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A TOPOLOGICAL STUDY OF PLANAR VECTOR FIELD SINGULARITIES

A tribute to Ivar Bendixson

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Abstract

In this paper one extends results of Bendixson [B] and Dumortier [D] about the germs of vector fields with an isolated singularity at the origin of $\mathbb{R}^2$, not accumulated by periodic orbits. As new tool, one introduces minimal curves, which are curves surrounding the origin, with a minimal number of contact points with the vector field. Moreover, the arguments are essentially topological, with no use of a desingularization theory, as it is the case in [D].

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Key words and phrases : Sectoral decomposition, minimal curve, contact index.

1. Introduction

In this paper we consider a germ $(X, 0)$ of vector field at the origin $0 \in I\!R^2$. This germ is represented by a vector field X, defined on an arbitrarily small neighborhood W of the origin. We will assume that the origin is an isolated singularity of $(X, 0)$. This means that we can choose W such that the origin is the unique singularity of X on W.

Ivar Bendixson, in his seminal work [B] published in 1903, uses the ideas of Henri Poincaré [P], to give general topological properties of the phase portrait of such a germ, also assumed to be isolated from periodic orbits. In the Part I of his article Bendixson proves general theorems for a germ of class $\mathcal{C}^1$, introducing the notions of nodal regions and pair of orbits crossing the origin (these notions are recalled in 2.3). It is in this Part I that he states and proves that is known as Theorem of Poincaré-Bendixson, which in modern language may be formulated as follows: *the limit set of an orbit contained in a bounded open region of the plane is a singular point, a periodic orbit or a graphic (a closed invariant topological curve made by orbits joining singular points).* In the Part II of his article Bendixson proves general theorems for analytic germs with an algebraically isolated singularity (he says, with components X, Y holomorphic without a common factor). He gives properties of orbits tending toward the origin (recalled in 3.2), and proves that there is a finite number of nodal regions and a finite number the pair of pairs of orbits crossing the origin. He relates these numbers with the topological index of the vector field at the origin.

In the present article I shall present some results in the spirit of the above mentioned results of Bendixson. In order to give a precise account of them, I want to introduce a more modern terminology. The following definitions are for germs of vector field at the origin, of class $\mathcal{C}^1$.

Definition 1 *A sector at the origin of $I\!R^2$, is the image by an homeomorphism sending* 0 *on* 0, *of an angular sector defined by* $\{0 \leq \theta \leq \theta_0 \mid r \leq r_0\}$ *in polar coordinates* (r, θ). *We assume that a sector is equiped with a vector field of one of the* 3 *different types :*

1. **Hyperbolic sector :** *The sides are two orbits L_+, L_- with $\omega(L_-) = \alpha(L_+) = \{0\}$. There are transverse sections σ to L_- and τ to L_+. The orbit γ_p through $p \in \sigma$ reaches τ at a first point $q(p)$. When $p \to L_- \cap \sigma$ we have that $q(p) \to L_+ \cap \tau$ and that the arc of orbit between p and $q(p)$ has a Hausdorff limit contained into $L_+ \cup L_-$. We can define as sector the union of these arcs and of their limit on $L_+ \cup L_-$.*

2. **Elliptic sector :** *A disk $D(\Gamma)$, singular at the origin, (I shall say a pinched disk), bounded by an orbit Γ such that $\omega(\Gamma) = \alpha(\Gamma) = \{0\}$. Each orbit γ inside $D(\Gamma)$ has the same limit properties and also bounds a pinched disk $D(\gamma)$. Moreover, if γ_1, γ_2 are two such orbits, we have that $\gamma_1 \subset D(\gamma_2)$ or $\gamma_2 \subset D(\gamma_1)$.*

3. **Parabolic sector :** *Every orbit has the origin as one of the limit sets. It is an attracting parabolic sector if each orbit γ tends towards 0 for positive time ($\omega(\gamma) = \{0\}$). It is a repulsing parabolic sector if each orbit γ tends towards 0 for negative time ($\alpha(\gamma) = \{0\}$).*

The different sectors are represented in Figure 1.

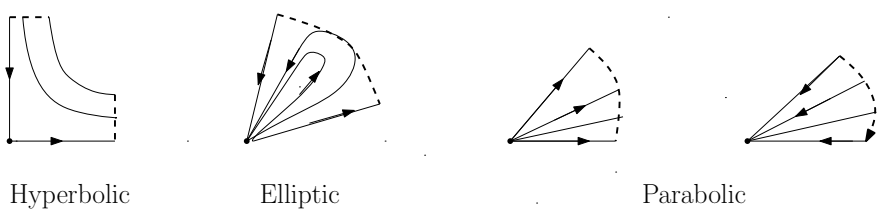


Figure 1 : *Sectors*

It is easy to find models for each of them : four hyperbolic sectors appear in an hyperbolic linear saddle; two elliptic sectors appear in the holomorphic differential equation $\dot{z} = z^2$, corresponding to the real differential equation : $\dot{x} = x^2 - y^2,\ \dot{y} = 2xy$; parabolic sectors appear in linear nodes (in fact a whole neighborhood of the origin may be considered as a unique generalized parabolic sector, taking $\theta_0 = 2\pi$).

Definition 2 *A topological node (at the origin) is a germ of $\mathcal{C}^1$ vector field represented by a vector field X on a neighborhood W of the origin, such that each orbit has the origin as ω-set or such that each orbit has the origin as α-set. In other words, if $(r(t), \theta(t))$ is the trajectory through any point $m \in W$, written in polar coordinates, we have that $r(t) \to 0 \in I\!R^2$ when $t \to +\infty$ for any $m \in W$ or that $r(t) \to 0 \in I\!R^2$ when $t \to -\infty$ for any $m \in W$ (nothing is asked for $\theta(t)$).*

Definition 3 *A $\mathcal{C}^1$ vector field X admits a sectoral decomposition on a neighborhood W of the origin homeomorphic to a disk if W is a finite union of sectors of the three types defined in Definition 1. When one turns around the origin, two successive elliptic or hyperbolic sectors are separated by a unique parabolic sector, which, between two successive hyperbolic sectors, may be reduced to a single orbit. See Figure* 2.

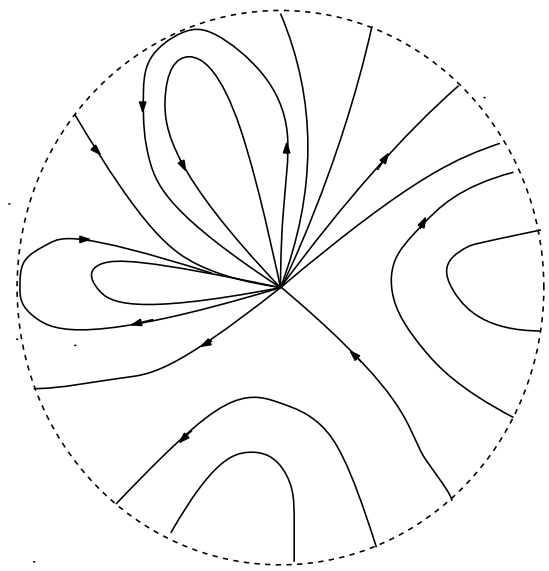


Figure 2 : *A sectoral decomposition*

The principal difference between the present work and the one of Bendixson is that it is made a systematic use of the contact points along closed curves embedded near the origin and surrounding it. More precisely one considers *minimal curves* and *contact indexes* $c(X, W), c(X, 0)$ associated to them (see 2.1). Orbits cutting minimal curves have remarkable properties given in 2.2 : for instance, an orbit cannot cut a minimal curve in more than two points. Using these properties, one derives general results for $\mathcal{C}^1$ germs of vector field, rather comparable to the results given in Part I of [B]. A new result (not existing in[B]) is proved in 2.5 : *a $\mathcal{C}^1$ germ with finite contact index $c(X, 0)$ and isolated from periodic orbits, is a topological node or admits a sectoral decomposition* (Theorem 2 in 2.5).

In Part II of [B], Bendixson has proved results rather similar to the ones stated in Theorem 2, under the stronger hypothesis that the germ of vector field is analytic (he said holomorphic), without common factor for the components.

In fact, these conditions imply that $c(X,0) < \infty$, that allows to use Theorem 2. But in order to have this condition, one does not need the assumption of analyticity, as it is shown in Theorem 3.

Theorem 2 and Theorem 3 are used together in order to prove the main theorem below. In order to state it, we need some well known definitions concerning the orbits tending to the origin for positive or negative time : a *characteristic orbit* is an orbit tending toward the origin with a limit tangent direction and a *spiral* is an orbit which tends toward the origin, while turning indefinitely around it. A *node* is a germ of vector field where all the orbits outside the origin are characteristic orbits and a *focus*, when the same orbits are spirals. Clearly, nodes and focuses are topological nodes. Precise definitions are recalled in 3.2.

We can now state the main Theorem :

Theorem 1 (Main Theorem)

For an integer $k \geq 1$, let $(X,0)$ be a germ of $\mathcal{C}^{2k+3}$ vector field at the origin of $I\!R^2$, with a (topologically) isolated singularity at the origin, not accumulated by periodic orbits (i.e., the origin is isolated from periodic orbits). Let also assume that at the origin, the $(k-1)$-jet of $(X,0)$ is zero but not the k-jet. Then, the germ $(X,0)$ is a focus, a node or admits a sectoral decomposition, where every orbit tending toward the origin, for positive or negative time, is a characteristic orbit.

Moreover, the number of sectors is bounded by $2(k+1)$. It follows that the topological index $i(X,0)$ verifies that $-k \leq i(X,0) \leq k+2$ and that the number of possible phase portraits for $(X,0)$ is roughly bounded by $2^{4(k+1)}$.

Remark 1 *A sufficient condition for a germ $(X,0)$ with an isolated singularity at the origin, to be isolated from periodic orbits, is to have at least a characteristic orbit. In this case $(X,0)$ admits a sectoral decomposition (maybe reduced to a node). This condition may be more easy to check.*

We can compare this result with the results obtained by Bendixson in Part II of [B]. As mentioned above, Bendixson supposed that the vector field is analytic with no common factor for the two components. But, looking at his proofs, it appears that Bendixson did not used the analyticity, but just that the germ has a non-zero jet and an isolated singularity at the origin, isolated from periodic orbits (exactly as in the statement of Theorem 1). Moreover, he did not introduced explicitly the notion of sectoral decomposition and he used a more general notion that the one of elliptic sector, that he called a *closed nodal region*. In a neighborhood of the origin where this point is the unique singularity, a closed nodal region is just a pinched disk $D(\Gamma)$ bounded an orbit Γ such that

$\omega(\Gamma) = \alpha(\Gamma) = 0$. It is an easy consequence of the Bendixson-Poincaré theory that each other orbit contained into $D(\Gamma)$ has the same property. Elliptic sectors are example of closed nodal region but the phase portrait inside closed nodal region $D(\Gamma)$ may be much more complicated than in the case of an elliptic sector. In fact one can introduced a partial order between the orbits in a closed nodal region $D(\Gamma)$, saying that γ is less than γ' if $D(\gamma) \subset D(\gamma')$. This order is total precisely when $D(\Gamma)$ is an elliptic sector. If not, the order is just partial and it may happen for instance inside a closed nodal region, that there are infinitely many orbits not comparable 2 by 2, . A part of the results of Bendixson may be also found in the Chapter X of Lefschetz's book [L].

