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Abstract. We consider the skew-product of an expanding map E on the circle $\mathbb{T}$ with an almot surely $\mathcal{C}^k$ random perturbation $\tau = \tau_0 + \delta\tau$ of a deterministic function τ_0:

$$F : \left\{ \begin{array}{ccc} \mathbb{T}\times\mathbb{R} & \longrightarrow & \mathbb{T}\times\mathbb{R} \\ (x,y) & \longmapsto & (E(x), y+\tau(x)) \end{array} \right. .$$

The associated transfer operator $\mathcal{L} : u \in C_c^\infty(\mathbb{T}\times\mathbb{R}) \mapsto u\circ F$ can be decomposed with respect to frequency in the y variable into a family of operators acting on dimension 1:

$$\mathcal{L}_\xi : \left\{ \begin{array}{ccc} C^\infty(\mathbb{T}) & \longrightarrow & C^\infty(\mathbb{T}) \\ u & \longmapsto & e^{i\xi\tau} u\circ E \end{array} \right. .$$

We show that the flat traces of $\mathcal{L}_\xi^n$ behave as normal distributions in the semiclassical limit $n, \xi \to \infty$ up to the Ehrenfest time $n \leq c_k \log \xi$.

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Contents

1. Introduction

This paper focuses on the distribution of the flat traces of iterates of the transfer operator of a simple example of partially expanding map. It is motivated by the Bohigas-Gianonni-Schmidt [BGS84] conjecture in quantum chaos (see below).
In chaotic dynamics, the transfer operator is an object of first importance linked to the asymptotics of the correlations. The collection of poles of its resolvent, called Ruelle-Pollicott spectrum, can be defined as the spectrum of the transfer operator in appropriate Banach spaces (see [Rue76] for analytic expanding maps, [Kit99],[BKL02],[BT07],[BT08],[GL06],[FRS08] for the construction of the spaces for Anosov diffeomorphisms.)

The study of the Ruelle spectrum for Anosov flows is more difficult because of the flow direction that is neither contracting nor expanding. Tsujii [Tsu10] constructed appropriate Hilbert spaces for the transfer operator of contact Anosov flows. Dolgopyat has shown in particular in [Dol98] the exponential decay of correlations (that is a spectral gap of the transfer operator) for the geodesic flow on negatively curved surfaces, and Liverani [Liv04] generalized this result to all contact Anosov flows. A simple example of Anosov flow is the suspension of an Anosov diffeomorphism, or the suspension semi-flow of an expanding map. Pollicott showed exponential decay of correlations in this setting under a weak condition in [Pol85] and Tsujii constructed suitable spaces for the transfer operator and gave an upper bound on its essential spectral radius in [Tsu08].

In this article we study a closely related discrete time model, the skew product of an expanding map of the circle. It is a particular case of compact group extension [Dol02], which are partially hyperbolic maps, with compact, isometric to each other leaves in the neutral direction. Dolgopyat showed in [Dol02] that in this case the correlation decrease generically rapidly. In our setting of skew-product of an expanding map of the circle, Faure [Fau11] has shown using semi-classical methods an upper bound on the essential spectral radius of the transfer operator under a condition shown to be generic by Nakano Tsujii and Wittsten [NTW16]. Arnoldi, Faure, and Weich [AFW17] and Faure and Weich [FW17] studied the case of some open partially expanding maps, iteration function schemes, for which they found an explicit bound on the essential spectral radius of the transfer operator in a suitable space, and obtained a Weyl law (upper bound on the number of Ruelle resonances outside the essential spectral radius). In the more general framework of random dynamical systems in which the transfer operator changes randomly at each iteration, for the skew product of an expanding map of the circle, Nakano and Wittsten [NW15] showed exponential decay of correlations.

Semiclassical analysis describes the link between quantum dynamics and the associated classical dynamics in a symplectic manifold. The transfer operator happens to be a Fourier integral operator and the semi-classical approach has thus shown to be useful. The famous Bohigas-Giannoni-Schmidt

[BGS84] conjecture of quantum chaos states that for quantum systems whose associated classical dynamic is chaotic, the spectrum of the Hamiltonian shows the same statistics as that of a random matrix (GUE, GOE or GSE according to the

symmetries of the system)(see also [Gut13] and [GVZJ91]). We are interested analogously in investigating the possible links between the Ruelle-Pollicott spectrum and the spectrum of random matrices/operators. At first we try to get informations about the spectrum using a trace formula. More useful results could follow from the use of a global normal form as obtained by Faure-Weich in [FW17].

1.1. **Expanding map.** Let us consider a smooth expanding map $E : \mathbb{T} \to \mathbb{T}$ on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, that is, satisfying $|E'| > 1$, of degree l, and let us call

$$m = \inf|E'| > 1$$

and

$$M = \sup|E'|.$$

Lemma 1.1. *For every integer n, E^n has $l^n - 1$ fixed points. The distance between two consecutive periodic points is bounded from below by $\frac{1}{M^n}$.*

Proof. See appendix A. □

We will write the set of periodic orbits of (non primitive) period n as

$$\mathrm{Per}(n) := \left\{\{x, E(x), \cdots, E^{n-1}(x)\}, E^n(x) = x, x \in \mathbb{T}\right\}, \tag{1.1}$$

and the set of periodic orbits of primitive period n as

$$\mathcal{P}_n := \left\{\{x, E(x), \cdots, E^{n-1}(x)\}, n = \min\{k \in \mathbb{N}^*, E^k(x) = x\}, x \in \mathbb{T}\right\}. \tag{1.2}$$

This way,

$$\mathrm{Per}(n) = \coprod_{m|n} \mathcal{P}_m. \tag{1.3}$$

For a function $\phi \in C(\mathbb{T})$ and a point $x \in \mathbb{T}$ we will denote by ϕ_x^n the Birkhoff sum

$$\phi_x^n := \sum_{k=0}^{n-1} \phi(E^k(x)). \tag{1.4}$$

Remark 1.2. If $O \in \mathrm{Per}(n)$, then the Birkhoff sum ϕ_x^n is the same for every $x \in O$. We might therefore refer to it as ϕ_O^n.

1.2. **Transfer operator.** Let us fix a smooth function $\tau \in C^\infty(\mathbb{T})$. We are interested in the partially expanding dynamical system on $\mathbb{T} \times \mathbb{R}$ defined by

$$F(x, y) = (E(x), y + \tau(x)) \tag{1.5}$$

We introduce the transfer operator

$$\mathcal{L}_\tau : \left\{ \begin{array}{ccc} L^2(\mathbb{T} \times \mathbb{R}) & \longrightarrow & L^2(\mathbb{T} \times \mathbb{R}) \\ u & \longmapsto & u \circ F \end{array} \right. .$$

1.3. **Reduction of the transfer operator.** Due to the particular form of the map F, the Fourier modes in y are invariant under $\mathcal{L}_\tau$: if for some $\xi \in \mathbb{R}$ and some $v \in L^2(\mathbb{T})$,

$$u(x, y) = v(x)e^{i\xi y},$$

then

$$\mathcal{L}_\tau u(x, y) = e^{i\xi\tau(x)} v(E(x))e^{i\xi y}.$$

Given $\xi \geq 0$ and a function τ, let us consequently consider the transfer operator $\mathcal{L}_{\xi,\tau}$ defined on smooth functions $v \in C^\infty(\mathbb{T})$ by

$$\forall x \in \mathbb{T}, \mathcal{L}_{\xi,\tau} v(x) := e^{i\xi\tau(x)} v(E(x)),$$

1.4. **Spectrum and flat trace.** The operator $\mathcal{L}_{\xi,\tau}$ can be extended to distributions $\mathcal{D}'(\mathbb{T})$ by duality. We will denote $H^m(\mathbb{T})$ the Sobolev space of order $m \in \mathbb{R}$.

Theorem 1.3 ([Rue86])**.** *For every $\epsilon > 0$, there exists a real number m_ϵ such that for every $m \geq m_\epsilon$, $\mathcal{L}_{\xi,\tau} : H^{-m}(\mathbb{T}) \to H^{-m}(\mathbb{T})$ is bounded and has an essential spectral radius smaller than ϵ. Moreover the discrete eigenvalues of modulus larger than ϵ and the associated eigenspaces do not depend on m.*

The collection of those discrete eigenvalues, is denoted by $\mathrm{Res}(\mathcal{L}_{\xi,\tau})$ and called the Ruelle-Pollicott spectrum of the transfer operator $\mathcal{L}_{\xi,\tau}$. We are interested in their statistical properties when τ is a random function. One way to get informations about the spectrum of such operators is using a trace formula. Although $\mathcal{L}_{\xi,\tau}$ is not trace-class, and although the Ruelle-Pollicott spectrum might not be summable, we can give a certain sense to the trace of $\mathcal{L}_{\xi,\tau}$.

Lemma 1.4 (Trace formula, [AB67], [G$^+$77])**.** *For any integer $n \geq 1$, $\mathcal{L}^n_{\xi,\tau}$ is a kernel integral operator with Schwartz kernel*

$$K^n_{\xi,\tau}(x,y) = e^{i\xi\tau^n_x} \delta(y - E^n(x)),$$

where τ^n_x is defined as in (1.4). The distribution $K^n_{\xi,\tau}$ can be integrated along the diagonal $\{(x,x), x \in \mathbb{T}\}$. The resulting quantity is called flat trace of $\mathcal{L}^n_{\xi,\tau}$ and can be expressed as a sum over periodic points:

$$\mathrm{Tr}^\flat \mathcal{L}^n_{\xi,\tau} := \int_{\mathbb{T}} K^n_{\xi,\tau}(x,x)dx = \sum_{x, E^n(x)=x} \frac{e^{i\xi\tau^n_x}}{(E^n)'(x) - 1} \tag{1.6}$$

Proof. Let $u \in L^2(\mathbb{T})$.

$$\begin{aligned} \mathcal{L}^n_{\xi,\tau} u(x) &= e^{i\xi\tau^n_x} u(E^n(x)) \\ &= \int_{\mathbb{T}} e^{i\xi\tau^n_x} \delta(y - E^n(x)) u(y) dy. \end{aligned}$$

Thus

$$K^n_{\xi,\tau}(x,x) = e^{i\xi\tau^n_x} \delta(x - E^n(x)).$$

Since E is expanding, $x \mapsto x - E^n(x)$ is a local diffeomorphism. We can therefore effectuate the change of variables $y = x - E^n(x)$ in the following integral:

$$\begin{aligned} \int_{\mathbb{T}} K^n_{\xi,\tau}(x,x)\mathrm{d}x &= \int \delta(y) \sum_{x - E^n(x) = y} \frac{e^{i\xi\tau^n_x}}{|1 - (E^n)'(x)|} dy \\ &= \sum_{x, E^n(x)=x} \frac{e^{i\xi\tau^n_x}}{(E^n)'(x) - 1}. \end{aligned}$$

□

This trace is related to the Ruelle-Pollicott spectrum defined above in the following way (see for instance Theorem 1.8 in [Jéz17]):

Proposition 1.5. *Let $\xi \in \mathbb{R}$, let $\epsilon > 0$ be such that $\mathcal{L}_{\xi,\tau}$ has no resonance of modulus ϵ, then*

$$\exists C > 0, \forall n \in \mathbb{N}, \left| \mathrm{Tr}^\flat \mathcal{L}^n_{\xi,\tau} - \sum_{\substack{\lambda \in \mathrm{Res}(\mathcal{L}_{\xi,\tau}) \\ |\lambda| > \epsilon}} \lambda^n \right| \leq C\epsilon^n. \tag{1.7}$$

1.5. **Gaussian random fields.** We define our random functions on the circle by means of their Fourier coefficients. We are only interested in $\mathcal{C}^1$ functions.

