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# Conformal boundary state for the rectangular geometry

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## Abstract

We discuss conformal field theories (CFTs) in rectangular geometries, and develop a formalism that involves a conformal boundary state for the  $1 + 1d$  open system. We focus on the case of homogeneous boundary conditions (no insertion of a boundary condition changing operator), for which we derive an explicit expression of the associated boundary state, valid for any arbitrary CFT. We check the validity of our solution, comparing it with known results for partition functions, numerical simulations of lattice discretizations, and coherent state expressions for free theories.

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## 1. Introduction

Boundary conformal field theory (BCFT) is a subject whose importance has grown over the years, both on the formal and on the applied side. It does for instance play a fundamental role in the axiomatization of conformal field theory (CFT) [1], in our growing understanding of logarithmic conformal field theory [2,3], or in the relationship between conformal field theory and the Schramm Loewner Evolution formalism [4]. It is also central to our understanding of the Kondo effect [5], of the physics of quantum impurities or the Fermi edge singularity [6], and, more recently, of local and global quenches in one-dimensional quantum systems [7,8].

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Two geometries are most naturally used in BCFT. One of them—more natural from the point of view of Euclidean field theory or statistical mechanics—is simply the upper half plane, with the theory defined *e.g.* only for  $\text{Im } z \geq 0$ , together with some boundary conditions at  $\text{Im } z = 0$ . The other involves a cylinder (sometimes considered instead as an annulus), and typically describes the physics of an open  $1D$  quantum system at finite temperature.

Of course, there are variants of these geometries. For instance, the physics of the cylinder can be described with an (imaginary) time evolution along the axis. In this case, the  $1D$  quantum system is closed, and the presence of the boundary is encoded in a boundary state. Such states have given rise to many developments, in the context of efforts to classify all conformal boundary conditions in particular.

We shall be concerned in this paper mostly with systems on a rectangle, which thus have trivial topology, sharp corners, and, in general, four different boundary conditions on their four edges. While in principle this situation can be tackled via conformal mappings of the half plane (see below), few of its features have actually been studied in detail. The situation is, on the other hand, clearly interesting in its own right. The rectangle geometry is, for instance, the natural one to consider in the case of quenches for  $1D$  quantum systems with open boundaries. In the  $2D$  point of view, it is the simplest geometry to study transport properties of network models for Anderson localization. More fundamentally, the rectangle provides a natural way to study and interpret the conformal blocks of four point functions by inserting four different fields at the four corners. This is particularly useful for instance to connect geometrical correlators in the Self Avoiding Walk (SAW) problem to CFT, in particular in the logarithmic (indecomposable) case.

We will focus on these and other geometrical features in subsequent and rather technical work to appear soon. The present paper is devoted to the exploration of the simplest—and yet very rich—aspects of this problem related with the boundary state for a theory defined on a segment.

## 2. Gluing condition

For a CFT in the complex plane  $\mathbb{C}$ , one can implement a boundary with the requirement that there is no energy flow across it. In other words, one of the components of the stress-tensor vanishes along the boundary:  $T_{\parallel\perp} = 0$ . In holomorphic/anti-holomorphic components this gives the constraint  $T(z) = \bar{T}(\bar{z})$  on the boundary. In radial quantization, a circular boundary at radius  $|z| = 1$  can be encoded in the form of a boundary state  $|B^p\rangle$ . Here the subscript  $p$  stands for periodic since the boundary is defined on a circle. The constraint is

$$(L_n - \bar{L}_{-n})|B^p\rangle = 0, \quad (1)$$

which is usually referred to as the conformal invariance of the boundary condition, or *gluing condition*, as it glues the modes of the chiral part of the CFT with the anti-chiral ones. A basis of solutions of the linear system of equations (1) is given by the so-called Ishibashi states, which are particular combinations of left–right symmetric Virasoro descendants of primary fields. Of course, to obtain the allowed boundary states  $|B^p\rangle$  themselves, more conditions have to be implemented (see for example [1,5] for reviews).

In this paper, we shall extend the formalism of boundary *states* to the case of an open system. Namely, we are now considering the CFT on a strip of width  $L$ , and we want to find a state in the theory that encodes the boundary perpendicular to the direction of the strip (see Fig. 1). In the  $1+1$  Hamiltonian description of the CFT, the boundary state now lives in the Hilbert space of the  $1d$  theory defined on a segment instead of a circle. To proceed, we choose coordinates  $z = x + iy$  in the plane and consider the semi-rectangular region  $0 \leq x \leq L$ ,  $y \geq 0$  of Fig. 1, and derive

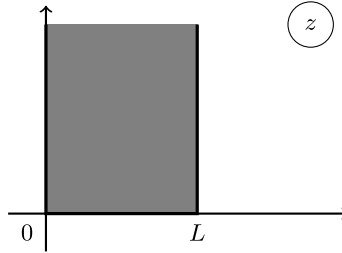


Fig. 1. The semi-rectangular geometry defining the boundary state.

the gluing condition for the stress–energy tensor in this geometry (for analogous discussions see [9–11]). Conformal invariance of the boundary implies then  $T_{xy} = 0$  on every side. This is realized at the boundaries  $z = iy$  and  $z = L + iy$ ,  $y > 0$ , by setting

$$\bar{T}(\bar{z}) = T(2L - \bar{z}), \tag{2}$$

and imposing periodic boundary conditions in the  $x$  direction of period  $2L$ :  $T(z + 2L) = T(z)$  (same for  $\bar{T}$ ).

Care is needed to treat properly the effect of corners. The stress tensor  $T$  has a singularity as its argument approaches the corners. To see this we start more generally by the singularity in the upper half plane when an operator of weight  $h$  is inserted at the origin. In this case the most singular term is

$$T(w) \approx \frac{h}{w^2}. \tag{3}$$

We then fold the upper half plane by the mapping  $z = w^{1/2}$  to have a corner in  $z = 0$ . After using the transformation law of the stress tensor  $T(z) = T(w)(dw/dz)^2 + (c/12)\{w; z\}$ , where  $\{w; z\}$  is the Schwarzian derivative

$$\{w; z\} \equiv \frac{d^3w/dz^3}{dw/dz} - \frac{3}{2} \left( \frac{d^2w/dz^2}{dw/dz} \right)^2, \tag{4}$$

we get

$$T(z) \approx \left( 4h - \frac{c}{8} \right) \frac{1}{z^2}. \tag{5}$$

Also if  $h = 0$ , at a corner there is an anomaly, which reflects itself in a non-trivial scaling dependence of physical quantities [12,13].

As a consequence the condition on  $T$  defining the boundary state  $|B^o\rangle$  at  $y = 0$  is

$$(T(x) - T(-x) + 4\pi i (\tilde{h}_l \delta'(x) + \tilde{h}_r \delta'(x - L)))|B^o\rangle = 0, \tag{6}$$

where we have used

$$\frac{1}{(x - i\epsilon)^2} - \frac{1}{(x + i\epsilon)^2} \approx \frac{4i\epsilon x}{(x^2 + \epsilon^2)^2} \rightarrow -2\pi i \delta'(x), \tag{7}$$

and defined the “effective conformal weight” at the corners

$$\tilde{h} := 2h - \frac{c}{16}, \tag{8}$$

$h_l$  ( $h_r$ ) being the weight of the operator inserted at the left (right) corner. Since the system is periodic with period  $2L$ , we can go to a mode expansion

$$T(x) = -\frac{\pi^2}{L^2} \left( \sum_n L_n e^{-i\pi nx/L} - \frac{c}{24} \right), \tag{9}$$

to get the gluing condition (for  $n \in \mathbb{Z}^{>0}$ )

$$(L_n - L_{-n} - 2n(\tilde{h}_l + (-)^n \tilde{h}_r)) |B^o\rangle = 0. \tag{10}$$

Note in particular that in this case one identifies not only  $L_n$  with  $\bar{L}_{-n}$  but also  $L_n$  with  $L_{-n}$ , since the semi-rectangular geometry is obtained by two consecutive folding of the plane.

Calling the boundary conditions on the sides of the semi-rectangle  $a$ ,  $b$ ,  $c$ , and the boundary condition changing operators sitting at the corners  $\phi_i^{ab}(0)$  and  $\phi_j^{bc}(L)$  (with opportune labels  $i, j$ ), the boundary state  $|B^o\rangle$  will generically live in a direct sum of vector spaces determined by the fusion  $\phi_i^{ab} \otimes \phi_j^{bc}$ . In this paper we consider the case of homogeneous boundary conditions  $a = b = c$ , where the identity operator sits at the corners ( $h_l = h_r = 0$ ), so that Eq. (10) becomes

$$\left( L_n - L_{-n} + \frac{nc}{8} [1 + (-1)^n] \right) |B^o\rangle = 0, \tag{11}$$

to be solved within the Verma module of the vacuum  $|0\rangle$  of the CFT. The solution of the homogeneous case will then be of the form  $G|0\rangle$ , with  $G$  a certain expression in terms of the Virasoro generators  $L_{-n}$ 's. When boundary condition changing operators are present, the boundary state will simply be  $G\phi_i^{ab}(w_1)\phi_j^{bc}(w_2)|0\rangle$ , where the two points  $w_1$  and  $w_2$  lie on the boundary and correspond to the images of the two corners at  $z = 0$  and  $z = L$  under a conformal mapping to the upper half plane. Further details about this general situation will be discussed in a sequel [14].