A more recent work about singularities of smooth planar vector fields, is the one of Dumortier [D] where the hypothesis that the singularity is topologically isolated is replaced by a strongest one : one considers a germ $(X, 0)$ with an algebraically isolated singularity at the origin. This means that, in the ring of smooth germs of functions at the origin, the ideal (P, Q) generated the components of the vector field $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$, contains a power $\mathcal{M}^k$ of the maximal ideal $\mathcal{M}$ (the germ is said algebraically isolated at order $k-1$, a condition which depends just of its $(k-1)$-jet). In fact Dumortier can replace this hypothesis by a weakest one : the vector field verifies in a neighborhood of the origin, a Łojasiewicz inequality of order L : $||X(m)|| = \sqrt{P^2 + Q^2} \geq c||m||^L$, for a constant $c > 0$ and some $L \in I\!N$. It is easy to see that *if a germ is algebraically isolated at order $k-1$, then, it verifies a Łojasiewicz inequality of order L, where $2L \geq k$, and this last property implies that the origin is a topologically isolated singularity.*

The study of Dumortier is much more algebraic than the present one. A theory of desingularization by successive blowing up was developed in order to prove a result of desingularization for the smooth germs of vector field verifying a Łojasiewicz inequality. Dumortier deduces from his desingularization theory a result which is stronger than Theorem 1 : *if the germ $(X, 0)$ verifies a Łojasiewicz inequality and has a characteristic orbit, then there is a k-jet of the germ $(X, 0)$ which determines its phase portrait.* Moreover this phase portrait admits a sectoral decomposition (may be reduced to a node) which is determined by the k-jet of $(X, 0)$. In contrast with the topological proof given in the present paper, the proof of Dumortier is constructive in the sense that the sectoral decomposition can be obtained from the singular points, all elementary, obtained for the *desingularized vector field.*

As the existence of a characteristic orbit implies that the origin is isolated from periodic orbits and as the origin is isolated if it verifies a Łojasiewicz inequality, we can apply Theorem 1 under the conditions of the Dumortier's

result. Moreover, if z_k is any non-zero k-jet of vector field, there is always a smooth germ $(X, 0)$ with the origin as algebraically isolated singularity and such that $j_k X(0) = z_k$, and there are infinitely many such germs with 2 by 2 distinct jets of arbitrarily high degree. On the other side Theorem 1 says that there exists a finite collection $\mathcal{C}_k$ of sectoral decompositions, depending just on the number k, such that if $(X, 0)$ has a characteristic through the origin and a non-zero k-jet, then its phase portrait belongs to $\mathcal{C}_k$. As an example, let us consider the 1-jet $z_1 = y\frac{\partial}{\partial y}$. The infinity of polynomial vector fields $X_{l,\pm} = y\frac{\partial}{\partial y} \pm x^l \frac{\partial}{\partial x}$, with $l \geq 2$, have z_1 as 1-jet and are algebraically isolated, each of them with characteristic orbits through the origin. They have just one of the three possible sectoral decompositions : a single node, two hyperbolic and one parabolic sectors or four hyperbolic sectors. This result is even true for a vector field as $X_\infty = y\frac{\partial}{\partial y} + f(x)\frac{\partial}{\partial x}$, where f is a smooth germ with an isolated singularity and a infinite jet equal to zero, for instance $f(x) = e^{-1/x^2}$. For such a vector field, the origin does not verify a Łojasiewicz inequality.

Acknowledgments *I am grateful to Etienne Ghys for having inspire this topological work.*

2. Phase portrait for a $\mathcal{C}^1$ vector field near an isolated singularity

In this section we consider a germ of vector field at $0 \in I\!R^2$, of class $\mathcal{C}^1$, such that the origin is an isolated singularity. We choose a representative for this germ : a $\mathcal{C}^1$ vector field X given on an arbitrary small neighborhood W_0 of $0 \in I\!R^2$, diffeomorphic to an Euclidean disk. We assume that 0 is an isolated singularity of X on W_0.

2.1. Minimal curves and contact indexes

Definition 4 *In the text, a* disk *will be a neighborhood of* 0, *contained into* W_0 *and* $\mathcal{C}^1$ *diffeomorphic to an Euclidean disk. The term* curve *will always design a curve* $\mathcal{C}^1$*-diffeomorphic to* S^1, *embedded in* $W_0 \setminus \{0\}$ *and surrounding the origin (i.e. generating the* 1*-homology of* $W_0 \setminus \{0\}$*). A curve* γ *is the boundary of a disk* $W(\gamma)$.

Let γ be a curve. A *contact point* on γ is a point where X is tangent at γ. If $\gamma(s) \ : S^1 \to W_0$ is a $\mathcal{C}^1$ parametrization of γ, the contact points on γ correspond to the zeros of the continuous function $n(s) \ : s \to < X(\gamma(s)), N(\gamma(s)) >$, where $< \cdot, \cdot >$ is the Euclidean scalar product and N is a $\mathcal{C}^0$ vector field along the

curve γ, orthogonal to it.

Definition 5 *An isolated contact point m on a curve γ is said even if the function $n(s)$ changes of sign at m and odd if not. The trajectory through an even contact point is locally on one side of γ (at an odd contact point the orbit is topologically transverse to γ). We say that a curve if generic if all its contacts points are even. These points are necessarily isolated and then are in finite number. We will call contact index $c(\gamma) \in I\!N$, the (finite) number of (even) contact points along a generic curve γ.*

Remark 2 *The notions of contact points, even or odd contact points, generic curve and then of of contact index are independent of the choice of the orthogonal vector field N. Let us also notice that for smooth vector fields, the contact points of finite order with a smooth curve are even or odd if their order of contact are respectively even or odd.*

even contact odd contact

Figure 3

For a generic curve γ, the number $c(\gamma)$ of even contact points is even (they are end points of an even number of intervals on γ where the function $n(s)$ is alternatively > 0 and < 0). .

By using coverings of curves by well-chosen $\mathcal{C}^1$ tubular neighborhoods of X, it is easy to show that the generic curves are dense in the space of $\mathcal{C}^1$-curves, for the $\mathcal{C}^0$ topology. For instance an odd point can be eliminated by a $\mathcal{C}^0$ perturbation of the curve, located in its neighborhood. In particular there exist generic curves on any disk.

As the orbit at an even contact point is locally on one side of the curve, we can distinguish two types of even contact points for a generic curve :

Definition 6 *Let γ be a generic curve. An (even) contact point m is said to be an interior contact point if locally the orbit of m is inside $W(\gamma)$. If not, the (even) contact point is said to be an exterior contact point. Let $c_i(\gamma)$ the the number of interior contact points and $c_e(\gamma)$ the number of exterior contact points. We have that $c(\gamma) = c_i(\gamma) + c_e(\gamma)$. The interlinking of the set of exterior*

contact points with the set of interior contact points along the curve γ traveled in the direct sense, will be called the arrangement of contact points on γ. It may be represented, up a cyclic permutation, by a list of $c(\gamma)$ symbols taken in the set $\{i, e\}$, with $c_i(\gamma)$ symbols i and $c_e(\gamma)$ symbols e (in Figure 2, for the disk boundary γ which is a generic curve, we have that $c_i(\gamma) = 2$, $c_e(\gamma) = 2$ and the arrangement (i, i, e, e)).

This distinction between interior and exterior contact points will be crucial for the topological analysis of the phase portrait presented in this paper. For this moment, I just want to indicate the relation with the *topological index* of the germ $(X, 0)$. I recall that if $(X, 0)$ is a germ of a vector field with the origin as isolated singularity, we can define the topological index $i(X, 0)$ in the following way. We choose a representative X of the germ on a neighborhood W_0, where 0 is the unique singularity. Next, we consider any curve $\gamma \subset W_0$, and take a parametrization $\gamma(s)$ with the direct orientation. Then the topological index $i(X, 0) \in \mathbb{Z}$ is the degree of the continuous map

$$s \in S^1 \to \frac{X}{||X||}\Big(\gamma(s)\Big) \in S^1.$$

Now, if γ is a generic curve, it is easy to see that :

$$i(X, 0) = \frac{1}{2}\Big(c_i(\gamma) - c_e(\gamma)\Big) + 1 \tag{1}$$

Let us notice that, as $c(\gamma) = c_i(\gamma) + c_e(\gamma)$ is even, we have that $c_i(\gamma) - c_e(\gamma)$ is also even and then the formula (1) defines, as expected, an an element of $\mathbb{Z}$.

We can now introduce the following *contact index* for the disks :

Definition 7 *Let W be a disk, contained into W_0. We call contact index of X in W, the minimum of the contact index $c(\gamma)$ among the generic curves $\gamma \subset W$ (as there exist generic curves in W, this index is finite and may even be equal to 0). We will write $c(X, W)$ this contact index of X in W. A generic curve γ in W, such that $c(\gamma) = c(X, W)$ will be called minimal curve in W (such a minimal curve always exists!).*

By definition, the number $c(\gamma) = c_i(\gamma)+c_e(\gamma)$ has the same value $c(X, W)$ for all the minimal curves in W. Using this remark and (1) we obtain the following :

Lemma 1 *Let γ be a minimal curve in a smooth disk W. Then,*

$$c_i(\gamma) = \frac{c(X, W)}{2} + i(X, 0) - 1 \text{ and } \; c_e(\gamma) = \frac{c(X, W)}{2} - i(X, 0) + 1. \tag{2}$$

The expressions (2) show that the numbers $c_i(\gamma), c_e(\gamma)$ are the same for any choice of minimal curve γ in W. We will see later that it is also true for the arrangement of contact points along a minimal curve γ.

Notation 1 *Let W be any disk in W_0. As we have write $c(X, W)$ the number of contact points of any minimal curve in W, Lemma 1 allows to introduce also numbers $c_i(X, W)$ and $c_e(X, W)$ for the number of interior contact points and exterior contact points on any minimal curve in W.*

If $W_1 \subset W_2$ are two disks as in Definition 7, we have that $c(X, W_1) \geq c(X, W_2)$. It follows that, if (W_n), with $n \in I\!N$, is a *nested sequence of disks* in W_0, i.e. such that $W_{n+1} \subset W_n$ for all $n \in I\!N$, the sequence of numbers $\Big(c(X, W_n)\Big)$ is increasing and has a limit in $\bar{I\!N} = I\!N \cup \{\infty\}$. Call $\mathrm{diam}(W)$ be the diameter of a disc W. If $(\mathrm{diam}(W_n)) \to 0$ when $n \to \infty$, the limit of the sequence $\Big(c(X, W_n)\Big)$ is clearly independent of the choice of the sequence of disks (W_n). This allows the following definition :

Definition 8 *The contact index $c(X, 0)$ of the germ $(X, 0)$ is the limit in $\bar{I\!N} = I\!N \cup \{\infty\}$ of the increasing sequence of numbers $\Big(c(X, W_n)\Big)$ for any choice of a nested sequence (W_n) of disks, whose sequence of diameters tends to zero. This index is independent of the choice of the sequence (W_n) and depends just of the vector field germ $(X, 0)$.*

It is easy to find germs $(X, 0)$ (even smooth) such that $c(X, 0) = \infty$. But this property is exceptional for smooth germs (in a sense made precise in Section 3). For this reason, we will be more interested in germs $(X, 0)$ with a finite contact index. The following proposition gives a characterization of this finiteness property :

Proposition 1 *Let us suppose that $c(X, 0) \in I\!N$ (i.e. is finite). Then we can find a representative X of the germ $(X, 0)$ on a disk W_0, such that $c(X, W_0) = c(X, 0)$, and a sequence $\Big(W_n\Big)$ of disks with the following properties :*
(a) (W_n) is a nested sequence and $(\mathrm{diam}(W_n)) \to 0$ for $n \to \infty$.
(b) For each n, the boundary $\gamma_n = \partial W_n$ is a minimal curve in W_0, i.e. $c(\gamma_n) = c(X, 0)$.

