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FLAT TRACE STATISTICS OF THE TRANSFER OPERATOR OF A RANDOM PARTIALLY EXPANDING MAP

LUC GOSSART

Institut Fourier, 100, rue des maths BP74 38402 Saint-Martin d'Heres France

ABSTRACT. We consider the skew-product of an expanding map E on the circle \mathbb{T} with an almost surely C^k random perturbation $\tau = \tau_0 + \delta\tau$ of a deterministic function τ_0 :

$$F : \begin{cases} \mathbb{T} \times \mathbb{R} & \longrightarrow & \mathbb{T} \times \mathbb{R} \\ (x, y) & \longmapsto & (E(x), y + \tau(x)) \end{cases} .$$

The associated transfer operator $\mathcal{L} : u \in C^k(\mathbb{T} \times \mathbb{R}) \mapsto u \circ F$ can be decomposed with respect to frequency in the y variable into a family of operators acting on functions on the circle:

$$\mathcal{L}_\xi : \begin{cases} C^k(\mathbb{T}) & \longrightarrow & C^k(\mathbb{T}) \\ u & \longmapsto & e^{i\xi\tau} u \circ E \end{cases} .$$

We show that the flat traces of \mathcal{L}_ξ^n behave as normal distributions in the semiclassical limit $n, \xi \rightarrow \infty$ up to the Ehrenfest time $n \leq c_k \log \xi$.

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1. INTRODUCTION

This paper focuses on the distribution of the flat traces of iterates of the transfer operator of a simple example of partially expanding map. It is motivated by the Bohigas-Gianonni-Schmidt [BGS84] conjecture in quantum chaos (see below). In chaotic dynamics, the transfer operator is an object of first importance linked to the asymptotics of the correlations. The collection of poles of its resolvent, called Ruelle-Pollicott spectrum, can be defined as the spectrum of the transfer operator in appropriate Banach spaces (see [Rue76] for analytic expanding maps, [Kit99],[BKL02],[BT07],[BT08],[GL06],[FRS08] for the construction of the spaces for Anosov diffeomorphisms.)

The study of the Ruelle spectrum for Anosov flows is more difficult because of the flow direction that is neither contracting nor expanding. Dolgopyat has shown in particular in [Dol98] the exponential decay of correlations for the geodesic flow on negatively curved surfaces, and Liverani [Liv04] generalized this result to all C^4 contact Anosov flows. His method involved the construction of anisotropic Banach spaces in which the transfer operator is quasicompact, and no longer relies on symbolic dynamics that prevented from using advantage of the smoothness of the flow. Tsujii [Tsu10] constructed appropriate Hilbert spaces for the transfer operator of C^r contact Anosov flows, $r \geq 3$ and gave explicit upper bounds for the essential spectral radii in terms of r and the expansion constants of the flow. Butterley and Liverani [BL07] and later Faure and Sjöstrand [FS11] constructed good spaces for Anosov flows, without the contact hypothesis. Weich and Bonthonneau defined in [BW17] Ruelle spectrum for geodesic flow on negatively curved manifolds with a finite number of cusps. Dyatlov and Guillarmou [DG16] handled the case of open hyperbolic systems. A simple example of Anosov flow is the suspension of an Anosov diffeomorphism, or the suspension semi-flow of an expanding map. Pollicott showed exponential decay of correlations in this setting under a weak condition in [Pol85] and Tsujii constructed suitable spaces for the transfer operator and gave an upper bound on its essential spectral radius in [Tsu08].

In this article we study a closely related discrete time model, the skew product of an expanding map of the circle. It is a particular case of compact group extension [Dol02], which are partially hyperbolic maps, with compact leaves in the neutral direction that are isometric to each other. Dolgopyat showed in [Dol02] that in this case the correlation decrease generically rapidly. In our setting of skew-product of an expanding map of the circle, Faure [Fau11] has shown using semi-classical methods an upper bound on the essential spectral radius of the transfer operator under a condition shown to be generic by Nakano Tsujii and Wittsten [NTW16]. Arnoldi, Faure, and Weich [AFW17] and Faure and Weich [FW17] studied the case of some open partially expanding maps, iteration function schemes, for which they found an explicit bound on the essential spectral radius of the transfer operator in a suitable space, and obtained a Weyl law (upper bound on the number of Ruelle resonances outside the essential spectral radius). Naud [Nau16] studied a model close to the one presented in this paper, in the analytic setting, in which the transfer operator is trace-class, and used the trace formula, in the deterministic and random case to obtain a lower bound on the spectral radius of the transfer operator. In the more general framework of random dynamical systems in which the transfer operator

changes randomly at each iteration, for the skew product of an expanding map of the circle, Nakano and Wittsten [NW15] showed exponential decay of correlations.

Semiclassical analysis describes the link between quantum dynamics and the associated classical dynamics in a symplectic manifold. The transfer operator happens to be a Fourier integral operator and the semi-classical approach has thus shown to be useful. The famous Bohigas-Giannoni-Schmidt [BGS84] conjecture of quantum chaos states that for quantum systems whose associated classical dynamic is chaotic, the spectrum of the Hamiltonian shows the same statistics as that of a random matrix (GUE, GOE or GSE according to the symmetries of the system)(see also [Gut13] and [GVZJ91]). We are interested analogously in investigating the possible links between the Ruelle-Pollicott spectrum and the spectrum of random matrices/operators. At first we try to get informations about the spectrum using a trace formula. More useful results could follow from the use of a global normal form as obtained by Faure-Weich in [FW17].

1.1. Expanding map. Let us consider a smooth orientation preserving expanding map $E : \mathbb{T} \rightarrow \mathbb{T}$ on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, that is, satisfying $E' > 1$, of degree l , and let us call

$$m := \inf E' > 1$$

and

$$M := \sup E'.$$

1.2. Transfer operator. Let us fix a function $\tau \in C^k(\mathbb{T})$ for some $k \geq 1$. We are interested in the partially expanding dynamical system on $\mathbb{T} \times \mathbb{R}$ defined by

$$(1.1) \quad F(x, y) = (E(x), y + \tau(x))$$

We introduce the transfer operator

$$\mathcal{L}_\tau : \begin{cases} C^k(\mathbb{T} \times \mathbb{R}) & \longrightarrow & C^k(\mathbb{T} \times \mathbb{R}) \\ u & \longmapsto & u \circ F \end{cases}.$$

1.3. Reduction of the transfer operator. Due to the particular form of the map F , the Fourier modes in y are invariant under \mathcal{L}_τ : if for some $\xi \in \mathbb{R}$ and some $v \in C^k(\mathbb{T})$,

$$u(x, y) = v(x)e^{i\xi y},$$

then

$$\mathcal{L}_\tau u(x, y) = e^{i\xi\tau(x)}v(E(x))e^{i\xi y}.$$

Given $\xi \geq 0$ and a function τ , let us consequently consider the transfer operator $\mathcal{L}_{\xi, \tau}$ defined on functions $v \in C^k(\mathbb{T})$ by

$$\forall x \in \mathbb{T}, \mathcal{L}_{\xi, \tau} v(x) := e^{i\xi\tau(x)}v(E(x)),$$

1.4. Spectrum and flat trace. In appropriate spaces, the transfer operator has a discrete spectrum outside a small disk, the eigenvalues are called Ruelle resonances. It is in general not trace-class, but one can define its flat trace (see Appendix C for a more precise discussion about Ruelle resonances, flat trace and their relationship).

Lemma 1.1 (Trace formula, [AB67], [G⁺77]). *For any \mathcal{C}^1 function τ on \mathbb{T} , the flat trace of $\mathcal{L}_{\xi, \tau}^n$ is well defined and*

$$(1.2) \quad \mathrm{Tr}^b \mathcal{L}_{\xi, \tau}^n = \sum_{x, E^n(x)=x} \frac{e^{i\xi\tau_x^n}}{(E^n)'(x) - 1},$$

where τ_x^n denotes the Birkhoff sum: For a function $\phi \in C(\mathbb{T})$ and a point $x \in \mathbb{T}$ we define

$$(1.3) \quad \phi_x^n := \sum_{k=0}^{n-1} \phi(E^k(x)).$$

1.5. Gaussian random fields. We define our random functions on the circle by means of their Fourier coefficients. We are only interested in \mathcal{C}^1 functions. We will denote by $\mathcal{N}(0, \sigma^2)$ (respectively $\mathcal{N}_{\mathbb{C}}(0, \sigma^2)$) the real (respectively complex) centered Gaussian law of variance σ^2 , with respective densities

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}x^2} \quad \text{and} \quad \frac{1}{\sigma\pi} e^{-\frac{1}{\sigma^2}|z|^2}.$$

With these conventions, a random variable of law $\mathcal{N}_{\mathbb{C}}(0, \sigma^2)$ has independent real and imaginary parts of law $\mathcal{N}(0, \frac{\sigma^2}{2})$, and the variance of its modulus is consequently σ^2 .

Definition 1.2. We will call centered stationary Gaussian random fields on \mathbb{T} the real random distributions τ whose Fourier coefficients $(c_p(\tau))_{p \geq 1}$ are independent complex centered Gaussian random variables, with variances growing at most polynomially, such that $c_0(\tau)$ is a real centered Gaussian variable independent of the $c_p(\tau)$, $p \geq 1$. The negative coefficients are necessarily given by

$$c_{-p}(\tau) = \overline{c_p(\tau)}.$$

The Gaussian fields are in general defined as distributions if their Fourier coefficients have variances with polynomial growth and the decay of the variances of the coefficients gives sufficient conditions for the regularity of the field.

Lemma 1.3. *If $\mathbb{E}[|c_p(\tau)|^2]$ has a polynomial growth, $\tau = \sum_p c_p(\tau) e^{2i\pi p \cdot}$ defines almost surely a distribution: almost surely*

$$\forall \phi = \sum c_p(\phi) e^{2i\pi p \cdot} \in \mathcal{C}^\infty(\mathbb{T}), \quad \langle \tau, \phi \rangle := \sum_p \overline{c_p(\tau)} c_p(\phi) < \infty.$$

Let $k \in \mathbb{N}$. If for some $\eta > 0$

$$(1.4) \quad \mathbb{E}[|c_p(\tau)|^2] = O\left(\frac{1}{p^{2k+2+\eta}}\right).$$

Then τ is almost surely \mathcal{C}^k .

Proof. See appendix B. □

In what follows we will always assume that (1.4) is satisfied, at least for $k = 1$, so that our random fields are random variables on $\mathcal{C}^1(\mathbb{T})$. This will ensure the existence of flat traces.

