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Adaptive identification of linear systems subject to gross errors

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Abstract

In this note, we investigate the convergence of a robust recursive identifier for linear models subject to impulsive disturbances. Under the assumption that the disturbance is unknown and can be of arbitrarily large magnitude, the analyzed algorithm attempts to minimize online the sum of absolute errors so as to achieve a sparse prediction error sequence. It is proved that the identifier converges exponentially fast into an euclidean ball whose size is determined by the richness properties of the estimation data, the frequency of occurrence of impulsive errors and the parameters of the algorithm.

Key words: robust estimation, outliers, system identification, adaptive algorithms.

1 Introduction

Consider a data model defined by

$$y(t) = x(t)^\top \theta^o + f(t), \quad (1)$$

where $y(t) \in \mathbb{R}$ is the output at time $t \in \mathbb{Z}$, $x(t) \in \mathbb{R}^n$ is the regressor; θ^o denotes an unknown parameter vector. The sequence $\{f(t)\}$ is a disturbance of impulsive nature. We may also call it a sparse sequence, an appellation by which it is meant here that the elements of the sequence $\{f(t)\}$ are nonzero only occasionally. However, whenever they are nonzero, they can take on values of arbitrarily large magnitudes. Denote with $t_1 < t_2 < \dots < t_k, \dots$, the time instants t for which $f(t)$ is different from zero and let $\mathcal{T} = \{t_1, t_2, t_3, \dots\}$. Such times t_i will be termed impulse times. Hence for any $t_k \in \mathcal{T}$, $f(t_k) \neq 0$ and for any $t \notin \mathcal{T}$, $f(t) = 0$. Let $\tau_{\min} = \min_i |t_{i+1} - t_i|$ denote the minimum time between consecutive impulse times and $\tau_{\max} = \max_i |t_{i+1} - t_i| < \infty$. Assuming that τ_{\max} is finite has the important implication that $\{f(t)\}$ is persistent, that is, it does not vanish when $t \rightarrow \infty$.

Such a model as (1) with impulsive additive signal arises in many engineering problems. For example, the impulsive disturbance $\{f(t)\}$ can model intermittent sensor failures. In networked control systems, it can account

for occasional data packets losses (or corruption) in the course of their transmission over the underlying communication network. Another important application is in the identification of switched linear systems. In this case, a useful representation for identification purpose consists of the equation of a single subsystem augmented with a sparse component accounting for the effect of the switchings. For more details on how these problems relate to Eq. (1), we refer to [3].

Model (1) can represent both static and dynamic systems. In the first case, the regression vector contains observations at time t and need not be structured. This finds applications in the fields of signal processing and machine learning (e.g. the problem of subspace clustering [2,18]). In the latter case the regression vector can have a structure of the form

$$x(t) = [y(t-1) \ \dots \ y(t-n_a) \ u(t)^\top \ \dots \ u(t-n_b)^\top]^\top \quad (2)$$

or

$$x(t) = [u(t)^\top \ u(t-1)^\top \ \dots \ u(t-n_b)^\top]^\top \quad (3)$$

where $u(t) \in \mathbb{R}^{n_u}$ denotes the input of the system and the integers n_a and n_b represent structural parameters of the model.

Given data $x(k), y(k)$, $k = 1, \dots, t$, generated by a system of the form (1), the goal of this paper is to provide an adaptive estimation scheme for the parameter vector θ^o . This is to be done under the assumptions that

- The values of the sequence $\{f(t)\}$ are unknown and can be arbitrarily large,

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- The impulse times t_k are not available,
- The sequence $\{f(t)\}$ is sparse.

Note that no particular probability distribution assumption is put on $\{f(t)\}$ and the question of whether $\{f(t)\}$ is stochastic or deterministic does not matter as long as the sparsity assumption holds.

The problem we pose is that of estimating the parameter vector θ^o from model (1) in the face of gross errors affecting the measurements. As such, it is, as already discussed earlier, related to the literature of compressed sensing [5,6,9], robust statistics [10,16] and hybrid system identification [1,13,14,12]. If the data were completely collected in a batch mode, the problem could be tackled through a nonsmooth optimization-based estimator, e.g. ℓ_1 -norm minimization (which is also known as the least absolute deviation estimator) [3,4,17,19], iterative reweighted least squares [8], least median of squares [15], least trimmed squares [16], M-estimation [10], etc. It is known in this case that when the number of gross errors is less than a certain threshold determined by the genericity properties of the data (usually expressed in terms of mutual coherence, restricted isometry, breakdown point, ...), the true parameter vector θ^o can be exactly recovered.

In the current setting however, the data are only sequentially measured and at each time an estimate must be provided for a certain online application (e.g., adaptive decision making system, adaptive control, ...). As a consequence, the available batch data-based estimation results obtained for example in [3,17,19] and the works cited above do not apply here. A significant difference is that the adaptive identifier to be designed takes the form of a time-varying dynamic system whose input is the incoming data. What matters then for the analysis is no longer the absolute number of nonzeros errors $f(t)$ but instead the frequency of appearance of nonzeros disturbances, that is, the (average) minimum time between consecutive nonzeros elements.

The contribution of this note is to provide a convergence analysis of a specific adaptive algorithm in the challenging situation where the data are corrupted by gross errors of arbitrarily large amplitude. More precisely, we show that this identification problem can be efficiently solved by a simple weighted recursive least squares identifier provided the weighting function is appropriately chosen. Sufficient conditions are derived for the estimation error to converge to an euclidean ball whose radius depends on the algorithm's parameters and the system data. The empirical behavior of the algorithm tend to suggest that it outperforms the traditional ℓ_1 -norm minimization solution (see Section 4).

The rest of this note is structured as follows. Section 2 presents the robust weighted recursive least squares (RW-RLS) algorithm and Section 3 provides the asso-

ciated analysis. To illustrate the behavior of the algorithm in extremely challenging circumstances, a numerical study is carried out in Section 4.

2 A weighted recursive least squares solution

In this section, we derive a recursive least squares estimator endowed with the ability to remove the undesirable effects of gross errors $f(t)$. Similarly to [8,5], the robustness properties of our method rely on an appropriate weighting strategy. The underlying idea of the current method is inspired by the sparsity-inducing optimization techniques and it was already discussed in [7]. The purpose of this paper is to provide a convergence analysis.