Conversely, given any germ $(X, 0)$, if there exists a nested sequence of smooth disks $\Big(W_n\Big)$ as above, with $(\mathrm{diam}(W_n)) \to 0$ for $n \to \infty$, and such that each boundary $\gamma_n = \partial W_n$ is a minimal curve in W_0, we have that $c(X, 0) = c(X, W_0)$ and then $c(X, 0)$ is finite.

Proof

Let us assume that $c(X, 0)$ is finite and choose a representative vector field

X on a disk W (where the origin is the unique singularity). Clearly, any finite subset in $I\!N$ contains its upper bound. As a consequence, there exists a disk $W_0 \subset W$, such that $c(X, W_0) = c(X, 0)$. Now, if $\tilde{W}$ is any smooth disk contained into W_0, by definition of the contact index $c(X, 0)$, we have also that that $c(X, \tilde{W}) = c(X, 0)$. This remark allows a construction by induction of a sequence (W_n) verifying the statements : if W_n is already constructed with $\mathrm{diam}(W_n) \leq \frac{1}{n}$, we take any disk $\tilde{W}_n$ in the interior of W_n, such that $\mathrm{diam}(\tilde{W}_n) \leq \frac{1}{n+1}$. Let us consider a minimal curve γ_{n+1} of $\tilde{W}_n$. As $c(\gamma_{n+1}) = c(X, \tilde{W}_n) = c(X, 0)$ (see above), the curve γ_{n+1} is the boundary of a disk $W_{n+1} \subset W_n$, such that $\mathrm{diam}(W_{n+1}) \leq \frac{1}{n+1}$.

Conversely, let (W_n) be a sequence of disks as in the statement. As the curves $\gamma_n = \partial W_n$ are minimal curves in W_0, they are à fortiori minimal curves in W_n. Then, the sequence $c(X, W_n)$ is stationary at the value $c(X, W_0)$ and, as $(\mathrm{diam}(W_n)) \to 0$ for $n \to \infty$, the proper definition of the index $c(X, 0)$ implies directly that $c(X, 0) = c(X, W_0)$. ∎

Passing to subsequences of curves, it is easy to adapt the proof of Proposition 1 in order to obtain the following useful sufficient condition for the finiteness of $c(X, 0)$:

Proposition 2 *Assume that there are a finite constant $C > 0$ and a sequence of generic curves (γ_n), converging in the Hausdorff sense toward the origin and such that $c(\gamma_n) \leq C$ for any n. Then $c(X, 0) \leq C$.*

Definition 9 *When $c(X, 0)$ is finite, a generic curve γ such that $c(\gamma) = c(X, 0)$ (as in Proposition 1) will be called : absolute minimal curve. A minimal curve in a disk W is absolute as soon as the diameter of W is small enough. Apart this condition, absolute minimal curves do not depend on the choice of a disk W, the reason for what they are called absolute.*

It follows from Proposition 1 that the number of interior and exterior contact points is the same for any absolute minimal curve of a germ with a finite contact index. This allows to introduce the following :

Notation 2 *Let $(X, 0)$ a vector field germ such that $c(X, 0)$ is finite. There exist integers $c_i(X, 0)$ and $c_e(X, 0)$ equal respectively to the number of interior contact points and exterior contact points of any absolute minimal curve.*

Remark 3 *As $c(\gamma)$ is an even number for any generic curve, so are the contact indexes $c(X, W)$ and $c(X, 0)$.*

2.2. **Relations between phase portrait and minimal curves**

In this subsection one will consider a generic curve γ, minimal in some disk $W \subset W_0$. One will prove some properties for the intersection of any orbit Γ (inside W) with γ. In order to prove these properties one will construct curves in a way very similar to the classical trick that is recalled in the following lemma :

Lemma 2 *Let Γ be an orbit of a 2-dimensional $\mathcal{C}^1$ vector field X. One assumes that Γ is neither periodic or reduced to a singular point and that there exists a transverse section Σ such that $\Gamma \cap \Sigma = \{p, q\}$, with $p \neq q$. Let $[p, q]_\Gamma$, respectively $[p, q]_\Sigma$, be the arc of Γ, respectively the arc of Σ between p and q. Then, one can $\mathcal{C}^0$-approximate the $\mathcal{C}^0$-curve $\gamma_0 = [p, q]_\Gamma \cup [p, q]_\Sigma$ by a $\mathcal{C}^1$-curve γ transverse to X.*

Proof

One chooses a flow box T, $\mathcal{C}^1$-diffeomorphic to a rectangle, having $[p, q]_\Gamma$ as one X-tangent side and $[p, p'], [q, q']$, disjoints subintervals of Σ, as transverse sides, with q' between q and p on Σ. See Figure 4.

Let be $\sigma \subset T$ the linear segment joining p and q'. This segment is transverse to X. It is easy to smooth $\gamma_1 = \sigma \cup [q', p]_\Sigma$ (where $[q', p]_\Sigma \subset \Sigma$) into a $\mathcal{C}^1$-curve γ transverse to X. We can construct γ_1, $\mathcal{C}^0$ close to γ_0 and γ, $\mathcal{C}^0$ close to γ_1, so that γ is $\mathcal{C}^0$ close to γ_0. ∎

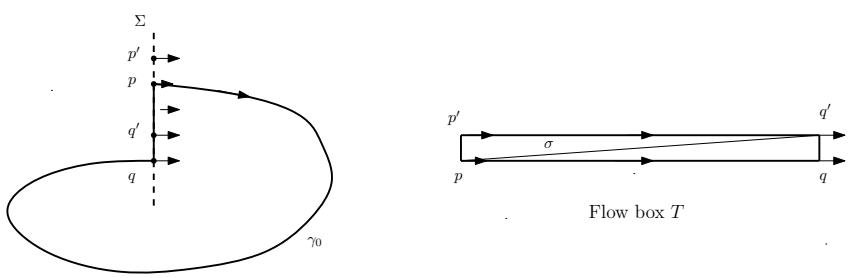


Figure 4

A first result for minimal curves is the following :

Proposition 3 *Let γ be a minimal curve of a disk $W \subset W_0$, contained in the interior of W. Let p be any contact point on γ and Γ the orbit through m in W. Then, $\gamma \cap \Gamma = \{p\}$.*

Proof The proof for an exterior contact point is similar to the proof for interior contact point. Then from now on, one will assume that m is an interior contact

point. Let $\varphi(t,m)$, for $m \in W$, be the flow of X. In contradiction with the statement, let us suppose that there exists a time $t_0 \neq 0$, such that if $q = \varphi(t_0,p) \in \gamma$, we have that $q \neq p$. One will assume that the point q is a first intersection of Γ with γ, i.e. that $\varphi(]0,t_0[,p) \cap \gamma = \emptyset$. Moreover, by taking a $\mathcal{C}^1$-perturbation of γ near q, one can also assume that γ remains minimal and is such that the intersection of γ with Γ is transverse at q.

Let $[p,q]_\Gamma = \varphi([0,t_0],p)$ be the arc of orbit between p and q. Let us notice that $[p,q]_\Gamma \subset W(\gamma)$. One and only one of the two arcs on γ with end points p and q, arc that one will denote $[p,q]_\gamma$, is such that $[p,q]_\gamma \cup [p,q]_\Gamma$ bounds a topological disk $B \subset W(\gamma)$, *which does not contain the origin*. One will denote by $[p,q]_\gamma^C$ the complementary arc on γ. As the arcs $[p,q]_\gamma$ and $[p,q]_\Gamma$ are homotopic in $W \setminus \{0\}$, relatively to their end points, the curve $\gamma_1 = [p,q]_\gamma^C \cup [p,q]_\Gamma$ is also a topological curve surrounding the origin (and contained into $W(\gamma)$)

Let Γ_+ be the half-orbit from p which contains the arc $[p,q]_\Gamma$, and Γ_- the other half-orbit. One will distinguish two cases : the *good case* when Γ_- starts outside B and the *bad case* when Γ_- starts inside B. See Figure 5

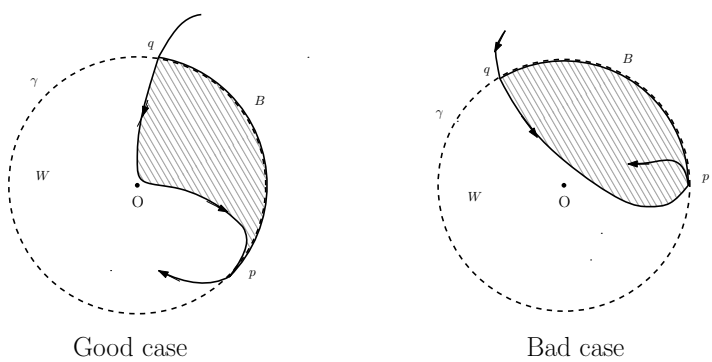


Figure 5

(a) One first considers the good case. The idea is that one can $\mathcal{C}^0$-approximate the curve γ_1 by a $\mathcal{C}^1$ generic curve γ_2 contained in W, with at least two contact points less than γ. But this fact will be in contradiction with the minimality of γ, proving that this good case is impossible.

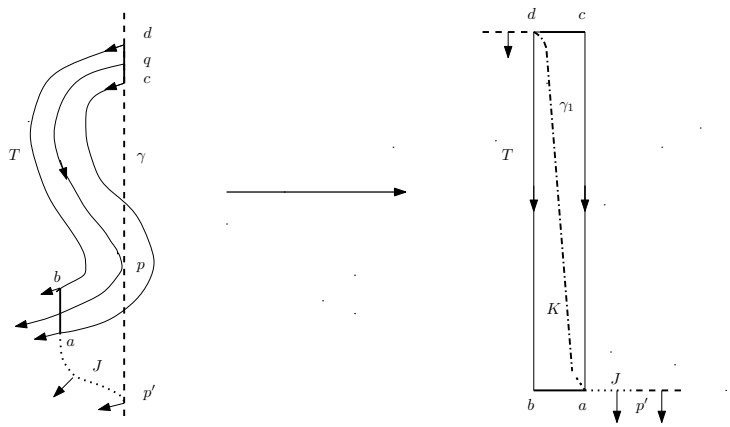


Figure 6 : *Elimination of contact points*

Let us construct the curve γ_2. Taking $\varepsilon > 0$ arbitrarily small, one considers a ε-thick $\mathcal{C}^1$ flow box T, containing the orbit arc $[p,q]_\Gamma$. This flow box is a thin rectangle $[a,b,c,d]$ where the long sides $[ac]$ and $[bd]$ are tangent to the flow and the arcs of orbit inside T are the segments parallel to these sides. The flow is transverse to the short segments of length ε, parallel to the short sides $[ab]$ and $[cd]$. One chooses ε small enough such that $T \subset W$.