### 3. Boundary state in boundary CFT, and the rectangular bottom

In this section we give the explicit form of the boundary state  $|B^o\rangle$  which solves the constraint (11), following the discussion in [15]. Similar tricks have been used, for instance, in [4,16].

#### 3.1. CFT in the half-plane, deformations of the boundary around the origin

Let us consider some boundary CFT defined in the half-plane  $\mathbb{H}$ . We make use of radial quantization in  $\mathbb{H}$ . The vacuum of the theory can be written formally as a path integral

$$|0\rangle = \int [d\phi (|z| < 1, z \in \mathbb{H})] e^{-S[\phi]} |\phi (|z| = 1)\rangle, \tag{12}$$

where  $\phi (|z| = 1)$  stands for the configurations of the fields of the theory on a semi-circle centered at the origin 0 and of radius 1. The weights of the different configurations is given by a Gibbs distribution  $e^{-S[\phi]}$ . In this section, we consider deformations of the half-plane around the origin, such as the one shown in Fig. 2. More precisely, we take some domain  $\mathbb{H}' = \mathbb{H} \setminus K$  with  $K$  a domain included in the semi-disc of radius 1 centered on the origin, such that  $\mathbb{H}'$  is simply connected. Our goal is to find an expression for the new state

$$|\mathbb{H}'\rangle = \int [d\phi (|z| < 1, z \in \mathbb{H}')] e^{-S[\phi]} |\phi (|z| = 1)\rangle. \tag{13}$$

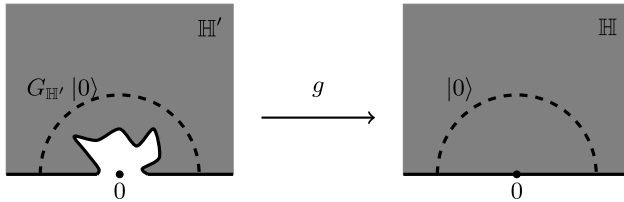


Fig. 2. A deformation of the boundary around the origin can be encoded in a conformal mapping  $g : \mathbb{H}' \rightarrow \mathbb{H}$ .

In this paper we do not consider insertions of boundary condition changing operators or other operators in the bulk. Therefore, we expect  $|\mathbb{H}'\rangle$  to be given by some linear combination of the vacuum  $|0\rangle$  and its descendants only. In other words, there must exist some operator  $G_{\mathbb{H}'}$  built out of the Virasoro generators such that

$$|\mathbb{H}'\rangle = G_{\mathbb{H}'}|0\rangle. \tag{14}$$

Our goal is to find how to construct  $G_{\mathbb{H}'}$ .

By Riemann’s mapping theorem, there is a conformal mapping from  $\mathbb{H}'$  onto  $\mathbb{H}$ . If one requires that the behavior of  $g(z)$  as  $z \rightarrow \infty$  be

$$g(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \dots \tag{15}$$

then the mapping  $g$  is unique.

Now, let us assume that we have a continuous family of deformations of the half-plane  $\{\mathbb{H}_t\}_{0 \leq t \leq 1}$ , such that  $\mathbb{H}_0 = \mathbb{H}$  and  $\mathbb{H}_1 = \mathbb{H}'$ . For each  $t$  there is a mapping  $g_t$  from  $\mathbb{H}_t$  onto  $\mathbb{H}$  with the above asymptotic behavior. The composition  $g_t^{-1} \circ g_{t+dt}$  gives us an infinitesimal mapping from  $\mathbb{H}_{t+dt}$  onto  $\mathbb{H}_t$ , which can be expanded as

$$g_t^{-1} \circ g_{t+dt}(z) = z + dt \left( \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3} + \dots \right). \tag{16}$$

This allows us to relate  $|\mathbb{H}_t\rangle$  and  $|\mathbb{H}_{t+dt}\rangle$ , using the definition of the stress-tensor in boundary CFT, namely the variation of the action  $S[\phi]$  under a small transformation  $z \mapsto z + \alpha(z)$  (see e.g. [17])

$$\delta S = \frac{1}{2\pi i} \oint \alpha(z) T(z) dz + \text{c.c.} \tag{17}$$

which gives for  $\alpha(z) = dt (b_1/z + b_2/z^2 + \dots)$

$$\begin{aligned} |\mathbb{H}_{t+dt}\rangle &= \int [d\phi (|z| < 1, z \in \mathbb{H}_t)] (1 - \delta S) e^{-S} |\phi (|z| = 1)\rangle \\ &= |\mathbb{H}_t\rangle - \frac{1}{2\pi i} \left( \oint \alpha(z) T(z) dz - \oint \bar{\alpha}(\bar{z}) \bar{T}(\bar{z}) d\bar{z} \right) |\mathbb{H}_t\rangle \\ &= |\mathbb{H}_t\rangle - dt (b_1 L_{-2} + b_2 L_{-3} + b_3 L_{-4} + \dots) |\mathbb{H}_t\rangle. \end{aligned}$$

We arrive at the differential equation

$$\frac{d}{dt} |\mathbb{H}_t\rangle = -(b_1 L_{-2} + b_2 L_{-3} + b_3 L_{-4} + \dots) |\mathbb{H}_t\rangle \tag{18}$$

or, in terms of the operator  $G_t \equiv G_{\mathbb{H}_t}$

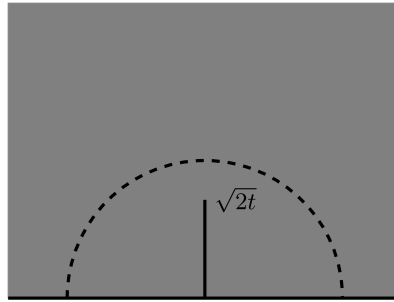


Fig. 3. The half-plane minus a slit of height  $\sqrt{2t}$ , namely  $\mathbb{H} \setminus [0, i\sqrt{2t}]$ , is the simplest example of a domain which can be encoded in a boundary state: it corresponds to  $e^{-tL_{-2}}|0\rangle$ .

$$\frac{d}{dt}G_t = -(b_1L_{-2} + b_2L_{-3} + b_3L_{-4} + \dots)G_t. \tag{19}$$

Note that (16) and (19) provide us with a differential equation which, at least formally, allows us to find the operator  $G_{\mathbb{H}'} = G_{t=1}$ . This differential equation will play an important role in the next sections.

### 3.2. The boundary state

In general, it is not possible to find an explicit solution to the above differential equation given some geometry  $\mathbb{H}'$ . For certain cases, however, solving the equation is easy.

Let us start with the case of a vertical slit of height  $\sqrt{2t}$  in the half-plane, namely the domain  $\mathbb{H} \setminus [0, i\sqrt{2t}]$ , see Fig. 3. It can be mapped onto the half-plane by  $g_t : z \mapsto \sqrt{z^2 + 2t}$ . Note that  $g_t(z) = z + \frac{t}{z} + \dots$  as  $z \rightarrow \infty$ , so it has the required asymptotic behavior. Differentiating this function with respect to  $t$ , we get

$$g_t^{-1} \circ g_{t+dt}(z) = z + \frac{dt}{z}. \tag{20}$$

According to the previous section, this corresponds to an operator  $G_t$  which is given by

$$\frac{d}{dt}G_t = -L_{-2}G_t \tag{21}$$

with the initial condition  $G_{t=0} = 1$ . We thus find  $G_t = e^{-tL_{-2}}$ . In particular, the state corresponding to a slit of height 1 is  $e^{-\frac{1}{2}L_{-2}}$ .

