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# LMI based approach for power flow analysis with uncertain power injection

Khaled Laib\* Anton Korniienko\*\* Florent Morel\*\*  
G rard Scorletti\*\*

\* GIPSA-lab, Univ. Grenoble Alpes, F-38000 Grenoble, France.  
(e-mail: khaled.laib@gipsa.lab.fr).

\*\* Univ. Lyon,  cole Centrale de Lyon, Laboratoire Amp re, 69134  cully, France  
(e-mail: <anton.korniienko, florent.morel, gerard.scorletti>@ec-lyon.fr)

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**Abstract:** This paper investigates the uncertain power flow analysis in distribution networks within the context of renewable power resources integration such as wind and solar power. The analysis aims to bound the worst-case voltage magnitude in any node of the network for a given uncertain power generation scenario. The major difficulty of this problem is the non-linear aspect of power flow equations. The proposed approach does not require the linearization of these equations and formulates the problem as an optimization problem with polynomial constraints. A new tool to investigate the feasibility of such problems is presented and it is obtained as an extension of the  $\mathcal{S}$ -procedure, a fundamental result in robustness analysis. A solution to the uncertain power flow analysis problem is proposed using this new tool. The different obtained results are expressed as LMI optimization problems which guarantees an efficient numerical resolution as it will be demonstrated through an illustrative example.

*Keywords:* Power flow analysis, uncertain power injection, voltage upper and lower bounds, polynomial constraints feasibility problem, LMI optimization.

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## 1. INTRODUCTION

The integration of renewable power resources such as wind and solar power into the existing distribution networks<sup>1</sup> has become a necessity in order to create an environmental responsible energy usage. Nevertheless, these renewable power resources are intermittent and difficult to predict accurately which make them a source of uncertainty in power systems. This paper focuses on the effect of this uncertain power integration on the network voltage magnitudes by computing their worst-case upper and lower bounds for a given renewable power generation scenario. This problem is known as the uncertain power flow analysis.

Uncertain power flow analysis considers the network performance in steady state by investigating if the different voltage magnitude bounds remain within the acceptable interval defined by power system operational requirements. Furthermore, since it is an off-line analysis, the uncertain power flow analysis is very beneficial in many operations which do not require fast responses. For instance, in authorizing further integration of renewable power resources, in scheduling network interventions and in defining power system operations across different time-scales: from day-ahead to long period scheduling. Therefore, the uncertain power flow analysis has received an important attention over the last decades and it is possible to distinguish two main approaches: probabilistic and deterministic.

In probabilistic approaches, e.g. Fan et al. (2013); Bienstock et al. (2014), the power generation uncertainty is

<sup>1</sup> Power distribution network is the terminal part of power network where residential buildings, schools, etc. are found.

modeled as random variables with predefined distribution functions. Probability theory is used to obtain probability distribution of the power flow solutions. However, using these approaches, no strict voltage magnitude bounds are obtained since the solutions are given as probability distributions and no worst-case warranty can be obtained.

In deterministic approaches, the uncertain power generation is characterized using sets such as polytopes and ellipsoids.

In the case when the generated (injected) power is characterized with polytopes, interval methods can be applied. These methods employ different techniques to deal with the non-linear aspects of power flow equations such as iterative techniques, see e.g. in Luo et al. (2014). However, the resulting computation complexity may be important due to some matrix interval inversions at each iteration.

In the general case when the injected power is characterized with ellipsoids, see e.g. Saric and Stankovic (2006), methods of Chen et al. (2011) can be applied. This approach consists in projecting the injected power ellipsoid into the voltage magnitude set using a linear model of power flow equations with the assumption that power generation variations are sufficiently small. However, because of the performed linearization, the obtained results are local and only valid around the operating point.

This paper focuses on the general case when the injected power is characterized with ellipsoids. In contrast with Chen et al. (2011), the linearization of power flow equations is not required in our approach and hence large injected power variations are allowed. We reveal that solv-

ing the uncertain power flow analysis problem requires the resolution of an optimization problem with non-linear constraints. More precisely, the constraints involved in this problem are polynomial.

The main contribution of this paper is Theorem 4.1 which is a new tool to investigate the feasibility of set of polynomial constraints using convex optimization constrained by linear matrix inequalities (LMI), see e.g. Boyd et al. (1994). This theorem represents an extension to the well-known  $\mathcal{S}$ -procedure, see e.g. Boyd et al. (1994); Jönsson (2001), in the case of polynomial constraints with complex variables. Hence, strong connections between uncertain power flow analysis and usual robustness analysis since the  $\mathcal{S}$ -procedure is a fundamental result in robustness analysis. Another contribution of this paper is Corollary 5.1 which is a new solution to the uncertain power flow analysis problem.

*Paper outline:* This paper is organized as follows. Section 2 presents some power network preliminaries followed by formulating the uncertain power flow analysis problem. Section 3 presents a reformulation of this problem within the context of optimization problems with polynomial constraints. Section 4 presents the main contribution of this paper while its application to solve the uncertain power flow analysis problem is presented in Section 5. The efficiency of the proposed solution is demonstrated through an illustrative example in Section 6. Conclusions and perspectives are presented in Section 7.

A long version of this paper which contains the missing proofs can be found in Laib et al. (2018).