Definition 1.6. We will call centered stationary Gaussian random fields on $\mathbb{T}$ the random functions $\tau : \mathbb{T} \longrightarrow \mathbb{R}$ whose Fourier coefficients $(c_p(\tau))_{p\in\mathbb{N}}$ are independent Gaussian random variables.

The decay of the variances of the coefficients gives sufficient conditions for the regularity of the field.

Lemma 1.7. *Let $k \in \mathbb{N}^*$. Let τ be a stationary centered Gaussian random field, such that for some $\eta > 0$*

$$\sigma_p^2 := \mathbb{E}\left[|c_p(\tau)|^2\right] = O(\frac{1}{p^{2k+2+\eta}}).$$

Then τ is almost surely $\mathcal{C}^k$.

Proof. See appendix B. □

A stationary centered Gaussian random field is characterized by its covariance function:

Definition 1.8. Let τ be a stationary centered Gaussian random field, and let $(\sigma_p^2)_{p\in\mathbb{Z}}$ be the family of variances of its Fourier coefficients. Its covariance function K is defined by

$$K(x) = \sum_p \sigma_p^2 e^{2i\pi px}. \tag{1.8}$$

In other words, for any couple of points $(x, y) \in \mathbb{T}^2$,

$$\mathbb{E}\left[\tau(x)\tau(y)\right] = K(x - y).$$

The following lemma follows immediately from the previous definition.

Lemma 1.9. *Let τ and τ' be two stationary centered Gaussian random fields, with repective covariance functions K_τ and $K_{\tau'}$. $\tau + \tau'$ is also a stationary centered Gaussian random field, and its covariance function $K_{\tau+\tau'}$ satisfies*

$$K_{\tau+\tau'} = K_\tau + K_{\tau'}$$

if, and only if τ and τ' are independent (that is if, and only if their Fourier coefficients are independent).

In our setting, the following theorem is straightforward.

Lemma 1.10 (Bochner's theorem)**.** *Let K be a continuous function on $\mathbb{T}$. K is the covariance function of a stationary Gaussian field if, and only if its Fourier coefficients are non-negative.*

We will denote by $\mathcal{N}(0,\sigma^2)$ (respectively $\mathcal{N}_{\mathbb{C}}(0,\sigma^2)$) the real (respectively complex) centered Gaussian law of variance σ^2, with respective densities

$$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2\sigma^2}x^2} \text{ and } \frac{1}{\sigma\pi}e^{-\frac{1}{\sigma^2}|z|^2}.$$

1.6. **Result.** Let us define for every $n\in\mathbb{N}$:

$$A_n := \left(\sum_{m|n} m^2 \sum_{O\in\mathcal{P}_m} \frac{1}{(e^{J_O^n}-1)^2}\right)^{-\frac{1}{2}} \tag{1.9}$$

Theorem 1.11. *Let $k\in\mathbb{N}^*$. Let $\tau_0\in\mathcal{C}^k(\mathbb{T})$. We suppose that the function τ that enters in (1.5) has the form*

$$\tau = \tau_0 + \delta\tau \tag{1.10}$$

with $\delta\tau$ a stationary Gaussian random field, whose covariance function K has big enough Fourier coefficients, in the sense that

$$\exists\epsilon>0, \exists C>0, \forall p\in\mathbb{Z}^*, c_p(K)\geq \frac{C}{p^{2k+2+\epsilon}}. \tag{1.11}$$

Then one has the convergence in law of the flat traces

$$A_n \mathrm{Tr}^\flat\left(\mathcal{L}^n_{\xi,\tau_0+\delta\tau}\right) \longrightarrow \mathcal{N}_{\mathbb{C}}(0,1) \tag{1.12}$$

as n,ξ go to infinity, with the constraint

$$\exists 0<c<1, \forall(n,\xi),\ \text{s.t. } n\leq c\frac{\log\xi}{\log l + (k+\frac{1}{2}+\frac{\epsilon}{2})\log M}. \tag{1.13}$$

Moreover, as stated in Lemma 1.7, if there is $0<\eta<\epsilon$ such that in addition

$$\exists C>0, \forall p\in\mathbb{Z}^*, c_p(K)\leq \frac{C}{p^{2k+2+\eta}},$$

then $\delta\tau$ is a.s. $\mathcal{C}^k$.

Remark 1.12. Condition (1.13) means that time n is smaller than a constant times the Ehrenfest time $\log\xi$, and this constant decreases with the regularity k of the field $\delta\tau$.

Remark 1.13. Let us denote by π_ϵ the spectral projector of the operator $\mathcal{L}_{\xi,\tau}$ outside the disk centered on 0 of radius ϵ. Then the sum of powers of eigenvalues in (1.7) can be written

$$\mathrm{Tr}(\pi_\epsilon\mathcal{L}^n_{\xi,\tau}) = \sum_{\substack{\lambda\in\mathrm{Res}(\mathcal{L}_{\xi,\tau})\\ |\lambda|\geq\epsilon}} \lambda^n.$$

Remark that the constant involved in the right hand side of the formula (1.7) might depend on ξ. But, if we assume that the right hand side is $C\xi\epsilon^n$, i.e.

$$\exists C>0, \forall\xi>0, \forall\epsilon>0, \forall n\in\mathbb{N}, \left|\mathrm{Tr}^\flat\mathcal{L}^n_{\xi,\tau} - \sum_{|\lambda_{j,\xi}|\geq\epsilon}\lambda^n_{j,\xi}\right| \leq C\xi\epsilon^n, \tag{1.14}$$

then for $\epsilon \ll \frac{1}{Ml}$, we have the same convergence as the flat trace in theorem 1.11:

$$A_n \mathrm{Tr}(\pi_\epsilon \mathcal{L}^n_{\xi,\tau}) \underset{\xi\to\infty}{\longrightarrow} \mathcal{N}_{\mathbb{C}}(0,1)$$

The hypothesis (1.14) is proved only in the case of a prequantum map (see lemma 9.3.4 p.175 in [FT12]). It can be conjectured for a generic map.

1.7. **Sketch of proof.** The proof is based around the following arguments:

(1) For ξ large enough, we show that the phases appearing in the trace formula (1.6) behave as uniform random variables. The phases corresponding to periodic points in different orbits are independent from each other. $\mathrm{Tr}^\flat(\mathcal{L}^n_{\xi,\tau})$ is thus similar to a random walk on $\mathbb{C}$ where each step has some deterministic modulus and a uniform random phase. We therefore adapt the proof of proposition C.1 in appendix C that concerns this random walk model.

(2) An important point is that if a random field τ satisfies the convergence of theorem 1.11, so will $\tau + \tilde{\tau}$ for any random field $\tilde{\tau}$ independent of τ. We can indeed adapt the following lemma:

Lemma 1.14. *Let X, X' be real independent random variables such that $e^{iX}, e^{iX'}$ are uniform on S^1. Let Y, Y' be real random variables such that X and X' are independent of Y, Y', then $e^{i(X+Y)}$ and $e^{i(X'+Y')}$ are still independent uniform random variables on S^1.*

See appendix D for the proof.

(3) We use the previous point to construct a random field $\delta\tau_0$ satisfying the convergence of theorem 1.11. To do so, we write

$$\delta\tau_0 = \sum_{j\geq 1} \delta\tau_j,$$

where the $\delta\tau_j$ are independent Gaussian random fields such that

$$\forall j \geq 1,\ \forall x \neq y \in \mathrm{Per}(j), \delta\tau_j(x) \text{ is independent of } \delta\tau_j(y).$$

Thus, for any n, we can write

$$\delta\tau_0 = \delta\tau_n + \delta\tilde{\tau},$$

so that $\delta\tilde{\tau}$ is independent of $\delta\tau_n$ and that for any pair of points $x \in O \in \mathrm{Per(n)}, y \in O' \in \mathrm{Per(n)}$, if $O \neq O'$, then the random variables $e^{i\xi(\tau_n)^n_x}$ and $e^{i\xi(\tau_n)^n_y}$ are independent and close to be uniform for large values of ξ.

(4) According to Lemma 1.1, the distance between periodic points decreases exponentially fast with n. Hence, if we want $\delta\tau_0$ to be $\mathcal{C}^k$, we need the amplitude $\sup|\delta\tau_n|$ to decrease also exponentially fast with n. Precisely we take the standard deviation $\sqrt{K_n(0)}$ of order $\frac{1}{M^{n(k+\frac{1}{2}+\epsilon/2)}}$. Then, for a periodic point x of period n the Birkhoff sum $(\delta\tau_n)^n_x$ is of typical size $n\sqrt{K_n(0)} \equiv \frac{n}{M^{n(k+\frac{1}{2}+\epsilon/2)}}$. To obtain a uniform phase, we want $\xi\tau^n_x \gg 1$, thus $\xi \gg M^{n(k+\frac{1}{2}+\epsilon/2)}$. If we want this condition to hold for most of the $l^n - 1$ periodic point, a factor l^n appears, and we obtain heuristically the condition:

$$\xi \gg l^n M^{n(k+\frac{1}{2}+\epsilon/2)}$$

that gives (1.13).

(5) The field $\delta\tau_0$ constructed this way has a covariance function K_0 satisfying

$$c_p(K_0) = O(\frac{1}{p^{2k+2+\epsilon}}),$$

and this implies that $\delta\tau_0$ is almost surely $\mathcal{C}^k$. Now, if a Gaussian field has a covariance function K whose Fourier coefficients satisfy

$$\exists C > 0, \forall p \in \mathbb{Z},\ c_p(K) \geq \frac{C}{p^{2k+2+\epsilon}}$$

then

$$\exists C > 0, K = CK_0 + K',$$

where K' has non negative Fourier coefficients, and is therefore from theorem 1.10 the covariant function of a centered Gaussian random field $\delta\tau'$ independent of $\delta\tau_0$. The Gaussian field $\delta\tau := \delta\tau_0 + \delta\tau'$ then defined satisfies the convergence of theorem 1.11 as we saw in step (2).