It turns out that one can easily extend this trick to a larger set of domains with slits. Let us define the half-plane minus  $k - 1$  slits of size  $2^{1/k}$

$$\mathbb{H}_k = \mathbb{H} \setminus \{z \mid z^k \in [-2, 2]\}. \tag{22}$$

See Fig. 4 for an illustration of the multi-slit geometry, and note that  $\mathbb{H}_1 = \mathbb{H}$ . The point of this definition is the following. We define the functions

$$g_k(z) = (z^k + 2)^{1/k} \tag{23}$$

and we make the observation that  $g_{2^N}$  is a conformal mapping from  $\mathbb{H}_{2^N}$  onto  $\mathbb{H}_{2^{N-1}}$ . Moreover, these mappings have the correct asymptotic behavior at infinity, and one can build a family of mappings  $g_{k,t} = (z^k + 2t)^{1/k}$  which gives rise to

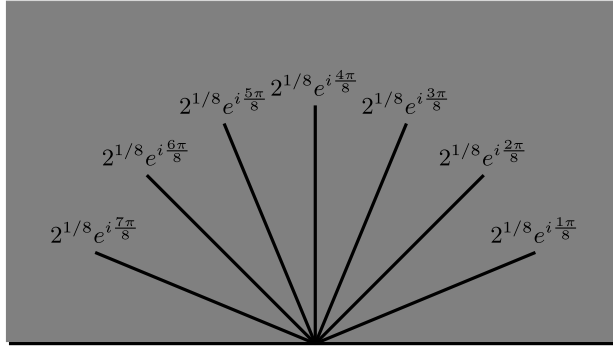


Fig. 4. The domain  $\mathbb{H}_8$  is the half-plane minus  $7 = 8 - 1$  slits of size  $2^{1/8}$ .

$$g_{k,t}^{-1} \circ g_{k,t+dt}(z) = z + \frac{2}{k} \frac{dt}{z^{k-1}} \tag{24}$$

and therefore to a differential equation for the operator  $G_{k,t} \equiv G_{\mathbb{H}_{k,t}}$

$$\frac{d}{dt} G_{k,t} = -\frac{2}{k} L_{-k} G_{k,t} \tag{25}$$

which gives in the end  $e^{-\frac{2t}{k} L_{-k}}$ .

As already mentioned,  $g_{2^N}$  is a mapping from  $\mathbb{H}_{2^N}$  onto  $\mathbb{H}_{2^{N-1}}$ , so by composing several of those, we see that  $g_2 \circ g_4 \circ \dots \circ g_{2^N}$  is a mapping from  $\mathbb{H}_{2^N}$  onto the half-plane  $\mathbb{H}$ . Thus, it is now straightforward to write down the boundary state  $|\mathbb{H}_{2^N}\rangle$  with  $2^N - 1$  slits of size  $2^{1/2^N}$

$$|\mathbb{H}_{2^N}\rangle = e^{-\frac{1}{2^{N-1}} L_{-2^N}} \dots e^{-\frac{1}{2} L_{-4}} e^{-L_{-2}} |0\rangle. \tag{26}$$

The connection between these states and the main topic of this paper appears when one considers the limit  $N \rightarrow \infty$ . Indeed, one can check that

$$\begin{aligned} g_2 \circ g_4 \circ \dots \circ g_{2^N}(z) &= \sqrt{\sqrt{\dots \sqrt{\sqrt{z^{2^N} + 2 + 2 + \dots + 2 + 2}}}} \\ &= z + \frac{1}{z} + O(z^{1-2^{N+1}}) \end{aligned} \tag{27}$$

when  $|z| \rightarrow \infty$ . We therefore have  $\lim_{N \rightarrow \infty} g_2 \circ g_4 \circ \dots \circ g_{2^N}(z) = z + z^{-1}$  when  $|z| > 1$ . The function  $z \mapsto z + z^{-1}$  is a conformal mapping from  $\mathcal{D} := \mathbb{H} \setminus \{z, |z| \leq 1\}$  to  $\mathbb{H}$ , see Fig. 5. It is thus tempting to conclude that the boundary state  $|B^o\rangle$  which corresponds to this domain (Fig. 6) is [15]

$$|B^o\rangle = \lim_{N \rightarrow \infty} e^{-\frac{1}{2^{N-1}} L_{-2^N}} \dots e^{-\frac{1}{2} L_{-4}} e^{-L_{-2}} |0\rangle. \tag{28}$$

At first sight, this expression might look ill-defined, however it is not. Indeed, it is obvious that, expanding the different exponentials in powers of the Virasoro modes, one gets a finite sum at each level. For instance, our boundary state is, up to level 4

$$|B^o\rangle = |0\rangle - L_{-2}|0\rangle - \frac{1}{2} L_{-4}|0\rangle + \frac{1}{2} L_{-2}^2|0\rangle + \dots \tag{29}$$



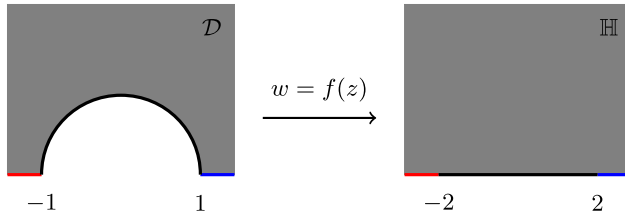


Fig. 5. Mapping of semicircular region  $\mathcal{D}$  to the upper half plane  $\mathbb{H}f(z) = z + z^{-1}$ .

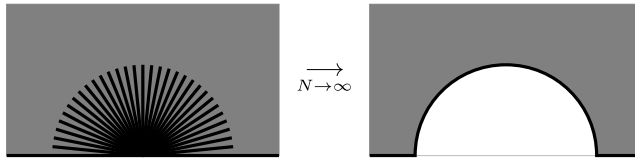


Fig. 6. The boundary state  $|B^0\rangle$  is obtained in the  $N \rightarrow \infty$  limit.

The foregoing argument leading to Eq. (28) is of course not entirely satisfactory, since it relies on an a priori uncomfortable identification of the half disk with an infinity of slits, supported by the asymptotic behavior (27). This can hold only in “some sense”—we believe, in the sense that all correlation functions in the CFT for the geometry of interest can be obtained using Eq. (28). We explore this further in what follows.

### 3.3. Deformed gluing conditions

In this section we will derive the gluing conditions satisfied by the stress tensor in the multi-slit geometries  $\mathbb{H}_{2N}$  introduced in Section 3.2. This will lead to a proof of the fact that the boundary state (28) satisfies the gluing condition for the semi-rectangular geometry, Eq. (11).

The region  $\mathbb{H}_{2N}$  can be mapped to the upper half plane by the conformal mapping (27), whose expansion at  $z \rightarrow \infty$  is

$$w = f_N(z) := z + \frac{1}{z} + \sum_{p=K}^{\infty} \frac{\alpha_k}{z^k}, \tag{30}$$

where we have introduced  $K := 2^{N+1} - 1$ . Its inverse has the following expansion at  $w \rightarrow \infty$ :

$$z = h_N(w) := \frac{w + \sqrt{w^2 - 4}}{2} + \sum_{p=K}^{\infty} \frac{\beta_k}{w^k}. \tag{31}$$

For the general argument given below we will not need the explicit expression of  $\alpha_k$ 's and  $\beta_k$ 's so we leave them unspecified. We note however that closed expressions of the above mappings can be found:

$$f_N(z) = 2 \cos\left(2^{-N} \arccos\left(\frac{z^{2^N}}{2}\right)\right), \tag{32}$$

$$h_N(w) = \left(2 \cos\left(2^N \arccos\left(\frac{w}{2}\right)\right)\right)^{2^{-N}}, \tag{33}$$

which eventually allow to write explicitly the coefficients appearing above and in the next computations. Let us now derive the gluing condition for the multi-slit geometry by using a slightly different point of view from the one of Section 2.

If we call  $X = \cdots \phi(z_1, \bar{z}_1) \cdots \psi(x) \cdots$  a chain of arbitrary bulk and boundary operators, by definition our boundary state (26) is such that

$$\langle 0|X|\mathbb{H}_{2N}\rangle = \langle 0|\tilde{X}|0\rangle, \tag{34}$$

where  $\tilde{X} = G_{\mathbb{H}_{2N}}^{-1} X G_{\mathbb{H}_{2N}}$  is the conjugation by the operators implementing the conformal mapping. Note that the out-vacuum  $\langle 0|$  is invariant under this mapping because  $f_N(z) \sim z$  when  $|z| \rightarrow \infty$ . Inserting the stress tensor in this correlator we have

$$\langle 0|XT(z)|\mathbb{H}_{2N}\rangle = (h'_N(w))^{-2} \left[ \langle 0|\tilde{X}T(w)|0\rangle - \frac{c}{12} \{h_N; w\} \langle 0|X|\mathbb{H}_{2N}\rangle \right], \tag{35}$$

where  $\{h_N; w\}$  is the Schwarzian derivative (4). We use this starting point to compute

$$\langle 0|X(L_n - L_{-n})|\mathbb{H}_{2N}\rangle \tag{36}$$

$$= \langle 0|X \oint_C \frac{dz}{2\pi i} z(z^n - z^{-n})T(z)|\mathbb{H}_{2N}\rangle \tag{37}$$

$$= \langle 0|\tilde{X} \oint_{\tilde{C}} \frac{dw}{2\pi i} J_{N,n}(w)T(w)|0\rangle - \frac{c}{12} \oint_{\tilde{C}} \frac{dw}{2\pi i} J_{N,n}(w) \{h_N; w\} \langle 0|X|\mathbb{H}_{2N}\rangle. \tag{38}$$