2.1 Recursive identifier

Let $\theta(t)$ denote the estimate at time t of the parameter vector θ^o based on the measurements $(x(k), y(k))$ available up to time t . Being at time t , we have full knowledge of the prior estimate $\theta(t-1)$ along with the measurements $x(t), y(t)$ and want to generate $\theta(t)$. The proposed recursive identifier does so through the following recursive equations,

$$\theta(t) = \arg \min_{\theta \in \mathbb{R}^n} \left[\frac{1}{2} w(t)^2 (y(t) - \theta^\top x(t))^2 + \frac{\lambda}{2} (\theta - \theta(t-1))^\top P^{-1}(t-1) (\theta - \theta(t-1)) \right] \quad (4)$$

where the sequence $P^{-1}(t)$ defined by

$$P^{-1}(t) = \lambda P^{-1}(t-1) + w(t)^2 x(t)x(t)^\top, \quad P^{-1}(0) \succ 0, \quad (5)$$

is called the covariance matrix sequence, $\lambda \in]0, 1[$ is called the forgetting factor and $w(t)$ is a positive weight to be defined. The inequality $P^{-1}(0) \succ 0$ in (5) means that $P^{-1}(0)$ is positive-definite.

Eqs. (4)-(5) describe a weighted recursive least squares (W-RLS) algorithm. The design of the weight plays a capital role in providing the algorithm with a relative insensitivity (or robustness) to the impulsive noise sequence $\{f(t)\}$. Of course, if we could choose the weight such that $w(t_k) = 0$ for any $t_k \in \mathcal{T}$ and $w(t) = 1$ for $t \notin \mathcal{T}$, then the effects of the impulsive errors could be completely removed and classical convergence results [11] would then apply. Unfortunately this is not possible since the impulse time sequence $\{t_k\}$ is not available. In this paper the weight function $w(t)$ will be defined by

$$w(t) = 1/|\varepsilon(t)|, \quad (6)$$

with

$$\varepsilon(t)^2 = [y(t) - \theta(t-1)^\top x(t)]^2 + \varepsilon_0 \quad (7)$$

and $\varepsilon_0 > 0$ is a "small" number which is intended essentially for avoiding division by zero. The rationale behind the choice (6)-(7) of the weights is, as argued in [7], to mimic the nonsmooth ℓ_1 -norm minimization-based estimator. Note that (4) can be rewritten in the more classical form,

$$q(t) = \frac{P(t-1)x(t)}{\lambda\varepsilon(t)^2 + x(t)^\top P(t-1)x(t)} \quad (8)$$

$$\theta(t) = \theta(t-1) + q(t)(y(t) - \theta(t-1)^\top x(t)) \quad (9)$$

$$P(t) = \frac{1}{\lambda} (I - q(t)x(t)^\top) P(t-1), \quad (10)$$

which will be termed the robust weighted recursive least squares (RW-RLS) algorithm with exponential forgetting factor.

3 Convergence analysis

The question we ask now is whether (8)-(10) can converge in some sense towards the true parameter vector θ° in the presence of the gross errors. To answer this question, introduce the estimation error $\tilde{\theta}(t) = \theta(t) - \theta^\circ$. Then,

$$\tilde{\theta}(t) = (I - q(t)x(t)^\top)\tilde{\theta}(t-1) + q(t)f(t). \quad (11)$$

The asymptotic convergence of the estimation algorithm can therefore be formulated in terms of stability of the point 0 for the time varying system (11). Note that $q(t)$ defined in (8), depends on the state $\tilde{\theta}(t-1)$ through the term $\varepsilon(t)^2 = [f(t) - \tilde{\theta}(t-1)^\top x(t)]^2 + \varepsilon_0$. Therefore Eq. (11) represents a nonlinear time-varying impulsive system whose stability analysis is known to be hard.

For the question of convergence to be well-posed, we must exclude the situations where there would exist a time τ and a constant vector $\theta^1 \in \mathbb{R}^n$ such that

$$f(t) = x(t)^\top \theta^1 \quad \forall t \geq \tau. \quad (12)$$

In effect, if this was to hold, then (11) would converge to θ^1 instead of zero that is, $\theta(t)$ would tend to $\theta^\circ + \theta^1$ under conditions that can be readily deduced from the classic analysis of recursive least squares (see e.g., [11]). Interestingly, as will be observed in Remark 2 below, such pathological situations are automatically excluded by the sparsity condition on $\{f(t)\}$ together with the property of persistence of excitation (PE) to be defined next.

Definition 1 A vector sequence $\{x(t)\}$ is said to be persistently exciting (PE) if there exist an integer T and

some real numbers $\alpha_1 > 0$, $\alpha_2 > 0$ and $\tau_0 \geq 0$ such that

$$\alpha_1 I \preceq \sum_{k=t+1}^{t+T} x(k)x(k)^\top \preceq \alpha_2 I \quad \forall t \geq \tau_0 \quad (13)$$

In Eq. (13), I stands for the identity matrix with appropriate dimensions. From [11], we know that the free motion part (i.e., when $f \equiv 0$) of the unweighted version of the system (11) converges to zero under the PE condition. Here however, the term $q(t)f(t)$, although somewhat only scarcely nonzero, might prevent the estimation error to go to zero unless we force somehow the subsequence $\{q(t_k)\}_k$ where $t_k \in \mathcal{T}$, to tend to zero asymptotically. A major difficulty in achieving this is that the impulse times $\{t_k\}$ are unknown. As a consequence, we must just drive the whole sequence $\{q(t)\}$ to zero. But then it must not decay more rapidly than $\{\tilde{\theta}(t)\}$ itself; otherwise since $q(t)$ is the learning rate, $\tilde{\theta}(t)$ will either evolve much slowly or stop changing before convergence occurs.

Remark 2 For any integers k and m , let

$\mathcal{T}_{[k+1, k+m]}^c = \{t \in [k+1, k+m] : f(t) = 0\}$ denote the set of discrete time instants t in the interval $[k+1, k+m]$ for which $f(t) = 0$. If there exists an integer m such that $\text{rank}[X_{\mathcal{T}_{[k+1, k+m]}^c}] = n \forall k$, where $X_{\mathcal{T}_{[k+1, k+m]}^c}$ is a matrix collecting the regressors $x(t)$ indexed by the set $\mathcal{T}_{[k+1, k+m]}^c$, then (12) cannot hold. In turn, it can be checked that this rank condition is satisfied for any $m \geq \tau_{\min} + \tau_{\max}$ provided that $\{x(t)\}$ fulfills the PE property (13), and that $\tau_{\min} > T$ and $\tau_{\max} < \infty$.