1.6. **Result.** If x is a periodic point, let us write its prime period

$$l_x := \min\{k \geq 1, E^k(x) = x\}.$$

Let us define for every $n \in \mathbb{N}$:

$$(1.5) \quad A_n := \left(\sum_{E^n(x)=x} \frac{l_x}{((E^n)'(x) - 1)^2} \right)^{-\frac{1}{2}}$$

Theorem 1.4. Let $k \in \mathbb{N}^*$. Let $\tau_0 \in \mathcal{C}^k(\mathbb{T})$. Let

$$\delta\tau = \sum_{p \in \mathbb{Z}} c_p e^{2i\pi p \cdot}$$

be a centered Gaussian random field, such that $\mathbb{E}[|c_p|^2] = O(p^{-4-\nu})$ for some $\nu > 0$. This way, $\delta\tau$ is a.s. \mathcal{C}^1 . If

$$(1.6) \quad \exists \epsilon > 0, \exists C > 0, \forall p \in \mathbb{Z}^*, \mathbb{E}[|c_p|^2] \geq \frac{C}{p^{2k+2+\epsilon}},$$

then one has the convergence in law of the flat traces

$$(1.7) \quad A_n \text{Tr}^b(\mathcal{L}_{\xi, \tau_0 + \delta\tau}^n) \longrightarrow \mathcal{N}_{\mathbb{C}}(0, 1)$$

as n and ξ go to infinity, under the constraint

$$(1.8) \quad \exists 0 < c < 1, \forall n, \xi, \quad n \leq c \frac{\log \xi}{\log l + (k + \frac{1}{2} + \frac{\epsilon}{2}) \log M}.$$

Note that condition (1.6) can allow τ to be \mathcal{C}^k by Lemma 1.3.

Remark 1.5. The statement implies that the convergence still holds if we multiply $\delta\tau$ by an arbitrarily small number $\eta > 0$. For instance for $\tau_0 = 0$,

$$A_n \text{Tr}^b(\mathcal{L}_{\xi, 0}^n) \longrightarrow \infty$$

at exponential speed, uniformly in ξ , but if $\delta\tau$ is an irregular enough Gaussian field in the sense of (1.6), then for any $\eta > 0$ and $c < 1$ holds

$$A_n \text{Tr}^b(\mathcal{L}_{\xi, \eta \cdot \delta\tau}^n) \longrightarrow \mathcal{N}_{\mathbb{C}}(0, 1)$$

under condition (1.8).

Remark 1.6. Condition (1.8) means that time n is smaller than a constant times the Ehrenfest time $\log \xi$, and this constant decreases with the regularity k of the field $\delta\tau$.

1.7. **Sketch of proof.** The proof is based around the following arguments:

- (1) Note first that the convergence (1.7) is satisfied if all the phases appearing in (1.2) are independent and uniformly distributed.

Remark 1.7. For sake of simplicity, in this sketch of proof, we will state pairwise independence for the phases in (1.2), while in fact we must pack them by orbits, since Birkhoff sums ϕ_x^n are the same on all the orbit, but this changes little to the problem. For instance this simplification would remove the factor l_x in the definition (3.21) of A_n corresponding to this multiplicity.

The convergence can be deduced from the standard proof of the central limit theorem showing pointwise convergence of the characteristic function. However, here, since the periodic points are dense in \mathbb{T} , requiring independence of the values $(\delta\tau(x))_{E^n(x)=x}$ would lead to very bad regularity of the field (it is not hard to see that it would be almost surely nowhere locally bounded).

- (2) We fix a Gaussian field $\delta\tau = \sum c_p e^{2i\pi p \cdot}$ fulfilling the hypothesis of Theorem 1.4 and start by constructing an auxiliary field with the same law and show that it satisfies the convergence (1.7). This is sufficient since the convergence in law only involves the law of the random field.
- (3) For each $j \geq 1$, we construct a smooth random field $\delta\tau_j$, such that for every periodic points $x \neq y$ of period j , $\delta\tau_j(x)$ and $\delta\tau_j(y)$ are independent. If moreover ξ is large enough, the variables $\xi(\delta\tau_n)_x^n$ are Gaussian with large variances, so $\xi(\delta\tau_n)_x^n \bmod 2\pi$ (and therefore the phases $e^{i\xi(\delta\tau_n)_x^n}$) are close to be uniform. Consequently, since by (1.2) $\text{Tr}^b(\mathcal{L}_{\xi, \delta\tau_n}^n)$ only involves points of period n , and thus independent almost uniform phases, the convergence (1.7) should hold for $\text{Tr}^b(\mathcal{L}_{\xi, \delta\tau_n}^n)$ under a certain relation between n and ξ that will be explained in number (8).
- (4) An important point is that if the phases $(e^{i\xi(\delta\tau_n)_x^n})_{\{x \in \mathbb{T}, E^n(x)=x\}}$ are independent and close to be uniform, then adding to $\delta\tau_n$ an independent field will not change this fact, as the following lemma suggests:

Lemma 1.8. *Let X, X' be real independent random variables such that $e^{iX}, e^{iX'}$ are uniform on S^1 . Let Y, Y' be real random variables such that X and X' are independent of both Y and Y' . Then $e^{i(X+Y)}$ and $e^{i(X'+Y')}$ are still independent uniform random variables on S^1 .*

Note that no independence between Y and Y' is needed. See appendix D for the proof.

- (5) Using this analogy, if the fields $\delta\tau_j$ are chosen independent, it should follow that the convergence (1.7) holds for $\text{Tr}^b(\mathcal{L}_{\xi, \sum_{j \geq 1} \delta\tau_j}^n)$ for large ξ .
- (6) The fields $\delta\tau_j$ are almost surely smooth. However, because the distance between periodic points decreases exponentially fast with n according to Lemma A.1, the requirement $\mathbb{E}[\delta\tau_n(x)\delta\tau_n(y)] = 0$ for all $x \neq y$ of period n leads to an exponential growth with respect to n of $\|\delta\tau_n\|_{\mathcal{C}^k}$ renormalized by the standard deviation (independent of the point x):

$$\|\delta\tau_n\|_{\mathcal{C}^k} \approx M^{kn} \sqrt{\mathbb{E}[|\delta\tau_n(x)|^2]}.$$

This can be deduced heuristically from the fact (see Definition 3.1 below) that

$$\mathbb{E}[\delta\tau_n(x)\delta\tau_n(y)] = \sum_p \mathbb{E}[|c_p(\delta\tau_n)|^2] e^{ip(x-y)} =: K_n(x-y)$$

and the incertitude principle: a localisation of K_n at a scale M^{-n} implies non negligible coefficients $\mathbb{E}[|c_p(\delta\tau_n)|^2]$ for p of order M^n . As a consequence we need

$$(1.9) \quad \sqrt{\mathbb{E}[|\delta\tau_n(x)|^2]} \leq \frac{C}{M^{n(k+\frac{1}{2}+\frac{\epsilon}{2})}}$$

to obtain a \mathcal{C}^k function $\sum_{j \geq 1} \delta\tau_j$.

- (7) This condition, together with (1.6) implies that the Fourier coefficients \tilde{c}_p of $\sum_{j \geq 1} \delta\tau_j$ satisfy

$$\mathbb{E}[|\tilde{c}_p|^2] \leq C\mathbb{E}[|c_p|^2].$$

This allows us to define a field $\delta\tau_0$, that we chose independent from the other $\delta\tau_j$, by

$$\mathbb{E}[|c_p(\delta\tau_0)|^2] = C\mathbb{E}[|c_p|^2] - \mathbb{E}[|\tilde{c}_p|^2],$$

so that $\frac{1}{C} \sum_{j \geq 0} \delta\tau_j$ has the same law as $\delta\tau$ and still satisfies the convergence (1.7) for ξ large enough from (4) of this sketch.

- (8) To get an idea of the origin of the relation (1.8) between n and ξ , let us assume that we want all the arguments $\xi(\delta\tau_n)_x$ in $\text{Tr}^b(\mathcal{L}_{\xi, \delta\tau_n}^n)$ to go uniformly to infinity in order to get approximate uniformity of the phases and thus convergence towards a Gaussian law. Note that for any x ,

$$(1.10) \quad \mathbb{P} \left[\frac{|(\delta\tau_n)_x|^n}{\sqrt{\mathbb{E}[|(\delta\tau_n)_x|^2]}} \leq \epsilon \right] \underset{\epsilon \rightarrow 0}{=} O(\epsilon).$$

Let (C_n) be a sequence going to infinity.

(1.10) implies

$$\begin{aligned} \mathbb{P} \left[\bigcap_{E^n(x)=x} \{\xi(\delta\tau_n)_x^n > C_n\} \right] &= 1 - \mathbb{P} [\exists x, E^n(x) = x, \xi(\delta\tau_n)_x^n \leq C_n] \\ &\geq 1 - \sum_{E^n(x)=x} \mathbb{P} [\xi(\delta\tau_n)_x^n \leq C_n] \\ &\stackrel{\text{Lemma A.1}}{=} 1 - (l^n - 1) \mathbb{P} [\xi(\delta\tau_n)_x^n \leq C_n] \\ &\stackrel{(1.10)}{\geq} 1 - Cl^n \frac{C_n}{\xi \sqrt{\mathbb{E}[|(\delta\tau_n)_x|^2]}} \end{aligned}$$

if x denotes any point and $\xi \gg \frac{C_n}{\sqrt{\mathbb{E}[|(\delta\tau_n)_x|^2]}}$. We deduce from (1.9) that

$$\sqrt{\mathbb{E}[|(\delta\tau_n)_x|^2]} \leq \frac{C_n}{M^{n(k+\frac{1}{2}+\frac{\epsilon}{2})}}.$$

Thus

$$\mathbb{P} [\xi(\delta\tau_n)_x^n \rightarrow \infty \text{ uniformly w.r.t. } x \text{ s.t. } E^n(x) = x] \rightarrow 1$$

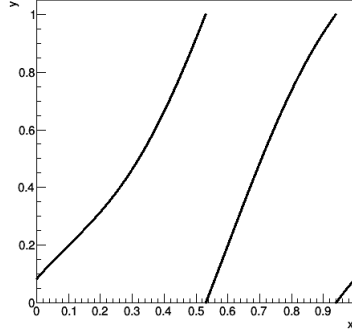
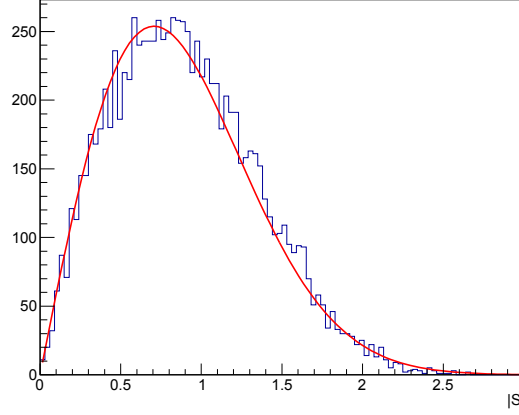
for $\xi \gg l^n M^{n(k+\frac{1}{2}+\frac{\epsilon}{2})}$, which gives (1.8).