The contour  $\tilde{C}$  above encircles the points  $w = \pm 2$  and lies in the region of validity of the Laurent expansion (31), and we have defined

$$J_{N,n} := \frac{h_N}{h'_N} (h_N^n - h_N^{-n}). \tag{39}$$

Further one can check that

$$J_{N,n}(w) = (w^2 - 4)U_{n-1}\left(\frac{w}{2}\right) + \sum_{p=K-n}^{\infty} \frac{\alpha'_p}{w^p}, \tag{40}$$

where

$$U_{n-1}\left(\frac{w}{2}\right) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n-1-k}{k} w^{n-1-2k} \tag{41}$$

is the Chebyshev polynomial of the second kind and  $\alpha'_p$  are opportune coefficients. Note then that as  $N \rightarrow \infty$  the function in Eq. (40) reduces to a polynomial of degree  $n + 1$ . We focus now on the integral in Eq. (38). Computing the residue at the origin, the polynomial part of  $J_{N,n}$  gives zero, while mapping the contribution of the other non-analytic part back to the original multi-slit geometry one finds

$$\begin{aligned} & \langle 0|\tilde{X} \oint_{\tilde{C}} \frac{dw}{2\pi i} J_{N,n}(w)T(w)|0\rangle \\ &= \sum_{p=K-n}^{\infty} \mu_p \langle 0|XL_{-p-1}|\mathbb{H}_{2N}\rangle + \frac{c}{12} \sum_{p=K-n}^{\infty} \alpha'_p \oint_{\tilde{C}} \frac{dw}{2\pi i} w^{-p} \{h_N; w\} \langle 0|X|\mathbb{H}_{2N}\rangle, \end{aligned} \tag{42}$$

where  $\mu_p$ 's are some coefficients whose explicit expression is not needed. Plugging this back in (38) we have

$$\begin{aligned} &\langle 0|X(L_n - L_{-n})|\mathbb{H}_{2N}\rangle \\ &= \sum_{p=K-n}^{\infty} \mu_p \langle 0|XL_{-p-1}|\mathbb{H}_{2N}\rangle - \frac{c}{12} \oint_{\tilde{C}} \frac{dw}{2\pi i} (w^2 - 4)U_{n-1}\left(\frac{w}{2}\right) \{h_N; w\} \langle 0|X|\mathbb{H}_{2N}\rangle. \end{aligned} \tag{43}$$

We now separate the  $N \rightarrow \infty$  result from the finite  $N$  contributions in the Schwarzian derivative to get an expression of the following form:

$$\{h_N; w\} = \frac{6}{(w^2 - 4)^2} + \sum_{p=K+3} \frac{\alpha''_p}{w^p}. \tag{44}$$

Both terms in the equation above give a non-vanishing contribution:

$$-\frac{c}{12} \oint_{\tilde{C}} \frac{dw}{2\pi i} (w^2 - 4)U_{n-1}\left(\frac{w}{2}\right) \{h_N; w\} \langle 0|X|\mathbb{H}_{2N}\rangle = -\frac{nc}{8} (1 + (-1)^n) + A_n, \tag{45}$$

where we called  $A_n$  the number resulting from the integration of the second term in (44). Clearly one has  $A_n = 0$  if  $n \leq K$ . In the end one finds the following gluing condition for the multi-slit geometry:

$$\left(L_n - L_{-n} + \frac{nc}{8} [1 + (-1)^n]\right) |\mathbb{H}_{2N}\rangle = \left(\sum_{p=K-n}^{\infty} \mu_p L_{-p-1} + A_n\right) |\mathbb{H}_{2N}\rangle. \tag{46}$$

This deformed constraint reduces to the one of Eq. (11) for the semi-rectangle when  $N \rightarrow \infty$ . This is enough to prove that the boundary state (28) obtained as limit of the geometries with  $2^N - 1$  slits indeed satisfies the gluing condition (11).

### 3.4. Comparison with partition functions

A next obvious step is to consider the amplitude associated to the boundary state Eq. (28) defined in the previous section and compare it with the known result for the universal part of the partition function on a rectangle. We recall first this result. Since we consider the case of the same boundary condition on each side, we note that the partition function for a rectangle of length  $L$  and width  $L'$  will have to be a modular form. If we call the universal part of the partition function  $Z_{\mathcal{R}}(L, L')$

$$Z = e^{f_b LL'} e^{f_s(L+L')} Z_{\mathcal{R}}(L, L'), \tag{47}$$

where  $f_b$  and  $f_s$  are bulk and surface energies, we have

$$Z_{\mathcal{R}}(L, L') = Z_{\mathcal{R}}(L', L). \tag{48}$$

Define as usual  $\tau := iL'/L$ ,  $q := e^{2\pi i\tau}$  and the Dedekind eta function as  $\eta(\tau) = q^{1/24} \times \prod_{n=1}^{\infty} (1 - q^n)$ . The partition function  $Z_{\mathcal{R}}(L, L')$  was computed in [18] and up to possible proportionality coefficients is:

$$Z_{\mathcal{R}}(L, L') = L^{c/4} \eta(\tau)^{-c/2} = L^{c/4} q^{-c/48} \left( 1 + \frac{c}{2} q + \frac{c(c+6)}{8} q^2 + \dots \right). \tag{49}$$

As it was observed in [19], this partition function can be also derived directly from modularity arguments. Indeed, on top of invariance under modular inversion, as we have discussed the presence of corners corresponds to an effective weight  $c/16$  for each corner, which fix the non-trivial scaling of the partition function.

Now we can take the scalar product of the boundary state with itself to form the amplitude

$$A_{\mathcal{R}}(\tau) = \left[ \begin{array}{c} \uparrow \\ \square \end{array} \right] := \langle B^o | \hat{q}^{L_0 - c/24} | B^o \rangle \tag{50}$$

$$= \hat{q}^{-c/24} \sum_{n \geq 0} c_n \hat{q}^n, \tag{51}$$

where  $\hat{q} := \sqrt{q}$  is the relevant combination of  $L'/L$  appearing in the transfer matrix on a strip, and the arrow is the direction of imaginary time. The following relation between this amplitude and the partition function  $Z_{\mathcal{R}}(L, L')$  should hold

$$A_{\mathcal{R}}(\tau) = L^{-c/4} Z_{\mathcal{R}}(L, L') = \eta(\tau)^{-c/2}. \tag{52}$$

We have verified this relation by computing the first coefficients  $c_n$  in (51) using the commutation relations of the Virasoro algebra and comparing them with the power series of  $\eta^{-c/2}$ , confirming the validity of our derivation up to a very high order ( $n = 52$ ) in  $\hat{q}$ . We note that as expected from the left–right symmetry of the boundary conditions of the problem, the boundary states couple only to descendants of the identity of even level (so that  $c_{2n+1} = 0$  in Eq. (51)).

### 3.5. Convergence and multi-slit geometry

In this subsection we discuss further the  $N \rightarrow \infty$  limit illustrated in Fig. 6, by computing amplitudes involving the finitized boundary state

$$|\mathbb{H}_{2^N}\rangle = e^{-\frac{1}{2^{N-1}}L_{-2^N}} \dots e^{-\frac{1}{2}L_{-4}} e^{-L_{-2}} |0\rangle \tag{53}$$

corresponding to  $2^N - 1$  slits. We are interested in the generating functions

$$P_N(q) = (\langle \mathbb{H}_{2^N} | q^{L_0/2} | \mathbb{H}_{2^N} \rangle)^{2/c} = \sum_{k=0}^{\infty} p_k^{(N)} q^k. \tag{54}$$

Since the expected limit is

$$\lim_{N \rightarrow \infty} P_N(q) = q^{1/24} \eta(\tau)^{-1} = \prod_{m=1}^{\infty} \frac{1}{1 - q^m} = \sum_{k=0}^{\infty} p_k q^k, \tag{55}$$

where  $p_k$  is the number of partitions of the integer  $k$ , we expect that the  $p_k^{(N)}$  will somehow converge to  $p_k$ . We shall see below how this occurs.