2. Numerical experiments

We consider an example with the non linear expanding map

$$E(x) = 2x + 0.9/(2\pi)\sin(2\pi(x + 0.4)) \tag{2.1}$$

plotted on Figure 1. In Figure 2, we have the histogram of the modulus $S = |A_n \mathrm{Tr}^\flat \left(\mathcal{L}^n_{\xi,\tau_0+\delta\tau}\right)|$ obtained after a sample of 10^4 random functions $\delta\tau$. We compare the histogram with the function $CS\exp(-S^2)$ in red, i.e. the radial distribution of a Gaussian function, obtained from the prediction of Theorem 1.11. We took $n = 11$, $\xi = 2.10^6$, $\tau_0 = \cos(2\pi x)$. We see a good agreement.

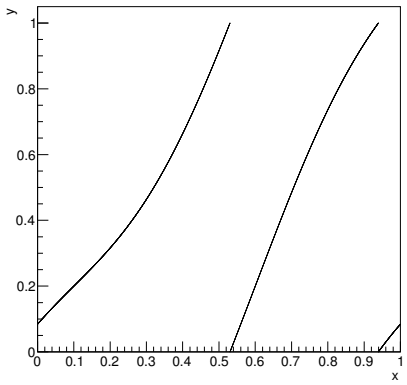

Figure 1. Graph of the expanding map $E(x)$ in Eq.(2.1)

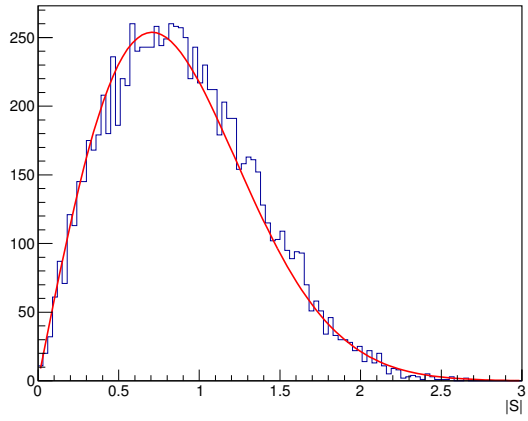

FIGURE 2. In blue, the histogram of $S = \left|A_n \mathrm{Tr}^\flat \left(\mathcal{L}^n_{\xi,\tau_0+\delta\tau}\right)\right|$ for $n = 11$, $\xi = 2.10^6$, $\tau_0 = \cos(2\pi x)$ and the sample 10^4 random functions $\delta\tau$. The histogram is well fitted by $CS\exp(-S^2)$ in red, as predicted by Theorem 1.11

3. Proof of theorem 1.11

3.1. **Definition of a Gaussian field satisfying theorem 1.11.** Let us define the Gaussian field mentioned in step (3) of the sketch of proof. Let $K_{\mathrm{init}} \in \mathcal{C}^\infty_c(\mathbb{R})$ be a smooth function supported in $\left[-\frac{1}{3}, \frac{1}{3}\right]$, with non negative Fourier transform, satisfying [1]

$$K_{\mathrm{init}}(0) = 1. \tag{3.1}$$

Let $k \geq 1$ be the integer involved in theorem 1.11 giving the regularity of the field. let $\epsilon > 0$ be the constant appearing in Theorem 1.11 and define for any integer $j \geq 1$

$$K_j(x) = \frac{1}{M^{j(2k+1+\epsilon)}} K_{\mathrm{init}}(M^j x). \tag{3.2}$$

Let

$$K_0(x) = \sum_{j\geq 1} K_j(x). \tag{3.3}$$

The Fourier transform of K_j is given by

$$\widehat{K_j}(\xi) = \frac{1}{M^{j(2k+2+\epsilon)}} \widehat{K_{\mathrm{init}}}\left(\frac{\xi}{M^j}\right) \geq 0 \tag{3.4}$$

Thus, $\widehat{K}_0 \geq 0$. The functions K_0 and K_j, for all $j \geq 0$, are supported in $\left[-\frac{1}{3}, \frac{1}{3}\right]$ and can then be seen as functions on the circle $\mathbb{T}$ by trivially periodizing them. The Fourier coefficients are given by $c_p(K_0) = \widehat{K}_0(2\pi p) \geq 0$. Hence, by Bochner's theorem 1.10, K_0 is the covariance function of a stationary Gaussian field $\delta\tau_0$ on

[1] To construct such a function, take a non zero even function $g \in \mathcal{C}^\infty_c(\mathbb{R})$. g has a real Fourier transform. Then $g * g \in \mathcal{C}^\infty_c(\mathbb{R})$ and its Fourier transform is $\hat{g}^2 \geq 0$. Moreover $g * g(0) = \int \hat{g}^2 > 0$.

$\mathbb{T}$. Similarly, each K_j defines a stationary Gaussian field $\delta\tau_j$ on $\mathbb{T}$, $\delta\tau_0 = \sum_j \delta\tau_j$ and (3.3) signifies that the fields $(\delta\tau_j)_j$ are pairwise independent.

Lemma 3.1. *With the choice (3.2), the field $\delta\tau_0$ is almost surely $\mathcal{C}^k$.*

Proof. We have seen in Eq.(3.4) that

$$\widehat{K_j}(\xi) = \frac{1}{M^{j(2k+2+\epsilon)}} \widehat{K_{\text{init}}}(\frac{\xi}{M^j}).$$

Since K_{init} is smooth, there exists a constant $C > 0$ such that

$$\forall \xi \in \mathbb{R}, \widehat{K_{\text{init}}}(\xi) \leq \frac{C}{\langle \xi \rangle^{2k+2+\frac{\epsilon}{2}}},$$

with the usual notation $\langle \xi \rangle = \sqrt{1+\xi^2} \geq |\xi|$. Thus, by equation (1.8), the variance of the p-th Fourier coefficient of $\delta\tau_j$ is

$$\sigma_{p,j}^2 = \widehat{K_j}(2\pi p)$$

and therefore

$$\begin{aligned}\sigma_{p,j}^2 &= \frac{1}{M^{j(2k+2+\epsilon)}} \widehat{K_{\text{init}}}(\frac{2\pi p}{M^j}) \\ &\leq \frac{C}{M^{j\frac{\epsilon}{2}}} \frac{1}{|2\pi p|^{2k+2+\frac{\epsilon}{2}}}.\end{aligned}$$

Consequently the p-th Fourier coefficient of $\delta\tau_0$ has variance

$$\sigma_p^2 = O(\frac{1}{|2\pi p|^{2k+2+\frac{\epsilon}{2}}}).$$

We can then apply lemma 1.7 to conclude.

□

3.2. **New expression of $\mathrm{Tr}^\flat(\mathcal{L}^n_{\xi,\tau})$.** Let us rewrite the sum $\mathrm{Tr}^\flat(\mathcal{L}^n_{\xi,\tau})$, where $\tau = \tau_0 + \delta\tau_0$ with a given τ_0 as in section 1.2. We know from (1.6) that

$$\begin{aligned}\mathrm{Tr}^\flat(\mathcal{L}^n_{\xi,\tau}) &= \sum_{E^n(x)=x} \frac{e^{i\xi\tau_x^n}}{(E^n)'(x)-1} \\ &= \sum_{E^n(x)=x} \frac{e^{i\xi\tau_x^n}}{e^{J_x^n}-1},\end{aligned}$$

where $J(x) = \log(E'(x)) > 0$ and J_x^n is the Birkhoff sum as defined in (1.4). If $\mathcal{P}_n$ denotes the set of primitive periodic orbits of period n and if f_O^n stands for the Birkhoff sum f_x^n for any $x \in O$, let us write

$$\mathrm{Tr}^\flat(\mathcal{L}^n_{\xi,\tau}) = \sum_{m|n} m \sum_{O\in\mathcal{P}_m} \frac{e^{i\xi\tau_O^n}}{e^{J_O^n}-1}. \tag{3.5}$$

For $O \in \mathrm{Per}(n)$ (set of non primitive periodic orbits of period n), we can write

$$\tau_O^n = (\delta\tau_n)_O^n + \sum_{j\neq n} (\delta\tau_j)_O^n + (\tau_0)_O^n.$$

Since the covariance function K_n is supported in $\left[-\frac{1}{3M^n}, \frac{1}{3M^n}\right]$, we deduce from Lemma 1.1 that the values taken by $\delta\tau_n$ at different periodic points of period dividing n, which have law $\mathcal{N}(0, K_n(0))$ are independent random variables.

Thus, for $O \in \mathcal{P}_m$, $(\delta\tau_n)_O^m$ is a centered Gaussian random variable of variance $mK_n(0)$, and $(\delta\tau_n)_O^n = \frac{n}{m}(\delta\tau_n)_O^m$ with variance $(\frac{n}{m})^2 mK_n(0) = \frac{n^2}{m}K_n(0)$ and has law $\mathcal{N}(0, \frac{n^2}{m}K_n(0))$.

We introduce the following Definition that will be used many times later.

Definition 3.2. We say that two families of real random variables $(X_O^n)_{\substack{n\geq 1\\ O\in Per(n)}}$ and $(Y_O^n)_{\substack{n\geq 1\\ O\in Per(n)}}$ satisfy condition (C) if

(1) X_O^n has law $\mathcal{N}(0, \frac{n^2}{m}K_n(0))$,
(2) for every $O' \neq O \in \mathrm{Per}(n)$ and every $O'' \in \mathrm{Per}(n)$, X_O^n is independent of $X_{O'}^n$ and $Y_{O''}^n$.

Writing $X_O^n = (\delta\tau_n)_O^n$ and $Y_O^n = \sum_{j\neq n}(\delta\tau_j)_O^n + (\tau_0)_O^n$, we have obtained

Lemma 3.3. *For each $O \in \mathcal{P}_m$, $m|n$,*

$$\tau_O^n = X_O^n + Y_O^n \tag{3.6}$$

with X_O^n, Y_O^n satisfying condition (C) of Definition (3.2)

In order to adapt the proofs of lemma C.1 and 1.14, we want to show that for large ξ, the random variables $e^{i\xi(X_O^n+Y_O^n)}$ are close to be independent and uniform on S^1.

Remark 3.4. We have

$$\mathrm{Tr}^\flat(\mathcal{L}_{\xi,\tau}^n) \underset{(3.5),(3.6)}{=} \sum_{m|n} m \sum_{O\in\mathcal{P}_m} \frac{e^{i\xi(X_O^n+Y_O^n)}}{e^{J_O^n}-1}. \tag{3.7}$$

Our aim is to approximate the characteristic function of $\mathrm{Tr}^\flat(\mathcal{L}_{\xi,\tau}^n)$ which is the expectation of

$$e^{i\left(\mu \mathrm{Re}(\mathrm{Tr}^\flat(\mathcal{L}_{\xi,\tau}^n))+\nu \mathrm{Im}(\mathrm{Tr}^\flat(\mathcal{L}_{\xi,\tau}^n))\right)} = \tag{3.8}$$

$$\prod_{m|n}\prod_{O\in\mathcal{P}_m} e^{im\left(\left(\frac{\mu}{e^{J_O^n}-1}\right)\cos(\xi(X_O^n+Y_O^n))+\left(\frac{\nu}{e^{J_O^n}-1}\right)\sin(\xi(X_O^n+Y_O^n))\right)}$$

The right hand side of (3.8) can be written as

$$\prod_{m|n}\left(\prod_{O\in\mathcal{P}_m} f_O\left(e^{i\xi(X_O^n+Y_O^n)}\right)\right)^m$$

with some continuous functions f_O. In the next Lemma we first consider indicator functions for f_O.