*Notations:*  $\mathbb{R}$  and  $\mathbb{C}$  are the sets of real and complex numbers respectively.  $\mathcal{N}$  denotes the finite set  $\{1, \dots, N\}$  and  $\mathbf{j}$  denotes the square root of -1. The transpose and the transpose conjugate of  $X$  are denoted  $X^T$  and  $X^*$  respectively. For several scalars  $\tau_i$  (respectively several matrices  $Q_i$ ),  $\text{diag}_i(\tau_i)$  (respectively  $\text{bdiag}_i(Q_i)$ ) denotes the diagonal matrix composed of  $\tau_i$  (respectively  $Q_i$ ). The vector  $u_k$  (respectively  $e_k$ ) is the  $(N^2 + N + 1)$  (respectively the  $N$ ) null row vector except the  $k^{\text{th}}$  entry which is equal to 1. At last and in order to avoid repetitions, the expression  $(\star)^* Mx$  (respectively  $(\star)^T Mx$ ) replaces any quadratic form such as  $x^* Mx$  (respectively  $x^T Mx$ ).

## 2. PRELIMINARIES AND PROBLEM FORMULATION

### 2.1 Preliminaries

*Generalities on power distribution networks:* Consider a power distribution network with  $N$  buses (nodes) connected through electrical lines. Each of these buses represents a power consumer (residential buildings, schools, etc.). The slack bus (reference bus) is denoted bus 0 and is located upstream of the  $N$  bus power distribution network.

We assume the following

- The power network three-phases form a balanced system i.e. the three phases have the same magnitude and are phase-shifted in time by one-third of the period. This assumption is required in order to

boil down the analysis of the three-phase network into the analysis of an equivalent one phase network.

- The power network steady state is established and the analysis does not concern the transient state.
- The bus  $k$ , with  $k \in \{1, \dots, N\}$  is connected to an uncertain power resource while the power consumption at this bus is known. The approach presented in this paper can be easily adapted to other cases<sup>2</sup>.
- The slack bus voltage is known and there are no loads or renewable power resource devices connected to it.

The quantities to be manipulated in this paper are

- The network admittance matrix  $Y$  defined as

$$Y = Y^T \in \mathbb{C}^{(N+1) \times (N+1)}$$

$$Y_{i,j} = \begin{cases} y_{\ell_i} + \sum_{j=1, j \neq i}^{N+1} y_{ij} & \text{if } i = j \\ -y_{ij} & \text{if } i \neq j \text{ and } i \sim j \\ 0 & \text{otherwise} \end{cases}$$

where  $y_{\ell_i}$  and  $y_{ij}$  denote the load admittance connected to bus  $i$  and the line admittance between bus  $i$  and bus  $j$  respectively. The symbol  $i \sim j$  means that bus  $i$  is connected to bus  $j$ .

- $\underline{v}_k$  and  $\underline{i}_k$ : the (complex) voltage and (complex) current at bus  $k$  respectively. The network voltages and currents are linked through the admittance matrix

$$\underline{i}_k = \sum_{j=1}^{N+1} Y_{(k+1),j} \underline{v}_{j-1} \quad (1)$$

- $\underline{s}_k = p_k + \mathbf{j}q_k$ : the (complex) power  $\underline{s}_k$ , real power  $p_k$  and reactive power  $q_k$  at bus  $k$ . The bus complex power is linked to its voltage and current through

$$\underline{s}_k = \underline{v}_k \underline{i}_k^* \quad (2)$$

- $\underline{s}_{g_k} = p_{g_k} + \mathbf{j}q_{g_k}$ : the (complex) generated power  $\underline{s}_{g_k}$ , generated real power  $p_{g_k}$  and generated reactive power  $q_{g_k}$  at bus  $k$ .
- $\underline{s}_{\ell_k} = p_{\ell_k} + \mathbf{j}q_{\ell_k}$ : the (complex) load power  $\underline{s}_{\ell_k}$ , load real power  $p_{\ell_k}$  and load reactive power  $q_{\ell_k}$  at bus  $k$ .

The power at each bus  $k$  is balanced between generation (injection) and load, that is

$$\underline{s}_k = \underline{s}_{g_k} - \underline{s}_{\ell_k} \quad (3)$$

Hence, for each bus  $k$  and by combining equations (1), (2) and (3), the power flow equations are given by

$$\underline{s}_{g_k} - \underline{s}_{\ell_k} = \underline{v}_k \left( \sum_{j=1}^{N+1} Y_{(k+1),j} \underline{v}_{j-1} \right)^*, \quad k \in \mathcal{N} \quad (4)$$

As it can be seen, the power flow equations (4) are non-linear with respect to the different  $\underline{v}_k$ .

Before presenting the characterization of injected powers  $\underline{s}_{g_k}$ , with  $k \in \mathcal{N}$ , an important phenomenon in electric circuit has to be taken into account. This phenomenon is the electric current magnitude limitations i.e. the current magnitude transmitted through a line is limited and cannot exceed some value. Therefore, the magnitude of current  $\underline{i}_k$  cannot exceed a given value  $I_k^{max}$ , that is

$$|\underline{i}_k| < I_k^{max}, \quad k \in \mathcal{N} \quad (5)$$

<sup>2</sup> Other cases such as uncertain power consumption or both powers (generation and consumption) are uncertain. Another case is when only some buses are connected to uncertain generation/consumption power.