The case  $N = 1$  is easily dealt with analytically. Using induction one can prove that

$$\langle 0|L_2^k L_{-2}^k|0\rangle = \frac{k!}{2^k} \prod_{p=0}^{k-1} (8p + c). \tag{56}$$

Developing the exponential  $e^{-L_{-2}}$  one then shows that

$$P_1(q) = (1 - 4q)^{-1/4} = 1 + q + \frac{5}{2}q^2 + \dots \tag{57}$$

In the next case,  $N = 2$ , a direct computation gives

$$P_2(q) = 1 + q + 2q^2 + 3q^3 + \frac{33}{4}q^4 + \dots \tag{58}$$

By automatizing the computations in the Virasoro algebra using MATHEMATICA we have been able to obtain the  $P_N(q)$  up to order  $q^{26}$ . In the case of  $P_2(q)$  the results are consistent with the conjecture

$$P_2(q) = \frac{(1 + 2q)^{1/2}(1 + 4q^2)^{5/8}}{(1 - 16q^4)^{3/4}}. \tag{59}$$

For higher  $N$  we have not been able to conjecture—let alone derive—such exact expressions. We find

$$P_3(q) = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + \frac{245}{8}q^8 + \dots + \frac{14988511}{512}q^{26} + \dots \tag{60}$$

and

$$P_4(q) = \sum_{k=0}^{15} p_k q^k + \frac{4005}{16}q^{16} + \dots + \frac{27657}{8}q^{26} + \dots, \tag{61}$$

while  $P_5(q)$  agrees with  $\sum_{k=0}^{\infty} p_k q^k$  at least up to order  $q^{26}$ .

So we observe from these examples that the coefficients  $p_k^{(N)}$  are non-negative rationals that coincide with the integers  $p_k$  for  $k = 0, 1, \dots, 2^N - 1$ . The first deviating coefficient occurs for  $k_0 = 2^N$ , and we have then  $p_{k_0}^{(N)} > p_{k_0}$ . We conjecture that these observed properties hold true for any  $N \geq 1$ .

#### 4. The case of free theories

While the expression (28) is general, there are more natural ways to think of the boundary state in the case of free theories. There, like when the boundary is a circle instead of a segment, expressions as coherent states over the bosonic/fermionic modes are possible. The comparison with results in the first section is intricate, and leads to remarkable identities. We note that some of the coherent states expressions presented in this section have an important overlap with [9–11].<sup>1</sup>

<sup>1</sup> We thank V. Schomerus for bringing these papers to our attention.

### 4.1. Free boson

The simplest case one can deal with is the free boson. We introduce then harmonic oscillators  $a_n$ ,  $n \in \mathbb{Z}$  satisfying  $[a_m, a_n] = m\delta_{m+n,0}$ ,  $a_m|0\rangle = 0$  if  $m > 0$ , and  $a_0|0\rangle = 0$ , with which we represent as usual the Virasoro generators:

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} a_{n-m} a_m, \quad \text{if } n \neq 0, \quad L_0 = \frac{1}{2} a_0^2 + \sum_{m \geq 1} a_{-m} a_m. \tag{62}$$

Substituting this expression in (10), we realize that the solution should be of the form:

$$|B_\phi^o\rangle = \exp\left(-\sum_{n>0} \frac{1}{2n} a_{-n}^2\right). \tag{63}$$

To verify this one simply has to use twice the following relation

$$(a_m + a_{-m})|B_\phi^o\rangle = 0, \quad m > 0. \tag{64}$$

The extra term  $n/8(1 + (-)^n)$  comes in because if  $n = 2p$  we have  $a_p^2|B_\phi^o\rangle = (a_{-p}^2 - p)|B_\phi^o\rangle$ . Another straightforward exercise is computing the amplitude associated with this boundary state. We have

$$\langle B_\phi^o | \hat{q}^{L_0 - 1/24} | B_\phi^o \rangle = q^{1/48} \prod_{m>0} \left( \sum_{s \geq 0} \frac{(2s)!}{s!s!} \left(\frac{q^m}{4}\right)^s \right) = \frac{1}{\sqrt{\eta(\tau)}}, \tag{65}$$

in agreement with (52). Alternatively one could also express  $L_n$  in (28) using (62) and check that the two expressions agrees for  $c = 1$ .

Further we note that this result for the partition function of the free boson can also be obtained by computing the determinant of the Laplacian with, say, Neumann boundary conditions on all four sides. To our knowledge this result appeared first in [20] (there is factor of  $LL'$  there due to the way the zero mode is subtracted):

$$\det(-\Delta) = L^{-1/2} \eta(\tau). \tag{66}$$

We note the explicit presence of the anomaly term  $L^{c/4}$  in the partition function given by  $Z_{\mathcal{R}}(L, L') = (\det(-\Delta))^{-1/2}$ .

It is interesting to observe that Eq. (66) corresponds, using the matrix-tree theorem, to the partition function of spanning trees, equivalent in turn to dense polymers or symplectic fermions—all variants of  $c = -2$  CFT. (More generally, partition functions for  $c = -2$  with different boundary conditions on the sides will be given by taking the power  $-2$  of the expressions for the free boson, since the boundary condition changing operators have dimensions  $1/16$  for  $c = 1$  and  $-1/8$  for  $c = -2$ . This will be exploited in a further paper [14].)

### 4.2. Majorana fermions

We consider now the case of Majorana fermions. We will go carefully through the derivation of the gluing condition which turns out to be more tricky in this case, and then derive the boundary state by computing the correlator in the rectangular geometry using a conformal mapping to the upper half plane.

First, we fix notations and conventions. The fermionic modes are defined via the expansion of the bulk fermionic fields

$$\psi(z) = \sum_{r \in I} \frac{\psi_r}{z^{r+1/2}}, \quad \bar{\psi}(\bar{z}) = \sum_{r \in I} \frac{\bar{\psi}_r}{\bar{z}^{r+1/2}} \tag{67}$$

where  $I = \mathbb{Z}$  in the Ramond (R) sector and  $I = \mathbb{Z} + 1/2$  in the Neveu–Schwartz (NS) sector. The OPEs

$$\psi(z)\psi(w) = \frac{1}{z-w}, \quad \bar{\psi}(\bar{z})\bar{\psi}(\bar{w}) = \frac{1}{\bar{z}-\bar{w}} \tag{68}$$

lead to

$$\{\psi_r, \psi_s\} = \{\bar{\psi}_r, \bar{\psi}_s\} = \delta_{r+s,0}. \tag{69}$$

Let us recall first the condition in the cylinder geometry with Dirichlet (D) or Neumann (N) boundary conditions on the bottom [21]. For simplicity we restrict to the NS case, where fermions are periodic in the plane, and thus antiperiodic on the cylinder. The cylinder can be obtained from the plane (with complex coordinate  $z$ ) by the mapping  $w = i \ln z$  or  $z = e^{-iw}$ . Set  $w = x + iy$ . The mode expansion now reads

$$\psi(x, y) = (-i)^{1/2} \sum_r \psi_r e^{irx-ry}, \tag{70}$$

$$\bar{\psi}(x, y) = i^{1/2} \sum_r \bar{\psi}_r e^{-irx-ry}. \tag{71}$$

The boundary conditions are then

$$\bar{\psi}(x, y=0) = \epsilon \psi(x, y=0), \tag{72}$$

where  $\epsilon = 1$  for N and  $-1$  for D. The boundary state associated to these equations is well known and has the form of (fermionic) coherent states:

$$|B_\psi^p\rangle \propto \exp\left(\epsilon i \sum_{p=0}^\infty \psi_{-p-1/2} \bar{\psi}_{-p-1/2}\right) |O\rangle, \tag{73}$$

where

$$\psi_{p+1/2}|O\rangle = 0, \quad \bar{\psi}_{p+1/2}|O\rangle = 0, \quad p \in \mathbb{Z}_{\geq 0}. \tag{74}$$

Now imagine doing the same problem in a geometry rotated by  $\pm\pi/2$ , obtained by the mapping  $w' = \pm iw$ . In this rotation, we use the general formulas for transforming Majorana fermions, which are objects of dimension  $1/2$ , and thus  $\psi(w')(dw')^{1/2} = \psi(w)(dw)^{1/2}$ , and similarly for  $\bar{\psi}$ . Hence the equations characterizing the boundary state now would read  $\bar{\psi} = \pm \epsilon i \psi$  instead of (72): when dealing with fermions, the D or N boundary condition is not represented by a unique equation, but depends on the orientation of the boundary.

Let us go back now to the geometry of interest of Fig. 1, and for definiteness choose, say, D boundary conditions on all sides and  $L = \pi$ . The case with N boundary conditions can be treated in a similar way, and lead in the end to the same result. According to our discussion we have

$$\begin{aligned} \bar{\psi}(x=0, y) &= i\psi(x=0, y), \quad y \in [-\infty, \infty], \\ \bar{\psi}(x=\pi, y) &= -i\psi(x=\pi, y), \quad y \in [-\infty, \infty]. \end{aligned} \tag{75}$$

Fully open boundary conditions require identification (up to monodromy conditions) of left and right modes from both the direct and crossed channels points of view. It is useful therefore to define

$$\psi(x, y) = -i\bar{\psi}(-x, y), \quad x \in [-\pi, 0] \tag{76}$$

so now we can re-express everything in terms of only one type of fermions—we choose  $\psi$ . Moreover, by construction,  $\psi$  is now regular at the origin. By symmetry we can also extend fermions on the other side of the right boundary by defining

$$\psi(2\pi - x, y) = i\bar{\psi}(x, y) \tag{77}$$

so  $\psi$  is now regular at  $x = \pi$ . Using (76) gives finally

$$\psi(x + 2\pi) = -\psi(x), \quad x \in [-\pi, \pi] \tag{78}$$

so that we now have a single species of fermions defined on the circle  $[-\pi, \pi]$ , with NS (anti-periodic) boundary conditions.