2. NUMERICAL EXPERIMENTS

We consider an example with the non linear expanding map

$$(2.1) \quad E(x) = 2x + 0.9/(2\pi) \sin(2\pi(x + 0.4))$$

plotted on Figure 1. In Figure 2, we have the histogram of the modulus $S = \left| A_n \text{Tr}^b \left(\mathcal{L}_{\xi, \tau_0 + \delta\tau}^n \right) \right|$ obtained after a sample of 10^4 random functions $\delta\tau$. We compare the histogram with the function $CS \exp(-S^2)$ in red, i.e. the radial distribution of a Gaussian function, obtained from the prediction of Theorem 1.4. We took $n = 11$, $\xi = 2.10^6$, $\tau_0 = \cos(2\pi x)$. We also observe a good agreement for the (uniform) distribution of the arguments that is not represented here.


 FIGURE 1. Graph of the expanding map $E(x)$ in Eq.(2.1)

 FIGURE 2. In blue, the histogram of $S = \left| A_n \text{Tr}^b \left(\mathcal{L}_{\xi, \tau_0 + \delta\tau}^n \right) \right|$ for $n = 11$, $\xi = 2.10^6$, $\tau_0 = \cos(2\pi x)$ and the sample 10^4 random functions $\delta\tau$. The histogram is well fitted by $CS \exp(-S^2)$ in red, as predicted by Theorem 1.4

3. PROOF OF THEOREM 1.4

A stationary centered Gaussian random field is characterized by its covariance function:

Definition 3.1. Let $\tau = \sum_{p \in \mathbb{Z}} c_p e^{2i\pi p \cdot}$ be a stationary centered Gaussian random field. Let us define its covariance function K by

$$(3.1) \quad K(x) := \sum_p \mathbb{E}[|c_p|^2] e^{2i\pi p x}.$$

For any couple of points $(x, y) \in \mathbb{T}^2$, we have

$$\mathbb{E}[\tau(x)\tau(y)] = K(x - y).$$

Proof of the last statement. Since we assume (1.4) to hold for some $k \geq 1$,

$$\begin{aligned} \mathbb{E}[\tau(x)\tau(y)] &= \sum_{p,q \in \mathbb{Z}} \mathbb{E}[c_p(\tau)c_q(\tau)]e^{2i\pi(px+qy)} \\ &= \sum_{p \in \mathbb{Z}} \left(\mathbb{E}[|c_p|^2]e^{2i\pi p(x-y)} + \mathbb{E}[c_p^2]e^{2i\pi p(x+y)} \right) \end{aligned}$$

from the independence relationships of the Fourier coefficients. Now, if we write $c_p = x_p + iy_p$

$$\mathbb{E}[c_p^2] = \mathbb{E}[x_p^2] - \mathbb{E}[y_p^2] + 2i\mathbb{E}[x_p y_p] = 0.$$

□

3.1. Definition of a Gaussian field satisfying theorem 1.4. Let us fix a random centered Gaussian field $\delta\tau = \sum_{p \in \mathbb{Z}} c_p e^{2i\pi p \cdot}$ satisfying the hypothesis of Theorem 1.4. Let us define the Gaussian fields mentioned in step (3) of the sketch of proof. Let $K_{\text{init}} \in \mathcal{C}_c^\infty(\mathbb{R})$ be a smooth function supported in $[-\frac{1}{3}, \frac{1}{3}]$, with non negative Fourier transform, satisfying ¹

$$(3.2) \quad K_{\text{init}}(0) = 1.$$

Let $k \geq 1$ be the integer involved in Theorem 1.4 giving the regularity of the field. Let $\epsilon > 0$ be the constant appearing in Theorem 1.4 and define for any integer $j \geq 1$

$$(3.3) \quad K_j(x) = \frac{1}{M^{j(2k+1+\epsilon)}} K_{\text{init}}(M^j x).$$

The Fourier transform of K_j is given by

$$(3.4) \quad \widehat{K}_j(\xi) = \frac{1}{M^{j(2k+2+\epsilon)}} \widehat{K}_{\text{init}}\left(\frac{\xi}{M^j}\right) \geq 0$$

The functions K_j , for all $j \geq 1$, are supported in $[-\frac{1}{3}, \frac{1}{3}]$ and can then be seen as functions on the circle \mathbb{T} by trivially periodizing them. Let $c_{p,j}$, for $p \geq 0, j \geq 1$ be independent centered Gaussian random variables of respective variances $\widehat{K}_j(2\pi p)$, and let us write

$$\delta\tau_j = \sum_p c_{p,j} e^{2i\pi p \cdot},$$

where $c_{-p,j} := \overline{c_{p,j}}, p \geq 1$.

Lemma 3.2. $\sum_{j \geq 1} \delta\tau_j$ is a centered Gaussian random field $\sum \tilde{c}_p e^{2i\pi p \cdot}$ and

$$\mathbb{E}[|\tilde{c}_p|^2] = O(\mathbb{E}[|c_p|^2]).$$

Proof. We have seen in Eq.(3.4) that

$$\widehat{K}_j(\xi) = \frac{1}{M^{j(2k+2+\epsilon)}} \widehat{K}_{\text{init}}\left(\frac{\xi}{M^j}\right).$$

Since K_{init} is smooth, there exists a constant $C > 0$ such that

$$\forall \xi \in \mathbb{R}, \widehat{K}_{\text{init}}(\xi) \leq \frac{C}{\langle \xi \rangle^{2k+2+\frac{\epsilon}{2}}},$$

¹To construct such a function, take a non zero even function $g \in \mathcal{C}_c^\infty(\mathbb{R})$. g has a real Fourier transform. Then $g * g \in \mathcal{C}_c^\infty(\mathbb{R})$ and its Fourier transform is $\hat{g}^2 \geq 0$. Moreover $g * g(0) = \int \hat{g}^2 > 0$.

with the usual notation $\langle \xi \rangle = \sqrt{1 + \xi^2} \geq |\xi|$. Thus,

$$\begin{aligned} \mathbb{E} [|c_{p,j}|^2] &= \frac{1}{M^{j(2k+2+\epsilon)}} \widehat{K_{\text{init}}}\left(\frac{2\pi p}{M^j}\right) \\ &\leq \frac{C}{M^{j\frac{\epsilon}{2}}} \frac{1}{|2\pi p|^{2k+2+\frac{\epsilon}{2}}}. \end{aligned}$$

Consequently, since by independence

$$\begin{aligned} \mathbb{E} [|\tilde{c}_p|^2] &= \mathbb{E} \left[\left| \sum_{j \geq 1} c_{p,j} \right|^2 \right] = \sum_{j \geq 1} \mathbb{E} [|c_{p,j}|^2], \\ \mathbb{E} [|\tilde{c}_p|^2] &= O\left(\frac{1}{|2\pi p|^{2k+2+\frac{\epsilon}{2}}}\right) \stackrel{(1.6)}{=} O(\mathbb{E} [|c_p|^2]). \end{aligned}$$

□

Thus, fixing a constant C such that

$$C\mathbb{E} [|c_p|^2] \geq \mathbb{E} [|\tilde{c}_p|^2],$$

we can define a random Gaussian field $\delta\tau_0 = \sum_{p \in \mathbb{Z}} c_{p,0} e^{2i\pi p \cdot}$ with coefficients $c_{p,0}$ independent from the $c_{p,j}$ such that

$$\mathbb{E} [|c_{p,0}|^2] = C\mathbb{E} [|c_p|^2] - \mathbb{E} [|\tilde{c}_p|^2].$$

This way $\frac{1}{C} \sum_{j \geq 0} \delta\tau_j$ and $\delta\tau$ have the same law. By this we mean that their Fourier coefficients have the same laws. By our hypothesis, the convergence of the Fourier series are almost surely normal, thus for any finite subset $\{x_k\}_k$ of \mathbb{T} , $(\frac{1}{C} \sum \delta\tau_j(x_k))_k$ and $(\delta\tau(x_k))_k$ have the same law. Therefore, the laws of $\text{Tr}^b(\mathcal{L}_{\xi, \tau_0 + \delta\tau}^n)$ and $\text{Tr}^b(\mathcal{L}_{\xi, \tau_0 + \frac{1}{C} \sum \delta\tau_j}^n)$ are the same, and the convergence of Theorem 1.4 is equivalent to

$$(3.5) \quad A_n \text{Tr}^b(\mathcal{L}_{\xi, \tau_0 + \sum \delta\tau_j}^n) \xrightarrow{\mathcal{L}} \mathcal{N}_{\mathbb{C}}(0, 1)$$

under condition (1.6). (The constant $\frac{1}{C}$ can be 'absorbed' in ξ up to the replacement of τ_0 by $C\tau_0$ that has no consequence.) In the rest of the paper we will show (3.5) and will write

$$(3.6) \quad \tau := \tau_0 + \sum_{j \geq 0} \delta\tau_j.$$

3.2. New expression for $\text{Tr}^b(\mathcal{L}_{\xi, \tau}^n)$. We will write the set of periodic orbits of (non primitive) period n as

$$(3.7) \quad \text{Per}(n) := \{\{x, E(x), \dots, E^{n-1}(x)\}, E^n(x) = x, x \in \mathbb{T}\},$$

and the set of periodic orbits of primitive period n as

$$(3.8) \quad \mathcal{P}_n := \{\{x, E(x), \dots, E^{n-1}(x)\}, n = \min\{k \in \mathbb{N}^*, E^k(x) = x\}, x \in \mathbb{T}\}.$$

This way, $\text{Per}(n)$ is the disjoint union

$$(3.9) \quad \text{Per}(n) = \coprod_{m|n} \mathcal{P}_m.$$

Let us rewrite the sum $\text{Tr}^b(\mathcal{L}_{\xi,\tau}^n)$, where τ is given by (3.6). We know from (1.2) that

$$\begin{aligned}\text{Tr}^b(\mathcal{L}_{\xi,\tau}^n) &= \sum_{E^n(x)=x} \frac{e^{i\xi\tau_x^n}}{(E^n)'(x) - 1} \\ &= \sum_{E^n(x)=x} \frac{e^{i\xi\tau_x^n}}{e^{J_x^n} - 1},\end{aligned}$$

where $J(x) = \log(E'(x)) > 0$ and J_x^n is the Birkhoff sum as defined in (1.3). If f_x^n stands for the Birkhoff sum f_x^n for any $x \in O$, let us write

$$(3.10) \quad \text{Tr}^b(\mathcal{L}_{\xi,\tau}^n) = \sum_{m|n} m \sum_{O \in \mathcal{P}_m} \frac{e^{i\xi\tau_O^n}}{e^{J_O^n} - 1}.$$

For $O \in \text{Per}(n)$, we can write

$$\tau_O^n = (\delta\tau_n)_O^n + \sum_{j \neq n} (\delta\tau_j)_O^n + (\tau_0)_O^n.$$

Since the covariance function K_n is supported in $[-\frac{1}{3M^n}, \frac{1}{3M^n}]$, we deduce from Lemma A.1 that the values taken by $\delta\tau_n$ at different periodic points of period dividing n , which have law $\mathcal{N}(0, K_n(0))$ are independent random variables.