*Characterization of the injected powers:* As explained above, the powers generated from renewable power resources are variable and difficult to predict with precision. A general characterization of these powers are ellipsoids, see Saric and Stankovic (2006).

The injected powers  $\underline{s}_{g_1}, \dots, \underline{s}_{g_N}$  are characterized with the ellipsoid  $\mathcal{S}_g$  can be expressed as

$$\left( \begin{array}{c} \underline{s}_{g_1} \\ \vdots \\ \underline{s}_{g_N} \end{array} \right) \in \mathcal{S}_g = \left\{ \left( \begin{array}{c} \underline{s}_{g_1} \\ \vdots \\ \underline{s}_{g_N} \end{array} \right) \in \mathbb{C}^N \mid \begin{array}{l} (\star) \Psi (\underline{S}_g - \underline{S}_g^0) < 1 \\ \text{with} \\ \underline{S}_g = (\underline{s}_{g_1} \dots \underline{s}_{g_N})^T \\ \underline{S}_g^0 = (\underline{s}_{g_1}^0 \dots \underline{s}_{g_N}^0)^T \end{array} \right\} \quad (6)$$

where

- $\underline{S}_g^0 = (\underline{s}_{g_1}^0 \dots \underline{s}_{g_N}^0)^T$  is the ellipsoid center and  $\underline{s}_{g_k}^0$  are the nominal values of injected powers;
- $\Psi \in \mathbb{C}^{N \times N}$  is a hermitian matrix describing how far the ellipsoid extends in every direction.

The main interest of ellipsoids is that it allows considering correlations between different powers in the network which is not possible with polytopes. Therefore, the ellipsoidal characterization (6) is adopted in this paper.

## 2.2 Problem formulation

In the uncertain power flow analysis, the objective is to determine bounds on the magnitude of each  $\underline{v}_k$ , that is

$$V_k^{min} < |\underline{v}_k| < V_k^{max} \quad k \in \mathcal{N}$$

such that constraints (5) and (6) are respected.

These  $2N$  voltage magnitude bounds inequalities can be rewritten as

$$(V_k^{min})^2 < \underline{v}_k^* \underline{v}_k < (V_k^{max})^2 \quad k \in \mathcal{N}. \quad (7)$$

Constraints (7) form a hyper-rectangle in  $\mathbb{R}^N$  where the different  $(V_k^{min})^2$  and  $(V_k^{max})^2$  are its vertices. This hyper-rectangle is denoted  $\mathcal{V}$  and is given by

$$\mathcal{V} = \left\{ \left( \begin{array}{c} \underline{v}_1^* \underline{v}_1 \\ \vdots \\ \underline{v}_N^* \underline{v}_N \end{array} \right) \in \mathbb{R}^N \mid \begin{array}{l} (V_1^{min})^2 < \underline{v}_1^* \underline{v}_1 < (V_1^{max})^2 \\ \vdots \\ (V_N^{min})^2 < \underline{v}_N^* \underline{v}_N < (V_N^{max})^2 \end{array} \right\}.$$

Therefore, in order to determine the tightest bounds  $V_k^{min}$   $V_k^{max}$ , it is required to find the smallest hyper-rectangle; hence the necessity to define a size measure.

We adopt in this paper the perimeter  $\mathcal{P}$  as a size measure for the hyper-rectangle  $\mathcal{V}$ . It is given by

$$\mathcal{P} = \vartheta \left( \sum_{k=1}^N (V_k^{max})^2 - (V_k^{min})^2 \right)$$

where  $\vartheta$  is a positive scalar which depends on  $N$ .

After introducing the different concepts of the uncertain power flow analysis problem and after clarifying its objective, it is now possible to announce the problem formally.

*Problem 2.1.* Consider a power distribution network with  $N$  buses and  $Y$  as its admittance matrix.

The voltage, current and injected power at bus  $k$ , with  $k \in \mathcal{N} = \{1, \dots, N\}$ , are  $\underline{v}_k$ ,  $\underline{i}_k$  and  $\underline{s}_{g_k}$  respectively.

Given

- the voltage  $\underline{v}_0$  at bus 0 (reference bus);
- the limitation  $I_k^{max}$  of  $\underline{i}_k$  at bus  $k$  with  $k \in \mathcal{N}$ ;
- the load power  $\underline{s}_{\ell_k}$  at bus  $k$  with  $k \in \mathcal{N}$ ;
- the nominal injected power  $\underline{s}_{g_k}^0$  at bus  $k$  with  $k \in \mathcal{N}$ ;
- the hermitian matrix  $\Psi \in \mathbb{C}^{N \times N}$ .

Find the different  $(V_k^{min})^2$  and  $(V_k^{max})^2$  which

$$\min_{(V_k^{min})^2, (V_k^{max})^2} \mathcal{P} = \vartheta \left( \sum_{k=1}^N (V_k^{max})^2 - (V_k^{min})^2 \right)$$

subject to

$$(V_k^{min})^2 < \underline{v}_k^* \underline{v}_k < (V_k^{max})^2 \quad k \in \mathcal{N}$$

for every  $\underline{v}_k$  such that

- $|\underline{i}_k| < I_k^{max}$ , for every  $k \in \mathcal{N}$ ;

