatrix} M_1 & \star \\ M_3 & -\gamma \mathbf{1} \end{bmatrix} < \mathbf{0} \quad (34)$$

with M_1 defined in Theorem 3.1, and

$$M_3 = \begin{bmatrix} B'_w & \mathbf{0} & -B'_w K' L \\ C W & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (35)$$

Then,

1. When $w = \mathbf{0}$, for any admissible initial conditions $(x(0), \Psi(0))$, the closed-loop system (32) is globally asymptotically stable with respect to the set \mathcal{S}_0 , where the set \mathcal{S}_0 is defined in (15).

2. When $w \neq \mathbf{0}$,

(a) The closed-loop trajectories (32) remain bounded in the set \mathcal{S}_∞ defined as

$$\mathcal{S}_\infty = \{x \in \mathbb{R}^n; x' P x \leq \gamma \delta^{-1} + x(0)' P x(0)\} \quad (36)$$

for any admissible initial conditions $(x(0), \Psi(0))$ such that $x(0) \in \mathbb{R}^n \setminus \mathcal{S}_0$ (i.e., $x(0)' P x(0) \geq 1$).

(b) The map from w to z is finite \mathcal{L}_2 -gain stable with

$$\int_0^T z(t)' z(t) dt \leq \gamma^2 \int_0^T w(t)' w(t) + \gamma x(0)' P x(0), \quad \forall T \geq 0 \quad (37)$$

with $P = W^{-1}$.

Proof. The proof of item 1 of Theorem 4.2 is directly obtained from the proof of Theorem 3.1. Indeed, in presence of disturbance we want to verify that there exists a class \mathcal{K} function α such that $\dot{V}(x) + \frac{1}{\gamma} z' z - \gamma w' w \leq -\alpha(V(x))$, for all x such that $x' P x \geq 1$, and for all nonlinearities Ψ satisfying Lemma 1 in [18]. Mimicking the previous arguments, by using the S-procedure, it is sufficient to check that $\mathcal{L}_w < 0$, where

$$\begin{aligned} \mathcal{L}_w &= \dot{V}(x) - \tau(1 - x' P x) \\ &\quad - \Psi' T_3 \Psi + \rho' L T_3 L \rho \\ &\quad - 2(\dot{\Psi} + L \dot{y}_c)' N_2 \dot{\Psi} + \frac{1}{\gamma} z' z - \gamma w' w \end{aligned} \quad (38)$$

with τ a positive scalar and T_3 a positive diagonal matrix. Then, it follows that $\mathcal{L}_w = \mathcal{L}_{w0} + \rho' L T_3 L \rho - \tau$ with

$$\mathcal{L}_{w0} = \begin{bmatrix} x \\ \Psi \\ \dot{\Psi} \\ w \end{bmatrix}' \left[\begin{array}{ccc|c} & & & \star \\ \hline [B'_w P & \mathbf{0} & -B'_w K' L N_2] & -\gamma \mathbf{1} \end{array} \right] \begin{bmatrix} x \\ \Psi \\ \dot{\Psi} \\ w \end{bmatrix}$$

By pre- and post-multiplying the matrix in the inequality above by $\begin{bmatrix} W & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 \end{bmatrix}$, with

$W = P^{-1}$ and $S_2 = N_2^{-1}$, and by using the Schur complement, the satisfaction of relations (34) and (13) implies both $\mathcal{L}_{w0} < 0$ and $\rho'LT_3L\rho - \tau \leq 0$, and then $\mathcal{L}_w < 0$, for all $(x, \Psi, \dot{\Psi}, w) \neq \mathbf{0}$. By integrating $\mathcal{L}_{w0} < 0$ between 0 and T , one gets:

$$V(x(T)) - V(x(0)) + \frac{1}{\gamma} \int_0^T z(t)'z(t)dt - \gamma \int_0^T w(t)'w(t) < 0$$

or still

$$V(x(T)) \leq V(x(0)) + \frac{1}{\gamma} \int_0^T z(t)'z(t)dt \leq \gamma \int_0^T w(t)'w(t) + V(x(0))$$

Therefore one obtains relation (37) and the definition of \mathcal{S}_∞ given in (36). That concludes the proof of item 2 of Theorem 4.2. ■

Algorithms 3.1 and 3.2 are updated in this case with the new condition (34) in plus of (13), but also considering for the optimization criterion $\min \gamma$ or a mixed criterion $\min \gamma + \text{trace}(W)$.