The following lemma states that if all the weights $w(t)$ in (4)-(5) were set to one, then the matrix $P(t)$ is uniformly bounded above and below when the sequence $\{x(t)\}$ is PE. This is a crucial property in our analysis.

Lemma 3 Assume that the sequence $\{x(t)\}$ satisfies the PE condition (13). Then the matrix sequence defined by

$$\bar{P}^{-1}(t) = \lambda \bar{P}^{-1}(t-1) + x(t)x(t)^\top, \quad \bar{P}^{-1}(0) \succ 0 \quad (14)$$

satisfies

$$\bar{\gamma}_1 I \preceq \bar{P}^{-1}(t) \preceq \bar{\gamma}_2 I, \quad \forall t \geq 0 \quad (15)$$

for some constant numbers $\bar{\gamma}_1 > 0$ and $\bar{\gamma}_2 > 0$.

PROOF. First note from (14) that

$$\bar{P}^{-1}(t) = \lambda^t \bar{P}^{-1}(0) + \sum_{k=1}^t \lambda^{t-k} x(k)x(k)^\top.$$

Assume $t \geq T$ and let $d = \lfloor t/T \rfloor$ that is d is the integer part of t/T . Then the following chain of inequalities hold

$$\begin{aligned} \bar{P}^{-1}(t) &\preceq \lambda^T \bar{P}^{-1}(0) + \alpha_2 I + \sum_{k=1}^{dT} \lambda^{dT-k} x(k)x(k)^\top \\ &\preceq \lambda^T \bar{P}^{-1}(0) + \alpha_2 I + \alpha_2 \lambda^{dT} \sum_{i=1}^d \sum_{k=(i-1)T+1}^{iT} \lambda^{-k} I \\ &\preceq \left[\lambda^T \lambda_{\max}[\bar{P}^{-1}(0)] + \alpha_2 \frac{2-\lambda}{1-\lambda} \right] I, \end{aligned}$$

where the notation λ_{\max} refers to the maximum eigenvalue. To derive the lower bound, we proceed similarly as above. With $t \geq T$, we have

$$\begin{aligned} \bar{P}^{-1}(t) &\succeq \lambda^{(d+1)T} \sum_{i=1}^d \sum_{k=(i-1)T+1}^{iT} \lambda^{-k} x(k)x(k)^\top \\ &\succeq \lambda^{(d+1)T} \sum_{i=1}^d \sum_{k=(i-1)T+1}^{iT} \lambda^{-(i-1)T-1} x(k)x(k)^\top \\ &\succeq \alpha_1 \lambda^{(d+1)T-1} \sum_{i=1}^d \lambda^{-(i-1)T} I \\ &= \alpha_1 \lambda^{T-1} \frac{1-\lambda^{dT}}{\lambda^{-T}-1} I \succeq \alpha_1 \lambda^{2T-1} I. \end{aligned}$$

Finally, we can see that the following numbers satisfy (15):

$$\begin{aligned} \bar{\gamma}_1 &= \min \left\{ \delta_1, \alpha_1 \lambda^{2T-1} \right\}, \\ \bar{\gamma}_2 &= \max \left\{ \delta_2, \lambda^T \lambda_{\max}[\bar{P}^{-1}(0)] + \alpha_2 \frac{2-\lambda}{1-\lambda} \right\}, \end{aligned}$$

where $\delta_1 = \min_{t=0, \dots, T-1} \lambda_{\min}[\bar{P}^{-1}(t)] > 0$ and $\delta_2 = \max_{t=0, \dots, T-1} \lambda_{\max}[\bar{P}^{-1}(t)] > 0$ with $\lambda_{\min}[\cdot]$ standing for the minimum eigenvalue. \square

A consequence of Lemma 3 is as follows. Assuming $\{x(t)\}$ is PE, if we could guarantee that the weight sequence $\{w(t)\}$ is bounded above and below with strictly positive lower bound, then the matrix defined in (5) would obey

$$\gamma_1 I \preceq P^{-1}(t) \preceq \gamma_2 I, \quad \forall t \geq 0 \quad (16)$$

with $\gamma_1 = \bar{\gamma}_1 w_{\min}^2 > 0$ and $\gamma_2 = \bar{\gamma}_2 w_{\max}^2 > 0$, $w_{\min} = \inf_{t \in \mathbb{N}} w(t)$ and $w_{\max} = \sup_{t \in \mathbb{N}} w(t)$. Unfortunately, since $\varepsilon(t)^2 = [\hat{\theta}(t-1)^\top x(t) - f(t)]^2 + \varepsilon_0$, $w(t)$ depends on the state of (11) whose boundedness is not secured yet at this point of our developments. For example, if $\hat{\theta}(t)$ was to go to infinity, then the lower bound in

(16), would tend to zero so that the desired property of uniform positive-definiteness for $P^{-1}(t)$ would be lost.

To overcome this difficulty, consider for the time being a finite horizon N_0 large enough in a sense that will become more clear latter. Then for any t , $0 \leq t \leq N_0$, (16) holds with a strictly positive lower bound. Introduce the notations

$$\mu(N_0) = \sup_{\substack{\tau=0, \dots, \tau_{\max} \\ t=\tau_{\max}, \dots, N_0}} \frac{\lambda_{\max}[P^{-1}(t-\tau)]}{\lambda_{\min}[P^{-1}(t)]} \quad (17)$$

$$\gamma(N_0) = \inf_{t=0, \dots, N_0} \lambda_{\min}[P^{-1}(t)]. \quad (18)$$

From the above discussion, $\gamma(N_0) > 0$ and $\mu(N_0)$ is well-defined since the PE condition ensures that $\lambda_{\min}[P^{-1}(t)] > 0$ on any finite horizon N_0 . For notational simplicity, we will use from now onwards the notation μ and γ for $\mu(N_0)$ and $\gamma(N_0)$ respectively.

Our main convergence result of the estimation error (11) will be stated in Theorem 6. The method of proof is as follows: (a) consider a sufficiently large but finite horizon N_0 on which the numbers (17)-(18) are well defined; (b) show that the error converges to a ball² before time N_0 under some assumptions; (c) once the estimation error enters the above mentioned ball in finite time, we get the assurance that the numbers (17)-(18) are indeed well-defined for any N_0 so that the analysis is valid independently of N_0 . We start by stating a key technical assumption.