$$\bullet \left( \begin{array}{c} \underline{s}_{g_1} \\ \vdots \\ \underline{s}_{g_N} \end{array} \right) \in \left\{ \left( \begin{array}{c} \underline{s}_{g_1} \\ \vdots \\ \underline{s}_{g_N} \end{array} \right) \in \mathbb{C}^N \mid \begin{array}{l} (\star) \Psi (\underline{S}_g - \underline{S}_g^0) < 1 \\ \text{with} \\ \underline{S}_g = (\underline{s}_{g_1} \dots \underline{s}_{g_N})^T \\ \underline{S}_g^0 = (\underline{s}_{g_1}^0 \dots \underline{s}_{g_N}^0)^T \end{array} \right\}.$$

with

$$\bullet \underline{i}_k = \sum_{j=1}^{N+1} Y_{(k+1),j} \underline{v}_{j-1};$$

$$\bullet \underline{s}_{g_k} = \underline{s}_{\ell_k} + \underline{v}_k \left( \sum_{j=1}^{N+1} Y_{(k+1),j} \underline{v}_{j-1} \right)^*.$$

## 3. PROPOSED APPROACH

The different constraints of Problem 2.1 are given in terms of voltages  $\underline{v}_k$ , injected powers  $\underline{s}_{g_k}$  and currents  $\underline{i}_k$ . Therefore, the first step toward the resolution of Problem 2.1 is to rewrite all of its constraints in an explicit form in terms of voltages  $\underline{v}_k$ .

The constraints (5) and (6) rewrite respectively as

$$\bullet \forall V \in \mathbb{C}^N \quad \begin{pmatrix} \star \\ \star \\ \star \end{pmatrix}^* Q_k^I \begin{pmatrix} V \otimes V^{*T} \\ V \\ 1 \end{pmatrix} < 0, \quad k \in \mathcal{N};$$

$$\bullet \forall V \in \mathbb{C}^N \quad \begin{pmatrix} \star \\ \star \\ \star \end{pmatrix}^* Q^{S_g} \begin{pmatrix} V \otimes V^{*T} \\ V \\ 1 \end{pmatrix} < 0$$

where

$$\bullet V = (\underline{v}_1 \dots \underline{v}_N)^T \in \mathbb{C}^N;$$

$$\bullet V \otimes V^{*T} = (\underline{v}_1 \times V^* \dots \underline{v}_N \times V^*)^T \in \mathbb{C}^{N^2};$$

- $Q^I_k$  and  $Q^{S_g}$  are  $(N^2 + N + 1)$  by  $(N^2 + N + 1)$  hermitian matrices and are given by (8) and (9).

$$Q_k^I = \begin{pmatrix} \star \\ \star \\ \star \end{pmatrix}^* \begin{pmatrix} e_k^T e_k & 0 \\ 0 & -(I_k^{max})^2 \end{pmatrix} \begin{pmatrix} (O_{N \times N^2} \quad Y_{2:(N+1),2:(N+1)} \quad (Y_{2,1\underline{v}_0} \dots Y_{(N+1),1\underline{v}_0})^T) \\ u_{N^2+N+1} \end{pmatrix} \quad (8)$$

$$Q^{S_g} = \begin{pmatrix} \star \\ \star \\ \star \end{pmatrix}^* \begin{pmatrix} \Psi & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} (\text{bdiag}(Y_{k+1,2:N+1}^*) \quad \text{diag}(Y_{k+1,1\underline{v}_0}^*) \quad (\underline{s}_{\ell_1} - \underline{s}_{g_1}^0 \dots \underline{s}_{\ell_N} - \underline{s}_{g_N}^0)^T) \\ u_{N^2+N+1} \end{pmatrix} \quad (9)$$

with  $O_{N \times N^2}$  the  $N$  by  $N^2$  null matrix,  $Y_{2:(N+1),2:(N+1)}$  the sub-matrix of  $Y$  which excludes the first row and the first column and  $Y_{k+1,2:N+1}^*$  the  $(k+1)^{\text{th}}$  row of the admittance matrix  $Y$  taken between columns 2 and  $N+1$ .

Please refer to Appendix A.1 in Laib et al. (2018) for more details.

In the sequel and in order to ease the notation, the matrices  $Q_1^I, \dots, Q_N^I, Q^{S_g}$  are collected in the set  $\mathcal{Q}$  and they will be denoted  $Q_i$  with  $i \in \{1, \dots, N+1\}$ , that is

$$\begin{aligned} \mathcal{Q} &= \{Q_1^I, \dots, Q_N^I, Q^{S_g}\} \\ &= \{Q_1, Q_2, \dots, Q_{N+1}\}. \end{aligned}$$

The  $2N$  constraints of (7) rewrite as

$$\begin{aligned} \begin{pmatrix} \star \\ \star \\ \star \end{pmatrix}^* Q_k^{min} \begin{pmatrix} V \otimes V^{*T} \\ V \\ 1 \end{pmatrix} &> 0 \\ \begin{pmatrix} \star \\ \star \\ \star \end{pmatrix}^* Q_k^{max} \begin{pmatrix} V \otimes V^{*T} \\ V \\ 1 \end{pmatrix} &> 0 \end{aligned} \quad k \in \mathcal{N}$$

where  $Q_k^{min}$  and  $Q_k^{max}$  are  $(N^2 + N + 1)$  by  $(N^2 + N + 1)$  symmetric real matrices and their expressions are given by (10) in Problem 3.1 below.

It is now possible to reformulate Problem 2.1 as follows.