Remark 4.3 *The presence of an additive disturbance has no direct impact on the set \mathcal{S}_0 in which converges the trajectory once the disturbance vanished. Then, during the design step, a mixed criterion $\min \gamma + \text{trace}(W)$ consisting in minimizing γ without neglecting the long term convergence to \mathcal{S}_0 , allows a trade-off between disturbance rejection and size of the attractor. At the inverse, in the analysis step, it is recommended to separate both optimization problems.*

5 Illustrative examples

Algorithms 3.1 and 3.2 (and their extensions to the uncertain and/or perturbed cases) have been implemented in the Matlab environment, using Yalmip [9] and Mosek [1].

5.1 Academic example

Let us consider to illustrate the approach a small-size example described by $(A_p, B_p, C_p, D_p) = (0.1, 1, 1, 0)$, with the stabilizing PI control $(A_c, B_c, C_c, D_c) = (0, -0.2, 1, -2)$ and a backlash element given by $(\rho, L) = (1.5, 1)$. Let us first compare the global asymptotic attractor \mathcal{S}_0 obtained without and with NSAW. Algorithm 3.1 is used both for the analysis of the system without NSAW, and to design an anti-windup gain E_c , whose the solution is $E_c = 0.11$. The results are illustrated on Figure 2. The ellipsoidal sets \mathcal{S}_0 and trajectories initiated from several initial conditions in the case without backlash (black dotted line), with backlash alone (dashed red line) and with backlash and NSAW (solid blue line) illustrate the positive effect of the NSAW. The simulations without NSAW also illustrate that the system may converge to a fixed point or to a limit cycle around the origin.

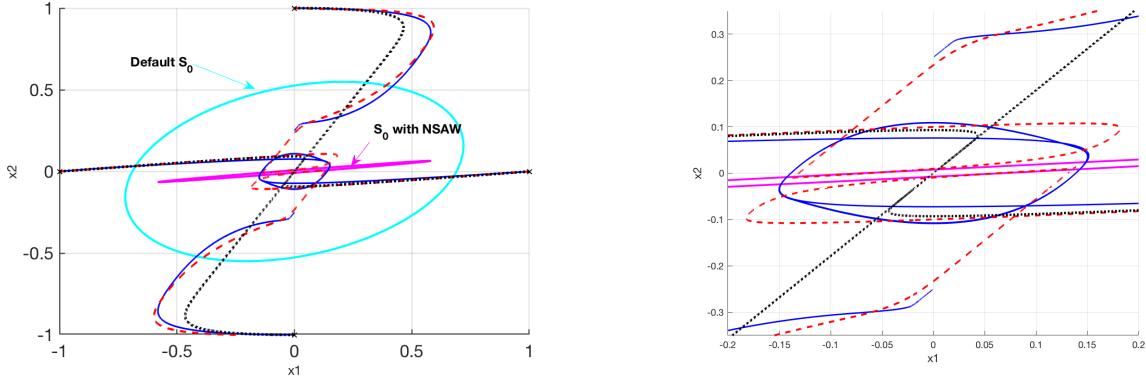


Figure 2: Academic example - Left: Plot of the ellipsoidal set \mathcal{S}_0 with (in magenta) and without (in cyan) NSAW. State-space evolution of the system initiated in $[1 \ 0]'$, $[-1 \ 0]'$, $[0 \ 1]'$ and $[0 \ -1]'$ without backlash (dotted black line), with backlash alone (solid blue line) and with backlash and NSAW (dashed red line). Right: zoom of the left figure.