Thus, for $n \in \mathbb{N}$, $m|n$ and $O \in \mathcal{P}_m$, $(\delta\tau_n)_O^n$ is a centered Gaussian random variable of variance $mK_n(0)$, and $(\delta\tau_n)_O^n = \frac{n}{m}(\delta\tau_n)_O^m$ has variance $(\frac{n}{m})^2 mK_n(0) = \frac{n^2}{m} K_n(0)$.

Definition 3.3. We say that two families of real random variables $(X_O^n)_{\substack{n \geq 1 \\ O \in \text{Per}(n)}}$

and $(Y_O^n)_{\substack{n \geq 1 \\ O \in \text{Per}(n)}}$ satisfy condition (C) if

- (1) for every $m|n$, and $O \in \mathcal{P}_m$, X_O^n has law $\mathcal{N}(0, \frac{n^2}{m} K_n(0))$,
- (2) for every $O' \neq O \in \text{Per}(n)$ and every $O'' \in \text{Per}(n)$, X_O^n is independent of $X_{O'}^n$ and $Y_{O''}^n$.

Writing $X_O^n = (\delta\tau_n)_O^n$ and $Y_O^n = \sum_{j \neq n} (\delta\tau_j)_O^n + (\tau_0)_O^n$, we have obtained

Lemma 3.4. *There exist families of random variables $(X_O^n), (Y_O^n)$ satisfying condition (C) of Definition (3.3) such that for every $n \geq 1$ and $O \in \text{Per}(n)$*

$$(3.11) \quad \tau_O^n = X_O^n + Y_O^n$$

In order to adapt the proof of lemma 1.8, we want to show that for large ξ , the random variables $e^{i\xi(X_O^n + Y_O^n)}$, $O \in \text{Per}(n)$ are close to be independent and uniform on S^1 .

Remark 3.5. We have

$$(3.12) \quad \text{Tr}^b(\mathcal{L}_{\xi,\tau}^n) \stackrel{(3.10),(3.11)}{=} \sum_{m|n} m \sum_{O \in \mathcal{P}_m} \frac{e^{i\xi(X_O^n + Y_O^n)}}{e^{J_O^n} - 1}.$$

Our aim is to approximate the characteristic function of $A_n \text{Tr}^b(\mathcal{L}_{\xi, \tau}^n)$ which is the expectation of

$$(3.13) \quad \exp \left(i A_n \left(\mu \text{Re}(\text{Tr}^b(\mathcal{L}_{\xi, \tau}^n)) + \nu \text{Im}(\text{Tr}^b(\mathcal{L}_{\xi, \tau}^n)) \right) \right) = \prod_{m|n} \prod_{O \in \mathcal{P}_m} \exp \left[i \frac{m A_n}{e^{J_O^n} - 1} (\mu \cos(\xi(X_O^n + Y_O^n)) + \nu \sin(\xi(X_O^n + Y_O^n))) \right]$$

for fixed, $\mu, \nu \in \mathbb{R}$. The right hand side of (3.13) can be written as

$$\prod_{m|n} \left(\prod_{O \in \mathcal{P}_m} f_O \left(e^{i\xi(X_O^n + Y_O^n)} \right) \right)^m$$

for some continuous functions $f_O : S^1 \rightarrow \mathbb{C}$ (depending on μ, ν):

$$f_O(z) = \exp \left[i \frac{A_n}{e^{J_O^n} - 1} (\mu \text{Re}(z) + \nu \text{Im}(z)) \right]$$

In the next Lemma we first consider indicator functions on S^1 for f_O .

Lemma 3.6. *Let $(X_O^n)_{\substack{n \geq 1 \\ O \in \text{Per}(n)}}$ and $(Y_O^n)_{\substack{n \geq 1 \\ O \in \text{Per}(n)}}$ be two families of real random variables satisfying condition (C) of Definition (3.3). Assume that n and ξ satisfy (1.8). Then there is a constant $C > 0$ such that for every $n \in \mathbb{N}$ and every real numbers $(\alpha_O)_{O \in \text{Per}(n)}, (\beta_O)_{O \in \text{Per}(n)}$ such that*

$$\forall O \in \text{Per}(n), \quad 0 < \beta_O - \alpha_O < 2\pi,$$

for every complex numbers $(\lambda_O)_{O \in \text{Per}(n)}$, if $A_O := e^{i[\alpha_O, \beta_O[} \subset S^1 \subset \mathbb{C}$ and $\mathbb{1}_{A_O} : S^1 \rightarrow \mathbb{C}$ is the characteristic function of A_O , we have

$$(3.14) \quad \left| \frac{\mathbb{E} \left[\prod_{m|n} \left(\prod_{O \in \mathcal{P}_m} \lambda_O \mathbb{1}_{A_O} \left(e^{i\xi(X_O^n + Y_O^n)} \right) \right)^m \right] - 1}{\prod_{m|n} \prod_{O \in \mathcal{P}_m} \lambda_O^m \left(\frac{\beta_O - \alpha_O}{2\pi} \right)} \right| \leq \xi^{c-1} n^{-\frac{1}{2}} \underset{(1.8)}{\rightarrow} 0.$$

Remark 3.7. In this expression, we compare the law of the family of random variables $(e^{i\xi(X_O^n + Y_O^n)})_{O \in \text{Per}(n)}$ to the uniform law on the torus of dimension $\#\text{Per}(n)$. The proof of this lemma is given in the next subsection.

3.3. A normal law of large variance on the circle is close to uniform. We will need the following lemma, which evaluates how much the law $\mathcal{N}(0, 1) \bmod \frac{1}{t}$ differs from the uniform law on the circle $\mathbb{R}/(\frac{1}{t}\mathbb{Z})$ for large values of t .

Lemma 3.8. *There exists a constant $C > 0$ such that for every real numbers α, β , such that $0 < \beta - \alpha < 2\pi$ and every real number $t \geq 1$,*

$$\left| \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \mathbb{1}_{\frac{\alpha + 2k\pi}{t} \leq x \leq \frac{\beta + 2k\pi}{t}} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} - \frac{\beta - \alpha}{2\pi} \right| \leq \frac{C}{t} (\beta - \alpha).$$

Proof. By mean value inequality, if $|x - y| \leq 1$, then

$$\left| e^{-\frac{x^2}{2}} - e^{-\frac{y^2}{2}} \right| \leq |x - y| f(y)$$

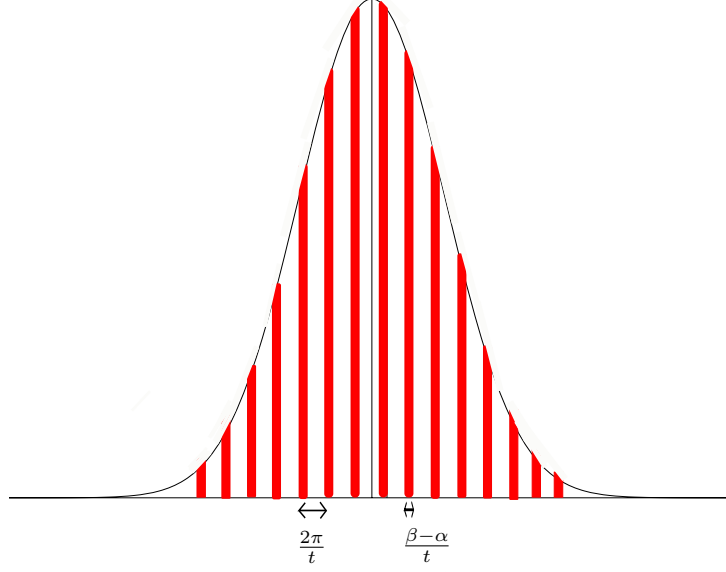


FIGURE 3. As t goes to infinity, the red area converges to $\frac{\beta-\alpha}{2\pi}$ with speed $O(\frac{\beta-\alpha}{t})$.

for the L^1 function

$$f(y) := \sup_{|u-y|\leq 1} |u|e^{-\frac{y^2}{2}}$$

Let us then write for $u \in [\frac{\alpha}{t}, \frac{\beta}{t}]$, $u_k := u + \frac{2k\pi}{t}$ and $I_k := [u_k, u_{k+1}]$. We have just seen that for $t \geq 2\pi$, for all $y \in I_k$,

$$\left| e^{-\frac{u_k^2}{2}} - e^{-\frac{y^2}{2}} \right| \leq \frac{C}{t} f(y).$$

Integrating over $y \in I_k$ of length $\frac{2\pi}{t}$ and summing over $k \in \mathbb{Z}$ yields

$$\left| \frac{2\pi}{t} \sum_{k \in \mathbb{Z}} e^{-\frac{u_k^2}{2}} - \sqrt{2\pi} \right| \leq \frac{C}{t}$$

(The value of the constant C changes at each line, but it depends neither on t , nor on α, β .) Averaging over $u \in [\frac{\alpha}{t}, \frac{\beta}{t}]$ gives

$$\left| \frac{2\pi}{\beta-\alpha} \sum_{k \in \mathbb{Z}} \int_{\frac{\alpha}{t}}^{\frac{\beta}{t}} \exp\left(-\frac{(u-2k\pi)^2}{2}\right) du - \sqrt{2\pi} \right| \leq \frac{C}{t}.$$

Consequently,

$$\left| \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \mathbf{1}_{\frac{\alpha+2k\pi}{t} \leq x \leq \frac{\beta+2k\pi}{t}} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} - \frac{\beta-\alpha}{2\pi} \right| \leq \frac{C}{t}(\beta-\alpha)$$

□

Proof of lemma 3.6. Let us denote by E the expectation

$$E := \mathbb{E} \left[\prod_{m|n} \left(\prod_{O \in \mathcal{P}_m} \lambda_O \mathbb{1}_{A_O} \left(e^{i\xi(X_O^n + Y_O^n)} \right) \right)^m \right].$$

If we write respectively \mathbb{P}_X , \mathbb{P}_Y and $\mathbb{P}_{X,Y}$ the probability laws of the variables $(\xi X_O^n)_{O \in \text{Per}(n)}$, $(\xi Y_O^n)_{O \in \text{Per}(n)}$ and $(\xi X_O^n)_{O \in \text{Per}(n)} \cup (\xi Y_O^n)_{O \in \text{Per}(n)}$ respectively, then condition (C) of Definition (3.3) implies

$$(3.15) \quad d\mathbb{P}_{X,Y}((x_O)_{O \in \text{Per}(n)}, (y_O)_{O \in \text{Per}(n)}) = \prod_{m|n} \prod_{O \in \mathcal{P}_m} e^{-\frac{x_O^2}{2\sigma_{n,\xi}^2}} \frac{dx_O}{\sigma_{n,\xi} \sqrt{2\pi}} \otimes d\mathbb{P}_Y((y_O)_{O \in \text{Per}(n)}).$$

with the variance $\sigma_{n,\xi}^2 := \xi^2 \frac{n^2}{m} K_n(0)$. We have

$$(3.16) \quad E = \int_{\mathbb{R}^{2\#\text{Per}(n)}} \prod_{O \in \text{Per}(n)} \left(\sum_{k \in \mathbb{Z}} \lambda_O^m \mathbb{1}_{\alpha_O + 2k\pi, \beta_O + 2k\pi}((x_O + y_O)) \right) d\mathbb{P}_{X,Y}((x_O)_{O \in \text{Per}(n)}, (y_O)_{O \in \text{Per}(n)}).$$