Lemma 3.5. *Let $(X_O^n)_{\substack{n\geq 1\\ O\in Per(n)}}$ and $(Y_O^n)_{\substack{n\geq 1\\ O\in Per(n)}}$ be two families of real random variables satisfying satisfying condition (C) of Definition (3.2). Assume that n and ξ satisfy (1.13). Then there is a constant $C > 0$ such that for every $n \in \mathbb{N}$ and every real numbers $(\alpha_O)_{O\in\mathrm{Per}(n)}, (\beta_O)_{O\in\mathrm{Per}(n)}$ such that*

$$\forall O \in \mathrm{Per}(n),\ 0 < \beta_O - \alpha_O < 2\pi,$$

for every complex numbers $(\lambda_O)_{O\in \mathrm{Per}(n)}$*, if* A_O *denotes the portion of circle* $e^{i]\alpha_O,\beta_O[}$ *and* $\mathbb{1}_{A_O}$ *is the characteristic function of* A_O*, we have*

$$\left| \frac{\mathbb{E}\left[\prod_{m|n}\left(\prod_{O\in\mathcal{P}_m}\lambda_O \mathbb{1}_{A_O}\left(e^{i\xi(X_O^n+Y_O^n)}\right)\right)^m\right]}{\prod_{m|n}\prod_{O\in\mathcal{P}_m}\lambda_O^m\left(\frac{\beta_O-\alpha_O}{2\pi}\right)} - 1\right| \leq \frac{Cl^n}{\xi\sqrt{n}M^{n(k+\frac{1}{2}+\frac{\epsilon}{2})}}. \tag{3.9}$$

Remark 3.6. In this complicated expression, we have compared the law of the family of random variables $\left(e^{i\xi(X_O^n+Y_O^n)}\right)_{O\in\mathrm{Per}(n)}$ to the uniform law on the torus of dimension $\#\mathrm{Per}(n)$. Remark that the right hand side is small for ξ large enough and later this will lead us to the final formula (1.13). The proof of this lemma is given in the next subsection.

3.3. **A normal law of large variance on the circle is close to uniform.** We will need the following lemma, which evaluates how much the law $\mathcal{N}(0,1) \mod \frac{1}{t}$ differs from the uniform law on the circle $\mathbb{R}/(\frac{1}{t}\mathbb{Z})$ for large values of t.

Lemma 3.7. *There exists a constant* $C>0$ *such that for every real numbers* α,β*, such that* $0<\beta-\alpha<2\pi$ *and every real number* $t\geq 1$*,*

$$\left|\int_{\mathbb{R}}\sum_{k\in\mathbb{Z}}\mathbb{1}_{\frac{\alpha+2k\pi}{t}\leq x\leq\frac{\beta+2k\pi}{t}}e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} - \frac{\beta-\alpha}{2\pi}\right| \leq \frac{C}{t}(\beta-\alpha).$$

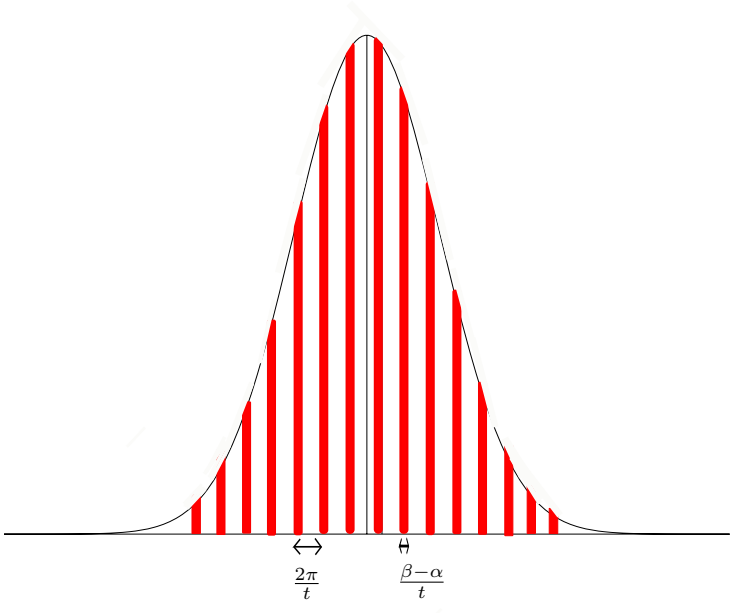

FIGURE 3. As t goes to infinity, the red area converges to $\frac{\beta-\alpha}{2\pi}$ with speed $O(\frac{\beta-\alpha}{t})$.

Proof. For $\frac{\alpha+2k\pi}{t} \leq x \leq \frac{\beta+2k\pi}{t}$,

$$\left|e^{-\frac{x^2}{2}} - e^{-\frac{1}{2}(\frac{\alpha+2k\pi}{t})^2}\right| \leq \left|x - \frac{\alpha+2k\pi}{t}\right| \sup_{u\in\left[\frac{\alpha+2k\pi}{t},x\right]} |u|e^{-\frac{u^2}{2}}.$$

By hypothesis

$$\left|x - \frac{\alpha+2k\pi}{t}\right| \leq \frac{\beta-\alpha}{t}.$$

Thus,

$$x - 1 \leq x - \frac{\beta-\alpha}{t} \leq \frac{\alpha+2k\pi}{t}$$

for t large enough and therefore

$$\sup_{\frac{\alpha+2k\pi}{t}\leq u\leq x} |u|e^{-\frac{u^2}{2}} \leq \sup_{x-\frac{\beta-\alpha}{t}\leq u\leq x} |u|e^{-\frac{u^2}{2}} \leq \sup_{u\in[x-1,x]} |u|e^{-\frac{u^2}{2}}.$$

Hence, for t large enough, we have an upper bound for the integrand

$$\begin{aligned}\left|e^{-\frac{x^2}{2}} - e^{\frac{1}{2}(\frac{\alpha+2k\pi}{t})^2}\right| &\leq \left|x - \frac{\alpha+2k\pi}{t}\right| \sup_{\frac{\alpha+2k\pi}{t}\leq u\leq x} |u|e^{-\frac{u^2}{2}} \\ &\leq \frac{\beta-\alpha}{t} \sup_{x-1\leq u\leq x} |u|e^{-\frac{u^2}{2}}.\end{aligned}$$

This upper bound is an integrable function of x. In consequence, we can exchange the sum and integral in the following expression

(3.10)

$$\begin{aligned}A_t &:= \left|\int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \mathbb{1}_{\frac{\alpha+2k\pi}{t}\leq x\leq\frac{\beta+2k\pi}{t}} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} - \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \mathbb{1}_{\frac{\alpha+2k\pi}{t}\leq x\leq\frac{\beta+2k\pi}{t}} e^{-\frac{1}{2}(\frac{\alpha+2k\pi}{t})^2} \frac{dx}{\sqrt{2\pi}}\right| \\ &\leq \sum_{k\in\mathbb{Z}} \int_{\frac{\alpha+2k\pi}{t}}^{\frac{\beta+2k\pi}{t}} \left|e^{-\frac{x^2}{2}} - e^{-\frac{1}{2}(\frac{\alpha+2k\pi}{t})^2}\right| \frac{dx}{\sqrt{2\pi}}.\end{aligned}$$

Then,

$$A_t \leq \frac{\beta-\alpha}{t} \int_{\mathbb{R}} \sup_{u\in[x-1,x]} |u|e^{-\frac{x^2}{2}} \frac{\mathrm{d}x}{\sqrt{2\pi}};$$

there is consequently a constant C independent of α, β and t such that

$$\forall t \geq 1, A_t \leq \frac{C}{t}(\beta-\alpha). \tag{3.11}$$

We notice that the term

$$B_t := \sum_{k\in\mathbb{Z}} \int_{\frac{\alpha+2k\pi}{t}}^{\frac{\beta+2k\pi}{t}} e^{-\frac{1}{2}(\frac{\alpha+2k\pi}{t})^2} \frac{dx}{\sqrt{2\pi}}$$

appearing in (3.10) is a Riemann sum up to a constant factor:

$$B_t = \frac{\beta-\alpha}{\sqrt{2\pi}t} \sum_{k\in\mathbb{Z}} e^{-\frac{1}{2}(\frac{\alpha+2k\pi}{t})^2}.$$

We know that there exists a constant C' such that

$$\left|\int_{\mathbb{R}} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} - \frac{2\pi}{t} \sum_{k\in\mathbb{Z}} \frac{e^{-\frac{1}{2}(\frac{\alpha+2k\pi}{t})^2}}{\sqrt{2\pi}}\right| \leq \frac{C'}{t}.$$

Thus,

$$\left|\frac{\beta-\alpha}{2\pi}\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} - B_t\right| \leq \frac{C'}{t}\frac{\beta-\alpha}{2\pi}. \tag{3.12}$$

Since

$$\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} = 1,$$

the combination of (3.11) and (3.12) furnishes the result. □

Proof of lemma 3.5. Let us denote by E the expectation

$$E := \mathbb{E}\left[\prod_{m|n}\left(\prod_{O\in\mathcal{P}_m}\lambda_O \mathbb{1}_{A_O}\left(e^{i\xi(X_O^n+Y_O^n)}\right)\right)^m\right].$$

If we write respectively $\mathbb{P}_X$, $\mathbb{P}_Y$ and $\mathbb{P}_{X,Y}$ the probability laws of the variables $(\xi X_O^n)_{O\in\mathrm{Per}(n)}$, $(\xi Y_O^n)_{O\in\mathrm{Per}(n)}$ and $(\xi X_O^n)_{O\in\mathrm{Per}(n)}\cup(\xi Y_O^n)_{O\in\mathrm{Per}(n)}$ respectively, then condition (C) of Definition (3.2) implies

$$d\mathbb{P}_{X,Y}((x_O)_{O\in\mathrm{Per}(n)},(y_O)_{O\in\mathrm{Per}(n)}) = \prod_{m|n}\prod_{O\in\mathcal{P}_m} e^{-\frac{{x_O}^2}{2\sigma_{n,\xi}^2}}\frac{dx_O}{\sigma_{n,\xi}\sqrt{2\pi}}\otimes d\mathbb{P}_Y((y_O)_{O\in\mathrm{Per}(n)}). \tag{3.13}$$

with the variance $\sigma_{n,\xi}^2 := \xi^2\frac{n^2}{m}K_n(0)$. We have

$$E = \int_{\mathbb{R}^{2\#\mathrm{Per}(n)}}\prod_{O\in\mathrm{Per}(n)}\left(\sum_{k\in\mathbb{Z}}\lambda_O^m\mathbb{1}_{]\alpha_O+2k\pi,\beta_O+2k\pi[}(x_O+y_O)\right) d\mathbb{P}_{X,Y}((x_O)_{O\in\mathrm{Per}(n)},(y_O)_{O\in\mathrm{Per}(n)}). \tag{3.14}$$