All the manipulations so far are there to handle the boundary conditions at  $x = 0$  and  $x = \pi$ . We have now, on top of this, to handle the boundary condition at the bottom of the system, that is for  $y = 0$ . Taking  $\epsilon = -1$  in (72), from (76) we have

$$[\psi(x, y = 0) + i\psi(-x, y = 0)]|B_\psi^o\rangle = 0, \quad x \in [0, \pi]. \tag{79}$$

Going through the definitions carefully shows that the relative sign of the two fermion terms in this equation switches in the interval  $x \in [-\pi, 0]$ . This is in line with the fact that fermion correlators change sign under reflection. So the final equation has to be for  $x \in [-\pi, \pi]$

$$[\psi(x, y = 0) + \text{sign}(x)i\psi(-x, y = 0)]|B_\psi^o\rangle = 0. \tag{80}$$

The sign function introduces serious complications with respect to the case of ordinary boundary states. Notice that it disappears when considering conditions for the Virasoro generators, since the stress–energy tensor is quadratic in the fermions. We represent the sign function through the Fourier series

$$\text{sign}(x) = \sum_{m=-\infty}^{\infty} a_m e^{imx} \tag{81}$$

with

$$a_{2m} = 0, \quad a_{2m+1} = -\frac{2i}{(2m+1)\pi}. \tag{82}$$

We can introduce the matrix  $A$  defined through

$$\begin{aligned} A_{m+1/2, n+1/2} &= \frac{1 - (-1)^{m+n+1}}{\pi(m+n+1)}, \quad m+n+1 \neq 0, \\ A_{m+1/2, -m-1/2} &= 0 \end{aligned} \tag{83}$$

so the equation satisfied by the boundary state reads, in terms of the fermionic modes

$$\left( \psi_{m+1/2} + \sum_n A_{m+1/2, n+1/2} \psi_{n+1/2} \right) |B_\psi^o\rangle = 0. \tag{84}$$

Now decompose the  $A$  matrix (whose labels are half odd integers) into four blocks corresponding to the label signs:

$$A = \begin{pmatrix} a_{m,n} & b_{m,n} \\ -b_{m,n} & -a_{m,n} \end{pmatrix}, \tag{85}$$



where we have defined blocks  $a_{m,n} \equiv A_{m+1/2,n+1/2}$ , and  $b_{m,n} \equiv A_{m+1/2,-n-1/2}$ , with  $m, n \geq 0$ . Take now  $m \geq 0$  (for  $m < 0$  one gets an equivalent condition) and consider Eq. (84), which we can rewrite as

$$\left( \psi_{m+1/2} + \sum_{n \geq 0} a_{mn} \psi_{n+1/2} + b_{mn} \psi_{-n-1/2} \right) |B_\psi^o\rangle = 0.$$

Collecting the positive and negative fermion modes and inverting the system gives the equation

$$\left( \psi_{m+1/2} + \sum_{n \geq 0} [(1+a)^{-1}b]_{mn} \psi_{-n-1/2} \right) |B_\psi^o\rangle = 0.$$

Now set  $G := -(1+a)^{-1}b = -G^T$ . The solution of this system is (up to proportionality coefficients) the coherent state

$$|B_\psi^o\rangle = \exp\left( \sum_{0 \leq m < n}^\infty G_{m,n} \psi_{-m-1/2} \psi_{-n-1/2} \right) |O\rangle. \tag{86}$$

With this approach the matrix  $G_{m,n}$  is somewhat undetermined. We present then a different strategy to compute it using correlation functions. As in Section 3 we define the boundary state in the geometry denoted by  $\mathcal{D}$  (Fig. 5 (left)), and consider the mapping to the upper half plane  $\mathbb{H}$  (Fig. 5 (right)), viz.  $w = z + z^{-1}$ . We thus consider D boundary conditions on all sides of the boundary of  $\mathcal{D}$ , so that  $|B_\psi^o\rangle$  should be the state representing this boundary. Calling  $|O\rangle$  the ground state of the model at infinity in  $\mathbb{H}$  we have

$$\langle \psi(z_1) \psi(z_2) \rangle_{\mathcal{D}} = \langle O | \psi(z_1) \psi(z_2) | B_\psi^o \rangle \tag{87}$$

$$= \left( \frac{\partial w_1}{\partial z_1} \frac{\partial w_2}{\partial z_2} \right)^{1/2} \langle \psi(w_1) \psi(w_2) \rangle_{\mathbb{H}}, \tag{88}$$

where on the rhs the correlator is evaluated in  $\mathbb{H}$  with the usual D boundary conditions. Using that this correlator must simply be  $1/(w_1 - w_2)$  (since it involves only right movers, which are not affected by the boundary), and evaluating derivatives gives straightforwardly that

$$\langle O | \psi(z_1) \psi(z_2) | B_\psi^o \rangle = \frac{1}{z_1 - z_2} \frac{\sqrt{1 - \frac{1}{z_1^2}} \sqrt{1 - \frac{1}{z_2^2}}}{1 - \frac{1}{z_1 z_2}}.$$

From the result (86) we can write the boundary state as

$$|B_\psi^o\rangle = : \exp\left( \oint \frac{dz}{2i\pi} \oint \frac{dz'}{2i\pi} \psi(z) G(z, z') \psi(z') \right) : |O\rangle, \tag{89}$$

where  $G(z, z')$  corresponds to the generating function of the numbers  $G_{m,n}$  introduced above. The left-hand side of Eq. (87) can now be evaluated using Wick’s theorem. We find in the end that

$$G(z_1, z_2) = \frac{1}{2(z_2 - z_1)} \left( \frac{\sqrt{1 - \frac{1}{z_1^2}} \sqrt{1 - \frac{1}{z_2^2}}}{1 - \frac{1}{z_1 z_2}} - 1 \right). \tag{90}$$

We need to evaluate this expression in the domain  $|z_1|, |z_2| > 1$  (because of the geometry of our problem), and expand it as

$$G(z_1, z_2) = \frac{1}{2z_1z_2} \sum_{m,n=0}^{\infty} \frac{G_{mn}}{z_1^m z_2^n}. \tag{91}$$

(Observe that on top of  $G_{mn} = -G_{nm}$ , we have  $G_{mn} = 0$  if  $m + n$  is even.) As a result, we now have the generating function for the quadratic form appearing in (86). The first few values of  $G_{m,n}$  read:

$$\begin{aligned} G_{01} &= \frac{1}{2}, \\ G_{03} &= \frac{1}{8}, & G_{12} &= \frac{5}{8}, \\ G_{05} &= \frac{1}{16}, & G_{14} &= \frac{3}{16}, & G_{23} &= \frac{5}{8}, \\ G_{07} &= \frac{5}{128}, & G_{16} &= \frac{13}{128}, & G_{25} &= \frac{25}{128}, & G_{34} &= \frac{81}{128}. \end{aligned}$$

We can then compare the result for the boundary state found in this section with the specialization at  $c = 1/2$  of formula (28) by expressing the Virasoro modes in a standard way in terms of fermionic ones:

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z} + \frac{1}{2}} k: \psi_{-k+n} \psi_k :, \quad n \in \mathbb{Z}. \tag{92}$$

We have verified the agreement for the first few descendants up to level 8 in the Virasoro modes of the two derivations for the Ising CFT.

Further by comparing the amplitudes of boundary states we find:

$$\langle B_{\psi}^o | \hat{q}^{L_0 - 1/48} | B_{\psi}^o \rangle = q^{-1/24} \left( 1 + \frac{q^2}{4} + \frac{13}{32}q^4 + \frac{55}{128}q^6 + \frac{1235}{2048}q^8 + \dots \right), \tag{93}$$

where  $|B_{\psi}^o\rangle$  as in Eq. (86) and  $L_0$  is expressed in terms of fermions using Eq. (92). Of course, we know on the other hand from the general case (52) that

$$\langle B_{\psi}^o | \hat{q}^{L_0 - 1/48} | B_{\psi}^o \rangle = [\eta(\tau)]^{-1/4}. \tag{94}$$

Matching (93) and (94) gives rise to intriguing combinatorial identities, which we have of course checked to high order, but are not able to prove.

## 5. Lattice models

### 5.1. Lattice discretization and numerics

We propose now a lattice discretization of the boundary state and verify numerically the contribution of the first levels descendants of the identity in (28) by computing the scaling limit of scalar products in the lattice model.