Thus, writing $u_O = \frac{x_O}{\sigma_{n,\xi}}$ for $O \in \mathcal{P}_m$,

$$(3.17) \quad E = \int_{\mathbb{R}^{\#\text{Per}(n)}} \prod_{m|n} \prod_{O \in \mathcal{P}_m} \left(\int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \lambda_O^m \mathbb{1}_{\frac{\alpha_O - y_O + 2k\pi}{\sigma_{n,\xi}}, \frac{\beta_O - y_O + 2k\pi}{\sigma_{n,\xi}}} \left[(u_O) e^{-\frac{u_O^2}{2}} \frac{du_O}{\sqrt{2\pi}} \right] \right) d\mathbb{P}_Y((y_O)_{O \in \text{Per}(n)}).$$

Let us write for $O \in \text{Per}(n)$

$$I_O = \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \lambda_O^m \mathbb{1}_{\frac{\alpha_O - y_O + 2k\pi}{\sigma_{n,\xi}}, \frac{\beta_O - y_O + 2k\pi}{\sigma_{n,\xi}}} \left[(u_O) e^{-\frac{u_O^2}{2}} \frac{du_O}{\sqrt{2\pi}} \right].$$

Lemma 3.8 yields

$$I_O = \lambda_O^m \frac{\beta_O - \alpha_O}{2\pi} (1 + \epsilon_O),$$

where

$$\begin{aligned} \exists C > 0, |\epsilon_O| &\leq \frac{C}{\sigma_{n,\xi}} \\ &\leq \frac{C}{\xi \sqrt{n K_n(0)}}. \end{aligned}$$

Let us remark that for every finite family $\{x_k\}_k \subset \mathbb{R}$, the expansion of the product and factorization after triangular inequality give

$$\left| \prod_k (1 + x_k) - 1 \right| \leq \prod_k (1 + |x_k|) - 1.$$

Thus,

$$\begin{aligned} \left| \frac{\prod_{m|n} \prod_{O \in \mathcal{P}_m} I_O}{\prod_{m|n} \prod_{O \in \mathcal{P}_m} \lambda_O^m \left(\frac{\beta_O - \alpha_O}{2\pi} \right)} - 1 \right| &= \left| \prod_{m|n} \prod_{O \in \mathcal{P}_m} (1 + \epsilon_O) - 1 \right| \\ &\leq \left(\left(1 + \frac{C}{\xi(nK_n(0))^{1/2}} \right)^{\#\text{Per}(n)} - 1 \right) \end{aligned}$$

From Lemma A.1 we have $\#\text{Per}(n) \leq l^n$.

Using hypothesis (1.8) we can bound the prefactor:

$$\begin{aligned} \left(1 + \frac{C}{\xi(nK_n(0))^{1/2}} \right)^{l^n} - 1 &\stackrel{(3.3), (3.2)}{=} \left(1 + \frac{CM^{n(k+\frac{1}{2}+\frac{\xi}{2})}}{\xi\sqrt{n}} \right)^{l^n} - 1 \\ &\leq \exp\left(l^n \frac{CM^{n(k+\frac{1}{2}+\frac{\xi}{2})}}{\xi\sqrt{n}} \right) - 1 \\ &\leq C' \frac{l^n M^{n(k+\frac{1}{2}+\frac{\xi}{2})}}{\xi\sqrt{n}} \end{aligned}$$

for some $C' > 0$ for n and ξ large enough and satisfying (1.8) since

$$(3.18) \quad l^n \frac{CM^{n(k+\frac{1}{2}+\frac{\xi}{2})}}{\xi\sqrt{n}} \stackrel{(1.8)}{\leq} C\xi^{c-1} n^{-\frac{1}{2}} \rightarrow 0.$$

□

3.4. End of proof. We can now easily extend the lemma 3.6 from characteristic functions to step functions.

Corollary 3.9. *Assume that n and ξ satisfy (1.8). For any families $(X_O^n)_{\substack{n \geq 1 \\ O \in \text{Per}(n)}}$ and $(Y_O^n)_{\substack{n \geq 1 \\ O \in \text{Per}(n)}}$ of real random variables satisfying condition (C) of Definition (3.3), there exists $C > 0$ such that, if $(f_{n,O})_{\substack{n \geq 1 \\ O \in \text{Per}(n)}}$ is a family of step functions $S^1 \rightarrow \mathbb{R}$, then*

$$(3.19) \quad \left| \mathbb{E} \left[\prod_{m|n} \prod_{O \in \mathcal{P}_m} f_{n,O}^m(e^{i\xi(X_O^n + Y_O^n)}) \right] - \prod_{m|n} \prod_{O \in \mathcal{P}_m} \int f_{n,O}^m d\text{Leb} \right| \leq C\xi^{c-1} n^{-\frac{1}{2}} \prod_{m|n} \prod_{O \in \mathcal{P}_m} \int |f_{n,O}^m| d\text{Leb}.$$

Proof. Let us write each $f_{n,O}$ as

$$f_{n,O} = \sum_{q=1}^{p_{n,O}} \lambda_{n,O,q} \mathbb{1}_{A_{n,O,q}},$$

where the $\lambda_{n,O,q}$ are complex numbers and the $A_{n,O,q}, 1 \leq q \leq p_{n,O}$ are disjoint intervals. We develop (3.19), we use Lemma 3.6 and factorize the result:

$$\begin{aligned} E &:= \mathbb{E} \left[\prod_{m|n} \prod_{O \in \mathcal{P}_m} f_{n,O}^m(e^{i\xi(X_O^n + Y_O^n)}) \right] \\ &= \sum_{(q_O) \in \prod_{m|n} \prod_{O \in \mathcal{P}_m} \{1, \dots, p_{n,O}\}} \mathbb{E} \left[\prod_{m|n} \prod_{O \in \mathcal{P}_m} \lambda_{n,O,q_O}^m \mathbf{1}_{A_{n,O,q_O}}(e^{i\xi(X_O^n + Y_O^n)}) \right]. \end{aligned}$$

Consequently,

$$\begin{aligned} &\left| E - \prod_{m|n} \prod_{O \in \mathcal{P}_m} \int f_{n,O}^m d\text{Leb} \right| \\ &\leq \sum_{(q_O) \in \prod_{m|n} \prod_{O \in \mathcal{P}_m} \{1, \dots, p_{n,O}\}} \left| \mathbb{E} \left[\prod_{m|n} \prod_{O \in \mathcal{P}_m} \lambda_{n,O,q_O}^m \mathbf{1}_{A_{n,O,q_O}}(e^{i\xi(X_O^n + Y_O^n)}) \right] - \right. \\ &\quad \left. \prod_{m|n} \prod_{O \in \mathcal{P}_m} \lambda_{n,O,q_O}^m \text{Leb}(A_{n,O,q_O}) \right| \\ &\leq C\xi^{c-1} n^{-\frac{1}{2}} \sum_{(q_O) \in \prod_{m|n} \prod_{O \in \mathcal{P}_m} \{1, \dots, p_{n,O}\}} \prod_{m|n} \prod_{O \in \mathcal{P}_m} |\lambda_{n,O,q_O}|^m \text{Leb}(A_{n,O,q_O}) \end{aligned}$$

from the previous lemma.

Hence,

$$\left| E - \prod_{m|n} \prod_{O \in \mathcal{P}_m} \int f_{n,O}^m d\text{Leb} \right| \leq C\xi^{c-1} n^{-\frac{1}{2}} \prod_{m|n} \prod_{O \in \mathcal{P}_m} \int |f_{n,O}|^m d\text{Leb}.$$

□

We can use this result in order to estimate the characteristic function of $\text{Tr}^b(\mathcal{L}_{\xi,\tau}^n)$, using remark (3.5).

Corollary 3.10. *Assume that n and ξ satisfy (1.8). Let $(X_O^n)_{\substack{n \geq 1 \\ O \in \text{Per}(n)}}$ and $(Y_O^n)_{\substack{n \geq 1 \\ O \in \text{Per}(n)}}$ be two families of real random variables satisfying condition (C) of Definition (3.3). There exists $C > 0$ such that for all $(\mu_O, \nu_O)_{O \in \text{Per}(n)} \in \mathbb{R}^{2\#\text{Per}(n)}$,*

$$\begin{aligned} &\left| \mathbb{E} \left[\prod_{m|n} \prod_{O \in \mathcal{P}_m} e^{im\mu_O \cos(\xi(X_O^n + Y_O^n)) + im\nu_O \sin(\xi(X_O^n + Y_O^n))} \right] \right. \\ &\quad \left. - \prod_{m|n} \prod_{O \in \mathcal{P}_m} \int_0^{2\pi} e^{i(m\mu_O \cos \theta + m\nu_O \sin \theta)} \frac{d\theta}{2\pi} \right| \leq C\xi^{c-1} n^{-\frac{1}{2}}, \end{aligned}$$

Proof. Let C be the constant from corollary 3.9. For $O \in \text{Per}(n)$, let f_O be the function defined on S^1 by

$$f_O(e^{i\theta}) = e^{i(\mu_O \cos \theta + \nu_O \sin \theta)}.$$

Each f_O is bounded by 1, we can consequently find for each $O \in \text{Per}(n)$ a family $(f_{j,O})_j$ of step functions uniformly bounded by 1 converging pointwise towards f_O . We have for n fixed, by dominated convergence

$$E_j := \mathbb{E} \left[\prod_{m|n} \prod_{O \in \mathcal{P}_m} f_{j,O}^m(e^{i\xi(X_O^n + Y_O^n)}) \right] \xrightarrow{j \rightarrow \infty} E := \mathbb{E} \left[\prod_{m|n} \prod_{O \in \mathcal{P}_m} f^m(e^{i\xi(X_O^n + Y_O^n)}) \right]$$

as well as

$$I_j := \prod_{m|n} \prod_{O \in \mathcal{P}_m} \int_0^{2\pi} f_{j,O}^m(e^{i\theta}) \frac{d\theta}{2\pi} \xrightarrow{j \rightarrow \infty} I := \prod_{m|n} \prod_{O \in \mathcal{P}_m} \int_0^{2\pi} f_O^m(e^{i\theta}) \frac{d\theta}{2\pi}.$$

It is thus possible to find an integer j_0 such that both

$$|E - E_{j_0}| \leq \xi^{c-1} n^{-\frac{1}{2}}$$

and

$$|I - I_{j_0}| \leq \xi^{c-1} n^{-\frac{1}{2}}$$

hold.