One supposes that the arc $[p,q]_\Gamma$ is contained in the arc of orbit in the middle of T, with p in the interior of T, at a distance of order ε of the side $[ab]$ and q on $[cd]$. One chooses a point p' in $[p,q]^C_\gamma$, outside T and that the flow is transverse to γ along the half-closed arc $[p',p[\subset \gamma$. See Figure 6.

One can now construct γ_2 as a union of 3 arcs : the sub-arc I of $[p,q]^C_\gamma$ between p' and d ; a small arc J joining p' and a, transverse to X and ε-near γ; an arc K inside T, going in diagonal from a to d, transverse to the flow in T. One can make this curve γ_2 to be $\mathcal{C}^1$ and $\mathcal{C}^0$ close to γ_1 by choosing ε small enough. In particular, one can construct γ_2 inside W.

If ε is small enough, the curve γ_2 is surrounding the origin, as the curve γ_1. There is no contact point on the $J \cup K$ and the contact points on I are the contact points of γ which belong to $[p,q]^C_\gamma \setminus \{p\}$. As γ has at least 2 contact points along the arc $[p,q]_\gamma$ (the point p and necessarily another contact point between p and q), is follows that γ_2 is a generic curve in W with at least 2 contact points less than γ, which is impossible by the minimality of γ.

(b) One considers now the bad case. Let us consider the half-orbit Γ_- starting inside B. One has the following alternative : Γ_- is contained entirely in B or Γ_- leaves B at a first point q' in the interior of $[p,q]_\gamma$. One will consider

the two terms of this alternative.

(1) One first supposes that $\Gamma_- \subset B$. As $\Gamma_- \setminus \{p\}$ does not intersect ∂B, it has a limit-set in the interior of B. By the Poincaré-Bendixson theory, this limit-set must contain a singular point or is a periodic orbit, which must bound a disk in B containing also a singular point. As B is free of singular point, none of these two possibilities can occur.

(2) Let us suppose now that Γ_- *leaves* B *at a first point* q' *in the interior of* $[p,q]_\gamma$. As above, one can assume (after a slight perturbation of γ), that this point is a transverse intersection of γ with Γ_-. As the sub-arc between p and q' inside $[p,q]_\gamma$ bounds in union with $[p,q']_\Gamma$ a disk B' contained into B, this arc is the arc $[p,q']_\gamma$ as defined above. But, as Γ_+ is outside B and then outside B', the pair (p,q') is of the good type. The impossibility follows from the part (a) above. ∎

It may happen that the contact property $\gamma \cap \Gamma = \{p\}$ which is obtained in Proposition 3 corresponds to a contact between a minimal curve γ and a periodic orbit Γ contained into W. This will imply a strong constraint on the contact index, as it appears in the following result :

Lemma 3 *Suppose that a smooth disk* $W \subset W_0$ *contains a periodic orbit in its interior. Then,* $c(X,W) \leq 2$.

Proof

Let Γ be a periodic orbit in the interior of W. Choose a $\mathcal{C}^1$ local transverse section Σ to Γ and consider the $\mathcal{C}^1$ return map $P(u)$ on it (u is a $\mathcal{C}^1$ coordinate on Σ such that $\Gamma \cap \Sigma = \{u=0\}$). One has two possibilities :

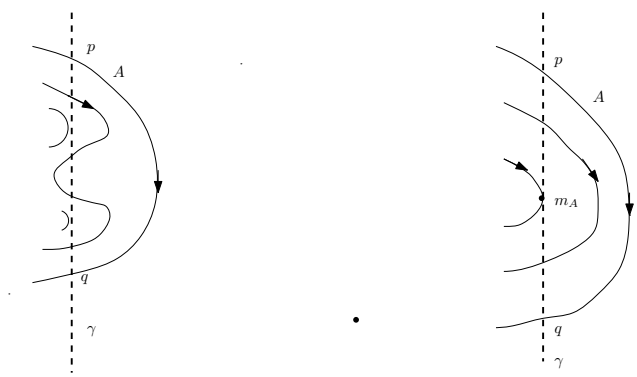
(1) If $P(u) \equiv u$ in a neighborhood of $u=0$, one can find a generic curve γ' which is $\mathcal{C}^0$-near γ, with just 2 contact points. One can see this in the following way. One chooses a $\mathcal{C}^1$ diffeomorphism $\psi(s,\rho)$ of the annulus $A_\delta = \{(s,\rho) \in S^1 \times]-\delta,\delta[\}$ onto tubular neighborhood of Γ, sending each orbit of the smooth vector field $X_0 = \frac{\partial}{\partial s}$ on an orbit of X. For any ε such that $0 < \varepsilon < \delta$, the curve of A_δ given by $\rho = \varepsilon \sin s$ has two quadratic contact points with X_0. Its image by ψ is a C^1 generic curve with two even contacts points with X. This implies that $c(X,W) \leq 2$.

(2) If $P(u) \not\equiv u$ in a neighborhood of $u=0$, one can find a u_0 arbitrarily near 0 such that $P(u_0) \neq u_0$. In this case, using Lemma 2, one can $\mathcal{C}^0$ approximate the piecewise curve γ_1, union of the interval of Σ between u_0 and $P(u_0)$, and the arc or orbit between the same points, by a $\mathcal{C}^1$ curve γ_2 transverse to X. In this case one has that $c(X,W) = 0$. ∎

In order to state the next result, one needs some definitions

Definition 10 *Let γ be a generic curve in a disk $W \subset W_0$. An arch attached to γ in W is an arc of orbit $A = [p,q]_\Gamma \subset W$ cutting transversally γ into two consecutive points p, q on the orbit (this means that $[p,q]_\Gamma \cap \gamma = \{p,q\}$). Let $[p,q]_\gamma$, called the base of the arch, the single arc of γ with end points p, q, such that $[p,q]_\Gamma \cup [p,q]_\gamma$ is the boundary of a topological disk D_A which does not contain the origin.*

The arch is interior if it is contained into $W(\gamma)$ and exterior if not. One will say that the arch is minimal if its base contains a unique contact point m_A, with a contact exterior at D_A (i.e. an interior contact of γ for an exterior arch, and an exterior contact of γ for an interior arch) and if $D_A \setminus \{m_A\}$ is filled by orbits joining a point between p and m_A to a point between m_A and q, on the base $[p,q]_\gamma$. See Figure 7.



General arch Minimal arch

Figure 7

Remark 4 *The foliation by the orbits in a minimal arch is $\mathcal{C}^1$ conjugate to the foliation by the intersections of parabolas $y = -x^2 + \alpha$, for $\alpha \in [0,1]$ with the topological disk $D = \{0 \leq y \leq 1 - x^2 | -1 \leq x \leq 1\} \subset I\!R^2$, (with coordinates (x,y) in $I\!R^2$).*

Proposition 4 *Let γ be a minimal curve in a smooth disk $W \subset W_0$. Then, any arch attached to γ in W is a minimal arch.*

Proof

One can limit the study to interior arches, the proof for exterior arches being quite similar. Then let $A = [p,q]_\Gamma \subset W(\gamma)$ be an interior arch attached to γ, with $[p,q]_\gamma$ and D_A as in Definition 10.

First, I claim that there is no interior contact m to γ (also interior contact to D_A) on $[p,q]_\gamma$. To see this, look for instance at a half-orbit Γ_+ starting at m.

This half-orbit cannot have a limit-set in D_A, as this disk contains no singular point of X (see the proof of the case (b-1) in the proof of Proposition 3) Then, Γ_+ must intersect $[p,q]_\gamma$. at a point $m' \neq m$. But, as γ is minimal in W, this is forbidden by Proposition 3.

Next, I claim that there exists one and only one exterior contact point m_A to γ on $[p,q]_\gamma$. In fact, if there exists c_e exterior contact points and no interior contact points on $[p,q]_\gamma$, it is easy to see that the topological index of X along ∂D_A is equal to $i(X, \partial D_A) = \frac{1-c_e}{2}$ (for a reason of orientation, c_e must be odd). As there is no singular point of X inside D_A, this index is zero, as a consequence of the Hopf-Poincaré formula, and it follows that $c_e = 1$.

Again, as D_A does not contain singular points of X, it is a direct consequence of the Poincaré-Bendixson theory that each orbit in $D_A \setminus \{m_A\}$ links on $[p,q]_\gamma$, a point on the left of m_A to a point on the right of m_A. This means that the arch A is a minimal one. ∎

Proposition 5 *Let γ be a minimal curve in a disk $W \subset W_0$, contained in the interior of W. Then, no orbit of X in the interior of W can cut γ into more than two distinct points.*

Proof

If $c(\gamma) = 0$, i.e. if X is transverse to γ, then each orbit Γ can cut γ at most one time. Then, the result is trivial in this case. From now on we will assume that $c(\gamma) = c(X, W) > 0$.

Let us suppose that an orbit Γ cuts γ in three distinct points p, q, r. One can always assume that these points are consecutive on Γ, and by a slight $\mathcal{C}^1$ perturbation of γ, one can also assume that the intersection of Γ with γ is transverse at each of these points.

One considers the two arches $A = [p,q]_\Gamma$ and $B = [q,r]_\Gamma$. One will assume that A is exterior at γ and then, that B is interior (the other case is completely similar). By Proposition 4, there exists a single contact point on $[p,q]_\gamma$, which is an interior contact point m_1 and a single contact point on $[q,r]_\gamma$, which is an exterior contact point m_2. Taking into account the different possible arrangements for the bases, it appears that there are four possible cases which are represented in Figure 8, where one has indicated the different possible positions of the disks D_A and D_B. One considers now these different cases.

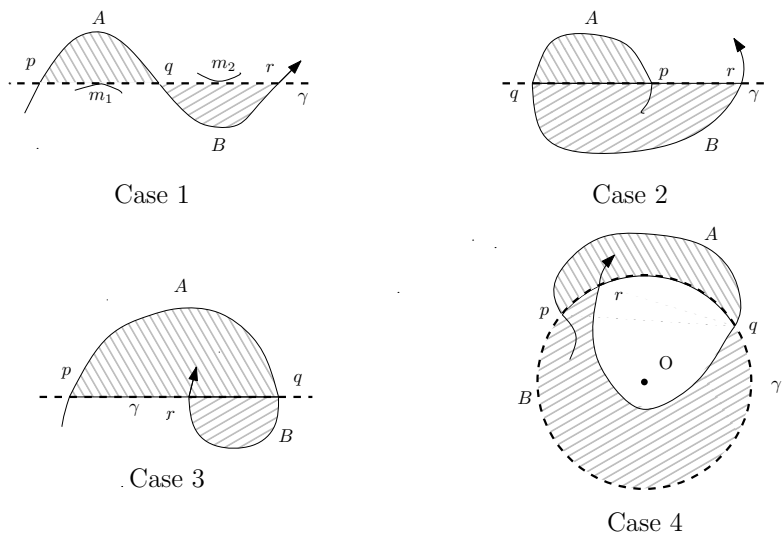


Figure 8

(1) In the case 1, one has that $[pq]_\gamma \cap [q,r]_\gamma = \{q\}$. This means that $[p,r]_\gamma = [pq]_\gamma \cup [q,r]_\gamma$ is a sub-arc of γ. Let $[p,r]_\gamma^C$ be the complementary sub-arc. One considers the topological curve $\gamma_1 = [p,r]_\gamma^C \cup [p,q]_\Gamma \cup [q,r]_\Gamma$ contained in the interior of W. This curve is homotopic to γ and then surrounds the origin. In similar way as in the proof of Proposition 3 for the good case, one can modify γ_1 in order to obtain a generic curve γ_2, which is $\mathcal{C}^0$ near γ_1 (and then it belongs to the interior of W and surrounds the origin), and has two contact points less than γ (the points m_1 and m_2). This contradicts the minimality of γ in W.