Thus, writing $u_O = \frac{x_O}{\sigma_{n,\xi}}$ for $O\in\mathcal{P}_m$,

$$E = \int_{\mathbb{R}^{\#\mathrm{Per}(n)}}\prod_{m|n}\prod_{O\in\mathcal{P}_m}\left(\int_{\mathbb{R}}\sum_{k\in\mathbb{Z}}\lambda_O^m\mathbb{1}_{\left]\frac{\alpha_O-y_O+2k\pi}{\sigma_{n,\xi}},\frac{\beta_O-y_O+2k\pi}{\sigma_{n,\xi}}\right[}(u_O)e^{-\frac{u_O^2}{2}}\frac{du_O}{\sqrt{2\pi}}\right) d\mathbb{P}_Y((y_O)_{O\in\mathcal{P}_n}). \tag{3.15}$$

Let us write for $O\in\mathrm{Per}(n)$

$$I_O = \int_{\mathbb{R}}\sum_{k\in\mathbb{Z}}\lambda_O^m\mathbb{1}_{\left]\frac{\alpha_O-y_O+2k\pi}{\sigma_{n,\xi}},\frac{\beta_O-y_O+2k\pi}{\sigma_{n,\xi}}\right[}(u_O)e^{-\frac{u_O^2}{2}}\frac{du_O}{\sqrt{2\pi}}.$$

Lemma 3.7 yields

$$I_O = \lambda_O^m\left(\frac{\beta_O-\alpha_O}{2\pi}+\epsilon_O\right),$$

where

$$\begin{aligned}\exists C>0, |\epsilon_O| &\leq \frac{C}{\sigma_{n,\xi}}\frac{\beta_O-\alpha_O}{2\pi}\\ &\leq \frac{C}{\xi\sqrt{nK_n(0)}}\frac{\beta_O-\alpha_O}{2\pi}.\end{aligned}$$

We will develop the product

$$\prod_{m|n}\prod_{O\in\mathcal{P}_m} I_O^m = \prod_{m|n}\prod_{O\in\mathcal{P}_m} \lambda_O^m\left(\frac{\beta_O-\alpha_O}{2\pi}+\epsilon_O\right).$$

The notation $\{O_1,\cdots,O_q\}\subset \mathrm{Per}(n)$ will implicitly signify that $O_1,\cdots,O_q$ are *distinct* elements of $\mathrm{Per}(n)$.

$$\begin{aligned}
&\left|\prod_{m|n}\prod_{O\in\mathcal{P}_m} I_O^m - \prod_{m|n}\prod_{O\in\mathcal{P}_m} \lambda_O^m\left(\frac{\beta_O-\alpha_O}{2\pi}\right)\right| \\
&= \left|\prod_{m|n}\prod_{O\in\mathcal{P}_m} \lambda_O^m\right| \left|\sum_{q=1}^{\#\mathrm{Per}(n)} \sum_{\{O_1,\cdots,O_q\}\subset\mathrm{Per}(n)} \prod_{p=1}^{q}\epsilon_{O_p} \prod_{O\in\mathrm{Per}(n)\setminus\{O_1,\cdots,O_q\}} \frac{\beta_O-\alpha_O}{2\pi}\right| \\
&\qquad \le \sum_{q=1}^{\#\mathrm{Per}(n)} \binom{\#\mathrm{Per}(n)}{q} \frac{C^q}{\xi^q (nK_n(0))^{q/2}} \prod_{m|n}\prod_{O\in\mathcal{P}_m} |\lambda_O^m| \frac{\beta_O-\alpha_O}{2\pi}
\end{aligned}$$

From Lemma 1.1 we have $\#\mathrm{Per}(n)\le l^n$ thus

$$\begin{aligned}
&\left|\prod_{m|n}\prod_{O\in\mathcal{P}_m} I_O - \prod_{m|n}\prod_{O\in\mathcal{P}_m} \lambda_O^m\left(\frac{\beta_O-\alpha_O}{2\pi}\right)\right| \\
&\qquad \le \left(\left(1+\frac{C}{\xi(nK_n(0))^{1/2}}\right)^{l^n}-1\right) \prod_{m|n}\prod_{O\in\mathcal{P}_m} |\lambda_O^m| \frac{\beta_O-\alpha_O}{2\pi}.
\end{aligned}$$

Using hypothesis (1.13) we can bound the prefactor:

$$\begin{aligned}
\left(1+\frac{C}{\xi(nK_n(0))^{1/2}}\right)^{l^n}-1 &\underset{(3.2),(3.1)}{=} \left(1+\frac{CM^{n(k+\frac{1}{2}+\frac{\epsilon}{2})}}{\xi\sqrt{n}}\right)^{l^n}-1 \\
&\le \exp(l^n\frac{CM^{n(k+\frac{1}{2}+\frac{\epsilon}{2})}}{\xi\sqrt{n}})-1 \\
&\le C'\frac{l^n M^{n(k+\frac{1}{2}+\frac{\epsilon}{2})}}{\xi\sqrt{n}}
\end{aligned}$$

for some $C'>0$ since

$$l^n\frac{CM^{n(k+\frac{1}{2}+\frac{\epsilon}{2})}}{\xi\sqrt{n}} \underset{(1.13)}{\le} Cl^{n(1-\frac{1}{c})}M^{n(k+\frac{1}{2}+\frac{\epsilon}{2})(1-\frac{1}{c})}n^{-\frac{1}{2}} \to 0. \tag{3.16}$$

□

3.4. $\mathrm{Tr}^\flat(\mathcal{L}^n_{\xi,\tau_0+\delta\tau_0})$ **satisfies the convergence from theorem 1.11.** When can now easily extend the lemma 3.5 from characteristic functions to step functions.

Corollary 3.8. *Assume that n and ξ satisfy (1.13). For any families $(X^n_O)_{\substack{n\ge1\\O\in Per(n)}}$ and $(Y^n_O)_{\substack{n\ge1\\O\in Per(n)}}$ of real random variables satisfying condition (C) of Definition (3.2), there exists $C>0$ such that, if $(f_{n,O})_{\substack{n\ge1\\O\in Per(n)}}$ is a family of step functions*

$S^1 \to \mathbb{R}$, *then*

$$\left|\mathbb{E}\left[\prod_{m|n}\prod_{O\in\mathcal{P}_m} f_{n,O}^m(e^{i\xi(X_O^n+Y_O^n)})\right] - \prod_{m|n}\prod_{O\in\mathcal{P}_m}\int f_{n,O}^m d\mathrm{Leb}\right| \leq C\frac{l^n M^{n(k+\frac{1}{2}+\frac{\epsilon}{2})}}{\xi\sqrt{n}}\prod_{m|n}\prod_{O\in\mathcal{P}_m}\int |f_{n,O}^m| d\mathrm{Leb}. \tag{3.17}$$

Proof. Let us write each $f_{n,O}$ as

$$f_{n,O} = \sum_{q=1}^{p_{n,O}} \lambda_{n,O,q}\mathbb{1}_{A_{n,O,q}},$$

where the $\lambda_{n,O,q}$ are complex numbers and the $A_{n,O,q}, 1 \leq q \leq p_{n,O}$ are disjoint intervals. We develop (3.17), we use Lemma 3.5 and factorize the result:

$$\begin{aligned} E &:= \mathbb{E}\left[\prod_{m|n}\prod_{O\in\mathcal{P}_m} f_{n,O}^m(e^{i\xi(X_O^n+Y_O^n)})\right] \\ &= \sum_{(q_O)\in\prod_{m|n}\prod_{O\in\mathcal{P}_m}\{1,\cdots,p_{n,O}\}} \mathbb{E}\left[\prod_{m|n}\prod_{O\in\mathcal{P}_m}\lambda_{n,O,q_O}^m\mathbb{1}_{A_{n,O,q_O}}(e^{i\xi(X_O^n+Y_O^n)})\right]. \end{aligned}$$

Consequently,

$$\begin{aligned} &\left|E - \prod_{m|n}\prod_{O\in\mathcal{P}_m}\int f_{n,O}^m d\mathrm{Leb}\right| \\ &\leq \sum_{(q_O)\in\prod_{m|n}\prod_{O\in\mathcal{P}_m}\{1,\cdots,p_{n,O}\}} \left|\mathbb{E}\left[\prod_{m|n}\prod_{O\in\mathcal{P}_m}\lambda_{n,O,q_O}^m\mathbb{1}_{A_{n,O,q_O}}(e^{i\xi(X_O^n+Y_O^n)})\right] - \right. \\ &\qquad \left. \prod_{m|n}\prod_{O\in\mathcal{P}_m}\lambda_{n,O,q_O}^m \mathrm{Leb}(A_{n,O,q_O})\right| \\ &\leq C\frac{l^n M^{n(k+\frac{1}{2}+\frac{\epsilon}{2})}}{\xi\sqrt{n}} \sum_{(q_O)\in\prod_{m|n}\prod_{O\in\mathcal{P}_m}\{1,\cdots,p_{n,O}\}} \prod_{m|n}\prod_{O\in\mathcal{P}_m}|\lambda_{n,O,q_O}|^m \mathrm{Leb}(A_{n,O,q_O}) \end{aligned}$$

from the previous lemma.
Hence,

$$\left|E - \prod_{m|n}\prod_{O\in\mathcal{P}_m}\int f_{n,O}^m d\mathrm{Leb}\right| \leq C\frac{l^n M^{n(k+\frac{1}{2}+\frac{\epsilon}{2})}}{\xi\sqrt{n}}\prod_{m|n}\prod_{O\in\mathcal{P}_m}\int |f_{n,O}|^m d\mathrm{Leb}.$$

□

We can use this result in order to estimate the characteristic function of $\mathrm{Tr}^\flat(\mathcal{L}_{\xi,\tau}^n)$, using remark (3.4).