From corollary 3.9, we know that for all $n \in \mathbb{N}$

$$|E_{j_0} - I_{j_0}| \leq C \xi^{c-1} n^{-\frac{1}{2}} \sup |f_{j_0}|.$$

Thus,

$$\begin{aligned} |E - I| &\leq |E - E_{j_0}| + |E_{j_0} - I_{j_0}| + |I - I_{j_0}| \\ &\leq (C + 2) \xi^{c-1} n^{-\frac{1}{2}}. \end{aligned}$$

□

We can now prove the final proposition :

Proposition 3.11. *Let $(X_O^n)_{\substack{n \geq 1 \\ O \in \text{Per}(n)}}$ and $(Y_O^n)_{\substack{n \geq 1 \\ O \in \text{Per}(n)}}$ be two families of real random variables satisfying condition (C) of Definition (3.3). If condition (1.8) is satisfied then we have the following convergence in law*

$$(3.20) \quad T_{n,\xi} := A_n \sum_{m|n} m \sum_{O \in \mathcal{P}_m} \frac{e^{i\xi(X_O^n + Y_O^n)}}{e^{J_O^n} - 1} \xrightarrow{n,\xi \rightarrow \infty} \mathcal{N}_{\mathbb{C}}(0, 1),$$

with the amplitude A_n defined in (1.5) by

$$(3.21) \quad A_n = \left(\sum_{m|n} m^2 \sum_{O \in \mathcal{P}_m} \frac{1}{(e^{J_O^n} - 1)^2} \right)^{-\frac{1}{2}}.$$

Proof. Let us fix two real numbers ξ_1 and ξ_2 and let ϕ_n be the characteristic function of $T_{n,\xi}$:

$$\begin{aligned} \phi_n(\xi_1, \xi_2) := \mathbb{E} \left[\exp \left(i A_n \left(\xi_1 \sum_{m|n} m \sum_{O \in \mathcal{P}_m} \frac{\cos(\xi(X_O^n + Y_O^n))}{e^{J_O^n} - 1} + \right. \right. \right. \\ \left. \left. \left. \xi_2 \sum_{m|n} m \sum_{O \in \mathcal{P}_m} \frac{\sin(\xi(X_O^n + Y_O^n))}{e^{J_O^n} - 1} \right) \right) \right]. \end{aligned}$$

We compute the limit of $\phi_n(\xi_1, \xi_2)$ as n goes to infinity. Corollary (3.10) yields

$$(3.22) \quad \left| \phi_n(\xi_1, \xi_2) - \prod_{m|n} \prod_{O \in \mathcal{P}_m} \int_0^{2\pi} e^{i \frac{mA_n}{e^{JO}-1} (\xi_1 \cos \theta + \xi_2 \sin \theta)} \frac{d\theta}{2\pi} \right| \leq C \xi^{c-1} n^{-\frac{1}{2}} \rightarrow 0$$

under the assumption (1.8).

Let

$$\psi(\xi_1, \xi_2) := \int_0^{2\pi} e^{i(\xi_1 \cos \theta + \xi_2 \sin \theta)} \frac{d\theta}{2\pi}.$$

We have the following Taylor's expansion in 0:

$$\psi(\xi_1, \xi_2) = 1 - \frac{1}{4}(\xi_1^2 + \xi_2^2) + o(\xi_1^2 + \xi_2^2).$$

In order to apply this to equation (3.22), we need to check that

Lemma 3.12.

$$nA_n \sup_{O \in \text{Per}(n)} \frac{1}{e^{JO} - 1} \xrightarrow{n \rightarrow \infty} 0.$$

Proof. See appendix E.3 □

We can now state that

$$\begin{aligned} & \prod_{m|n} \prod_{O \in \mathcal{P}_m} \int_0^{2\pi} e^{i \frac{mA_n}{e^{JO}-1} (\xi_1 \cos \theta + \xi_2 \sin \theta)} \frac{d\theta}{2\pi} = \prod_{m|n} \prod_{O \in \mathcal{P}_m} \psi \left(\xi_1 \frac{mA_n}{e^{JO}-1}, \xi_2 \frac{mA_n}{e^{JO}-1} \right) \\ & = \prod_{m|n} \prod_{O \in \mathcal{P}_m} \left(1 - \frac{\xi_1^2 + \xi_2^2}{4} \frac{(mA_n)^2}{(e^{JO}-1)^2} + o \left(\frac{(mA_n)^2}{(e^{JO}-1)^2} \right) \right) \\ & = \exp \left(\sum_{m|n} \sum_{O \in \mathcal{P}_m} \log \left(1 - \frac{\xi_1^2 + \xi_2^2}{4} \frac{(mA_n)^2}{(e^{JO}-1)^2} + o \left(\frac{(mA_n)^2}{(e^{JO}-1)^2} \right) \right) \right) \\ & = \exp \left(\sum_{m|n} \sum_{O \in \mathcal{P}_m} - \frac{\xi_1^2 + \xi_2^2}{4} \frac{(mA_n)^2}{(e^{JO}-1)^2} + o \left(\frac{(mA_n)^2}{(e^{JO}-1)^2} \right) \right) \\ & \stackrel{(3.21)}{=} e^{-\frac{\xi_1^2 + \xi_2^2}{4} + o(1)}, \end{aligned}$$

We deduce that

$$\phi_n(\xi_1, \xi_2) \xrightarrow{n \rightarrow \infty} e^{-\frac{\xi_1^2 + \xi_2^2}{4}}$$

which is the characteristic function of a Gaussian variable of law $\mathcal{N}_{\mathbb{C}}(0, 1)$. □

4. DISCUSSION

In this paper we have considered a model where the roof function τ is random. However, the numerical experiments suggest a far stronger result: for a fixed function τ and a semiclassical parameter ξ chosen according to a uniform random distribution in a small window at high frequencies, the result seems to remain true, as shown in the following figures for $\tau(x) = \sin(2\pi x)$. The moduli also seem to become uniform. We define C_e by $n = C_e \frac{\log \xi_0}{\log 2}$.

It would be interesting to understand what informations about the Ruelle resonances can be recovered from the convergence (1.7). We know from the Weyl law from [AFW17] established in a similar context that the number of resonances of $\mathcal{L}_{\xi,\tau}$ outside the essential spectral radius, for a given τ , are of order $O(\xi)$. A complete characterization would thus require a knowledge of the traces of $\mathcal{L}_{\xi,\tau}^n$ up to times of order $O(\xi)$, while we only have information for $n = O(\log \xi)$.

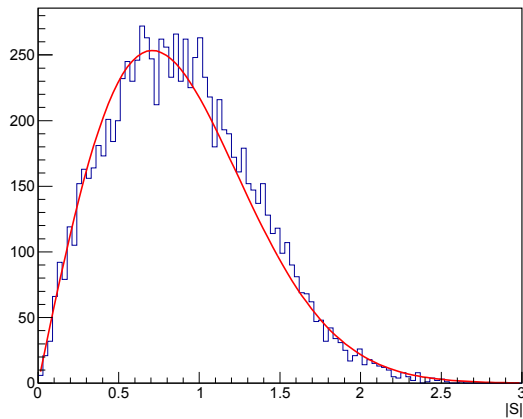


FIGURE 4. Histogramm of $S = \left| A_n \text{Tr}^b \left(\mathcal{L}_{\tau_0, \xi}^n \right) \right|$ for a sample of 10^4 random values of ξ uniformly distributed in $[\xi_0, \xi_0 + 10]$ with $\xi_0 = 2 \cdot 10^6$ and $n = 11$ giving $C_e = 0.5$. This is well fitted by $CS \exp(-S^2)$ in red.

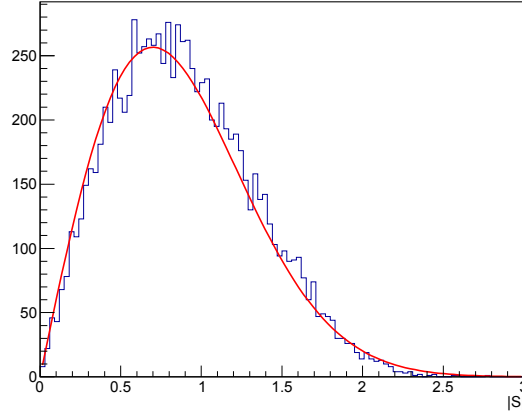


FIGURE 5. Histogramm of $S = \left| A_n \text{Tr}^b \left(\mathcal{L}_{\tau_0, \xi}^n \right) \right|$ for a sample of 10^4 random values of ξ uniformly distributed in $[\xi_0, \xi_0 + 10]$ with $\xi_0 = 2000$ and $n = 11$ giving $C_e = 1.0$. This is well fitted by $CS \exp(-S^2)$ in red.

APPENDIX A. PROOF OF LEMMA A.1

Lemma A.1. *For every integer n , E^n has $l^n - 1$ fixed points. The distance between two distinct periodic points is bounded from below by $\frac{1}{M^n - 1}$.*

Proof. E is topologically conjugated to the linear expanding map of same degree $x \mapsto lx \pmod{1}$, (see [KH97], p.73). Thus E^n has $l^n - 1$ fixed points. Let $\tilde{E} : \mathbb{R} \rightarrow \mathbb{R}$ be a lift of E , $x \neq y$ be two fixed points of E^n and $\tilde{x}, \tilde{y} \in \mathbb{R}$ be representatives of x and y respectively. Note that

$$d(x, y) = \inf |\tilde{x} - \tilde{y}|$$

where the infimum is taken over all couples of representatives (\tilde{x}, \tilde{y}) . Since $E^n(x) = x$ and $E^n(y) = y$, $\tilde{E}^n(\tilde{y}) - \tilde{E}^n(\tilde{x}) - (\tilde{y} - \tilde{x})$ is an integer, different from 0 because \tilde{E}^n is expanding. Thus,

$$\left| \tilde{E}^n(\tilde{y}) - \tilde{E}^n(\tilde{x}) - (\tilde{y} - \tilde{x}) \right| \geq 1,$$

that is

$$\left| \int_{\tilde{x}}^{\tilde{y}} \left((\tilde{E}^n)'(t) - 1 \right) dt \right| \geq 1$$

Finally,

$$|\tilde{y} - \tilde{x}|(M^n - 1) \geq 1.$$

Taking the infimum gives the result. \square

APPENDIX B. PROOF OF LEMMA 1.3 ON THE LINK BETWEEN REGULARITY OF A GAUSSIAN FIELD AND VARIANCE OF THE FOURIER COEFFICIENTS

Let us recall the following classical estimate:

Lemma B.1. *If $(X_p)_{p \in \mathbb{Z}}$ is a family of independent centered Gaussian random variables of variance 1, then, almost surely,*

$$\forall \delta > 0, X_p = o(p^\delta).$$

Proof. Let $n \in \mathbb{N}^*$. Let us use Borel-Cantelli lemma:

$$\forall p \in \mathbb{Z}, \mathbb{P}(|X_p| > p^{\frac{1}{n}}) = \int_{|x| > p^{\frac{1}{n}}} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}.$$

Now, we have the upper bound

$$p^{\frac{1}{n}} \int_{p^{\frac{1}{n}}}^{+\infty} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \leq \int_{p^{\frac{1}{n}}}^{+\infty} x e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = \frac{e^{-\frac{p^{\frac{2}{n}}}{2}}}{\sqrt{2\pi}}.$$

Thus,

$$\forall p \in \mathbb{Z}^*, \mathbb{P}(|X_p| > p^{\frac{1}{n}}) \leq \frac{2}{\sqrt{2\pi} p^{\frac{1}{n}}} e^{-\frac{p^{\frac{2}{n}}}{2}}.$$

Consequently,

$$\forall n \geq 1, \sum_p \mathbb{P}(|X_p| > p^{\frac{1}{n}}) < \infty$$

and by Borel-Cantelli, almost surely, for all $n \in \mathbb{N}$,

$$\#\{p \in \mathbb{Z}, |X_p| > p^{\frac{1}{n}}\} < \infty.$$

□

With this in mind, we can see that if a real random function τ has random Fourier coefficients $(c_p)_{p \in \mathbb{Z}}$, pairwise independent (for non-negative values of p), with variance

$$\sigma_p^2 := \mathbb{E}[|c_p|^2] = O\left(\frac{1}{p^{2k+2+\eta}}\right),$$

for some $\eta > 0$, then by the previous lemma, for all p , almost surely, for all $\delta > 0$,

$$\frac{c_p}{\sigma_p} = o(p^\delta),$$

and thus for $\delta = \frac{\eta}{2}$,

$$c_p = o\left(\frac{1}{p^{k+1+\frac{\eta}{2}}}\right) \text{ a.s.}$$

As a consequence,

$$\sum_p c_p (2i\pi p)^k e^{2i\pi p x}$$

converges normally and thus τ is almost surely \mathcal{C}^k .