(2) In the case 2, one has that $[p,q]_\gamma \subset [q,r]_\gamma$. But, as $m_1 \neq m_2$ (one is exterior and the other one is interior), the base $[q,r]_\gamma$ contains two distinct contact points. This is impossible by Proposition 4.

(3) In the case 3, one has that $[q,r]_\gamma \subset [p,q]_\gamma$. Then, the same proof as in case 2 shows that this case is impossible.

(4) In the case 4, at the point r which belongs to $[p,q]_\gamma$, the field X has the same transverse orientation as at the point p. Then, the point r is between p and m_1 on the base $[p,q]_\gamma$ and the field X is transverse to γ all along the arc $[p,r]_\gamma$. As in Lemma 2, we can $\mathcal{C}^0$-approximate the piecewise $\mathcal{C}^1$-curve $\gamma_1 = [p,r]_\Gamma \cup [p,r]_\gamma$, which is in the interior of W and surrounds the origin by a $\mathcal{C}^1$-curve γ_2 with the same properties and which is transverse to X. But the existence of such a curve means that $c(X,W) = 0$, in contradiction with the assumption that

$c(X, W) > 0$ made in this part of the proof. ∎

2.3. **Some topological results**

As consequence of Propositions 4 and 5 I want to deduce now some global properties for a germ $(X, 0)$ of $\mathcal{C}^1$ vector field with an isolated singularity at the origin. One will also assume that *the origin is not accumulated by periodic orbits.*

One can represent the germ by a $\mathcal{C}^1$ vector field X on a disk W_0, which contains no singular point other that the origin nor periodic orbit. Moreover one assumes that $c(X, W_0) > 0$: if a germ is isolated from periodic orbits and such that $c(X, 0) = 0$, one will see in subsection 2.5 that it is a topological node. Let γ be a minimal generic curve in W_0. As $c(X, W_0) > 0$), this curve contains at least two contact points. I want to give properties of the orbits of X on the disk $W(\gamma)$. These results are comparable (up the formulation) to the results obtained by Bendixson in the Part I of [B], but here the proofs are based on the notion of minimal curves, which was not the case in [B]. I shall recall and use the terminology introduced in [B].

Proposition 6 *One considers a minimal curve in W_0 for a field X as above. Let L be an half-orbit of X through some point m of γ and assume that it is contained into $W(\gamma)$. Then the limit set of L is the origin.*

Proof

Let us assume that L is a positive half-orbit by $m \in \Gamma$, i.e. defined for the positive time. We want to prove that $\omega(L) = \{0\}$. The proof for a negative half-orbit is exactly the same, replacing X by $-X$, and will be omitted. Let $\varphi(t)$ the trajectory by m. The positive half-orbit L is image of the trajectory for the positive times. As L is contained in the compact $W(\gamma)$, the trajectory is defined for all positive times in $I\!R^+$ and its ω-set is contained into $W(\gamma)$. It follows from the Poincaré-Bendixson theory that this limit set is reduced to a singular point (necessarily the origin) or must contain at least a regular point (which may belong to a periodic orbit or a graphic).

Let us prove that this second case is impossible. If $\omega(L)$ contains a regular point r, that we can assume to belong to the interior of $W(\gamma)$, the trajectory $\varphi(t, m)$ accumulates on r for $t \to +\infty$. It is easy to find a transverse section Σ in a neighborhood of r, which is cut just in two points p, q by L, i.e. such that $L \cap \Sigma = \{p, q\}$. Using Lemma 2, one finds a curve γ' which is transverse to X and also contained into the interior of $W(\gamma)$. This curve must surround the origin, which is the unique singular point in $W(\gamma)$. Moreover, as it is transverse to X,

this curve is generic and such that $c(\gamma') = 0$. But this value 0 is strictly less than the value $c(X, W(\gamma)) \geq c(X, W_0) > 0$, which contradicts the assumption made at the beginning of this subsection 2.3. As a consequence, one has that $\omega(\Gamma) = \{0\}$. ∎

Corollary 1 *Let γ be a minimal curve in W_0 for a field X as above and Γ be the orbit of X through some interior contact point m of γ. Then $\omega(\Gamma) = \alpha(\Gamma) = \{0\}$.*

Proof It follows from Proposition 3 that each half-orbit through the interior contact point m is contained into $W(\gamma)$. Then one can apply Proposition 6 to obtain the result. ∎

An orbit Γ as in Corollary 1 bounds a piecewice $\mathcal{C}^1$ disk $D(\Gamma)$, singular at the origin : a *pinched disk*, in short. Each orbit $\tilde{\Gamma}$ inside $D(\Gamma)$ has the same limit property that Γ, i.e. $\omega(\tilde{\Gamma}) = \alpha(\tilde{\Gamma}) = \{0\}$. The proof is exactly the same as in Proposition 6 and follows from non-existence of singular point of X in the interior of $D(\Gamma)$. Following Bendixson in [B], we define

Definition 11 (Bendixson) *A closed nodal region is a pinched disk $D(\Gamma)$ bounded by an orbit Γ such that $\omega(\Gamma) = \alpha(\Gamma) = \{0\}$, filled by orbits with the same limit property.*

Elliptic sectors introduced in Introduction are simple examples of closed nodal region. But the phase portrait inside a general closed nodal region may be much more complicated than inside an elliptic sector. One can introduce an order in the set of orbits contained into $D(\Gamma)$, putting that $\tilde{\Gamma}_1$ is less than $\tilde{\Gamma}_2$ if and only if $\tilde{\Gamma}_1 \subset D(\tilde{\Gamma}_2)$. This order is total for an elliptic sector but is just partial in general. As a consequence, the phase portrait inside a general closed nodal region may be much more complicated than the 1-parameter family of orbits that one finds inside an elliptic sector. See Figure 9 for an example.

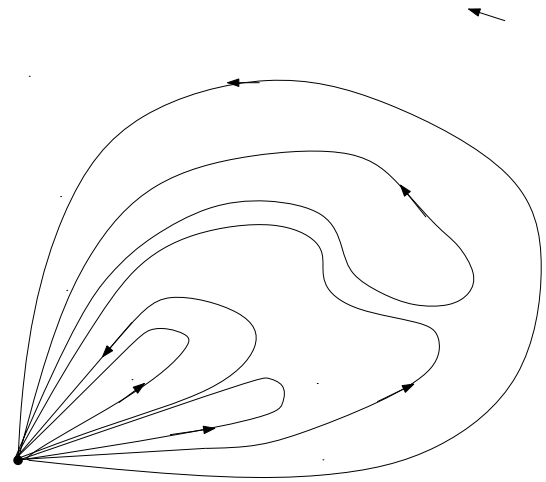


Figure 9 : *Closed nodal region (following Bendixson)*

Inside $W(\gamma)$ one has $c_i(\gamma)$ disjoint closed nodal regions associated to the $c_i(\gamma)$ interior contact points of γ. One also associates to each exterior contact point of γ, a remarkable type of orbit pair, also introduced by Bendixson :

Definition 12 (Bendixson) *Let (L, L') be a pair of orbits such that $\omega(L) = \alpha(L') = \{0\}$, at the boundary of a sector S. This pair (L, L') is said to cross the origin (on the side of S) if one has the following. One considers a section σ, topologically transverse to L and a section σ', topologically transverse to L', these two sections pointing in the direction of S with a base point respectively p_0 on L and p'_0 on L'. One assume each positive half-orbit $\Gamma(p)$ through a point $p \in \sigma \setminus \{p_0\}$ cuts $\sigma' \setminus \{p'_0\}$ a first time at a point $p' = h(p)$, if σ is chosen short enough. One assumes that this transition map h, which is continuous on $\sigma \setminus \{p_0\}$, can be extended by continuity at p_0 by $h(p_0) = p'_0$, i.e. $h(p) \to p'_0$ when $p \to p_0$ on $\sigma \setminus \{p_0\}$.*

Remark 5 *If the Definition 12 is verified for a choice of sections σ, σ', it is also verified for any other choices. This is easily shown by using well-chosen tubular neighborhoods.*

If L and L' are the sides of an hyperbolic sector S at the origin, as defined in Introduction, it is clear that this pair of orbits is crossing the origin (on the side

of S) . But, as it was noticed by Bendixson the converse is not always true. It may exist pair of orbits crossing the origin which bound a sector containing closed nodal regions, possibly in an infinite number. See Figure 10.

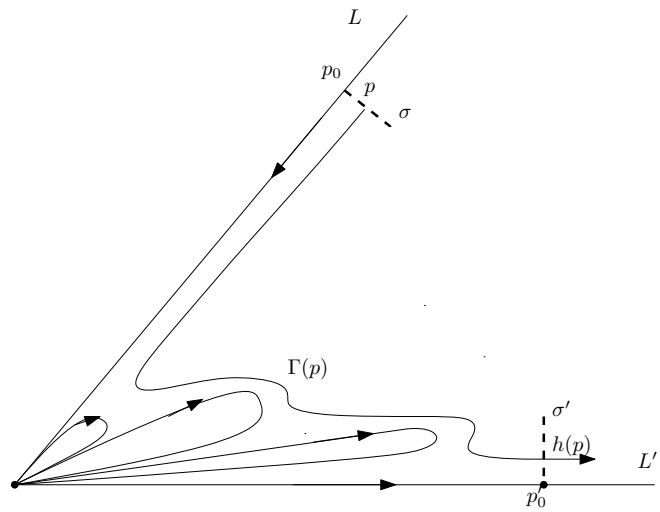


Figure 10 :*Pair of orbits crossing the origin (following Bendixson)*

In order to state the next result we need the following definition :

Definition 13 *Let m be an exterior contact point on the minimal curve γ. Then, the maximal interval $]q_-(m), q_+(m)[$ of m, is the open interval of γ, union of the bases of all minimal arches associated to m (the notation is chosen such that X has an inward direction at points of $]q_-(m), q_+(m)[$ near $q_-(m)$, and the converse near $q_+(m)$).*

Remark 6 *As arch disks associated to two different exterior contact points of γ are disjoint, it follows that maximal intervals associated to different exterior contact points are also disjoint.*

The following result associates a pair of orbits crossing the origin at each exterior contact point :

Proposition 7 *Let m be an exterior contact point and $]q_-(m), q_+(m)[$ its maximal interval. Let L_- be the positive half-orbit by $q_-(m)$ and L_+ be the negative half-orbit by $q_+(m)$. Then $\omega(L_-) = \alpha(L_+) = \{0\}$. Let S be the sector with sides L_-, L_+ and containing the arches associated to m. Then, the pair (L_-, L_+) is crossing the origin (on the side of S).*

Proof

(1) If a point $q \in \gamma$ is an end point of an arch associated to an exterior contact m', it is the same for points in a whole neighborhood of q in γ. It follows that $q_-(m), q_+(m)$ cannot be end points of an arch associated to m, by maximality,

and cannot be end points of an arch associated to an exterior contact point $m' \neq m$, because in this case there would exist an arch associated to two distinct exterior contact points m and m', that is not possible. As a consequence, the positive half-orbit L_- through $q_-(m)$ and the negative half-orbit L_+ through $q_+(m)$ are contained into $W(\gamma)$. It follows from Proposition 6 that $\omega(L_-) = \alpha(L_+) = \{0\}$. Let be S the sector with sides L_-, L_+, containing the arches associated to m.