Assumption 1 *The design parameters λ , ε_0 , the minimum dwell time τ_{\min} and the sequence $\{x(t)\}$ are such that*

$$\mu \lambda^{\tau_{\min}} < \frac{\lambda}{1 + \frac{1}{\lambda \gamma \varepsilon_0}} \quad (19)$$

for some N_0 sufficiently large such that $t_1 + K \tau_{\max} \leq N_0$, where

$$K = 1 + \left\lceil \frac{1}{\ln(\lambda)} \ln \frac{r_0^2}{\|\tilde{\theta}(t_1)\|_2^2} \right\rceil \quad (20)$$

with

$$r_0^2 = \frac{1}{\gamma \left[\lambda - \mu \lambda^{\tau_{\min}} \left(1 + \frac{1}{\lambda \varepsilon_0 \gamma} \right) \right]}. \quad (21)$$

The notation $\lceil \cdot \rceil$ in (20) refers to the ceiling function, $\|\cdot\|_2$ denotes the Euclidean norm and t_1 is the first impulse time.

The condition (19) involves three different categories of parameters: the user-specified parameters λ , ε_0 of the algorithm; two parameters characterizing the properties

² A vector sequence $\{w(t)\}$ in \mathbb{R}^n is said to converge to a set $\mathcal{S} \subset \mathbb{R}^n$ if $\lim_{t \rightarrow \infty} \text{dist}(w(t), \mathcal{S}) = 0$, where $\text{dist}(w, \mathcal{S}) = \min_{z \in \mathcal{S}} \|z - w\|$ for some norm $\|\cdot\|$.

of the data sequence $\{x(t)\}$, such as μ , γ ; and a third parameter τ_{\min} which measures the frequency of occurrence of the gross errors. The parameters μ and γ can be viewed essentially as some quantitative measures of the richness of the regression data. Note that they depend also on the algorithm's parameters to a certain extent. Richest data are expected to have a small μ and a large γ . Hence, condition (19) requires on the one hand, that the frequency of the impulsive errors be relatively small and on the other hand, that the regression data be rich enough. We will show in Lemmas 4 and 5 that the subsequence $\{\tilde{\theta}(t_i)\}_i$ converges to a ball. As a consequence, the whole sequence $\{\tilde{\theta}(t)\}$ will, as stated in Theorem 6, have its values confined also to a ball when $t \rightarrow \infty$.

Lemma 4 *Assume that the regressor $\{x(t)\}$ is PE and $\|x(t)\|_2 = 1$ for all t . Then, under Assumption 1, the following holds: $\forall t_{i-1}, t_i \leq N_0$,*

$$\|\tilde{\theta}(t_{i-1})\|_2^2 > r_0^2 \Rightarrow \|\tilde{\theta}(t_i)\|_2^2 < \lambda \|\tilde{\theta}(t_{i-1})\|_2^2 \quad (22)$$

PROOF. Define a Lyapunov-like function V as

$$V(t) = \tilde{\theta}(t)^\top P(t)^{-1} \tilde{\theta}(t).$$

By means of some algebraic calculations, we obtain

$$\begin{aligned} V(t) - V(t-1) &= -(1-\lambda)V(t-1) \\ &\quad - \frac{\lambda(f(t) - x(t)^\top \tilde{\theta}(t-1))^2}{\lambda\varepsilon(t)^2 + x(t)^\top P(t-1)x(t)} + \frac{f(t)^2}{\varepsilon(t)^2} \end{aligned} \quad (23)$$

from which it can be inferred that

$$V(t) \leq \lambda V(t-1) + \frac{f(t)^2}{\varepsilon(t)^2}. \quad (24)$$

We see that for any $t \notin \mathcal{T}$, $V(t) \leq \lambda V(t-1)$ and for any $t_i \in \mathcal{T}$,

$$V(t_i) \leq \lambda^{t_i - t_{i-1}} V(t_{i-1}) + \frac{f(t_i)^2}{\varepsilon(t_i)^2}.$$

This, by using the definitions (17)-(18), implies that

$$\|\tilde{\theta}(t_i)\|_2^2 \leq \mu \lambda^{\tau_{\min}} \|\tilde{\theta}(t_{i-1})\|_2^2 + \frac{1}{\gamma} \frac{f(t_i)^2}{\varepsilon(t_i)^2}. \quad (25)$$

We wish to find sufficient conditions for $\|\tilde{\theta}(t_i)\|_2^2 < \lambda \|\tilde{\theta}(t_{i-1})\|_2^2$. Proceeding from (25), a sufficient condition

for that is

$$-\gamma(\lambda - \mu \lambda^{\tau_{\min}}) \|\tilde{\theta}(t_{i-1})\|_2^2 + \frac{f(t_i)^2}{(f(t_i) - x(t_i)^\top \tilde{\theta}(t_{i-1}))^2 + \varepsilon_0} < 0.$$

Note from (19) that $\gamma(\lambda - \mu \lambda^{\tau_{\min}}) > 0$. For convenience of manipulation let us drop time indexation and pose simply $z = x(t_i)^\top \tilde{\theta}(t_{i-1})$, $\eta^2 = \gamma(\lambda - \mu \lambda^{\tau_{\min}}) \|\tilde{\theta}(t_{i-1})\|_2^2$ and $f(t_i) = f$. Then the previous inequality reads as the following quadratic inequality

$$\begin{bmatrix} f \\ 1 \end{bmatrix}^\top \begin{bmatrix} (1-\eta^2) & \eta^2 z \\ \eta^2 z & -\eta^2(z^2 + \varepsilon_0) \end{bmatrix} \begin{bmatrix} f \\ 1 \end{bmatrix} < 0. \quad (26)$$

Since no condition is imposed on the sequence $\{f(t)\}$, the above inequality must be true for any $f \in \mathbb{R}$. By invoking a Schur complement argument, (26) holds for all $f \in \mathbb{R}$ if and only if

$$\begin{cases} \eta^2 > 1 \\ \eta^2(z^2 + (1-\eta^2)\varepsilon_0) < 0 \end{cases} \quad (27)$$

which in turn is equivalent to

$$z^2 < (\eta^2 - 1)\varepsilon_0. \quad (28)$$

On the other hand, let us observe that under Assumption 1, it necessarily holds that $\tau_{\min} \geq 2$. This is because Eq. (19) cannot hold for $\tau_{\min} = 1$. Consequently, $t_i - 1$ is not an impulse time. Therefore by using the Cauchy-Schwarz inequality and (24), we can write

$$z^2 \leq \|\tilde{\theta}(t_i - 1)\|_2^2 \leq \mu \lambda^{\tau_{\min} - 1} \|\tilde{\theta}(t_{i-1})\|_2^2. \quad (29)$$