APPENDIX C. RUELLE RESONANCES AND FLAT TRACE

C.1. Ruelle spectrum. If $\tau \in \mathcal{C}^k(\mathbb{T})$, the operator $\mathcal{L}_{\xi, \tau}$ can be extended to distributions $(\mathcal{C}^k(\mathbb{T}))'$ by duality. We will denote $H^s(\mathbb{T})$ the Sobolev space of order $s \in \mathbb{R}$.

Theorem C.1 ([Rue86],[Bal18] Thm 2.15 and Lemma 2.16). *Let $k \geq 1$. If τ belongs to \mathcal{C}^k , then for every $0 \leq s < k$, $\mathcal{L}_{\xi,\tau} : H^{-s}(\mathbb{T}) \rightarrow H^{-s}(\mathbb{T})$ is bounded and its essential spectral radius r_{ess} satisfies*

$$r_{\text{ess}} \leq \frac{e^{\text{Pr}(-\frac{1}{2}J)}}{m^s},$$

where $m = \inf E'$, $J(x) = \log E'(x)$ and $\text{Pr}(-\frac{1}{2}J)$ is defined in E.1.

The discrete set of eigenvalues of finite multiplicities outside a given disk of radius $r \geq \frac{e^{\text{Pr}(-\frac{1}{2}J)}}{m^s}$, and the associated eigenspaces remain the same in every space $H^{-s'}(\mathbb{T})$ for $s' \geq s$. This can be deduced for example from the fact that these spectral elements give the asymptotic behaviour of the correlation functions: for any smooth functions f, g on \mathbb{T} , for any s large enough, if $\mathcal{L}_{\xi,\tau} : H^{-s}(\mathbb{T}) \rightarrow H^{-s}(\mathbb{T})$ has no eigenvalue of modulus r ,

$$(C.1) \quad \int \mathcal{L}_{\xi,\tau}^n f \cdot g = \sum_{\substack{\lambda \in \sigma(\mathcal{L}_{\xi,\tau}) \\ |\lambda| > r}} \int \mathcal{L}_{\xi,\tau}^n(\Pi_\lambda f) \cdot g + O_{n \rightarrow \infty}(r^n),$$

where Π_λ is the spectral projector associated to λ . We are interested in the statistical properties of these eigenvalues, called Ruelle-Pollicott spectrum or Ruelle resonances, when τ is a random function. One way to get informations about the spectrum of such operators is using a trace formula. Although $\mathcal{L}_{\xi,\tau}$ is not trace-class, we can give a certain sense to the trace of $\mathcal{L}_{\xi,\tau}$.

C.2. Flat trace. This section is an adaptation of section 3.2.2 in [Bal18] In order to motivate the definition of flat trace, let us first recall the following fact:

Lemma C.2. *Let $m > \frac{1}{2}$. (Then the Dirac distributions belong to $H^{-m}(\mathbb{T})$). If $T : H^{-m}(\mathbb{T}) \rightarrow H^m(\mathbb{T})$ is a bounded operator, then it has a continuous Schwartz kernel K and*

$$K(x, y) = \langle \delta_x, T\delta_y \rangle.$$

If moreover T is class-trace, then

$$\text{Tr } T = \int_{\mathbb{T}} K(x, x) dx.$$

Let ρ be a smooth compactly supported function such that $\int_{\mathbb{R}} \rho = 1$. For $\epsilon > 0$ and $y \in \mathbb{T}$ we write

$$\rho_{\epsilon,y}(t) = \frac{1}{\epsilon} \rho\left(\frac{t-y}{\epsilon}\right).$$

Periodizing this function gives rise to a smooth function $\rho_{\epsilon,y}$ on \mathbb{T} satisfying

$$\rho_{\epsilon,y} \xrightarrow{\epsilon \rightarrow 0} \delta_y$$

as distributions.

Definition C.3. Let $T : H^{-m}(\mathbb{T}) \rightarrow H^{-m}(\mathbb{T})$ be a bounded operator. The formula

$$K_\epsilon(x, y) := \langle \rho_{\epsilon,x}, T\delta_y \rangle$$

defines for every $\epsilon > 0$ a continuous function on \mathbb{T}^2 . Let

$$\text{Tr}_\epsilon^b(T) := \int_{\mathbb{T}} K_\epsilon(x, x) dx.$$

We say that T admits a flat trace $\text{Tr}^b(T)$ if $\text{Tr}_\epsilon^b(T) \rightarrow \text{Tr}^b(T)$ as ϵ goes to zero.

Lemma C.4 (Trace formula, [AB67], [G⁺77]). *Let $\tau \in \mathcal{C}^k(\mathbb{T})$, $k \geq 1$. For any integer $n \geq 1$, $\mathcal{L}_{\xi, \tau}^n$ has a flat trace*

$$(C.2) \quad \mathrm{Tr}^b \mathcal{L}_{\xi, \tau}^n = \sum_{x, E^n(x)=x} \frac{e^{i\xi\tau_x}}{(E^n)'(x) - 1}$$

Proof.

$$\mathrm{Tr}_\epsilon^b(\mathcal{L}_{\xi, \tau}^n) = \int_{\mathbb{T}} \langle \rho_{\epsilon, x}, \mathcal{L}_{\xi, \tau}^n \delta_x \rangle dx.$$

By definition of the action of $\mathcal{L}_{\xi, \tau}^n$ on distributions,

$$\langle \rho_{\epsilon, x}, \mathcal{L}_{\xi, \tau}^n \delta_x \rangle = (\mathcal{L}_{\xi, \tau}^n)^* \rho_{\epsilon, x}(x),$$

where $(\mathcal{L}_{\xi, \tau}^n)^*$ is the L^2 -adjoint of $\mathcal{L}_{\xi, \tau}^n$. Let us recall that, if $\phi : \mathbb{T} \rightarrow \mathbb{T}$ is a local diffeomorphism, for every continuous functions u, v on \mathbb{T} ,

$$(C.3) \quad \int u(\phi(y))v(y)dy = \int u(x) \sum_{\phi(y)=x} \frac{v(y)}{|\phi'(y)|} dx.$$

Thus,

$$(\mathcal{L}_{\xi, \tau}^n)^* v(x) = \sum_{E^n(y)=x} \frac{v(y)e^{i\xi\tau_y}}{(E^n)'(y)}.$$

Therefore

$$\begin{aligned} \mathrm{Tr}_\epsilon^b(\mathcal{L}_{\xi, \tau}^n) &= \int_{\mathbb{T}} (\mathcal{L}_{\xi, \tau}^n)^* \rho_{\epsilon, x}(x) dx \\ &= \int_{\mathbb{T}} \sum_{E^n(y)=x} \frac{\rho_{\epsilon, 0}(y - E^n(y))e^{i\xi\tau_y}}{(E^n)'(y)} dx \\ &= \int_{\mathbb{T}} \rho_{\epsilon, 0}(y - E^n(y))e^{i\xi\tau_y} dy \end{aligned}$$

by the change of variables $x = E^n(y)$. Now, since E is expansive, $y \mapsto y - E^n(y)$ is a local diffeomorphism, so applying (C.3) once again gives

$$\begin{aligned} \mathrm{Tr}_\epsilon^b(\mathcal{L}_{\xi, \tau}^n) &= \int_{\mathbb{T}} \rho_{\epsilon, 0}(z) \sum_{y - E^n(y)=z} \frac{e^{i\xi\tau_y}}{(E^n)'(y) - 1} dz \\ &\xrightarrow{\epsilon \rightarrow 0} \sum_{E^n(y)=y} \frac{e^{i\xi\tau_y}}{(E^n)'(y) - 1}. \end{aligned}$$

□

If E and τ are analytic, it is well known that \mathcal{L} is trace-class and that $\mathrm{Tr}^b(\mathcal{L}_{\xi, \tau}) = \mathrm{Tr}(\mathcal{L}_{\xi, \tau})$ (see for instance [Jéz17]). In the smooth setting however the decay of the Ruelle-Pollicott spectrum can be arbitrarily slow ([Jéz17], Proposition 1.10). The flat trace is however related to the Ruelle-Pollicott spectrum defined above in the following way (This is a consequence of Thm 3.5 in [Bal18] and Thm 2.4 in [Jéz17]):

Proposition C.5. *Assume that $\tau \in \mathcal{C}^k(\mathbb{T})$ for some $k \geq 1$. Let $\xi \in \mathbb{R}$, $0 \leq s < k$, and $r > \frac{e^{\mathrm{Pr}(-\frac{1}{2}J)}}{m^s}$ be such that $\mathcal{L}_{\xi, \tau} : H^{-s}(\mathbb{T}) \rightarrow H^{-s}(\mathbb{T})$ has no eigenvalue of*

modulus r , then

$$(C.4) \quad \exists C > 0, \forall n \in \mathbb{N}, \left| \operatorname{Tr}^b \mathcal{L}_{\xi, \tau}^n - \sum_{\substack{\lambda \in \sigma(\mathcal{L}_{\xi, \tau}) \\ |\lambda| > r}} \lambda^n \right| \leq Cr^n,$$

where the eigenvalues are counted with multiplicity.