5.2 Four-dimensional unstable F-8 aircraft

Let us now consider an unstable MIMO plant model with four states and two inputs. It corresponds to the longitudinal dynamics of an F-8 aircraft, slightly modified in [21] to make it unstable. This model has been frequently used in the past to study the influence of saturating inputs. In this paper, we do not consider saturations but concentrate on the presence of backlash phenomenon in the input. Let us then consider the system defined by the following matrices

$$A_p = \begin{bmatrix} -0.8 & -0.006 & -12 & 0 \\ 0 & -0.014 & -16.64 & -32.2 \\ 1 & -0.0001 & -1.5 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}; B_{pw} = \begin{bmatrix} -19 & -3 \\ -0.66 & -0.5 \\ -0.16 & -0.5 \\ 0 & 0 \end{bmatrix}; C_p = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The state variables correspond to the pitch rate, the forward velocity, the angle of attack and the pitch angle. The elevator and flaperon angles are the input variables. The output measurements are set as the pitch angle and the flight path angle (see [21] for details). External disturbances are considered as acting at the input of the system, with $B_{pw} = B_{pu}$, mimicking the effect of noise on the actuator signal.

5.2.1 State-feedback control (SF)

This system has already been used in [18] to illustrate the influence of backlash defined as follows

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

interconnecting the system with a state-feedback controller

$$K = \begin{bmatrix} 0.2672 & 0.0059 & -0.8323 & 0.8089 \\ 0.5950 & -0.1534 & 2.1168 & 0.0525 \end{bmatrix}.$$

In [18], the control problem was to compute a new state-feedback controller allowing to minimize the global asymptotic attractor \mathcal{S}_0 for the closed-loop system. In the current paper, the control problem is to compute a NSAW gain F_c allowing to minimize \mathcal{S}_0 , but without modifying the controller gain K . Algorithm 3.2 is used, initialized with $\tau = 0.5$.

- First, Step 1 is performed for the nominal case, with conditions (12)-(13). It gives the nominal value of \mathcal{S}_0 , independently of the presence of additive disturbance (See Remark 4.3). The volume of the ellipsoid obtained without NSAW is $v(W_0) = \sqrt{\det(W_0)} = 19.8668$.
- Secondly, a NSAW F_c is computed in Step 2 for the perturbed system with $\delta = 20$, considering conditions (13) and (34) in the optimization problem, with the cost function $\gamma + \text{trace}(W)$. One then obtains for the NSAW gain:

$$F_c = \begin{bmatrix} -0.9798 & -0.2004 \\ 0.1946 & -0.9794 \end{bmatrix}$$

Solutions with and without NSAW are compared in two directions.

- Ellipsoidal sets are compared in both cases. The analysis of the system with NSAW is performed for the nominal case ($B_{pw} = 0$). It gives a volume $v(W_1) = \sqrt{\det(W_1)} = 0.0852$ to be compared to $v(W_0) = \sqrt{\det(W_0)} = 19.8668$. Note that it corresponds more or less to the volume obtained in [18] when building a new state-feedback gain.
- \mathcal{L}_2 gains may be also compared in both cases, solving the analysis problem with cost function γ . One obtains:

$$\begin{aligned} \text{--without NSAW:} & \quad \gamma_0 = 40.8 \\ \text{--with NSAW:} & \quad \gamma_1 = 27.1 \end{aligned}$$

Results are also illustrated in Figures 3 to 6. The time evolution of the angle of attack (x_3) and the pitch angle (x_4) are plotted in Figures 3 and 4, respectively, for the system without backlash (dotted black line), with backlash alone (solid blue line) and with backlash and NSAW (dashed red line), initiated from $x_0 = [0 \ 0 \ 1 \ 0]'$ (angle of attack equal to 1).