*Problem 3.1.* Given the data of Problem 2.1 and the set of matrices  $\mathcal{Q} = \{Q_1, \dots, Q_{N+1}\}$ . Let the matrices  $Q_k^{min}$  and  $Q_k^{max}$ , with  $k \in \{1, \dots, N\}$ , given by

$$\begin{aligned} Q_k^{min} &= u_{N^2+k}^T u_{N^2+k} - (V_k^{min})^2 u_{N^2+N+1}^T u_{N^2+N+1} \\ Q_k^{max} &= -u_{N^2+k}^T u_{N^2+k} + (V_k^{max})^2 u_{N^2+N+1}^T u_{N^2+N+1} \end{aligned} \quad (10)$$

Find the different  $(V_k^{min})^2$  and  $(V_k^{max})^2$  which

$$\min_{(V_k^{min})^2, (V_k^{max})^2} \mathcal{P} = \vartheta \left( \sum_{k=1}^N (V_k^{max})^2 - (V_k^{min})^2 \right)$$

subject to

$$\begin{aligned} \begin{pmatrix} \star \\ \star \\ \star \end{pmatrix}^* Q_k^{min} \begin{pmatrix} V \otimes V^{*T} \\ V \\ 1 \end{pmatrix} &> 0 \\ \begin{pmatrix} \star \\ \star \\ \star \end{pmatrix}^* Q_k^{max} \begin{pmatrix} V \otimes V^{*T} \\ V \\ 1 \end{pmatrix} &> 0 \end{aligned} \quad k \in \{1, \dots, N\}$$

for every  $V$  in  $\mathbb{C}^N$  satisfying

$$\begin{pmatrix} \star \\ \star \\ \star \end{pmatrix}^* Q_i \begin{pmatrix} V \otimes V^{*T} \\ V \\ 1 \end{pmatrix} < 0 \quad i \in \{1, \dots, N+1\}$$

with

- $V = (v_1 \dots v_N)^T \in \mathbb{C}^N$ ;
- $V \otimes V^{*T} = (v_1 \times V^* \dots v_N \times V^*)^T \in \mathbb{C}^{N^2}$ .

In Problem 3.1 and due to the non-linear aspect of the power flow equations (4), developing the different inequalities results in a set of polynomial constraints each of which is of the form  $\sum_{a=1}^N \sum_{b=1}^N \sum_{c=1}^N \sum_{d=1}^N \alpha_{abcd} v_a^* v_b v_c v_d^* < 0$  with  $\alpha_{abcd} \in \mathbb{C}$ .

Even without attempting to minimize  $\mathcal{P}$  in Problem 3.1, finding the different  $(V_k^{min})^2$  and  $(V_k^{max})^2$  which satisfy all those polynomial constraints at the same time is a challenging task. For this reason, we will attempt to solve Problem 3.1 in two steps.

- Test if there exist some values (feasible set) of  $(V_k^{min})^2$  and  $(V_k^{max})^2$  for which all the polynomial constraints (without the cost function) are satisfied.
- Search within the feasible set for the values of  $(V_k^{min})^2$  and  $(V_k^{max})^2$  which give the smallest value for the perimeter  $\mathcal{P}$ .

Testing the existence of a set of values for  $(V_k^{min})^2$  and  $(V_k^{max})^2$  for which all the polynomial constraints of Problem 3.1 are satisfied is a feasibility problem. This problem can be decomposed into  $2N$  feasibility problems each of which consists in testing if

$$\begin{pmatrix} \star \\ \star \\ \star \end{pmatrix}^* Q_0 \begin{pmatrix} V \otimes V^{*T} \\ V \\ 1 \end{pmatrix} > 0$$

is respected for every  $V$  in  $\mathbb{C}^N$  satisfying

$$\begin{pmatrix} \star \\ \star \\ \star \end{pmatrix}^* Q_i \begin{pmatrix} V \otimes V^{*T} \\ V \\ 1 \end{pmatrix} < 0, \quad i \in \{1, \dots, N+1\}$$

where  $Q_0$  is either equal to  $Q_k^{min}$  or  $Q_k^{max}$  for a given  $k \in \{1, \dots, N\}$  depending on the constraint to be tested.

We define thus the following feasibility problem with polynomial constraints.

*Problem 3.2.* Let the  $(N^2 + N + 1)$  by  $(N^2 + N + 1)$  complex hermitian matrices  $Q_0$  and  $Q_i$ , with  $i \in \{1, \dots, N+1\}$ .

Test if

$$\begin{pmatrix} \star \\ \star \\ \star \end{pmatrix}^* Q_0 \begin{pmatrix} V \otimes V^{*T} \\ V \\ 1 \end{pmatrix} > 0$$

is respected for every  $V$  in  $\mathbb{C}^N$  satisfying

$$\begin{pmatrix} \star \\ \star \\ \star \end{pmatrix}^* Q_i \begin{pmatrix} V \otimes V^{*T} \\ V \\ 1 \end{pmatrix} < 0, \quad i \in \{1, \dots, N+1\}$$

with

- $V = (v_1 \dots v_N)^T \in \mathbb{C}^N$ ;
- $V \otimes V^{*T} = (v_1 \times V^* \dots v_N \times V^*)^T \in \mathbb{C}^{N^2}$ .

A new tool to solve Problem 3.2 is presented in the next section.

## 4. MAIN RESULT

Theorem 4.1 which is the main contribution of this paper is stated in this section. It gives sufficient conditions to solve Problem 3.2 as an optimization problem with LMI constraints.