(2) Using notations of Definition 12, we take $p_0 = q_-(m)$ and $p'_0 = q_+(m)$. We take sections $\sigma = [p_0, q_0[$ and $\sigma' = [p'_0, q'_0[$ for points $q_0 \in]p_0, m[\subset]p_0, p'_0[$ and $q'_0 \in]p'_0, m[\subset]p'_0, p_0[$, such that q_0, q'_0 are the two end points of a same arch. The transition map h is defined by the condition that, for any $p \in]p_0, q_0]$, $h(p)$ is the second end point of the arch starting at p. The map h is an homeomorphism from $]p_0, q_0]$ onto $]p'_0, q'_0]$, reversing the orientation. The continuous extension at p_0 follows from the following observation : if p' is any point in $]p'_0, q'_0[$, then h sends the interval $]p_0, h^{-1}(p')[$ onto the interval $]p'_0, p'[$. It follows that the pair (L_-, L_+) is crossing the origin on the side of S ∎

Remark 7 *Using Proposition 5 and the minimality of γ, it is easy to prove that if $]q_-(m), q_+(m)[$ is the maximal interval of an exterior contact point m on γ, then $q_-(m) \neq q_+(m)$.*

The half-orbits which enter into $W(\gamma)$, starting at points of γ outside the maximal intervals of the exterior contact points, cannot return on γ. Then, they are contained into $W(\gamma)$ and it follows of Proposition 6 that their limit set is the origin. They belong to parabolic sectors (possibly reduced to a single orbit).

As a consequence we obtain that the disk $W(\gamma)$ is the finite union of $c_i(\gamma)$ closed nodal regions, $c_e(\gamma)$ sector associated to pair of orbits crossing the origin and $c(\gamma)$ parabolic sectors (possibly reduced to a single orbit located between two sectors associated to pairs of orbits crossing the origin).

2.4. Phase portrait inside a minimal annulus

Definition 14 *Let W be a smooth disk in W_0. A minimal annulus A in W is an annulus whose boundary is the union of two minimal curves γ_1, γ_2 in W such that γ_2 is contained into the interior of $W(\gamma_1)$. The curve γ_1 is the exterior side and the curve γ_2 is the interior side of the boundary ∂A.*

Proposition 8 *Let A be a minimal annulus with boundary $\gamma_1 \cup \gamma_2$ where γ_1 is the exterior boundary side and γ_2 the interior one, as defined above. We assume*

that A contains no periodic orbit.

Then, each interior contact of γ_1 is on a minimal arch with base on γ_2 and each exterior contact of γ_2 is on a minimal arch with base on γ_1. The arch-disks associated with these arches are contained in A and are 2 by 2 disjoints. All the contact points of γ_1 and γ_2 belong to the union of these arch-disks. In the complement of the arch-disks the orbits are arcs transverse to ∂A, between a point on γ_1 and a point on γ_2. See Figure 11.

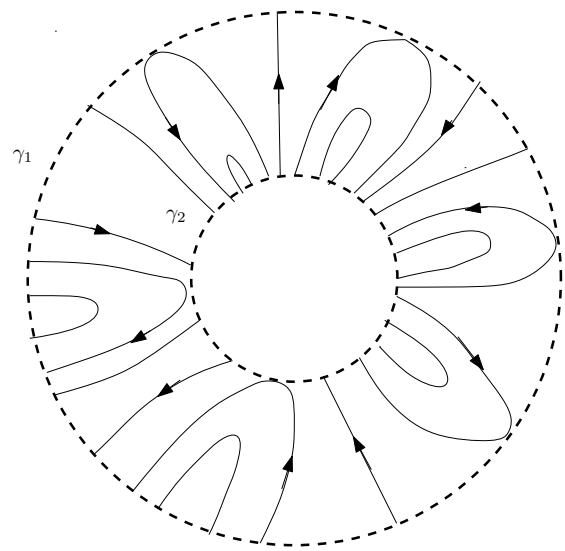


Figure 11 :*A minimal annulus*

Proof

Let us consider any interior contact point m on γ_1. Two half-orbits Γ_+ and Γ_- are starting from m, and enter into A. One want to prove that each of them has a first intersection with γ_2, which is transverse. The proof is the same for the two half-orbits, then we will just consider Γ_+, image of the trajectory $\varphi(t, m)$ for $t \geq 0$. As there exists no singular point nor periodic orbit inside A, $\varphi(t, m)$ cannot remain into the interior of A for all $t > 0$. Then, there exists a first $t_0 > 0$ such that $p(m) = \varphi(t_0, m) \in \partial A$.

Let us suppose first that $p(m) \in \gamma_1$. It could happen that $p(m) = m$. But this would mean that the orbit by m in A is periodic, which is excluded by hypothesis. On the other side, as γ_1 is minimal in W, the fact that $p(m) \neq m$ is excluded by Proposition 3.

It follows that $p(m) \in \gamma_2$. It could happens that $p(m)$ is an exterior contact point on γ_2. In this case Γ_+ coincide with one of the two half-orbits starting from $p(m)$ before entering into A. Let Γ'_+ the other half-orbit starting from $p(m)$. By

the same argument as above, we conclude that Γ'_+ must reach γ_1 at a point $q(m)$. If $q(m) = m$, we have again that the orbit through m is a periodic orbit inside A, which is excluded by hypothesis. If $q(m) \neq m$, this means that the orbit of m in A return on γ_1 at a point distinct from m, which is excluded by Proposition 3. The only possibility is that Γ_+ reaches γ_2 at a transverse point.

Then, there is an arch in A attached to γ_2, passing through the interior contact point $m \in \gamma_1$. Its base $]p, q[\subset \gamma_2$ contains a unique contact point of γ_2, which is an interior contact and the arch is minimal, as it follows from Proposition 4. As the arch and its base belong to A, its arch-disk also belongs to A.

The same proof can be repeated for each interior contact point of γ_1 and for each exterior contact point of γ_2. We obtain $c_i(X, W)$ arches attached to γ_2 and $c_e(X, W)$ arches attached to γ_1. All these arches are minimal, with arch-disks in A and 2 by 2 disjoint precisely because they are minimal. Moreover, the $2c(X, W)$ contact points located on ∂A are contained in the union of the arch-disks (half of them on the arches and half of them on the bases of the arches).

Consider now any point $m \in \gamma_1$ and outside the arch-disks. The vector field X is transverse to γ_1 at m (no contact points is outside the union of arch-disks), and the same argument as above shows that the orbit by m is an arc joining m to a point $p \in \gamma_2$ where X is transverse to γ_2. Moreover, this arc lies outside the union of arch-disks. ∎

The result stated in Proposition 8 is trivial if X is smooth and if we consider a smooth minimal curve γ with quadratic contact points and another smooth curve γ', sufficiently $\mathcal{C}^2$ close to γ and disjoint from it. The interest of the Proposition is that it is valid in class $\mathcal{C}^1$, for any pair of minimal curves in W.

Remark 8 *As follows from Lemma 3, a sufficient condition in order to have no periodic orbit in A, is that $c(X, W) \geq 3$.*

Corollary 2 *Consider two disjoint minimal curves in a smooth disk $W \subset W_0$. Then, the arrangement of contact points in the same for the two curves.*

Proof

The two minimal curves are the components γ_1, γ_2 of a minimal annulus, as above. If $c(X, W) \leq 2$, the result is obviously true (even if the annulus between the two curves contains a periodic orbit). Then one can assume that that $c(X, W) \geq 3$, and then that there exists no periodic orbit in W, and in particular between γ_1 and γ_2. Then, one can apply the Proposition 8.

Let $p_1, \ldots, p_k$ with $k = c_i(X, W)$ be the interior contact points on γ_1 and $q'_1, \ldots, q'_l$ with $l = c_e(X, W)$ be the exterior contact points on γ_2. At each p_i is

associated the base $I_i \subset \gamma_2$ of the arch through p_i. In the same way, at each q'_j is associated the base $J_j \subset \gamma_j$ of the arch through q'_j. The I_i are 2 by 2 disjoints and each I_i contains a unique contact point of γ_2, which is the interior point p'_i. In the same way, the J_j are 2 by 2 disjoints and each J_j contains a unique contact point of γ_1, which is the exterior point q_j.

Now, as the arch-disks are 2 by 2 disjoint, it is clear that the arrangement of $\{p_1, \ldots, p_k, q_1, \ldots, q_l\}$ is the same as the arrangement of $\{p_1, \ldots, p_k, J_1, \ldots, J_l\}$, which is the same as the arrangement of $\{I_1, \ldots, I_k, q'_1, \ldots, q'_l\}$, which is the same as the arrangement of $\{p'_1, \ldots, p'_k,'_1, \ldots, q'_l\}$. See Figure 12. ∎

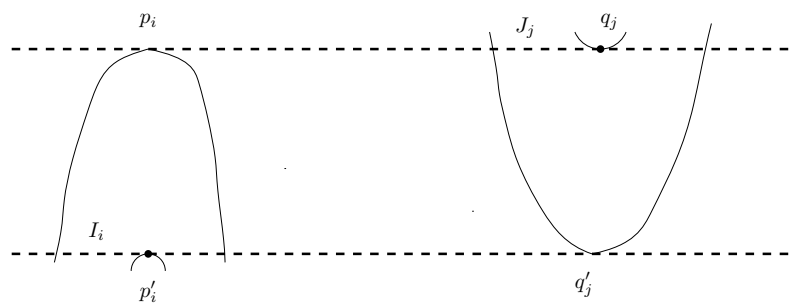


Figure 12

2.5. Vector field germs of class $\mathcal{C}^1$ with finite contact index

In this part we suppose that $(X, 0)$ is a $\mathcal{C}^1$ germ at the origin, isolated from the periodic orbits and such that $c(X, 0) \in I\!N$ (i.e. is finite).