The backlash characteristics $\Phi[y_c]$ of the closed-loop systems without (left side) and with NSAW (right side) are plotted in Figures 5 and 6 for the elevator and flaperon angles, respectively.

5.2.2 Dynamic output-feedback control (DOF)

Let us now consider the same F-8 aircraft but controlled by a dynamic output feedback, issued from [16]:

$$A_c = \begin{bmatrix} -4.2676 & 0.0362 & -11.7964 & -31.7599 \\ -1.7022 & -0.0182 & 2703.8 & 470.5213 \\ -0.9265 & -0.0066 & -7.0109 & 6.2734 \\ 1 & 0 & 0.0801 & -3.2993 \end{bmatrix}; \quad B_c = \begin{bmatrix} 0.3711 & 1.9263 \\ -3217.1 & 2717.5 \\ 7.6163 & -9.1867 \\ 3.2192 & 0.0801 \end{bmatrix}$$

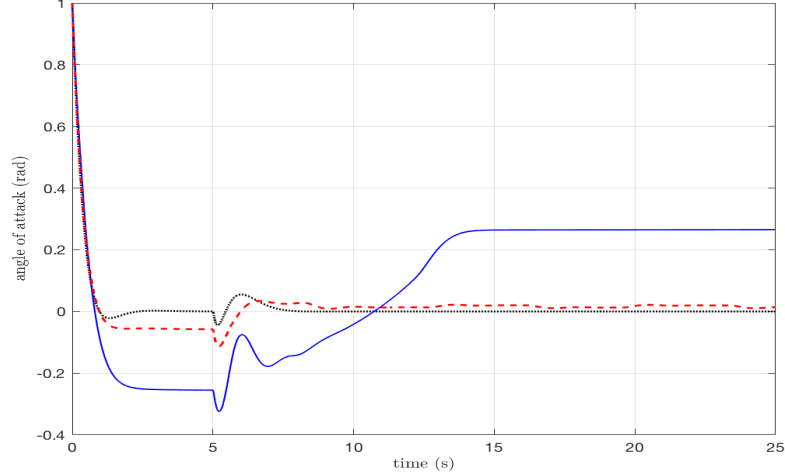


Figure 3: F-8 example (SF) - Time evolution of the angle of attack (x_3). System without backlash (dotted black line), with backlash alone (solid blue line) and with backlash and NSAW (dashed red line).

$$C_c = \begin{bmatrix} -0.4485 & -0.0045 & 1.3181 & 3.1974 \\ 3.9966 & 0.0144 & -7.7734 & -10.4292 \end{bmatrix}; D_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Consider also that the exact dead-zone ρ of the backlash is unknown, with an uncertainty of $\pm 10\%$ around the nominal $\rho_N = [0.15 \ 0.15]'$ and that the system is prone to disturbance limited in energy with bound $\delta = 20$. A NSAW gain E_c may be computed by combining the conditions (28), (29) and (34). One obtains:

$$E_c = \begin{bmatrix} -18.6757 & -2.9305 \\ -421.7373 & -119.8615 \\ 1.7229 & -0.0357 \\ 2.0566 & 0.7154 \end{bmatrix}$$

The time evolution of the angle of attack (x_3) for the system initiated with $x_3(0) = 1$ is plotted in Figure 7 for the system without backlash (dotted black line), with backlash alone (blue line) and with backlash and NSAW (red line). Simulations with backlash are provided with $\rho = \rho_N + \mu$ (solid line) and $\rho = \rho_N - \mu$ (dashed line). Once again, one can check that the NSAW significantly reduces the effect of the backlash on the response of the closed-loop system.