Hence, for Eq. (28) to hold, it suffices that

$$\mu \lambda^{\tau_{\min} - 1} \|\tilde{\theta}(t_{i-1})\|_2^2 < [\gamma(\lambda - \mu \lambda^{\tau_{\min}}) \|\tilde{\theta}(t_{i-1})\|_2^2 - 1] \varepsilon_0$$

that is, $\|\tilde{\theta}(t_{i-1})\|_2^2 > r_0^2$. Therefore $\|\tilde{\theta}(t_i)\|_2^2 < \lambda \|\tilde{\theta}(t_{i-1})\|_2^2$ provided $\|\tilde{\theta}(t_{i-1})\|_2^2 > r_0^2$. \square

Before proceeding further, let us comment briefly on the assumptions of Lemma 4. Since Assumption 1 has already been discussed, let us consider just the requirement that $\|x(t)\|_2 = 1$ for all t . The primary purpose of this last assumption is to simplify the analysis. Indeed this is without loss of generality. In effect, if there is any zero regressor $x(t)$, it can be removed from the data used to feed the algorithm without incurring any loss of information (since the updating gain (8) in this

case is zero). For the nonzero remaining regressors, Eq. (1) can be divided by $\|x(t)\|_2$ so as to achieve the unit norm requirement. Note further that the precise feature exploited in the proof (see Eq. (29)) is $\|x(t)\|_2 \leq 1$ which can be fulfilled simply by multiplying Eq. (1) for example, by $(1 + \|x(t)\|_2^2)^{-1/2}$. A secondary potential benefit of such a normalization is to control the relative contribution of each of the error terms $(y(t) - x(t)^\top \theta)^2$ in the cost function associated with (4).

Lemma 5 *Under the assumptions of Lemma 4, the sequence $\{\tilde{\theta}(t_i)\}$ converges exponentially fast in finite time to the euclidean ball $\mathbf{B}(0, r_0) = \{x \in \mathbb{R}^n : \|x\|_2 \leq r_0\}$.*

PROOF. 1) $\{\tilde{\theta}(t_i)\}$ enters the ball in finite time. We adopt a contradiction argument here. Assume that $\{\tilde{\theta}(t_i)\}$ never enters the ball $\mathbf{B}(0, r_0)$ over the time horizon N_0 . Then for all $t_i \leq N_0$, $\|\tilde{\theta}(t_i)\|_2^2 > r_0^2$. By applying Lemma 4, this implies that $\|\tilde{\theta}(t_i)\|_2^2 \leq \lambda^{i-1} \|\tilde{\theta}(t_1)\|_2^2$ for all $t_i \leq N_0$, that is, the norm of $\tilde{\theta}(t_i)$ decreases exponentially fast. It follows that if N_0 is sufficiently large as required by Assumption 1, then there is necessarily an i^* such that $t_{i^*} \leq N_0$ and $\|\tilde{\theta}(t_{i^*})\|_2^2 \leq r_0^2$. For example, take $i^* = K$ with K defined by (20). This constitutes a contradiction with the initial thesis. Hence $\{\tilde{\theta}(t_i)\}$ enters the ball at a time prior to t_K .

2) $\{\tilde{\theta}(t_i)\}$ cannot exit the ball. We need now to show that once the sequence $\{\tilde{\theta}(t_i)\}$ enters the ball $\mathbf{B}(0, r_0)$ it stays in it for ever. We proceed again by contradiction. Assume that the sequence can exit the ball after entering it. Then there exist some t_{i-1} and t_i such that $\|\tilde{\theta}(t_{i-1})\|_2^2 \leq r_0^2$ but $\|\tilde{\theta}(t_i)\|_2^2 > r_0^2$. From (25), it follows that

$$\begin{aligned} r_0^2 < \|\tilde{\theta}(t_i)\|_2^2 &\leq \mu\lambda^{\tau_{\min}} \|\tilde{\theta}(t_{i-1})\|_2^2 + \frac{1}{\gamma} \frac{f(t_i)^2}{\varepsilon(t_i)^2} \\ &\leq \mu\lambda^{\tau_{\min}} r_0^2 + \frac{1}{\gamma} \frac{f(t_i)^2}{\varepsilon(t_i)^2} \end{aligned}$$

so that

$$\gamma(1 - \mu\lambda^{\tau_{\min}}) r_0^2 < \frac{f(t_i)^2}{\varepsilon(t_i)^2}.$$

Expanding this as in (26), we see that

$$\begin{bmatrix} f \\ 1 \end{bmatrix}^\top \begin{bmatrix} (\eta^2 - 1) & -\eta^2 z \\ -\eta^2 z & \eta^2(z^2 + \varepsilon_0) \end{bmatrix} \begin{bmatrix} f \\ 1 \end{bmatrix} < 0 \quad (30)$$

must hold for any f , with $\eta^2 = \gamma(1 - \mu\lambda^{\tau_{\min}})r_0^2$, $z = |x(t_i)^\top \tilde{\theta}(t_i - 1)|$, $f = f(t_i)$. Now we observe the following two facts: (i) $\eta^2 > 1$ and (ii) $z^2 \leq (\eta^2 - 1)\varepsilon_0$. The statement (i) is immediate by simple calculations. The

second is a consequence of (29) from which we can infer that

$$z^2 \leq \mu\lambda^{\tau_{\min}-1} \|\tilde{\theta}(t_{i-1})\|_2^2 \leq \mu\lambda^{\tau_{\min}-1} r_0^2.$$

Finally the statement (ii) is obtained by checking that $\mu\lambda^{\tau_{\min}-1} r_0^2 < (\eta^2 - 1)\varepsilon_0$. These two facts together show, by a Schur complement argument, that the 2×2 matrix appearing in (30) is positive semi-definite, hence rendering the corresponding inequality impossible. We therefore get a contradiction to the initial thesis according to which $\{\tilde{\theta}(t_i)\}$ could exit the ball. In conclusion, the sequence $\{\tilde{\theta}(t_i)\}$ converges in finite time into the ball $\mathbf{B}(0, r_0)$ and remains in it. \square