The following theorem describes the phase portrait of $(X, 0)$:

Theorem 2 *Let $(X, 0)$ a $\mathcal{C}^1$ germ of vector field at the origin of $I\!R^2$, isolated from periodic orbits and with a finite index $c(X, 0)$. Let X a representative of $(X, 0)$ on a disk W_0 such that $c(X, W_0) = c(X, 0)$, and γ an absolute minimal curve in W_0, as above. Then, on $W(\gamma)$, the vector field X is a topological node or admits a sectoral decomposition, as defined in Definition 3. There are $c_i(X, 0)$ elliptic sectors, $c_e(X, 0)$ hyperbolic sectors, which are separated by $c(X, 0)$ parabolic sectors (some of them may be reduced to a single orbit).*

Proof

The general idea of the proof will be to associate to γ a sequence of other absolute minimal orbits (γ_n) in the interior of $W(\gamma)$, two by two disjoint, with

diameter tending to zero, as in Propostion 1. One begins by the simpler case $c(X,0)=0$.

(1) Case $c(X,0)=0$. In this case the vector field X is transverse to the minimal curves γ and γ_n. Assume that the vector field is pointing inside along γ (for the converse, one just replaces X by $-X$). As there exists no periodic orbit in the annulus bounded by γ and γ_n, the positive half-orbit $\Gamma(m)$ starting at a point $m \in \gamma$ arrives to a point m_n on γ_n When $n \to +\infty$, one has that $m_n \to 0 \in I\!R^2$. Then, one has that $\omega(\Gamma(m)) = \{0\}$ (to be precise, one has to use that the segment of orbit between m_n and m_{n+1} is contained into the annulus between γ_n and γ_{n+1} for any n). As the result is true for any $m \in \gamma$, one has that X is a topological node on $W(\gamma)$.

(2) Case $c(X,0)>0$. One will start with the general results of Subsection 2.3, taking into account the existence of the sequence (γ_n) and considering the interior contact points on γ. and next the exterior contact points.

(a) Interior contact points. Let m be any interior contact point on γ. It follows from Corollary 1 that the orbit Γ passing through m bounds a closed nodal region $D(\Gamma)$ in $W(\gamma)$. recall that this means that any orbit $\tilde{\Gamma}$ in $D(\Gamma)$ verifies that $\omega(\tilde{\Gamma}) = \alpha(\tilde{\Gamma}) = \{0\}$. One wants to prove that $D(\Gamma)$ is in fact an elliptic sector, i.e. that if $\tilde{\Gamma}_1, \tilde{\Gamma}_2$ are two orbits in $D(\Gamma)$, then $\tilde{\Gamma}_1 \subset D(\tilde{\Gamma}_2)$ or $\tilde{\Gamma}_2 \subset D(\tilde{\Gamma}_1)$.

One first notices that there exists a number n_0 such that if $n \geq n_0, n$ the curve γ_n intersects $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$, each at two tranverse points (if $M_i > 0$ is the maximum of the distance between a point on $\tilde{\Gamma}_i$ and the origin it suffices to take n_0 such that $\mathrm{diam}(W(\gamma_{n_0})) < \mathrm{Inf}(M_1, M_2)$). One takes any $n \geq n_0$ and let A_n be the annulus between γ and γ_n. One uses now Proposition 8 : there exists a minimal arch passing through m and containing the two arches $\alpha_1^n = \tilde{\Gamma}_1 \cap A_n$ and $\alpha_2^n = \tilde{\Gamma}_2 \cap A_n$. Let $[q_{1,-}^n, q_{1,+}^n]$ and $[q_{2,-}^n, q_{2,+}^n]$ be the bases on γ_n of these two arches. One knows that one of them is included in the other and using again Proposition 8 for the annulus between any pair of curves γ_n one sees that this order of inclusion is independent of n. To fix the ideas, let us assume that $[q_{1,-}^n, q_{1,+}^n] \subset [q_{2,-}^n, q_{2,+}^n]$. This implies that $\tilde{\Gamma}_1 \cap A_n \subset D(\tilde{\Gamma}_2) \cap A_n$ for any $n \geq n_0$. Taking the limit for $n \to +\infty$ one obtains the result : $\tilde{\Gamma}_1 \subset D(\tilde{\Gamma}_2)$. To conclude, inside $W(\gamma)$ one has $c_i(X,0)$ elliptic sectors, which are 2 by 2 distinct, associated to the $c_i(X,0)$ interior contact points on γ.

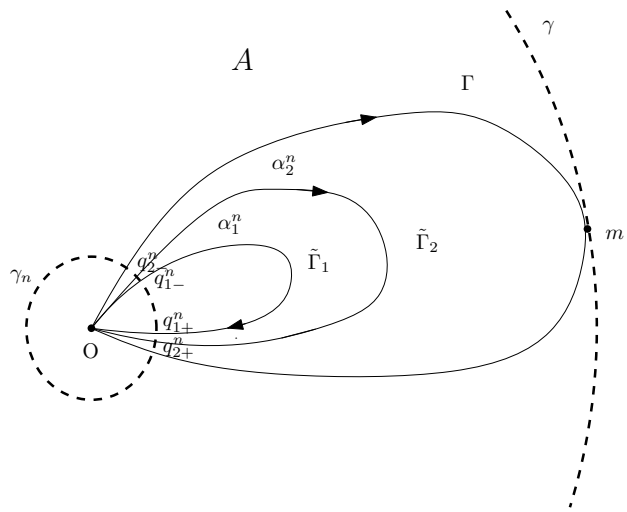


Figure 13

(b) Exterior contact points. Let m be any exterior contact point. Using Proposition 7, one can associate to m a sector S containing all arches associated to m, with sides a pair (L_-, L_+) of orbits crossing the origin on the side of S. Moreover L_- and L_+ cut γ respectively at the end points $q_-(m)$ and $q_+(m)$, where $]q_-(m), q_+(m)[$ is the maximal interval associated to m. Choose now any $\varepsilon > 0$ and a number n such that $\mathrm{diam}(W(\gamma_n)) < \varepsilon$. From now on, ε and n are maintained fixed.

Put $q_-^n(m) = L_- \cap \gamma_n$ and $q_+^n(m) = L_+ \cap \gamma_n$. For a point $p \in]q_-(m), m[\subset]q_-(m), q_+(m)[$ one calls $A(p)$ the arch starting at the point p and ending at $h(p)$, the second end point of $A(p)$. If ε is small enough, the arch $A(p)$ cuts γ_n at two successive points p^n and q^n. It follows the fact that (L_-, L_+) is crossing the origin and directly from the proof of Proposition 7 that $h(p) \to q_+(m)$ when $p \to q_-(m)$.

This implies that there exists a $\delta = \delta(\varepsilon) > 0$ such that if $\mathrm{dist}(p, q_-(m)) < \delta$ (for a given distance on γ) then the distance between the arc $[p, p^n]_{A(p)}$ on $A(p)$ and the arc $[q_-(m), q_-^n(m)]_{L_-}$ on L_- is smallest than ε and the same for the distance between the arc $[h(p), q^n]_{A(p)}$ on $A(p)$ and the arc $[q_+(m), q_+^n(m)]_{L_+}$ on L_+. As the distance from $A(p) \cap W(\gamma_n)$ to the origin is less than ε (by the choice of n), we have that the distance from $A(p)$ to $L \cup L'$ is less than ε. As conclusion, the arches $A(p)$ fill up the sector S and then this sector is an hyperbolic one.

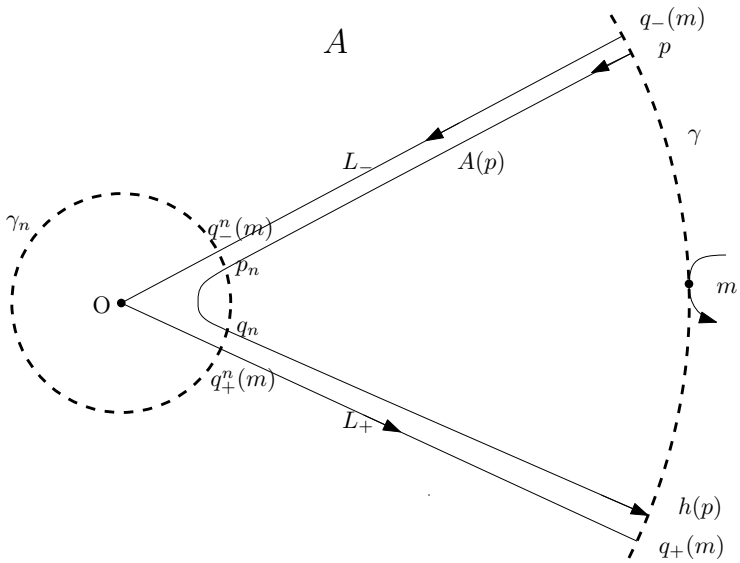


Figure 14

One has already observed at the end of Part 2.3 that the complement set in $W(\gamma)$ of the union of the $c_i(X,0)$ elliptic sectors and the $c_e(X,0)$ hyperbolic sectors, that we have found above, is filled up by parabolic sectors (some of then possibly reduced to a single orbit). This finished the proof in the case $c(X,0) > 0$. ∎

If a germ of vector field is a topological node or admits a sectoral decomposition, then the origin is an isolated singularity which cannot be accumulated by periodic orbits and it is rather easy to construct curves, at arbitrarily small distance of the origin, which are generic with a number of contact points equal to the number of elliptic sectors plus the number of hyperbolic sectors, i.e. the number index $c(X,0)$ is finite. This means that Theorem 2 admits a reverse. As the numbers of elliptic and hyperbolic sectors are equal respectively to $c_i(X,0)$ and $c_e(X,0)$ we obtain as consequence of Theorem 2, of the above observation and of formula (1), a formula given in [B] for the index $i(X,0)$:

Corollary 3 *Let $(X,0)$ be a $\mathcal{C}^1$ germ of vector field which is a topological node or admits a sectorial decomposition with $E(X,0)$ elliptic sectors and $H(X,0)$ hyperbolic sectors. Then the index of $(X,0)$ at the origin is given by :*

$$i(X,0) = \frac{E(X,0) - H(X,0)}{2} + 1 \tag{3}$$

Remark 9 *Mind that the convention is different in [B] and the index given by Bendixson has an opposite value that the one given here in (3).*

One can associate to a sectoral decomposition, its arrangement of elliptic and hyperbolic sectors and also take into account that some parabolic sectors may be reduced to a single orbit. This is coded by a sequence of $c(X,0)$ symbols taken in the set $\{E,H\}$ (for the elliptic and hyperbolic sectors), interlinked by a sequence of $c(X,0)$ symbols in the set $\{P,S\}$: P for true parabolic and S for a single orbit between two hyperbolic sectors. By this way one obtains a sequence of $4k$ symbols defined up a circular permutation, as an example the sequence (E,P,E,P,H,S,H,P) for the sectoral decomposition in Figure 2. One can associate the trivial sequence (P) to a topological node. It is rather trivial that two sectoral decompositions which are topologically equivalent (i.e., with the same phase portrait) have the same associated sequence. The reverse is true, but the proof, which can be founded in [D], is not so easy.