The influence of the backlash uncertainty is illustrated in Figure 8, relative to the time evolution of the angle of attack. It is compared in this figure the evolution of the angle of attack, in presence of a backlash with dead-zone $\rho = \rho_N + \mu$, with a NSAW using either the nominal dead-zone ρ_N (dashed red line) or the real dead-zone ρ (dashdot green line). One can check that even with an approximate knowledge of the backlash dead-zone, one can expect to reduce its effect on the system response thanks to a non-standard anti-windup strategy.

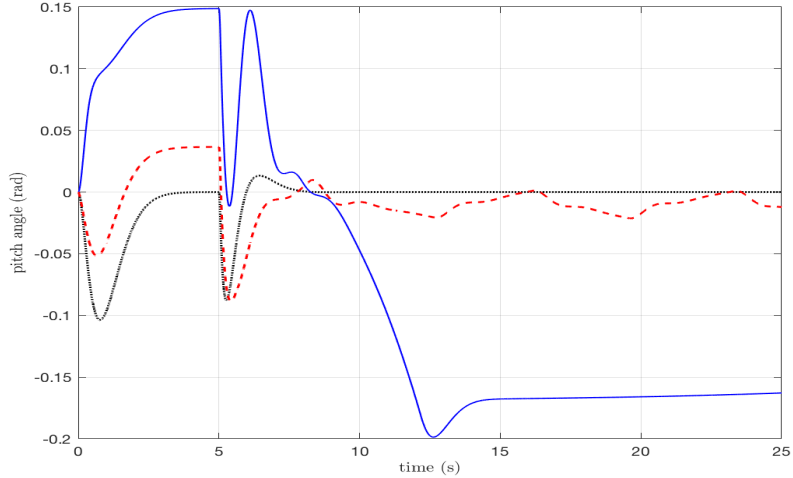


Figure 4: F-8 example (SF) - Time evolution of the pitch angle (x_4). System without backlash (dotted black line), with backlash alone (solid blue line) and with backlash and NSAW (dashed red line).

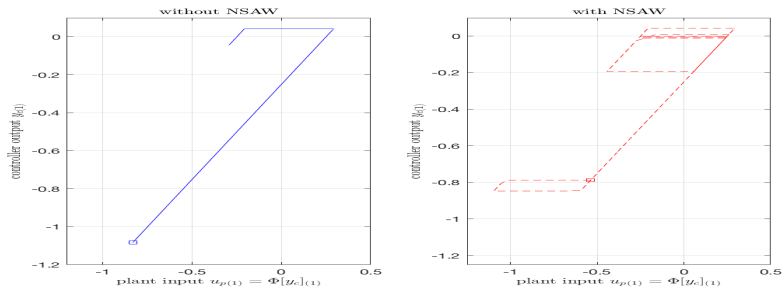


Figure 5: F-8 example (SF) - Backlash characteristics $\Phi[y_c]$ for the elevator angle (u_1). Left: system without NSAW. Right: system with NSAW.

6 Conclusion

Nonlinear control systems, such as those modeled with a backlash operator in the input, have been studied in this paper. Under the assumption that the linear closed-loop system is asymptotically stable, it was shown that some numerically tractable conditions hold so that it is possible to estimate the attractor of the backlash control system. Moreover conditions allow for the design of anti-windup loops to improve some performance, such as sensitivity with respect to backlash uncertainty, or \mathcal{L}_2 -gain with respect to external disturbances. Both academic and aeronautical examples illustrate the obtained results. The use of a non-standard anti-windup loop showed its benefit in a similar way as in the presence of saturation.