Theorem 6 (Convergence in finite time) *Assume that the regressor sequence $\{x(t)\}$ is PE and $\|x(t)\|_2 = 1$ for all t . If Assumption 1 holds, then $\{\tilde{\theta}(t)\}$ converges exponentially fast in finite time to the euclidean ball $\mathbf{B}(0, r_1)$, where*

$$r_1 = r_0 \max[1, \sqrt{\mu\lambda}]. \quad (31)$$

PROOF. The proof follows from Lemma 5. In effect, since the assumptions of the lemma are fulfilled, we can conclude that the sequence $\{\tilde{\theta}(t_i)\}$ converges in $\mathbf{B}(0, r_0)$. Now for any $t \notin \mathcal{T}$ such that $t_i < t < t_{i+1}$,

$$\|\tilde{\theta}(t)\|_2^2 \leq \mu\lambda^{t-t_i} \|\tilde{\theta}(t_i)\|_2^2 \leq \mu\lambda \|\tilde{\theta}(t_i)\|_2^2.$$

Since $\{\tilde{\theta}(t_i)\}$ converges into $\mathbf{B}(0, r_0)$, $\{\tilde{\theta}(t)\}_{t \notin \mathcal{T}}$ converges into $\mathbf{B}(0, r_0\sqrt{\mu\lambda})$. Hence $\{\tilde{\theta}(t)\}_t$ converges into $\mathbf{B}(0, r_1)$ as claimed. \square

The result stated in Theorem 6 guarantees boundedness of the estimation error sequence $\{\tilde{\theta}(t)\}$. Once $\{\tilde{\theta}(t)\}$ enters the ball $\mathbf{B}(0, r_1)$, we get that $w(t)^2 \geq 1/(r_1^2 + \varepsilon_0)$ whenever $t \notin \mathcal{T}$ with $w(t)$ referring to the weight defined in (6). An implication of this is that the weights in (6) will not vanish asymptotically hence preserving the PE condition for $\{w(t)x(t)\}$. Therefore, we can redefine the parameters μ and γ on an infinite horizon and the reasoning carried out so far still holds unchanged.

Discussion on the significance of the analysis. A question one may ask regarding the proposed analysis is whether the size of the attraction ball of the estimation error will shrink as time goes by. Here we provide an informal discussion. As we will see in the next section, there is an empirical evidence showing that the radius of the attraction ball can indeed be very small. The intuitive reason why this is very likely is as follows. Instead

of reasoning only on the interval $[0, N_0]$, we can consider a time interval $[N_k, N_{k+1}]$ with k possibly changing over time. Define on $[N_k, N_{k+1}]$ the parameters γ_k and μ_k respectively as in (17) and (18) with 0 and N_0 replaced with N_k and N_{k+1} . Then what the theorem says is that if the condition (19) holds over any time interval $[N_k, N_{k+1}]$ with respect to μ_k and γ_k , then the estimation error will rest in $B(0, r_{1k})$ where r_{1k} is defined in a similar way as r_1 in (31) and (21). Therefore, the more the estimation error shrinks, the larger the weights related to the samples indexed by \mathcal{T}^c (the complement of \mathcal{T} in \mathbb{N}) will increase hence causing γ_k to get large as well. The consequence of this is that r_{1k} will get smaller and smaller but potentially in a non monotone way.

4 Numerical study

4.1 Dynamic models

We first check the performance of the algorithm. For this purpose, we consider a model of the form (1)-(2), with $n_a = n_b = 2$ and a randomly generated parameter vector. The input signal is chosen as a realization of a zero-mean white noise process with unit variance. $\{f(t)\}$ is selected as a sequence of large numbers drawn for a Gaussian distribution with variance 1000^2 and mean 10. To challenge the identification method to an extreme extent, we let $\{f(t)\}$ be *non sparse*, and allow $f(t) = 0$ only occasionally, e.g. $f(t) = 0$ for one sample out of T for $T \in \{1, 2, 4, 5, 10, 20, \dots, 100\}$. To be more clear, we mean that only the samples $f(kT)$ are set to zero, all the others are nonzero and arbitrarily large. This means that rather than having only few nonzeros as generally assumed in the paper, the experiment is conducted in a very challenging case where there are only a few zeros in the sequence $\{f(t)\}$. For each value of T , the identification algorithm is run on 100 independent realizations of input-output data of size $N = 5000$ and an average relative error is measured out of the experiment. For the sake of comparison we also compute:

- The more classical sparsity-inducing estimate

$$\hat{\theta}_{L_1} = \arg \min_{\theta \in \mathbb{R}^n} \sum_{t=1}^N |y(t) - \theta^\top x(t)| \quad (32)$$

by minimizing the sum of absolute values of the errors. $\hat{\theta}_{L_1}$ is called the ℓ_1 -norm batch estimator.

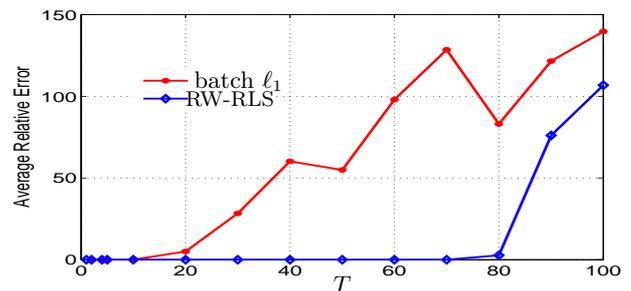
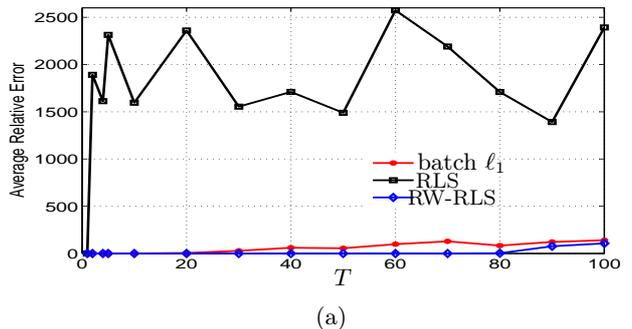
- The ordinary (unweighted) recursive least squares (RLS) estimate with exponential forgetting factor. This corresponds to setting $w(t) = 1$ for all t in Eqs. (4)-(5) which produces the same effect as fixing $\varepsilon(t)$ to 1 in Eq. (8).