*Theorem 4.1.* Given the data of Problem 3.2. Let  $\mathcal{E}$  be the set of hermitian matrices  $\tilde{Q}_\ell \in \mathcal{E}$  given by

$$\mathcal{E} = \left\{ \tilde{Q}_\ell \mid \tilde{Q}_\ell \in \mathcal{E}^1 \cup \mathcal{E}^2 \cup \mathcal{E}^3 \cup \mathcal{E}^4 \cup \mathcal{E}^5 \right\} \quad (11)$$

where

$$\begin{aligned} \bullet \mathcal{E}^1 &= \left\{ \tilde{Q}_\ell \mid \begin{array}{l} \exists (a, b, c, d) \in \mathcal{N} \times \mathcal{N} \times \mathcal{N} \times \mathcal{N} \\ \tilde{Q}_\ell = \begin{pmatrix} \star \\ \star \\ \star \\ \star \end{pmatrix}^T \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} u_{(a-1)N} \\ u_{(c-1)N+d} \\ u_{(d-1)N+b} \\ u_{(c-1)N+a} \end{pmatrix} \end{array} \right\}; \\ \bullet \mathcal{E}^2 &= \left\{ \tilde{Q}_\ell \mid \begin{array}{l} \exists (a, b, c) \in \mathcal{N} \times \mathcal{N} \times \mathcal{N} \\ \tilde{Q}_\ell = \begin{pmatrix} \star \\ \star \\ \star \\ \star \end{pmatrix}^T \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} u_{N^2+a} \\ u_{(b-1)N+c} \\ u_{N^2+c} \\ u_{(b-1)N+c} \end{pmatrix} \end{array} \right\}; \\ \bullet \mathcal{E}^3 &= \left\{ \tilde{Q}_\ell \mid \begin{array}{l} \exists (a, b) \in \mathcal{N} \times \mathcal{N} \\ \tilde{Q}_\ell = \begin{pmatrix} \star \\ \star \\ \star \\ \star \end{pmatrix}^T \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_{N^2+N+1} \\ u_{(a-1)N+b} \\ u_{(b-1)N+a} \end{pmatrix} \end{array} \right\}; \\ \bullet \mathcal{E}^4 &= \left\{ \tilde{Q}_\ell \mid \begin{array}{l} \exists a \in \mathcal{N} \\ \tilde{Q}_\ell = \begin{pmatrix} \star \\ \star \\ \star \\ \star \end{pmatrix}^T \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} u_{N^2+N+1} \\ u_{(a-1)N+a} \\ u_{N^2+a} \end{pmatrix} \end{array} \right\}; \end{aligned}$$

$$\bullet \mathcal{E}^5 = \left\{ \tilde{Q}_\ell \left| \begin{array}{l} \exists a \in \mathcal{N} \\ \tilde{Q}_\ell = \begin{pmatrix} * \\ * \\ * \end{pmatrix}^T \begin{pmatrix} 0 & \mathbf{j} \\ -\mathbf{j} & 0 \end{pmatrix} \begin{pmatrix} u_{(a-1)N+a} \\ u_{N^2+N+1} \end{pmatrix} \end{array} \right. \right\}.$$

The constraint

$$\begin{pmatrix} * \\ * \\ * \end{pmatrix}^* Q_0 \begin{pmatrix} V \otimes V^{*T} \\ V \\ 1 \end{pmatrix} > 0$$

is respected for every  $V$  in  $\mathbb{C}^N$  satisfying

$$\begin{pmatrix} * \\ * \\ * \end{pmatrix}^* Q_i \begin{pmatrix} V \otimes V^{*T} \\ V \\ 1 \end{pmatrix} < 0, \quad i \in \{1, \dots, N+1\}$$

if there exist  $N+1$  positive scalars  $\tau_i$  and  $N^E$  scalars  $\tilde{\tau}_\ell$  such that

$$Q_0 + \sum_{i=1}^{N+1} \tau_i Q_i + \sum_{\ell=1}^{N^E} \tilde{\tau}_\ell \tilde{Q}_\ell > 0. \quad (12)$$

with  $N^E = N^4 + N^3 + N^2 + 2N$  and  $\tilde{Q}_\ell \in \mathcal{E}$ .

Finding the positive scalars  $\tau_1, \dots, \tau_{N+1}$  and the scalars  $\tilde{\tau}_1, \dots, \tilde{\tau}_{N^E}$  which satisfy constraint (12) is a feasibility problem subject to LMI constraints.