The lattice model we consider is the dense loop model based on the adjoint representation of the Temperley–Lieb algebra acting on  $N$  (even) strands  $TL_N(\beta)$  ( $\beta$  is the so-called weight

of loops), and imposing free boundary conditions at both boundaries. At the critical point, the anisotropic version of the model is defined by the Hamiltonian:

$$H = - \sum_{i=0}^{N-2} e_i, \tag{95}$$

where  $e_i$  are the TL generators. This Hamiltonian acts on link states  $|s\rangle_N$ , reduced TL diagrams keeping track of connectivities of sites. The Hamiltonian can be put in a triangular block form, with  $j = 0, 1, \dots, N$ , the number of through lines, indexing each block [22]. On the lattice we define the transpose  ${}_N\langle s|$  of a state by turning it upside down and the bilinear form which we call loop scalar product  ${}_N\langle s_1|s_2\rangle_N$ , associating to the link states the diagram obtained by gluing  $|s_2\rangle_N$  with  ${}_N\langle s_1|$ . The result is given by  $\beta^n$ , with  $n$  the number of loops thus formed, and we get zero if we contract two strings. For instance we have

$$\langle \cup \cup \cup | \cup \cup \cup \rangle = \text{loop} = \beta^2. \tag{96}$$

This is the usual bilinear form used for the TL algebra [23]. For generic  $\beta$  the loop scalar product is not positive definite. There is no reason to expect otherwise, as the theories we are dealing with are not unitary. The continuum limit of this loop model, when we parametrize the loop weight as  $\beta = 2 \cos(\pi/(p + 1))$  ( $p \geq 1$  is a real parameter), is a CFT with central charge  $c = 1 - 6/(p(p + 1))$ . Further the bilinear form introduced above flows in the continuum limit to the Virasoro bilinear form, as already observed in [16]. This will allow us to measure the boundary state on the lattice. We claim that the lattice state renormalizing in the scaling limit to our boundary state introduced above is the following link pattern

$$|B^o\rangle_N := \frac{1}{\sqrt{\beta^N}} | \cup \dots \cup \rangle \xrightarrow{N \rightarrow \infty} \alpha |B^o\rangle, \tag{97}$$

where in the continuum limit procedure we discard non-universal contributions, and introduce the proportionality constant  $\alpha$ . Note that the lattice state is normalized according to the form  ${}_N\langle \cdot | \cdot \rangle_N$  and that only the sector  $j = 0$  without through lines couples to this state.

To check this relation we have computed numerically the scaling limits of lattice scalar products  ${}_N\langle B^o | k \rangle_N$  of the boundary state with the  $k$ -th eigenvector in the sector  $j = 0$ , ordered according to its energy ( $|0\rangle_N$  is the ground state). The vectors  $|k\rangle_N$  are normalized to 1 using the loop scalar product. In the  $j = 0$  sector, the continuum theory has field content given generically by the quotient of Virasoro Verma modules  $V_{1,1}/V_{1,-1}$  (with the usual Kac table notation  $V_{r,s}$  for Verma modules) [3], and the scaling limit  $|k\rangle$  of  $|k\rangle_N$  will be given by a combination of Virasoro descendants at a level determined by the energy of  $|k\rangle$ . We identify then obviously  $|k = 0\rangle$  as the vacuum  $|0\rangle$  and (due to normalization)  $|1\rangle = \sqrt{2/c}L_{-2}|0\rangle$ ,  $|2\rangle = \sqrt{1/2c}L_{-3}|0\rangle$ . Unfortunately we do not know a priori which combination of Virasoro descendants contributes to  $|k\rangle$ , for  $k > 2$ . We however do know the number of descendants at each level, since the above quotient amounts to setting  $L_{-1}|0\rangle = 0$  in the generic Verma module.

Due to the anomaly at the corners, we expect the following scaling form of the lattice scalar products:

$$-\log({}_N\langle B^o | k \rangle_N) = a_0 N + a_1 \log N + a_2 + \frac{a_3}{N} + \frac{a_4}{N^2} + O\left(\frac{1}{N^3}\right), \tag{98}$$

with

Table 1  
 Numerical results for the loop model obtained by fitting the lattice scalar products with (98) for system sizes  $N = 8 \rightarrow 24$ .

	$p$	Numerics	CFT
$a_1$	3	$-0.06238 \pm 0.00137$	-0.0625
	4	$-0.08659 \pm 0.00214$	-0.0875
	5	$-0.09515 \pm 0.00033$	-0.1
	$\infty$	$-0.11706 \pm 0.00762$	-0.125
$\alpha$	3	$0.97642 \pm 0.00447$	
	4	$0.94890 \pm 0.00679$	
	5	$0.94000 \pm 0.00010$	
	$\infty$	$0.88731 \pm 0.00327$	
$\langle B^o 1\rangle$	3	$0.49994 \pm 0.00229$	0.5
	4	$0.58652 \pm 0.00422$	$\approx 0.591608$
	5	$0.61994 \pm 0.00067$	$\approx 0.632456$
	$\infty$	$0.65330 \pm 0.01789$	$\approx 0.707107$
$\langle B^o 2\rangle$	3	0	0
	4	0	
	5	0	
	$\infty$	0	

$$a_1 = -\frac{c}{8}. \tag{99}$$

Note the difference between formula (98) and the one for the scaling in the case of periodic boundary states  $|B^p\rangle$ , for which there is no logarithmic term and excited states corresponding to descendant fields decouple.

In the following we take anyway  $a_1$  as a free parameter and fit our numerical results with (98) where we drop the term  $O(N^{-3})$ .  $\langle B^o|k\rangle$  can then be extracted from  $a_2 = -\log(\langle B^o|k\rangle/\alpha)$ . As usual we cannot determine the sign of the scalar product, which however depends on an arbitrary phase of  $|k\rangle_N$ . The numerical results are presented in Table 1.

The value of  $a_1$  and  $\alpha$  are obtained from the scaling of  $\langle B^o|0\rangle$ . We get good agreement of the numerical result for  $a_1$  with the CFT prediction when  $p = 3$ , and the discrepancy increases with  $p$ . The constant  $\alpha$  clearly depends on  $p$ , but we have not been able to determine if it is a universal quantity or not. Simulations for the critical XY spin chain in a transverse field show that it is independent of the parameter  $r$  tuning from  $c = 1/2$  to  $c = 1$ , suggesting that it could be universal. Further it is shown in [14] that the ratio of two proportionality constants  $\alpha$  for lattice boundary states corresponding to different boundary conditions is universal, and so we believe that  $\alpha$  should contain a universal part. For computing  $\langle B^o|k\rangle$ , we actually fit

$$-\log\left(\frac{N\langle B^o|k\rangle_N}{N\langle B^o|0\rangle_N}\right) = a_2 + \frac{a_3}{N} + \frac{a_4}{N^2} + \frac{a_5}{N^3}, \tag{100}$$

and drop first points. According to the identification  $|1\rangle = \sqrt{2/c}L_{-2}|0\rangle$ , the numbers  $\langle B^o|1\rangle$  in the column CFT are then  $\sqrt{c/2}$ . As for the comparison of  $a_1$ , when  $p = 3$  the agreement is very good, while it gets worse for greater values of  $p$ .  $|2\rangle$  is a state with conformal dimension 3, so that our formula of the boundary state predicts its decoupling. Remarkably,  $\langle B^o|2\rangle$  is exactly zero within numerical precision for sizes  $N \geq 10$ . For higher excited states numerical simulations do not give results accurate enough to allow to clearly identify the combinations of Virasoro descendants to which they correspond, so that we cannot compare further the prediction of our

formula for the boundary state (28). Finally, we note that for  $p = 1, 2$  corresponding respectively to dense polymers ( $c = -2$ ) and percolation ( $c = 0$ ), the presence of null vectors in the module of the identity require smart tricks for computing scalar products, such as those employed in [16], and we have not studied numerically these cases.

## 5.2. Ising chain

In this section we give evidence that the fermionic coherent state of Eq. (86) is the universal part of the continuum limit of a lattice state expressed in terms of Ising spins. Similar computations appear in [24]. D boundary conditions considered in the previous derivation correspond to fixed boundary conditions for the spins. As we have already observed, result (86) holds more generally for homogeneous boundary conditions, and here we will consider instead free boundary conditions (as done for the loop model in Section 5.1).

We consider the Hamiltonian limit of the 2D critical Ising model on a strip with free/free boundary conditions:

$$H = -\frac{1}{2} \left( \sum_{i=1}^N \sigma_i^z + \sum_{i=1}^{N-1} \sigma_i^x \sigma_{i+1}^x \right), \quad (101)$$

acting on an Hilbert space made of  $N$  (even) copies of the fundamental representation of  $\mathfrak{su}(2)$ . This model has been extensively studied in the past (see for example [25]). After briefly reviewing the solution, we will compute the limit of lattice scalar products which we will compare with the CFT predictions.