Corollary 3.9. *Assume that n and ξ satisfy (1.13). Let $(X_O^n)_{\substack{n\geq 1\\ O\in Per(n)}}$ and $(Y_O^n)_{\substack{n\geq 1\\ O\in Per(n)}}$ be two families of real random variables satisfying condition (C) of Definition (3.2). There exists $C>0$ such that for all $(\mu_O,\nu_O)_{O\in \mathrm{Per}(n)}\in \mathbb{R}^{2\#\mathrm{Per}(n)}$,*

$$\left|\mathbb{E}\left[\prod_{m|n}\prod_{O\in\mathcal{P}_m} e^{im\mu_O\cos(\xi(X_O^n+Y_O^n))+im\nu_O\sin(\xi(X_O^n+Y_O^n))}\right] - \prod_{m|n}\prod_{O\in\mathcal{P}_m}\int_0^{2\pi} e^{i(m\mu_O\cos\theta+m\nu_O\sin\theta)}\frac{d\theta}{2\pi}\right| \leq C\frac{l^nM^{n(k+\frac{1}{2}+\frac{\epsilon}{2})}}{\xi\sqrt{n}},$$

Proof. Let C be the constant from corollary 3.8. For $O\in \mathrm{Per}(n)$, let f_O be the function defined on S^1 by

$$f_O(e^{i\theta}) = e^{i(\mu_O\cos\theta+\nu_O\sin\theta)}.$$

Each f_O is bounded by 1, we can consequently find for each $O\in\mathrm{Per}(n)$ a family $(f_{j,O})_j$ of step functions uniformly bounded by 1 converging pointwise towards f_O. We have for n fixed, by dominated convergence

$$E_j := \mathbb{E}\left[\prod_{m|n}\prod_{O\in\mathcal{P}_m} f_{j,O}^m(e^{i\xi(X_O^n+Y_O^n)})\right] \xrightarrow[j\to\infty]{} E := \mathbb{E}\left[\prod_{m|n}\prod_{O\in\mathcal{P}_m} f^m(e^{i\xi(X_O^n+Y_O^n)})\right]$$

as well as

$$I_j := \prod_{m|n}\prod_{O\in\mathcal{P}_m}\int_0^{2\pi} f_{j,O}^m(e^{i\theta})\frac{d\theta}{2\pi} \xrightarrow[j\to\infty]{} I := \prod_{m|n}\prod_{O\in\mathcal{P}_m}\int_0^{2\pi} f_O^m(e^{i\theta})\frac{d\theta}{2\pi}.$$

It is thus possible to find an integer j_0 such that both

$$|E-E_{j_0}|\leq \frac{l^nM^{n(k+\frac{1}{2}+\frac{\epsilon}{2})}}{\xi\sqrt{n}}$$

and

$$|I-I_{j_0}|\leq \frac{l^nM^{n(k+\frac{1}{2}+\frac{\epsilon}{2})}}{\xi\sqrt{n}}$$

hold.
From corollary 3.8, we know that for all $n\in\mathbb{N}$

$$|E_{j_0}-I_{j_0}|\leq C\frac{l^nM^{n(k+\frac{1}{2}+\frac{\epsilon}{2})}}{\xi\sqrt{n}}\sup|f_{j_0}|.$$

Thus,

$$\begin{aligned}|E-I| &\leq |E-E_{j_0}|+|E_{j_0}-I_{j_0}|+|I-I_{j_0}|\\ &\leq (C+2)\frac{l^nM^{n(k+\frac{1}{2}+\frac{\epsilon}{2})}}{\xi\sqrt{n}}.\end{aligned}$$

□

We can know prove the final proposition :

Proposition 3.10. *Let* $(X_O^n)_{\substack{n\geq 1\\ O\in Per(n)}}$ *and* $(Y_O^n)_{\substack{n\geq 1\\ O\in Per(n)}}$ *be two families of real random variables satisfying condition (C) of Definition (3.2). If condition (1.13) is satisfied then we have the following convergence in law*

$$T_{n,\xi} := A_n \sum_{m|n} m \sum_{O\in\mathcal{P}_m} \frac{e^{i\xi(X_O^n+Y_O^n)}}{e^{J_O^n}-1} \underset{n,\xi\to\infty}{\longrightarrow} \mathcal{N}_{\mathbb{C}}(0,1), \tag{3.18}$$

with the amplitude A_n *defined in (1.9) by*

$$A_n = \left(\sum_{m|n} m^2 \sum_{O\in\mathcal{P}_m} \frac{1}{(e^{J_O^n}-1)^2}\right)^{-\frac{1}{2}}. \tag{3.19}$$

Proof. Let us fix two real numbers ξ_1 and ξ_2 and let ϕ_n be the characteristic function of $T_{n,\xi}$:

$$\phi_n(\xi_1,\xi_2) := \mathbb{E}\left[e^{iA_n\left(\xi_1 \sum_{m|n} m \sum_{O\in\mathcal{P}_m} \frac{\cos(\xi(X_O^n+Y_O^n))}{e^{J_O^n}-1} + \xi_2 \sum_{m|n} m \sum_{O\in\mathcal{P}_m} \frac{\sin(\xi(X_O^n+Y_O^n))}{e^{J_O^n}-1}\right)}\right].$$

We compute the limit of $\phi_n(\xi_1,\xi_2)$ as n goes to infinity. Corollary (3.9) yields

$$\left|\phi_n(\xi_1,\xi_2) - \prod_{m|n}\prod_{O\in\mathcal{P}_m} \int_0^{2\pi} e^{i\frac{mA_n}{e^{J_O^n}-1}(\xi_1\cos\theta+\xi_2\sin\theta)}\frac{d\theta}{2\pi}\right| \leq C\frac{l^n M^{n(k+\frac{1}{2}+\frac{\varepsilon}{2})}}{\xi\sqrt{n}} \to 0 \tag{3.20}$$

under the assumption (1.13).

Let

$$\psi(\xi_1,\xi_2) := \int_0^{2\pi} e^{i(\xi_1\cos\theta+\xi_2\sin\theta)}\frac{d\theta}{2\pi}.$$

We have the following Taylor's expansion in 0:

$$\psi(\xi_1,\xi_2) = 1 - \frac{1}{4}(\xi_1^2+\xi_2^2) + o(\xi_1^2+\xi_2^2).$$

In order to apply this to equation (3.20), we need to check that

Lemma 3.11.

$$nA_n \sup_{O\in \mathrm{Per}(n)} \frac{1}{e^{J_O^n}-1} \underset{n\to\infty}{\longrightarrow} 0.$$

Proof. See appendix E.3 □

We can now state that

$$\prod_{m|n}\prod_{O\in\mathcal{P}_m}\int_0^{2\pi} e^{i\frac{mA_n}{e^{J_O}-1}(\xi_1\cos\theta+\xi_2\sin\theta)}\frac{d\theta}{2\pi} = \prod_{m|n}\prod_{O\in\mathcal{P}_m}\psi(\xi_1\frac{mA_n}{e^{J_O}-1},\xi_2\frac{mA_n}{e^{J_O}-1})$$
$$= \prod_{m|n}\prod_{O\in\mathcal{P}_m}\left(1-\frac{\xi_1^2+\xi_2^2}{4}\frac{(mA_n)^2}{(e^{J_O}-1)^2}+o(\frac{(mA_n)^2}{(e^{J_O}-1)^2}\right)$$
$$= \exp\left(\sum_{m|n}\sum_{O\in\mathcal{P}_m}\log\left(1-\frac{\xi_1^2+\xi_2^2}{4}\frac{(mA_n)^2}{(e^{J_O}-1)^2}+o(\frac{(mA_n)^2}{(e^{J_O}-1)^2})\right)\right)$$
$$= \exp\left(\sum_{m|n}\sum_{O\in\mathcal{P}_m}-\frac{\xi_1^2+\xi_2^2}{4}\frac{(mA_n)^2}{(e^{J_O}-1)^2}+o(\frac{(mA_n)^2}{(e^{J_O}-1)^2})\right)$$
$$\underset{(3.19)}{=} e^{-\frac{\xi_1^2+\xi_2^2}{4}+o(1)},$$

We deduce that

$$\phi_n(\xi_1,\xi_2)\underset{n\to\infty}{\longrightarrow} e^{-\frac{\xi_1^2+\xi_2^2}{4}}$$

which is the characteristic function of a Gaussian variable of law $\mathcal{N}_{\mathbb{C}}(0,1)$. □

3.5. **End of the proof.** To end the proof of theorem 1.11, we use Bochner's theorem 1.10.

Proof of theorem 1.11. Let us first show that

Lemma 3.12. *The covariance function K_0 defined in (3.3) satisfies as $p\to\infty$*

$$c_p(K_0) = O\left(\frac{1}{p^{2k+2+\epsilon}}\right),$$

where k is as usual the regularity parameter from theorem 1.11.

Proof. We know from formula (3.4) that

$$\forall\xi\in\mathbb{R},\ \widehat{K_0}(\xi) = \sum_{j\leq 1} M^{-j(2k+2+\epsilon)}\widehat{K_{\text{init}}}(\frac{\xi}{M^j}).$$

Now, since K_{init} is smooth,

$$\forall N\in\mathbb{R}_+, \exists C_N>0, \forall\xi\in\mathbb{R}, \widehat{K_{\text{init}}}(\xi)\leq\frac{C_N}{\langle\xi\rangle^N},$$

where the japanese bracket

$$\langle\xi\rangle = \sqrt{1+\xi^2}.$$

Thus, for all $N\in\mathbb{R}_+$, there exists $C_N>0$, such that for all $p\in\mathbb{Z}$,

$$|c_p(K_0)| = \sum_{j\leq 1} M^{-j(2k+2+\epsilon)}\widehat{K_{\text{init}}}(\frac{2\pi p}{M^j})$$
$$\leq \frac{C_N}{\langle 2\pi p\rangle^N}\sum_{j\geq 1} M^{j(N-2k-2-\epsilon)}.$$

The last sum is finite if, and only if $N-2k-2-\epsilon<0$. □

Thus if a stationary Gaussian random field $\delta\tau$ has as in the statement of theorem 1.11 a covariance function K, such that

$$\exists C > 0, \forall p \in \mathbb{Z},\ c_p(K) \geq \left| \frac{C}{p^{2k+2+\epsilon}} \right|,$$

then

$$\exists C > 0, \forall p \in \mathbb{Z},\ c_p(K) \geq C c_p(K_0),$$

thus one can write

$$K = CK_0 + \tilde{K},$$

where $\tilde{K}$ has non-negative Fourier coefficients. Thus, $\tilde{K}$ is the covariance function of a centered stationary Gaussian random process $\delta\tilde{\tau}$. We can consequently write $\delta\tau$ as the sum of two independent stationary Gaussian random fields:

$$\delta\tau = \delta\tau_0 + \delta\tilde{\tau}.$$

Therefore, we can express the flat trace of the associated transfer operator as

$$\mathrm{Tr}^\flat(\mathcal{L}^n_{\xi,\tau_0+\delta\tau}) \underset{(3.7)}{=} \sum_{m|n} m \sum_{O\in\mathcal{P}_m} \frac{e^{i\xi(X^n_O+Y^n_O)}}{e^{J^n_O}-1},$$

where

$$X^n_O = X^n_O = (\delta\tau_n)^n_O$$

and

$$Y^n_O = \sum_{j\neq n} (\delta\tau_j)^n_O + (\tau_0)^n_O + \delta\tilde{\tau}^n_O$$

satisfy condition (C) of Definition (3.2). We can then conclude with proposition 3.10. □

4. Discussion

In this paper we have considered a model where the roof function τ is random. However, the numerical experiments suggest a far stronger result: for a fixed function τ and a semiclassical parameter ξ chosen according to a uniform random distribution in a small window at high frequencies, the result seems to remain true, as shown in the following figures for $\tau(x) = \sin(2\pi x)$. We define C_e by $n = C_e \frac{\log \xi_0}{\log 2}$.