*Proof 1.* See Appendix A.3 in Laib et al. (2018).  $\square$

*Remark 4.1.* Theorem 4.1 represents an extension to the well-known  $\mathcal{S}$ -procedure, see Boyd et al. (1994); Jönsson (2001), in the case of polynomial constraints and with complex variables. For a set of Quadratic Constraints (QC), and Integral Quadratic Constraints (IQC) in general, the  $\mathcal{S}$ -procedure is used to test if  $X^T Q_0 X > 0$  is respected for every  $X \in \mathbb{R}^N$  satisfying  $X^T Q_i X < 0$  with  $i \in \{1, \dots, N+1\}$ . The  $\mathcal{S}$ -procedure allows to perform this test by finding some positive scalars  $\tau_i$  with  $i \in \{1, \dots, N+1\}$  such that  $Q_0 + \sum_{i=1}^{N+1} \tau_i Q_i > 0$ . Nevertheless, this result is only valid when all components of  $X$  are independent which is not the case with the vector  $X = ((V \otimes V^{*T})^T V^T 1)^T$ . Theorem 3.2 represents an extension for the  $\mathcal{S}$ -procedure by introducing the scalars  $\tilde{\tau}_\ell$  and the matrices  $\tilde{Q}_\ell$  such that  $Q_0 + \sum_{i=1}^{N+1} \tau_i Q_i + \sum_{\ell=1}^{N^E} \tilde{\tau}_\ell \tilde{Q}_\ell > 0$  where the matrices  $\tilde{Q}_\ell$  characterize important links between the components of  $X$ , see Appendix A.2 in Laib et al. (2018) for the details.

*Remark 4.2.* Theorem 4.1 represents an alternative to Sum Of Square (SOS) techniques, see Parrilo (2003), which can be used to obtain the different links between the components of  $X = ((V \otimes V^{*T})^T V^T 1)^T$ . In this case, the number  $N^E$  of these links is  $\frac{1}{2} \frac{(k+m)!}{k!m!} \left( \frac{(k+m)!}{k!m!} + 1 \right) - \frac{1}{2} \frac{(2m+k)!}{k!m!}$  where  $m = 4$  and  $k = 2(N^2 + N)$ , see Parrilo (2003). Therefore, SOS techniques will be time consuming when solving Problem 3.2 due to the important number of decision variables. Theorem 4.1 represents an alternative by considering only important links between the components of  $X$  and the resulting  $N^E$  is equal to  $N^4 + N^3 + N^2 + 2N$ . The consequence will be an important reduction in computation time since the number of decision variables is significantly reduced.

## 5. APPLICATION TO THE UNCERTAIN POWER FLOW ANALYSIS PROBLEM

As stated above in Section 3, the major difficulty in Problem 3.1 is its polynomial constraints due to the

non-linear aspect of the power flow equations (4). After proposing Theorem 4.1 as a new tool to test the feasibility of a set of polynomial constraints, we present in this section Corollary 5.1 as a new solution to the uncertain power flow analysis problem.

Let  $\mathcal{P}_{opt}$  be the optimal perimeter of Problem 3.1. An upper bound  $\widetilde{\mathcal{P}}_{opt}$  on  $\mathcal{P}_{opt}$  can be found using the following corollary.

*Corollary 5.1.* Given the data of Problem 3.1 and let  $\mathcal{E}$  be the set of matrices  $\tilde{Q}_\ell$  given by (11). Let  $N^E = N^4 + N^3 + N^2 + 2N$ .

An upper bound  $\widetilde{\mathcal{P}}_{opt}$  on the optimal bound of Problem 3.1 can be obtained by finding for every  $k \in \{1, \dots, N\}$  the scalars

- $(V_k^{min})^2$  and  $(V_k^{max})^2$ ;
- $(\tau_{min})_i^k$  and  $(\tau_{max})_i^k$  with  $i \in \{1, \dots, N+1\}$ ;
- $(\tilde{\tau}_{min})_\ell^k$  and  $(\tilde{\tau}_{max})_\ell^k$  with  $\ell \in \{1, \dots, N^E\}$ .

which minimize

$$\text{trace} \left( \text{diag}_{k=1, \dots, N} \left( (V_k^{max})^2 - (V_k^{min})^2 \right) \right)$$

subject to

- (i)  $\text{bdiag}_{k=1, \dots, N} \left( Q_k^{min} + \sum_{i=1}^{N+1} (\tau_{min})_i^k Q_i + \sum_{\ell=1}^{N^E} (\tilde{\tau}_{min})_\ell^k \tilde{Q}_\ell \right) > 0$ ;
- (ii)  $\text{bdiag}_{k=1, \dots, N} \left( Q_k^{max} + \sum_{i=1}^{N+1} (\tau_{max})_i^k Q_i + \sum_{\ell=1}^{N^E} (\tilde{\tau}_{max})_\ell^k \tilde{Q}_\ell \right) > 0$ ;
- (iii)  $\text{diag}_{k=1, \dots, N} \left( \text{diag}_{i=1, \dots, N+1} \left( (\tau_{min})_i^k \right) \right) > 0$ ;
- (iv)  $\text{diag}_{k=1, \dots, N} \left( \text{diag}_{i=1, \dots, N+1} \left( (\tau_{max})_i^k \right) \right) > 0$ ;
- (v)  $\text{diag}_{k=1, \dots, N} \left( \text{diag} \left( (V_k^{min})^2, (V_k^{max})^2 \right) \right) > 0$ ;
- (vi)  $\text{diag}_{k=1, \dots, N} \left( (V_k^{max})^2 - (V_k^{min})^2 \right) > 0$ .

The upper bound  $\widetilde{\mathcal{P}}_{opt}$  is given by

$$\widetilde{\mathcal{P}}_{opt} = \vartheta \text{ argmin trace} \left( \text{diag}_{k=1, \dots, N} \left( (V_k^{max})^2 - (V_k^{min})^2 \right) \right)$$

such that conditions (i), (ii), (iii), (iv), (v) and (vi) are respected.