This work lets some research lines open. In particular, it would be relevant to study the effect of the backlash when this nonlinearity affects the output. For such control problems,

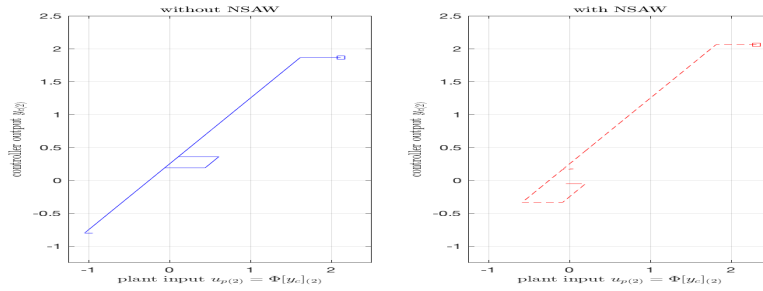


Figure 6: F-8 example (SF) - Backlash characteristics $\Phi[y_c]$ for the flaperon angle (u_2). Left: system without NSAW. Right: system with NSAW.

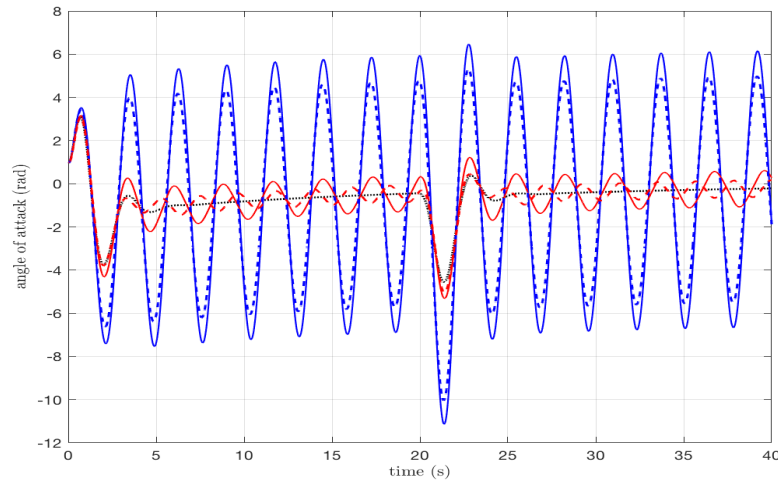


Figure 7: F-8 example (DOF) - Time evolution of the angle of attack (x_3). System without backlash (dotted black line), with backlash alone (solid blue line) and with backlash and NSAW (dashed red line).

it may be necessary to exploit an observer of the nonlinearity map in order to be able to build the NSAW loop. Moreover it could be interesting to see the effect of the backlash on infinite dimensional systems, maybe in a similar way as in [15] for the wave equation with input saturation. Also some works are under progress on a complementary subject, namely when both saturation and backlash nonlinearities are active in the closed loop.

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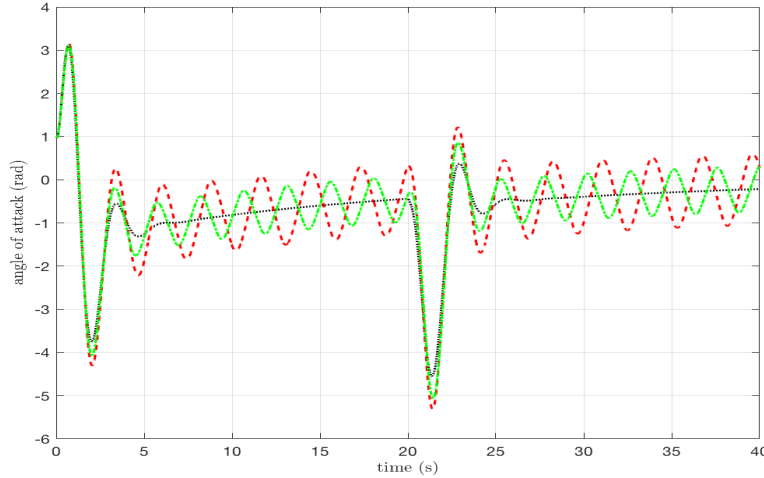


Figure 8: F-8 example (DOF) - Time evolution of the angle of attack (x_3). NSAW using the nominal dead-zone ρ_N (dashed red line) and the real dead-zone $\rho_N + \mu$ (dashdot green line).

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