Appendix A. Proof of lemma 1.1

Proof. E is topologically conjugated to the linear expanding map of same degree $x \mapsto lx \mod 1$, (see [KH97], p.73). Thus E^n has l^n-1 fixed points. Let $\tilde{E} : \mathbb{R} \longrightarrow \mathbb{R}$ be a lift of E, $x \neq y$ be two fixed points of E^n and $\tilde{x}, \tilde{y} \in \mathbb{R}$ be representatives of x and y respectively. Note that

$$d(x, y) = \inf |\tilde{x} - \tilde{y}|$$

where the infimum is taken over all couples of representatives $(\tilde{x}, \tilde{y})$. Now,

$$\begin{aligned} |\tilde{x} - \tilde{y}| M^n &\geq \left| \int_{\tilde{x}}^{\tilde{y}} (\tilde{E}^n)'(t) dt \right| \\ &= |\tilde{E}^n(\tilde{x}) - \tilde{E}^n(\tilde{y})|. \end{aligned}$$

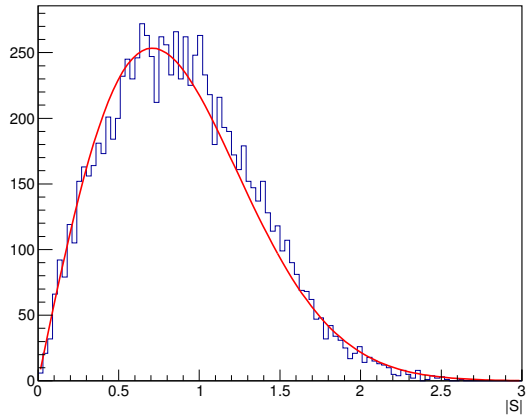

FIGURE 4. Histogramm of $S = \left|A_n \mathrm{Tr}^\flat \left(\mathcal{L}^n_{\tau_0,\xi}\right)\right|$ for a sample of 10^4 random values of ξ uniformly distributed in $[\xi_0, \xi_0 + 10]$ with $\xi_0 = 2.10^6$ and $n = 11$ giving $C_e = 0.5$. This is well fitted by $CS\exp(-S^2)$ in red.

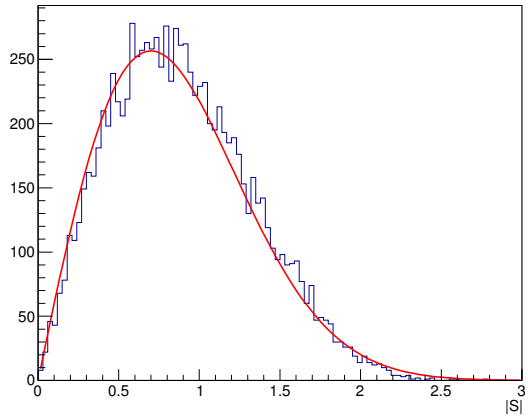

FIGURE 5. Histogramm of $S = \left|A_n \mathrm{Tr}^\flat \left(\mathcal{L}^n_{\tau_0,\xi}\right)\right|$ for a sample of 10^4 random values of ξ uniformly distributed in $[\xi_0, \xi_0 + 10]$ with $\xi_0 = 2000$ and $n = 11$ giving $C_e = 1.0$. This is well fitted by $CS\exp(-S^2)$ in red.

Since $E^n(x) = x$ and $E^n(y) = y$, $\tilde{E}^n(\tilde{x}) - \tilde{E}^n(\tilde{y})$ is an integer, different from 0 because $\tilde{E}^n$ is expanding, thus one-to-one. Finally,

$$|\tilde{x} - \tilde{y}|M^n \geq 1.$$

Taking the infimum gives the result. □

Appendix B. Proof of lemma 1.7 on the link between regularity of a Gaussian field and variance of the Fourier coefficients

Let us recall the following classical estimate:

Lemma B.1. *If $(X_p)_{p\in\mathbb{Z}}$ is a family of independent centered Gaussian random variables of variance 1, then, almost surely,*

$$\forall \delta > 0, X_p = o(p^\delta).$$

Proof. Let $n \in \mathbb{N}$. Let us use Borel-Cantelli lemma:

$$\forall p \in \mathbb{Z}, \mathbb{P}(|X_p| > p^{\frac{1}{n}}) = \int_{|x|>p^{\frac{1}{n}}} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}.$$

Now, we have the upper bound

$$p^{\frac{1}{n}} \int_{p^{\frac{1}{n}}}^{+\infty} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \leq \int_{p^{\frac{1}{n}}}^{+\infty} x e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = \frac{e^{-p^{\frac{2}{n}}}}{\sqrt{2\pi}}.$$

Thus,

$$\forall p \in \mathbb{Z}^*, \mathbb{P}(|X_p| > p^{\frac{1}{n}}) \leq \frac{2}{\sqrt{2\pi} p^{\frac{1}{n}}} e^{-p^{\frac{2}{n}}}.$$

Consequently,

$$\forall n \geq 1, \sum_p \mathbb{P}(|X_p| > p^{\frac{1}{n}}) < \infty$$

and by Borel-Cantelli, almost surely, for all $n \in \mathbb{N}$,

$$\#\{p \in \mathbb{Z}, |X_p| > p^{\frac{1}{n}}\} < \infty.$$

□

With this in mind, we can see that if a real random function τ has random Fourier coefficients $(c_p)_{p\in\mathbb{Z}}$, pairwise independent (for non-negative values of p), with variance

$$\sigma_p^2 := \mathbb{E}\left[|c_p|^2\right] = O(\frac{1}{p^{2k+2+\eta}}),$$

for some $\eta > 0$, then by the previous lemma, for all p, almost surely, for all $\delta > 0$,

$$\frac{c_p}{\sigma_p} = o(p^\delta),$$

and thus for $\delta = \frac{\eta}{2}$,

$$c_p = o(\frac{1}{p^{k+1+\frac{\eta}{2}}}) \text{ a.s.}$$

As a consequence,

$$\sum_p c_p (2i\pi p)^k e^{2i\pi p x}$$

converges normally and thus τ is almost surely $\mathcal{C}^k$.

Appendix C. Random walk on $\mathbb{C}$

Proposition C.1. *If $(X_{k,n})_{n\in\mathbb{N},0\leq k\leq n-1}$ is a family of pairwise independent identically distributed real random variables such that $e^{iX_{k,n}}$ is uniform on $\mathbb{T}$ and if the $c_{k,n}$ are real coefficients such that*

$$\frac{\sup_k c_{k,n}^2}{\sum_{k=0}^{n-1} c_{k,n}^2} \underset{n\to\infty}{\longrightarrow} 0,$$

then we have the following convergence in law :

$$\frac{1}{\sqrt{\sum_k c_{k,n}^2}} \left(\sum_{k=0}^{n-1} c_{k,n} e^{iX_{k,n}} \right) \longrightarrow \mathcal{N}_{\mathbb{C}}(0,1).$$

Proof. Let $(X_{k,n})_{n\in\mathbb{N},0\leq k\leq n-1}$ be a family of pairwise independent identically distributed real random variables such that $e^{iX_{k,n}}$ is uniform on $\mathbb{T}$ and $(c_{k,n})_{n\in\mathbb{N},0\leq k\leq n-1}$ be real coefficients such that

$$\frac{\sup_k c_{k,n}^2}{\sum_{j=0}^{n-1} c_{j,n}^2} \underset{n\to\infty}{\longrightarrow} 0.$$

Let

$$S_n := \sum_{0\leq k\leq n-1} c_{k,n} e^{iX_{k,n}}$$

and let us compute the characteristic function ϕ_n of $\frac{S_n}{\sqrt{\sum_{k=0}^{n-1} c_{k,n}^2}}$:

$$\phi_n(\xi_1,\xi_2) := \mathbb{E}\left[e^{i\left(\frac{\xi_1}{\sqrt{\sum_{k=0}^{n-1} c_{k,n}^2}} \mathrm{Re}(S_n) + \frac{\xi_2}{\sqrt{\sum_{k=0}^{n-1} c_{k,n}^2}} \mathrm{Im}(S_n) \right)} \right].$$

$$\begin{aligned}\phi_n(\xi_1,\xi_2) &= \mathbb{E}\left[\prod_{0\leq k\leq n-1} e^{i\left(\frac{c_{k,n}\xi_1}{\sqrt{\sum_{j=0}^{n-1} c_{j,n}^2}} \cos(X_{k,n}) + \frac{c_{k,n}\xi_2}{\sqrt{\sum_{j=0}^{n-1} c_{j,n}^2}} \sin(X_{k,n}) \right)} \right] \\ &= \prod_{0\leq k\leq n-1} \mathbb{E}\left[e^{i\left(\frac{c_{k,n}\xi_1}{\sqrt{\sum_{j=0}^{n-1} c_{j,n}^2}} \cos(X_{k,n}) + \frac{c_{k,n}\xi_2}{\sqrt{\sum_{j=0}^{n-1} c_{j,n}^2}} \sin(X_{k,n}) \right)} \right]\end{aligned}$$

by independence. Let us denote by $\psi(\xi_1,\xi_2)$ the quantity

$$\int_0^{2\pi} e^{i(\xi_1\cos\theta + \xi_2 \sin\theta)} \frac{d\theta}{2\pi}.$$

By derivating under the integral, we have the following Taylor's expansion in 0:

$$\psi(\xi_1,\xi_2) = 1 - \frac{1}{4}(\xi_1^2+\xi_2^2) + o(\xi_1^2+\xi_2^2).$$

By hypothesis,

$$\frac{\sup_k c_{k,n}^2}{\sum_{j=0}^{n-1} c_{j,n}^2} \underset{n\to\infty}{\longrightarrow} 0.$$

Thus,

$$\begin{aligned}
\phi_n(\xi_1,\xi_2) &= \prod_{0\le k\le n-1} \psi\left(\frac{c_{k,n}\xi_1}{\sqrt{\sum_{j=0}^{n-1} c_{j,n}^2}}, \frac{c_{k,n}\xi_2}{\sqrt{\sum_{j=0}^{n-1} c_{j,n}^2}}\right) \\
&= \prod_{0\le k\le n-1} \left(1 - \frac{1}{4}\frac{c_{k,n}^2(\xi_1^2+\xi_2^2)}{\sum_{j=0}^{n-1} c_{j,n}^2} + o(\frac{c_{k,n}^2}{\sum_{j=0}^{n-1} c_{j,n}^2})\right) \\
&= \exp\left(\sum_{0\le k\le n-1} \log\left(1 - \frac{1}{4}\frac{c_{k,n}^2(\xi_1^2+\xi_2^2)}{\sum_{j=0}^{n-1} c_{j,n}^2} + o(\frac{c_{k,n}^2}{\sum_{j=0}^{n-1} c_{j,n}^2})\right)\right) \\
&= \exp\left(\sum_{0\le k\le n-1} -\frac{1}{4}\frac{c_{k,n}^2(\xi_1^2+\xi_2^2)}{\sum_{j=0}^{n-1} c_{j,n}^2} + o(\frac{c_{k,n}^2(\xi_1^2+\xi_2^2)}{\sum_{j=0}^{n-1} c_{j,n}^2})\right) \\
&= \exp\left(-\frac{\xi_1^2+\xi_2^2}{4} + o(1)\right).
\end{aligned}$$