One can use this sequence in order to obtain an estimation of the number of different phase portraits in terms of $c(X,0)$. It is also easy to obtain an estimation for the topological index $i(X,0)$:

Proposition 9 *Let $(X,0)$ be a germ of $\mathcal{C}^1$ vector field as in Theorem 2. Then, we have that $-c(X,0)/2+1 \leq i(X,0) \leq c(X,0)/2+1$ and that the number of different phase portraits is less than $2^{2c(X,0)}$.*

Proof

To obtain the estimation on $i(X,0)$ we have just to notice $-c(X,0) \leq c_i(X,0)-c_e(X,0) \leq c(X,0)$, and then that

$$-c(X,0)/2+1 \leq i(X,0) \leq c(X,0)/2+1$$

To count the different possible phase portraits, we have just to count the number of sequences associated above to sectoral decompositions. The number of possible arrangements of $c_i(X,0)$ elliptic sectors among $c_e(X,0)$ hyperbolic sectors is less than $2^{c(X,0)}$. The number of occurrences of a single orbit in place of a true parabolic sector is also surely less than $2^{c(X,0)}$. Then the number of distinct phase portraits is bounded by $2^{2c(X,0)}$. ∎

3. Differentiable germs with an isolated singularity

In the Part II of [B], Bendixson considers what he calls "holomorphic vector fields" germs at the origin of $I\!R^2$, whose components have no common factor at $\{0\}$. Clearly, he wants to speak of real analytic germs with an algebraically isolated singularity at the origin. But it appears that Bendixson did not used the analyticity, nor the fact that the origin is algebraically isolated. In fact,

his arguments simply use that the germ is sufficiently differentiable, with a non zero jet at the origin and has the origin as a (topologically) isolated singularity. For this reason, in this Section I shall consider germs with a finite class of differentiability and a non zero finite jet at the origin.

3.1. **A sufficient condition for the finiteness of** $c(X,0)$

For an integer $k \geq 1$, we consider a $\mathcal{C}^{2k+3}$ vector field germ $(X,0)$, represented by a $\mathcal{C}^{2k+3}$ vector field X on a disk W_0, neighborhood of the origin. We assume that this vector field can be written :

$$\begin{cases} \dot{x} = P_k(x,y) + o(||(x,y)||^k) \\ \dot{y} = Q_k(x,y) + o(||(x,y)||^k), \end{cases} \tag{4}$$

where P_k, Q_k are homogeneous polynomials of degree k, such that at least one of the two is non-zero ($(X,0)$ has a zero $(k-1)$-jet but a non-zero k-jet). The remainders are functions of class $\mathcal{C}^{2k+3}$.

The main result of this Section is the following one :

Theorem 3 *For an integer $k \geq 1$, let $(X,0)$ be a $\mathcal{C}^{2k+3}$ germ of vector field, such that the origin is an isolated singularity. Assume moreover that $j_{k-1}X(0) = 0$, but that $j_kX(0) \neq 0$, as expressed in (4). Then $c(X,0) \leq 2(k+1)$*

Proof

For this proof one will use some ideas introduced by of Bendixson in [B], for another aim that is recalled in 3.2. The germ is represented by a vector field X on W_0, as above.

(1) Ones first assumes that the polynomial $R(x,y) = xP_k(x,y) + yQ_k(x,y)$ is not zero. One considers the circles γ^ρ given by $\gamma^\rho(\theta) = (\rho\cos\theta, \rho\sin\theta)$, which are contained into W_0 for ρ small enough. Along these circles one chooses the orthogonal vector field $N(\theta) = \cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y}$. Taking into account the homogeneity of P_k, Q_k, the function $n^\rho(\theta) =< X(\gamma^\rho(\theta)), N(\theta) >$ is given by :

$$n^\rho(\theta) = \rho^{k+1}\Big(R(\cos(\theta),\sin(\theta)) + \rho h(\theta,\rho)\Big), \tag{5}$$

where the function $h(\theta,\rho)$ is of class $\mathcal{C}^{k+1}$ (this function is obtained through the division of a remainder of class $\mathcal{C}^{2k+3}$ in (θ,ρ) by the function ρ^{k+2}). The trigonometrical polynomial $R(\cos(\theta),\sin(\theta))$ is non-zero and homogeneous of degree $k+1$. Then, it has $2(k+1)$ zeros counted with their multiplicity. As the function $\theta \to R(\cos(\theta),\sin(\theta)) + \rho h(\theta,\rho)$ converges in the $\mathcal{C}^{k+1}$-topology towards $R(\cos(\theta),\sin(\theta))$ when $\rho \to 0$, we have that the function $n^\rho(\theta)$ has

also less than $2(k+1)$ zeros (counted with their multiplicity) for $0 < \rho < \rho_0$, where $\rho_0 > 0$ is small enough. For any ρ, such that $0 < \rho < \rho_0$, it is easy to approximate in the $\mathcal{C}^{k+1}$-topology, the curve γ^ρ by a curve with $2(k+1)$ quadratic (and then even) contact points. Then, such a curve is generic with less than $2(k+1)$ (even) contact points. By this way we can construct a sequence (γ_n) of generic curves, converging toward the origin in the Hausdorff sense and such that $c(\gamma_n) \leq 2(k+1)$. It follows from Proposition 2, that $c(X,0) \leq 2(k+1)$.

(2) Assume now that $R(x,y)$ is the zero polynomial. It follows from the polynomial identity $xP_k(x,y) \equiv yQ_k(x,y)$ that there exists a non-zero homogenuous polynomial S_{k-1} of degree $k-1$ such that $P_k(x,y) = -yS_{k-1}(x,y)$ and $Q_k(x,y) = xS_{k-1}(x,y)$. Taking fixed constants $a > 0$ and $b > 0$, with $a \neq b$, one replaces the circles used in case (1) by the ellipses $\gamma^\rho(\theta) = (a\rho\cos(\theta), b\rho\sin(\theta))$. Along γ^ρ, one takes the orthogonal field to γ^ρ : $N(\theta) = b\cos(\theta)\frac{\partial}{\partial x} + a\sin(\theta)\frac{\partial}{\partial y}$. Putting $c = \cos(\theta)$ and $s = \sin(\theta)$, the function $n^\rho(\theta)$ is now given by :

$$n^\rho(\theta) = \rho^{k+1}\Big(bcP_k(c,s) + asQ_k(c,s) + \rho h(\theta,\rho)\Big)\Big)$$

which reduces to :

$$n^\rho(\theta) = \rho^{k+1}\Big((a-b)csS_{k-1}(c,s) + \rho h(\theta,\rho)\Big) \tag{6}$$

As the trigonomical polynomial $(a-b)csS_{k-1}(c,s)$ is non-zero and homogeneous of degree $k+1$, one can finish the proof as in case (1). ∎

Remark 10 *In the proof of Theorem 3, one obtains the existence of a sequence γ_n of generic curves with a contact index bounded by $C = 2(k+1)$, converging toward the origin. For $n \to \infty$, their shape tends to be circular. But the curves γ_n are not absolute minimal curves and Proposition 2 gives no information about the asymptotic shape of absolute minimal curves converging toward the origin. This asymptotic shape could be obtained using the desingularization theory of [D], in case of a smooth vector field verifying a Łojasiewicz condition. In this case, absolute minimal curves converging toward the origin can be obtained as direct images through the desingularization mapping, of curves converging toward the critical locus of the desingularization. For instance, for a smooth germ $\frac{1}{4}y\frac{\partial}{\partial x} - \frac{1}{2}x^3\frac{\partial}{\partial y} + \cdots$, with the origin as an algebraically isolated singularity, a ρ-family of absolute minimal curves is given by $\{ax^4 + by^2 = \rho^2\}$, for $a \neq b$ and $\rho \to 0$. They are associated to the desingularization mapping $(\bar{x}, \bar{y}, u) \in S^1 \times I\!R^+ \to (u\bar{x}, u^2\bar{y}) \in I\!R^2$.*

3.2. **Some results of Bendixson**

Let Γ be an orbit image of $\varphi(t)$, trajectory of a vector field X, which tends

toward $0 \in I\!R^2$ for $t \to +\infty$ (if the limit is for $t \to -\infty$ we can make the same considerations, changing X by $-X$). To study the limit, one can work in polar coordinates (r, θ) where the flow is given by $\varphi(t) = (r(t), \theta(t))$. To say that $\varphi(t)$ tends to $0 \in I\!R^2$ is just to say that $r(t) \to 0$. depending on the behavior of $\theta(t)$ we have different type of trajectory :

Definition 15 *We consider that $t \to +\infty$. If $\theta(t) \to \theta_0 \in S^1$ the orbit γ is said an (attracting) characteristic orbit. If $\theta(t) \to +\infty$ or $-\infty$ (in the universal covering of S^1) the orbit is said an (attracting) spiral in the direct or the clock-ward direction. If all points of a neighborhood of the origin the orbits are attracting spirals the field germ is said to be an attracting focus (spiraling in the direct or clock-ward direction). If the orbits by all the points of neighborhood of the origin are attracting characteristic orbits (with perhaps different angle limits) the field germ is said to be an attracting node.*

We have the repulsing notions (repulsing spiral, charasteristic orbit, focus, node) by changing X in $-X$.

Remark 11 *A characteristic orbit is an orbit with a limit tangent $\{\theta = \theta_0\}$. This terminology was not used by Bendixson in [B] but was used by Dumortier in [D]. In fact, for Bendixson, characteristic curve is synonymous of orbit.*

The behavior of an orbit which tends toward the origin may be much wilder than the behaviors described in Definition 15. But, we can found in [B] a result which can be restated for differentiable fields as follows :

Proposition 10 *For an integer $k \geq 1$, we consider a germ of $\mathcal{C}^{2k+3}$ vector field at the origin with a non-zero k-jet as in (4) and such that the origin is isolated. If $\varphi(t)$ is a trajectory tending toward the origin when t tends toward $+\infty$ or toward $-\infty$, then the corresponding orbit is a spiral or a charasteristic orbit.*

In order to obtain this result, Bendixson proved, that for an orbit tending toward the origin, the angular component $\theta(t)$ must have a limit, finite or infinite. For proving this, he used for the tangential component $-yP + xQ$ the ideas that I have adapted to the radial component in the proof of Theorem 3 above. Also, as I have mentioned above, Bendixson supposed that the vector field is analytic and that the origin is algebraically isolated, but the assumptions made in Proposition 10 are sufficient. Another result of [B] can be restated as follows :

Proposition 11 *Consider a germ of differentiable vector field as in Proposition 10. If there is at least one characteristic orbit , then every other orbit tending toward the origin (for positive or negative times) is also a characteristic orbit.*

3.3. **Proof of the Main Theorem**

The proof is a direct consequence of the previous results. Applying Theorem 3 we have that $c(X,0) \leq 2(k+1)$. As the origin is supposed to be isolated from periodic orbits, we can also apply Theorem 2 : the germ $(X,0)$ is a topological node or admits a sectoral decomposition with less than $2(k+1)$ sectors. It follows from Propositions 10 and 11 that a topological node for a differentiable germ with non-zero jet, as in the statement of Theorem 1, is a focus or a node and that, in the case of a non trivial sectoral decomposition, each orbit tending toward the origin is a characteristic orbit.

To estimate the index $i(X,0)$ and the number of different phase portraits, we can apply Proposition 9. As $c(X,0) \leq 2(k+1)$, we obtain that $-k \leq i(X,0) \leq k+2$ and that the number of different phase portraits is less than $2^{4(k+1)}$. ∎

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