The results are depicted in Figure 1 in term of the relative error

$$\tilde{\theta}_r(N) = \frac{\|\theta(N) - \theta^\circ\|_2}{\|\theta^\circ\|_2} \quad (33)$$

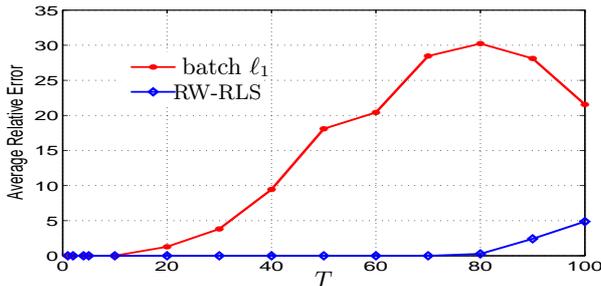
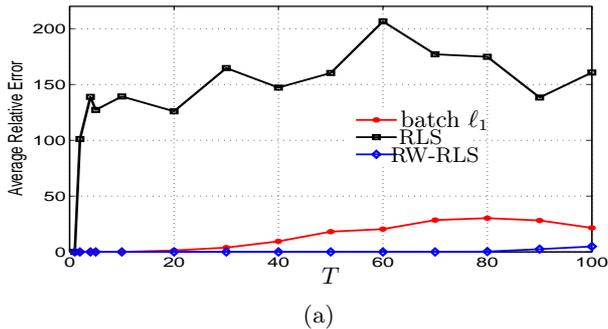
where $\theta(N)$ denotes the estimate at the last time N and θ° is the true parameter vector in Eq. (1). A striking observation is that the Robust Weighted-RLS algorithm discussed in this paper performs dramatically well in comparison to the batch ℓ_1 -norm based optimization and the RLS algorithm. To better judge how impressive and surprising these results are, recall that they have been obtained with a *non sparse* error sequence $\{f(t)\}$. On the dynamic data example (see Fig. 1), RW-RLS proves able to handle the impulsive noise in situations where only one sample of $\{f(t)\}$ out of $T = 70$ is equal to zero. Meanwhile, the ℓ_1 norm estimator is unsuccessful above $T = 10$ and the RLS completely breaks down once T is larger than one.

As it turns out the proposed algorithm returns accurate estimates even when the conditions of the paper are not satisfied. This confirms that the convergence conditions of the paper (which are only sufficient) are somewhat conservative. But the conservativeness in question here should be balanced by the fact that the proposed conditions concern worst case situations independently of magnitude and sign of the sequence $\{f(t)\}$.



(b) Zoom on sub-figure (a) after a removal of the curve corresponding to RLS.

Fig. 1. Dynamic model: Average relative error $\tilde{\theta}_r(N)$ over 100 simulations for each value of T . Algorithm's parameters: $\varepsilon_0 = 10^{-10}$, $\lambda = 0.99$, $P_0 = 10I$.



(b) Zoom on sub-figure (a) after a removal of the curve corresponding to RLS.

Fig. 2. Static model: Average relative error $\tilde{\theta}_r(N)$ over 100 simulations for each value of T . Here, the regression vectors $\{x(t)\}$ are drawn from a unit-variance and zero-mean Gaussian distribution on \mathbb{R}^4 . Algorithm's parameters: $\varepsilon_0 = 10^{-10}$, $\lambda = 0.99$, $P_0 = 10I$.

4.2 Static models

On static models the performance of the algorithm is even better as shown by Figure 2. This can be informally justified by the fact that the static data are more generic than dynamic data. While the static data are drawn from all directions without any constraint (e.g., from a Gaussian distribution), the directions spanned by the dynamic data structured as in (2) are constrained by the equation of the underlying system. As a consequence, the number γ defined in (18) is expected to be larger in the first case than in the second.

4.3 Discussions on the size of the attraction set

As is usual in this type of analysis, the derived conditions are not testable a priori. But they do provide a useful insight into the behavior of the identification method.

To achieve convergence of the estimation error in the sense of Theorem 6, it is desired that γ be as large and $\mu\lambda^{\tau_{\min}}$ as small as possible. From the definition of the size of the attraction ball in (21) and (31), large γ and small $\mu\lambda^{\tau_{\min}}$ would also lead to a small estimation error. To give an idea as to how small r_0 and r_1 may be, we can

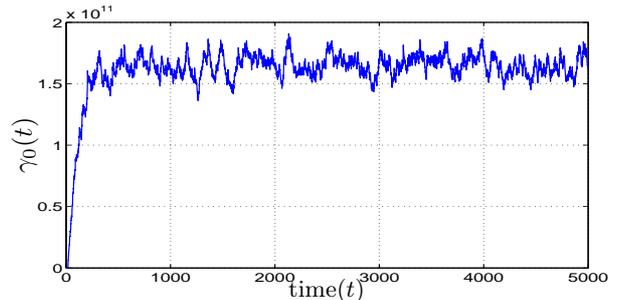


Fig. 3. Evolution of $\gamma_0(t) = \lambda_{\min}[P^{-1}(t)]$ over time. Algorithm's parameters: $\varepsilon_0 = 10^{-10}$, $\lambda = 0.99$, $P_0 = 10I$.

carry out a numerical experiment in which we compute r_0 and r_1 online while the algorithm is operating. For example, let the impulse time sequence $\{t_k\}$ be periodic, i.e., $t_k = t_{k-1} + T$. In this case, $\tau_{\min} = \tau_{\max} = T$. Fix $T = 30$ and choose the algorithm's parameters to be $\lambda = 0.8$, $P_0 = 100I$ and $\varepsilon_0 \in \{10^{-2}, 10^{-5}, 10^{-10}, 10^{-15}\}$. Then for each value of ε_0 , we carry out a Monte-Carlo simulation of size 100 and report the average values in Table 1. The results are very instructive. First they suggest that the condition (19) can indeed be fulfilled by lowering λ . Second, the estimation error converges effectively as stated by Theorem 6 to the aforementioned ball. Third, the attraction ball has a very small radius which keeps shrinking as ε_0 gets smaller. Also, it can be remarked that the numbers r_0 and r_1 are generally different but they tend to become closer as ε_0 is taken smaller.

Now, to give a sense of the magnitudes of the parameters γ and μ involved in condition (19) and the size of the convergence ball, we plot in Figures 3 and 4 some signals $\gamma_0(t)$ and $\mu_0(t)$ from which they are derived (see the captions of the respective figures for definition). As it turns out, μ is typically slightly larger than one and γ is roughly ten times larger than $1/\varepsilon_0$.