Minimizing the trace of  $\text{diag}_{k=1, \dots, N} \left( (V_k^{max})^2 - (V_k^{min})^2 \right)$  in Corollary 5.1 subject to conditions (i), (ii), (iii), (iv), (v) and (vi) is a problem of minimizing a linear cost function subject to LMI constraints.

*Proof 2.* See Appendix A.4 in Laib et al. (2018).  $\square$

## 6. ILLUSTRATION EXAMPLE

We consider a 3 bus distribution network with injected and load powers  $\underline{s}_{g_k}$  and  $\underline{s}_{\ell_k}$  at each bus  $k$  as shown in Fig.1. This example and its numerical data are taken from Chen et al. (2011).

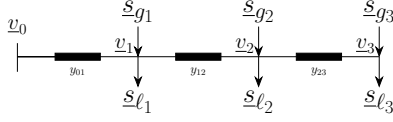


Fig. 1. Example of a 3 bus distribution network.

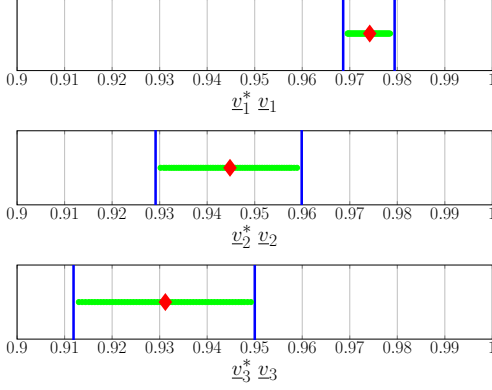


Fig. 2. Visualization of the sampling of  $\underline{v}_k^* \underline{v}_k$  (green),  $(V_k^{min})^2$  and  $(V_k^{max})^2$  (blue); and  $(\underline{v}_k^0)^* \underline{v}_k^0$  (red).

In this example, none of the renewable power resources inject reactive power into the network, that is  $q_{g_k} = 0$  with  $k \in \{1, 2, 3\}$ .

The data are normalized and given per unit

- the voltage  $\underline{v}_0$  is equal to  $0.995 e^{j0^\circ}$ ;
- the load powers  $\underline{s}_{l_1}$ ,  $\underline{s}_{l_2}$  and  $\underline{s}_{l_3}$  are  $0.8 + 0.25j$ ,  $0.5 + 0.1j$  and  $0.9 + 0.5j$  respectively;
- the current magnitude limitations  $I_1^{max}$ ,  $I_2^{max}$  and  $I_3^{max}$  are 0.48, 0.23 and 0.66 respectively;
- the nominal values of the three voltages are denoted  $\underline{v}_1^0$ ,  $\underline{v}_2^0$  and  $\underline{v}_3^0$ ; and are equal to  $0.987 e^{-j0.124^\circ}$ ,  $0.972 e^{-j0.273^\circ}$  and  $0.965 e^{-j0.302^\circ}$  respectively.

The injected power vector  $\underline{S}_g = (\underline{s}_{g_1} \ \underline{s}_{g_2} \ \underline{s}_{g_3})^T$  belongs to the ellipsoid  $\mathcal{S}_g$  given by

$$\mathcal{S}_g = \{ \underline{S}_g \in \mathbb{C}^3 \mid (\star)^* \Psi (\underline{S}_g - \underline{S}_g^0) < 1 \}$$

where  $\Psi = (\text{diag}(0.08^2, 0.06^2, 0.1^2))^{-1}$  and  $\underline{S}_g^0 = (0.4 \ 0.3 \ 0.5)^T$ .

Corollary 5.1 is applied to find the different  $(V_k^{min})^2$  and  $(V_k^{max})^2$  with  $k \in \{1, 2, 3\}$ . The results are presented in Fig. 2 where

- the green dots represent a sampling of the variation intervals of  $\underline{v}_k^* \underline{v}_k$  such that the different constraints on the injected powers and currents are respected;
- the blue lines represent the different  $(V_k^{min})^2$  and  $(V_k^{max})^2$ ;
- the red diamond shapes represent the different  $(\underline{v}_k^0)^* \underline{v}_k^0$ .

The obtained results present few conservatism as shown in Fig. 2 and it is possible to obtain the following bounds

$$\begin{aligned} 0.9842 &< |\underline{v}_1| < 0.9896 \\ 0.9639 &< |\underline{v}_2| < 0.9797 \\ 0.9549 &< |\underline{v}_3| < 0.9747 \end{aligned}$$

which demonstrates the efficiency of the proposed solution.

For comparison, the obtained results in Chen et al. (2011) were given as an ellipsoid containing all the voltage magnitudes and independent bounds cannot be obtained directly while in our approach it is possible to obtain independent bounds directly. Furthermore, the obtained results of Chen et al. (2011) are only valid around the operating point while our results do not depend on the operating point since no linearization is required in our approach.

## 7. CONCLUSION

In this paper, the uncertain power flow analysis problem is investigated. The major difficulty in this problem is the non-linear aspects of the power flow equations. To overcome this difficulty, and to avoid solving the problem locally around an operating point, our approach reformulates the problem as an optimization problem with polynomial constraints. The main contribution of this paper was proposing a new tool to solve the feasibility problem of set of polynomial constraints. Another contribution was proposing a new solution to the uncertain power flow analysis problem. The efficiency of this solution is illustrated through an illustrative example.

As perspective to this work, we propose the application of our result on large power network data in order to validate the efficiency of our results on large scale networks.

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