We introduce the fermions  $c_i, c_i^\dagger$  and perform the Jordan–Wigner transformation (redefining  $\sigma_i^z$  as  $-\sigma_i^z$ )

$$\sigma_i^z = 1 - 2c_i^\dagger c_i, \quad \sigma_i^x = \prod_{j < i} (1 - 2c_j^\dagger c_j) (c_i + c_i^\dagger), \quad (102)$$

to obtain an expression of  $H$  quadratic in  $c_i^\dagger$ :

$$H = \sum_{i=1}^N 2c_i^\dagger c_i - \frac{1}{2} \sum_{i=1}^{N-1} c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}^\dagger + c_{i+1} c_i, \quad (103)$$

which can readily be diagonalized through a canonical transformation

$$\psi_k \pm \psi_k^\dagger = \sum_{i=1}^N \phi_{ki}^\pm (c_i \pm c_i^\dagger). \quad (104)$$

We have

$$H = \sum_{k=1}^N \Lambda_k \left( \psi_k^\dagger \psi_k - \frac{1}{2} \right) \quad (105)$$

with

$$\Lambda_k = 2 \sin \left( \frac{2k-1}{2(2N+1)} \pi \right), \quad (106)$$

$$\phi_{ki}^\pm = (-)^i \frac{2}{\sqrt{2N+1}} \cos \left( \frac{2k-1}{2N+1} \pi \left( i - \frac{1}{2} \right) \right), \quad (107)$$

$$\phi_{ki}^- = (-)^{i+1} \frac{2}{\sqrt{2N+1}} \sin\left(\frac{2k-1}{2N+1}\pi i\right). \tag{108}$$

Note that  $\phi^\pm$  are orthogonal matrices. The ground state of the Hamiltonian is  $|0\rangle_N$ , with  $\psi_k|0\rangle_N = 0$  for every  $k$ . When  $N$  is large, we have

$$\sum_{k=1}^N \Lambda_k \psi_k^\dagger \psi_k \simeq \frac{\pi}{N} \sum_{k=1}^N \left(k - \frac{1}{2}\right) \psi_k^\dagger \psi_k. \tag{109}$$

If now we introduce the notation  $\psi_{-k} \equiv \psi_k^\dagger$  for  $k > 0$ , and redefine  $\psi_k \rightarrow \psi_{k-1/2}$ , we identify  $H$  with  $L_0$ , where  $L_0$  is defined in terms of rescaled fermions as in (92). We recover then that the system in the continuum limit is described by Ising CFT. For the free boundary conditions chosen the states propagating in the strip are the descendants of the identity (sector with an even number of fermions) and the energy  $\psi$  (sector with an odd number of fermions).

Now recall that the Hamiltonian (101) is related to the 2D Ising model in the  $\sigma^x$  basis. Then to make the contact with the 2D model, we should rotate 90 degrees clockwise the spins of the chain in the  $x$ - $z$  plane, that is, we define the free boundary state  $|B_\psi^o\rangle_N$  as

$$|B_\psi^o\rangle_N = \frac{1}{\sqrt{2^N}} \sum_{\{\mu_i^x = \rightarrow, \leftarrow\}} |\mu_1^x \cdots \mu_N^x\rangle = |\uparrow \cdots \uparrow\rangle. \tag{110}$$

We want to compute  ${}_N\langle B_\psi^o | k \rangle_N$ , where  $|k\rangle_N$  is the  $k$ -th excited state. One way to deal with that is writing the projector onto  $|B_\psi^o\rangle_N$  in terms of  $c_i$ 's (the same trick was used for example in [26])

$$|B_\psi^o\rangle_N \langle B_\psi^o| = \prod_{i=1}^N \left(\frac{1 - \sigma_i^z}{2}\right) = \prod_{i=1}^N c_i^\dagger c_i, \tag{111}$$

so that  ${}_N\langle B_\psi^o | k \rangle_N^2$  can be evaluated efficiently using Wick's theorem. Let us start dealing with the ground state. Rearranging the factors of  $c_i^\dagger$  and  $c_i$  in a convenient way, we have to compute

$${}_N\langle 0 | \prod_{i=1}^N c_i^\dagger \prod_{i=1}^N c_i | 0 \rangle_N = \text{Pf} \begin{pmatrix} C & D \\ -D & -C \end{pmatrix}, \tag{112}$$

where (see also [27])

$$C_{ij} = -C_{ji} = \langle c_i^\dagger c_j^\dagger \rangle = \frac{G_{ij} - G_{ji}}{4}, \tag{113}$$

$$D_{ij} = D_{ji} = \langle c_i^\dagger c_j \rangle = \frac{2\delta_{ij} - G_{ij} - G_{ji}}{4} \tag{114}$$

and we have introduced the function

$$G_{ij} = - \sum_{k=1}^N \phi_{ki}^- \phi_{kj}^+. \tag{115}$$

Computing the Pfaffian from the square of the determinant and using properties of block matrices, we have

$${}_N\langle B_\psi^o | k \rangle_N^2 = \det_{1 \leq i, j \leq N} \left(\frac{\delta_{ij} + G_{ij}}{2}\right). \tag{116}$$

Table 2  
 Scalar product of  $|B_{\psi}^o\rangle$  and  $|k\rangle$  from finite size scaling (numerics) and comparison with CFT prediction.  $h_k$  is the conformal dimension of the field  $|k\rangle$ .

	$h_k$	Numerics	CFT
$\langle B_{\psi}^o   1 \rangle$	1/2	0	0
$\langle B_{\psi}^o   2 \rangle$	3/2	0	0
$\langle B_{\psi}^o   3 \rangle$	2	$0.499994 \pm 0.000003$	1/2
$\langle B_{\psi}^o   4 \rangle$	5/2	0	0
$\langle B_{\psi}^o   5 \rangle$	3	0	0
$\langle B_{\psi}^o   6 \rangle$	7/2	0	0
$\langle B_{\psi}^o   7 \rangle$	4	$0.124995 \pm 0.000003$	$1/8 = 0.125$
$\langle B_{\psi}^o   8 \rangle$	4	$0.624989 \pm 0.000003$	$5/8 = 0.625$
$\langle B_{\psi}^o   9 \rangle$	9/2	0	0
$\langle B_{\psi}^o   10 \rangle$	9/2	0	0

The scalar product with the excited state  $\psi_{k_1} \cdots \psi_{k_s} |0\rangle_N$  can be computed by regarding this state as the vacuum state of a new set of  $\psi$ 's, where  $\psi_k$  and  $\psi_k^\dagger$  are interchanged for  $k_1, \dots, k_s$  [25]. For these  $k$ 's we have  $\Lambda_k \rightarrow -\Lambda_k$  and  $\phi_{ki}^- \rightarrow -\phi_{ki}^-$ , so that (116) holds also for  $\psi_{k_1} \cdots \psi_{k_s} |0\rangle_N$  if we replace  $G_{ij}$  with

$$G_{ij}(k_1, \dots, k_s) = - \sum_{k \text{ unexc.}} \phi_{ki}^- \phi_{kj}^+ + \sum_{k \text{ exc.}} \phi_{ki}^- \phi_{kj}^+. \tag{117}$$

We have not been able to find a closed formula for the scalar products, but Eq. (116) allows for very efficient numerical computations of the scalar products. We have fitted results for  $N = 2 \rightarrow 500$  (dropping the first points for higher excited states), using the fit function (98). The value  $a_1 = -1/16$  is always found with a precision of  $10^{-6}$  and for the proportionality constant  $\alpha$  we find with a high precision  $\alpha = 1.01937$ . For  ${}_N \langle B_{\psi}^o | k \rangle_N$  we have the results in Table 2. We note that obviously the sector of the energy  $\psi$  ( $h_k$  half integer) decouples. The agreement is very good. It remains intriguing of course, that (94) does not seem to be easily obtainable from the explicit solution of the lattice model. This is in contrast with similar expressions for other geometries, such as the torus.

### 6. Conclusion

This general exploration of the boundary states for theories defined on a segment will be used in our next paper to discuss in particular properties of geometrical problems on a rectangle. Of course, the set-up can find many other applications as well, in condensed matter or string theory [9–11,28]. Another field where our boundary state (28) might be relevant is the calculation of quantum information quantities such as entanglement entropies or overlaps in 1d or 2d systems. For example, in some 2d conformally invariant wave functions [26,29], the calculation of the entanglement entropy boils down to the study of expansions like (98). Some Refs. [30,31] have focused on the universal logarithmic term  $a_1 \log L$  in (98), but we have shown that the next term  $a_2$  contains also a universal piece, and it would be interesting to extend their analysis using our present results.

## Note added in proof

After the completion of this work, we became aware of a similar result published in [32–34], which is consistent with our main formula Eq. (28).

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