ε_0	10^{-2}	10^{-5}	10^{-10}	10^{-15}
A1	92%	94%	97%	98%
r_0^2	0.3	9.9×10^{-5}	3.3×10^{-9}	3.7×10^{-14}
r_1^2	7.4	10^{-3}	6.8×10^{-8}	7.9×10^{-13}
$\tilde{\theta}_r(N)$	3.6×10^{-7}	2.1×10^{-10}	7.5×10^{-15}	1.8×10^{-16}

Table 1

Some tests of the derived sufficient conditions. The line A1 gives the percentages of situations where Assumption (19) holds. Results obtained with $P_0 = 100I$, $\lambda = 0.8$, $T = 30$.

4.4 On the behavior in the presence of dense noise

Before closing this section let us make a last remark concerning the situation where both dense zero-mean noise $\{e(t)\}$ and impulsive gross sequence $\{f(t)\}$ are present in the system equation (1) so that it is now defined by $\tilde{y}(t) = x(t)^\top \theta^o$, $y(t) = \tilde{y}(t) + f(t) + e(t)$. To illustrate

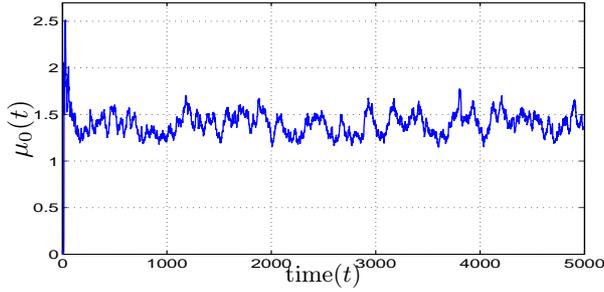
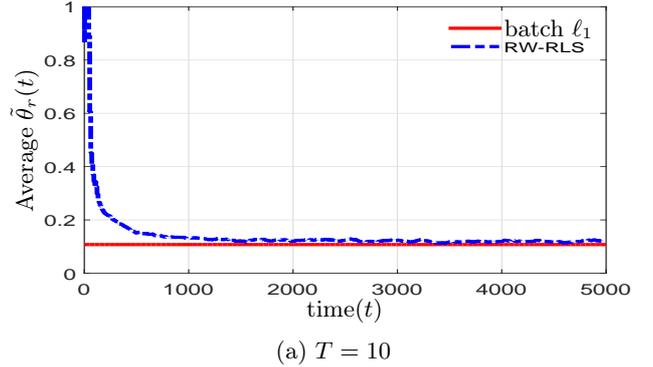


Fig. 4. Evolution of $\mu_0(t) = \sup_{\tau=0, \dots, \tau_{\max}} \frac{\lambda_{\max}[P^{-1}(t-\tau)]}{\lambda_{\min}[P^{-1}(t)]}$ over time. Algorithm's parameters: $\varepsilon_0 = 10^{-10}$, $\lambda = 0.99$, $P_0 = 10I$.

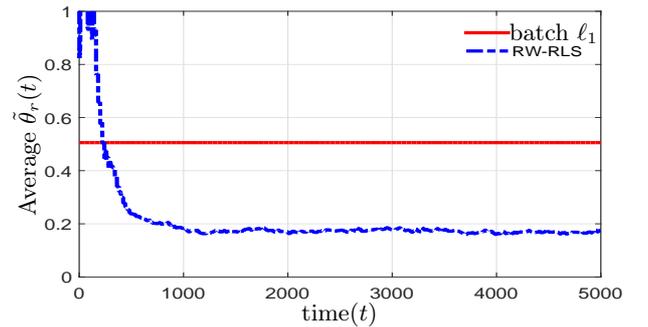
the behavior of the RW-RLS identifier in this case, consider the following numerical study on static systems: 100 different systems are generated at random and the output is computed as in the equation above with $\{e(t)\}$ being, for each system, a zero-mean noise sequence sampled independently from a Gaussian distribution and the impulsive sequence $\{f(t)\}$ being nonzero for all t except at times kT , with T a constant integer. The amount of noise $\{e(t)\}$ is such that the signal to noise ratio

$$10 \log_{10} \left[\frac{\sum_{t=1}^N \tilde{y}(t)^2}{\sum_{t=1}^N e(t)^2} \right]$$

is about 10 dB. This experiment is carried out successively for two fixed values of T : $T = 10$ and $T = 20$. Figure 5 compares the batch ℓ_1 estimator $\hat{\theta}_{L_1}$ defined in (32) and the RW-RLS identifier in terms of the average relative error $\{\tilde{\theta}_r(t)\}$ (see Eq. (33) for a definition). Since $\hat{\theta}_{L_1}$ is estimated once off-line for each run, the resulting average error is represented as a horizontal straight line while the average value of $\tilde{\theta}_r(t)$ over the 100 independent runs oscillate over time as a result of the dense noise acting on the recursive estimate. What this reveals is that the performance of the recursive identifier is robust to the joint effect of dense and impulsive noises. We remark in particular that when T is small, i.e., when $\{f(t)\}$ is sparse enough, the ℓ_1 batch estimator tends to perform better. But as T increases, this trend is reversed as shown by Figure 5-(b). Overall the numerical experiment suggests that when the sparsity level of $\{f(t)\}$ is sufficient enough to allow both the ℓ_1 and the RW-RLS estimators to yield θ^o in case $e \equiv 0$, the performance of RW-RLS tends to deteriorate more than that of the ℓ_1 -estimator after addition of the dense noise $\{e(t)\}$. But this observation is quite consistent with the batch nature of ℓ_1 -estimator which implies averaging of noise effect over the entire data sequence and the adaptive nature of RW-RLS which processes one sample at a time.



(a) $T = 10$



(b) $T = 20$

Fig. 5. Average relative estimation error versus time for the case of static models. The regression vectors $\{x(t)\}$ are drawn from a unit-variance and zero-mean Gaussian distribution on \mathbb{R}^4 . Algorithm's parameters: $\varepsilon_0 = 10^{-10}$, $\lambda = 0.985$, $P_0 = 100I$.

5 Conclusion

In this note we have studied the convergence properties of a recursive identifier that is designed to be robust to impulsive sensor failures. The failures are modeled as an unknown and sparse error sequence of arbitrary magnitude. The main theoretical result roughly says that if the failures do not occur too frequently, then the estimation error converges into a ball whose size is determined by some richness properties of the identification data and the algorithm's design parameters. Empirical evidence on generic data tend to suggest that the derived conditions, because they intend to characterize worst case situations, are somewhat conservative. Indeed, the proposed robust identifier can achieve a very good performance in practice even when the sequence of impulsive gross errors is not sparse.

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