□

Appendix D. Proof of lemma 1.14

Proof. Let X, X', Y, and Y' be as in the statement of the lemma real random variables such that $e^{iX}, e^{iX'}$ are uniform on S^1 and so that X ad X' are both independent of all three other random variables. Let us write $\mathbb{P}_Z$ the law of a random variable Z. To show that $e^{i(X+Y)}$ and $e^{i(X'+Y')}$ are independent and uniform on S^1, it suffices to show that for any continuous functions $f, g : S^1 \longrightarrow \mathbb{R}$,

$$\mathbb{E}\left[f(e^{i(X+Y)})g(e^{i(X'+Y')})\right] = \int_0^{2\pi}\int_0^{2\pi} f(e^{i\theta})g(e^{i\theta'})\frac{d\theta}{2\pi}\frac{d\theta'}{2\pi}.$$

$$\mathbb{E}\left[f(e^{i(X+Y)})g(e^{i(X'+Y')})\right] = \int_{(S^1)^4} f(e^{i(x+y)})g(e^{i(x'+y')})d\mathbb{P}_{(X,Y,X',Y')}(x,y,x',y').$$

By hypothesis,

$$d\mathbb{P}_{(X,Y,X',Y')}(x,y,x',y') = \frac{dx}{2\pi}\frac{dx'}{2\pi}d\mathbb{P}_{(Y,Y')}(y,y').$$

Thus,

$$\begin{aligned}
&\mathbb{E}\left[f(e^{i(X+Y)})g(e^{i(X'+Y')})\right] \\
&\quad= \int_{(S^1)^2}\left(\int_0^{2\pi}\int_0^{2\pi} f(e^{i(x+y)})g(e^{i(x'+y')})\frac{dx}{2\pi}\frac{dx'}{2\pi}\right)d\mathbb{P}(Y,Y')(y,y') \\
&\quad\underset{\theta=x+y,\theta'=x'+y'}{=} \int_{(S^1)^2}\left(\int_0^{2\pi}\int_0^{2\pi} f(e^{i\theta})g(e^{i\theta'})\frac{d\theta}{2\pi}\frac{d\theta'}{2\pi}\right)d\mathbb{P}(Y,Y')(y,y') \\
&\quad= \int_0^{2\pi}\int_0^{2\pi} f(e^{i\theta})g(e^{i\theta'})\frac{d\theta}{2\pi}\frac{d\theta'}{2\pi}.
\end{aligned}$$

□

Appendix E. Topological pressure

E.1. **Definition.**

Definition E.1. Let $\phi : \mathbb{T} \longrightarrow \mathbb{R}$ be a continuous function. The limit

$$\Pr(\phi) := \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{E^n(x)=x} e^{\phi_x^n} \right) \tag{E.1}$$

exists and is called the topological pressure of ϕ (see [KH97] Proposition 20.3.3 p.630).

In other words

$$\sum_{E^n(x)=x} e^{\phi_x^n} = e^{n\Pr(\phi)+o(n)}. \tag{E.2}$$

The particular case $\phi = 0$ gives the topological entropy $\Pr(0) = h_{top}$.

E.2. **Variational principle.** Another definition of the pressure is given by the variational principle. Let us denote by $h(\mu)$ the entropy of a measure μ invariant under E (see [KH97] section 4.3 for a definition of entropy). For the next theorem, see [KH97], sections 20.2 and 20.3. The last sentence comes from Proposition 20.3.10.

Theorem E.2 (Variational principle)**.** *Let $\phi : \mathbb{T} \longrightarrow \mathbb{R}$ be a continuous function.*

$$\Pr(\phi) = \sup_{\mu \ E-invariant} \left(\int \phi \ d\mu + h(\mu) \right).$$

This supremum is moreover attained for a unique E-invariant measure μ, called equilibrium measure. In addition, if we note $J = \log|E|'$ and μ_β the equilibrium measure of $-\beta J$, $\beta \mapsto \mu_\beta$ is one-to-one.

Corollary E.3. *The function*

$$F : \left\{ \begin{array}{ccc} \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ \beta & \longmapsto & \frac{1}{\beta}\Pr(-\beta J) \end{array} \right. \tag{E.3}$$

is decreasing.

Proof. Let $\beta' > \beta > 0$. By the previous theorem, with the same notations,

$$\int -\beta J \ d\mu_\beta + h(\mu_\beta) > \int -\beta J \ d\mu'_\beta + h(\mu'_\beta)$$

and thus

$$F(\beta) = \int -J \ d\mu_\beta + \frac{h(\mu_\beta)}{\beta} > \int -J \ d\mu'_\beta + \frac{h(\mu'_\beta)}{\beta} \geq \int -J \ d\mu'_\beta + \frac{h(\mu'_\beta)}{\beta'} = F(\beta').$$

□

E.3. **Proof of lemma 3.11.** Let $\phi : \mathbb{T} \to \mathbb{R}$ be a $\mathcal{C}^1$ function. Let as before ϕ_x^n be the Birkhoff sum (1.4). By subadditivity of the sequence $(\inf_{x\in\mathbb{T}} \phi_x^n)_n$ and Fekete's Lemma we can define the following quantity:

Definition E.4. Let us define

$$\phi_{\min} := \lim_{n\to\infty} \inf_{x\in\mathbb{T}} \frac{1}{n}\phi_x^n. \tag{E.4}$$

Lemma E.5. *The infimum in (E.4) can be taken over periodic points:*

$$\phi_{min} = \lim_{n\to\infty} \inf_{x, E^n(x)=x} \frac{1}{n}\phi_x^n. \tag{E.5}$$

Proof. By lifting the expanding map to $\mathbb{R}$, we easily see that E has at least a fixed point x_0. This point has l^n preimages by E^n, defining $l^n - 1$ intervals I_k^n such that for all $1 \leq k \leq l^n - 1$

$$E^n : I_k^n \to \mathbb{T}\backslash\{x_0\}$$

is a diffeomorphism. Thus, there exists $C > 0$ such that for all k, if $x, y \in \overline{I_k^n}$,

$$\forall 0 \leq j \leq n, d(E^j(x), E^j(y)) \leq \frac{C}{m^{n-j}},$$

with $m = \inf|E'| > 1$. Each $\overline{I_k^n}$ contains moreover a periodic point $y_{k,n}$ of period n given by $E^n(y_{k,n}) = y_{k,n} + k$. Hence let $n \in \mathbb{N}$, let $x_n \in \mathbb{T}$ be such that

$$\phi_{x_n}^n = \inf_{x\in\mathbb{T}} \phi_x^n,$$

and suppose that $x_n \in \overline{I_k^n}$. We have

$$\begin{aligned}\left|\phi_{y_{k,n}}^n - \phi_{x_n}^n\right| &= \left|\sum_{j=0}^{n-1} \phi(E^j(x_n)) - \phi(E^j(y_{k,n}))\right| \\ &\leq C \max|\phi'| \sum_{k=0}^{\infty} \frac{1}{m^k}\end{aligned}$$

is bounded independently of n. Consequently

$$\lim_{n\to\infty} \inf_{x, E^n(x)=x} \frac{1}{n}\phi_x^n = \phi_{\min}.$$

□

Lemma E.6.

$$F(\beta) \underset{\beta\to+\infty}{\longrightarrow} -\phi_{min}.$$

Proof. Let $\beta > 0$. Let us write

$$F_n(\beta) = \frac{1}{n\beta} \log\left(\sum_{E^n(x)=x} e^{-\beta\phi_x^n}\right),$$

so that

$$F(\beta) \underset{(E.1,E.3)}{=} \lim_{n\to\infty} F_n(\beta).$$

Let $\epsilon > 0$. By definition of $\phi_{\min}$, for n large enough,

$$\forall x \in \mathrm{Per}(n), \phi_x^n \geq n(\phi_{\min} - \epsilon)$$

and

$$\exists x \in \mathrm{Per}(n), \phi_x^n \leq n(\phi_{\min} + \epsilon).$$

Thus,

$$e^{-\beta n(\phi_{\min}+\epsilon)} \leq \sum_{E^n(x)=x} e^{-\beta\phi_x^n} \leq l^n e^{-\beta n(\phi_{\min}-\epsilon)}$$

and consequently

$$-\phi_{\min} - \epsilon \leq F_n(\beta) \leq \frac{\log l}{\beta} - \phi_{\min} + \epsilon.$$

Hence, letting $\epsilon \to 0$, we get

$$-\phi_{\min} \leq F(\beta) \leq \frac{\log l}{\beta} - \phi_{\min}.$$

When β goes to infinity, the result follows. □

Proof of lemma 3.11. Now we take $\phi = J = \log(E')$. By the definition of $J_{\min}$

$$\inf_{O\in\mathrm{Per}(n)} J_O^n = nJ_{\min} + o(n),$$

thus

$$\sup_{O\in\mathrm{Per}(n)} \frac{1}{e^{J_O} - 1} = e^{-nJ_{\min}+o(n)} = e^{n\lim_\infty F+o(n)}. \tag{E.6}$$

We have

(E.7)

$$\frac{1}{\sqrt{n}} \left(\sum_{m|n} m \sum_{O\in\mathcal{P}_m} \frac{1}{(e^{J_O^n} - 1)^2} \right)^{-\frac{1}{2}} \leq A_n \underset{(1.9)}{=} \left(\sum_{m|n} m^2 \sum_{O\in\mathcal{P}_m} \frac{1}{(e^{J_O^n} - 1)^2} \right)^{-\frac{1}{2}} \leq \left(\sum_{m|n} m \sum_{O\in\mathcal{P}_m} \frac{1}{(e^{J_O^n} - 1)^2} \right)^{-\frac{1}{2}}.$$

Since

$$\sum_{m|n} m \sum_{O\in\mathcal{P}_m} \frac{1}{(e^{J_O^n} - 1)^2} = e^{n\mathrm{Pr}(-2J)+o(n)},$$

Eq.(E.7) gives

$$\frac{1}{\sqrt{n}} e^{-\frac{n}{2}\mathrm{Pr}(-2J)+o(n)} \leq A_n \leq e^{-\frac{n}{2}\mathrm{Pr}(-2J)+o(n)}$$

hence

$$nA_n = e^{-\frac{n}{2}\mathrm{Pr}(-2J)+o(n)} = e^{-nF(2)+o(n)}.$$

Finally,

$$nA_n \sup_{O\in\mathrm{Per}(n)} \frac{1}{e^{J_O} - 1} \underset{(E.6)}{=} e^{n(\lim_\infty F - F(2)+o(n))} \to 0$$

from Corollary E.3. □

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