APPENDIX D. PROOF OF LEMMA 1.8

Proof. Let X, X', Y , and Y' be as in the statement of the lemma real random variables such that $e^{iX}, e^{iX'}$ are uniform on S^1 and so that X and X' are both independent of all three other random variables. Let us write \mathbb{P}_Z the law of a random variable Z . To show that $e^{i(X+Y)}$ and $e^{i(X'+Y')}$ are independent and uniform on S^1 , it suffices to show that for any continuous functions $f, g : S^1 \rightarrow \mathbb{R}$,

$$\mathbb{E} \left[f(e^{i(X+Y)}) g(e^{i(X'+Y')}) \right] = \int_0^{2\pi} \int_0^{2\pi} f(e^{i\theta}) g(e^{i\theta'}) \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi}.$$

$$\mathbb{E} \left[f(e^{i(X+Y)}) g(e^{i(X'+Y')}) \right] = \int_{(S^1)^4} f(e^{i(x+y)}) g(e^{i(x'+y')}) d\mathbb{P}_{(X, Y, X', Y')}(x, y, x', y').$$

By hypothesis,

$$d\mathbb{P}_{(X, Y, X', Y')}(x, y, x', y') = \frac{dx}{2\pi} \frac{dx'}{2\pi} d\mathbb{P}_{(Y, Y')}(y, y').$$

Thus,

$$\begin{aligned} & \mathbb{E} \left[f(e^{i(X+Y)}) g(e^{i(X'+Y')}) \right] \\ &= \int_{(S^1)^2} \left(\int_0^{2\pi} \int_0^{2\pi} f(e^{i(x+y)}) g(e^{i(x'+y')}) \frac{dx}{2\pi} \frac{dx'}{2\pi} \right) d\mathbb{P}_{(Y, Y')}(y, y') \\ & \stackrel{\theta=x+y, \theta'=x'+y'}{=} \int_{(S^1)^2} \left(\int_0^{2\pi} \int_0^{2\pi} f(e^{i\theta}) g(e^{i\theta'}) \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi} \right) d\mathbb{P}_{(Y, Y')}(y, y') \\ &= \int_0^{2\pi} \int_0^{2\pi} f(e^{i\theta}) g(e^{i\theta'}) \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi}. \end{aligned}$$

□

APPENDIX E. TOPOLOGICAL PRESSURE

E.1. Definition.

Definition E.1. Let $\phi : \mathbb{T} \rightarrow \mathbb{R}$ be a Hölder-continuous function. The limit

$$(E.1) \quad \operatorname{Pr}(\phi) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{E^n(x)=x} e^{\phi_x^n} \right)$$

exists and is called the topological pressure of ϕ (see [KH97] Proposition 20.3.3 p.630).

In other words

$$(E.2) \quad \sum_{E^n(x)=x} e^{\phi_x^n} = e^{n\text{Pr}(\phi)+o(n)}.$$

The particular case $\phi = 0$ gives the topological entropy $\text{Pr}(0) = h_{top}$.

Remark E.2. Note that the expression $e^{n\text{Pr}(\phi)+o(n)}$ describes a large class of sequences, since for instance for any $k \in \mathbb{N}$,

$$n^k e^{n\text{Pr}(\phi)} = e^{n\text{Pr}(\phi)+o(n)}.$$

E.2. Variational principle. Another definition of the pressure is given by the variational principle. Let us denote by $h(\mu)$ the entropy of a measure μ invariant under E (see [KH97] section 4.3 for a definition of entropy). For the next theorem, see [KH97], sections 20.2 and 20.3. The last sentence comes from Proposition 20.3.10.

Theorem E.3 (Variational principle). *Let $\phi : \mathbb{T} \rightarrow \mathbb{R}$ be a Hlder function.*

$$\text{Pr}(\phi) = \sup_{\mu \text{ } E\text{-invariant}} \left(\int \phi \, d\mu + h(\mu) \right).$$

*This supremum, taken over the invariant **probability** measures, is moreover attained for a unique E -invariant measure μ , called equilibrium measure. In addition, if we note $J = \log E'$ and μ_β the equilibrium measure of $-\beta J$, $\beta \mapsto \mu_\beta$ is one-to-one.*

Corollary E.4. *The function*

$$(E.3) \quad F : \begin{cases} \mathbb{R}_+^* & \longrightarrow \mathbb{R} \\ \beta & \longmapsto \frac{1}{\beta} \text{Pr}(-\beta J) \end{cases}$$

is strictly decreasing.

Proof. Let $\beta' > \beta > 0$. By the previous theorem, with the same notations,

$$\int -\beta J \, d\mu_\beta + h(\mu_\beta) > \int -\beta J \, d\mu_{\beta'} + h(\mu_{\beta'})$$

and thus

$$F(\beta) = \int -J \, d\mu_\beta + \frac{h(\mu_\beta)}{\beta} > \int -J \, d\mu_{\beta'} + \frac{h(\mu_{\beta'})}{\beta} \geq \int -J \, d\mu_{\beta'} + \frac{h(\mu_{\beta'})}{\beta'} = F(\beta').$$

□

E.3. Proof of lemma 3.12. Let $\phi : \mathbb{T} \rightarrow \mathbb{R}$ be a \mathcal{C}^1 function. Let as before ϕ_x^n be the Birkhoff sum (1.3). By subadditivity of the sequence $(\inf_{x \in \mathbb{T}} \phi_x^n)_n$ and Fekete's Lemma we can define the following quantity:

Definition E.5. Let us define

$$(E.4) \quad \phi_{\min} := \lim_{n \rightarrow \infty} \inf_{x \in \mathbb{T}} \frac{1}{n} \phi_x^n.$$

Lemma E.6. *The infimum in (E.4) can be taken over periodic points:*

$$(E.5) \quad \phi_{\min} = \lim_{n \rightarrow \infty} \inf_{x, E^n(x)=x} \frac{1}{n} \phi_x^n.$$

Proof. By lifting the expanding map to \mathbb{R} , we easily see that E has at least a fixed point x_0 . This point has l^n preimages by E^n , defining $l^n - 1$ intervals I_k^n such that for all $1 \leq k \leq l^n - 1$

$$E^n : I_k^n \rightarrow \mathbb{T} \setminus \{x_0\}$$

is a diffeomorphism. Thus, there exists $C > 0$ such that for all k , if $x, y \in \overline{I_k^n}$,

$$\forall 0 \leq j \leq n, d(E^j(x), E^j(y)) \leq \frac{C}{m^{n-j}},$$

with $m = \inf|E'| > 1$. Each $\overline{I_k^n}$ contains moreover a periodic point $y_{k,n}$ of period n given by $E^n(y_{k,n}) = y_{k,n} + k$. Hence let $n \in \mathbb{N}$, let $x_n \in \mathbb{T}$ be such that

$$\phi_{x_n}^n = \inf_{x \in \mathbb{T}} \phi_x^n,$$

and suppose that $x_n \in \overline{I_k^n}$. We have

$$\begin{aligned} \left| \phi_{y_{k,n}}^n - \phi_{x_n}^n \right| &= \left| \sum_{j=0}^{n-1} \phi(E^j(x_n)) - \phi(E^j(y_{k,n})) \right| \\ &\leq C \max|\phi'| \sum_{k=0}^{\infty} \frac{1}{m^k} \end{aligned}$$

is bounded independently of n . Consequently

$$\lim_{n \rightarrow \infty} \inf_{x, E^n(x)=x} \frac{1}{n} \phi_x^n = \phi_{\min}.$$

□

Lemma E.7.

$$F(\beta) \xrightarrow{\beta \rightarrow +\infty} -\phi_{\min}.$$

Proof. Let $\beta > 0$. Let us write

$$F_n(\beta) = \frac{1}{n\beta} \log \left(\sum_{E^n(x)=x} e^{-\beta\phi_x^n} \right),$$

so that

$$F(\beta) \stackrel{(E.1, E.3)}{=} \lim_{n \rightarrow \infty} F_n(\beta).$$

Let $\epsilon > 0$. By definition of ϕ_{\min} , for n large enough,

$$\forall x \in \text{Per}(n), \phi_x^n \geq n(\phi_{\min} - \epsilon)$$

and

$$\exists x \in \text{Per}(n), \phi_x^n \leq n(\phi_{\min} + \epsilon).$$

Thus,

$$e^{-\beta n(\phi_{\min} + \epsilon)} \leq \sum_{E^n(x)=x} e^{-\beta\phi_x^n} \leq l^n e^{-\beta n(\phi_{\min} - \epsilon)}$$

and consequently

$$-\phi_{\min} - \epsilon \leq F_n(\beta) \leq \frac{\log l}{\beta} - \phi_{\min} + \epsilon.$$

Hence, letting $\epsilon \rightarrow 0$, we get

$$-\phi_{\min} \leq F(\beta) \leq \frac{\log l}{\beta} - \phi_{\min}.$$

When β goes to infinity, the result follows. \square

Proof of lemma 3.12. Now we take $\phi = J = \log(E')$. By the definition of J_{\min}

$$\inf_{O \in \text{Per}(n)} J_O^n = nJ_{\min} + o(n),$$

thus

$$(E.6) \quad \sup_{O \in \text{Per}(n)} \frac{1}{e^{J_O} - 1} = e^{-nJ_{\min} + o(n)} = e^{n \lim_{\beta \rightarrow \infty} F(\beta) + o(n)}.$$

We have

$$(E.7) \quad \frac{1}{\sqrt{n}} \left(\sum_{m|n} m \sum_{O \in \mathcal{P}_m} \frac{1}{(e^{J_O^n} - 1)^2} \right)^{-\frac{1}{2}} \leq A_n \stackrel{(1.5)}{=} \left(\sum_{m|n} m^2 \sum_{O \in \mathcal{P}_m} \frac{1}{(e^{J_O^n} - 1)^2} \right)^{-\frac{1}{2}} \\ \leq \left(\sum_{m|n} m \sum_{O \in \mathcal{P}_m} \frac{1}{(e^{J_O^n} - 1)^2} \right)^{-\frac{1}{2}}.$$

Since

$$\begin{aligned} \sum_{m|n} m \sum_{O \in \mathcal{P}_m} \frac{1}{(e^{J_O^n} - 1)^2} &= \sum_{E^n(x)=x} \frac{1}{(e^{J_x^n} - 1)^2} \\ &= \sum_{E^n(x)=x} e^{-2J_x^n} \left(1 + O(e^{-J_x^n}) \right) \\ &= \left(\sum_{E^n(x)=x} e^{-2J_x^n} \right) (1 + o(1)) \\ &= e^{n\text{Pr}(-2J) + o(n)}, \end{aligned}$$

Eq.(E.7) gives

$$\frac{1}{\sqrt{n}} e^{-\frac{n}{2}\text{Pr}(-2J) + o(n)} \leq A_n \leq e^{-\frac{n}{2}\text{Pr}(-2J) + o(n)}$$

hence from Remark E.2

$$nA_n = e^{-\frac{n}{2}\text{Pr}(-2J) + o(n)} = e^{-nF(2) + o(n)}.$$

Finally,

$$nA_n \sup_{O \in \text{Per}(n)} \frac{1}{e^{J_O} - 1} \stackrel{(E.6)}{=} e^{n(\lim_{\infty} F - F(2) + o(n))} \rightarrow 0$$

from Corollary E.4. \square

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