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# A Theorem of Fermat on Congruent Number Curves

Lorenz Halbeisen and Norbert Hungerbühler

*To the memory of S. Srinivasan*

**Abstract.** A positive integer  $A$  is called a *congruent number* if  $A$  is the area of a right-angled triangle with three rational sides. Equivalently,  $A$  is a *congruent number* if and only if the congruent number curve  $y^2 = x^3 - A^2x$  has a rational point  $(x, y) \in \mathbb{Q}^2$  with  $y \neq 0$ . Using a theorem of Fermat, we give an elementary proof for the fact that congruent number curves do not contain rational points of finite order.

**Keywords.** congruent numbers, Pythagorean triples.

**2010 Mathematics Subject Classification.** primary 11G05; secondary 11D25.

## 1. Introduction

A positive integer  $A$  is called a **congruent number** if  $A$  is the area of a right-angled triangle with three rational sides. So,  $A$  is congruent if and only if there exists a rational Pythagorean triple  $(a, b, c)$  (i.e.,  $a, b, c \in \mathbb{Q}$ ,  $a^2 + b^2 = c^2$ , and  $ab \neq 0$ ), such that  $\frac{ab}{2} = A$ . The sequence of integer congruent numbers starts with

$$5, 6, 7, 13, 14, 15, 20, 21, 22, 23, 24, 28, 29, 30, 31, 34, 37, \dots$$

For example,  $A = 7$  is a congruent number, witnessed by the rational Pythagorean triple

$$\left(\frac{24}{5}, \frac{35}{12}, \frac{337}{60}\right).$$

It is well-known that  $A$  is a congruent number if and only if the cubic curve

$$C_A : y^2 = x^3 - A^2x$$

has a rational point  $(x_0, y_0)$  with  $y_0 \neq 0$ . The cubic curve  $C_A$  is called a **congruent number curve**. This correspondence between rational points on congruent number curves and rational Pythagorean triples can be made explicit as follows: Let

$$C(\mathbb{Q}) := \{(x, y, A) \in \mathbb{Q} \times \mathbb{Q}^* \times \mathbb{Z}^* : y^2 = x^3 - A^2x\},$$

where  $\mathbb{Q}^* := \mathbb{Q} \setminus \{0\}$ ,  $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$ , and

$$P(\mathbb{Q}) := \{(a, b, c, A) \in \mathbb{Q}^3 \times \mathbb{Z}^* : a^2 + b^2 = c^2 \text{ and } ab = 2A\}.$$

Then, it is easy to check that

$$\begin{aligned} \psi : P(\mathbb{Q}) &\rightarrow C(\mathbb{Q}) \\ (a, b, c, A) &\mapsto \left(\frac{A(b+c)}{a}, \frac{2A^2(b+c)}{a^2}, A\right) \end{aligned} \tag{1.1}$$

is bijective and

$$\begin{aligned} \psi^{-1} : \quad C(\mathbb{Q}) &\rightarrow P(\mathbb{Q}) \\ (x, y, A) &\mapsto \left( \frac{2xA}{y}, \frac{x^2 - A^2}{y}, \frac{x^2 + A^2}{y}, A \right). \end{aligned} \quad (1.2)$$

For positive integers  $A$ , a triple  $(a, b, c)$  of non-zero rational numbers is called a **rational Pythagorean  $A$ -triple** if  $a^2 + b^2 = c^2$  and  $A = \left| \frac{ab}{2} \right|$ . Notice that if  $(a, b, c)$  is a rational Pythagorean  $A$ -triple, then  $A$  is a congruent number and  $|a|, |b|, |c|$  are the lengths of the sides of a right-angled triangle with area  $A$ . Notice also that we allow  $a, b, c$  to be negative.

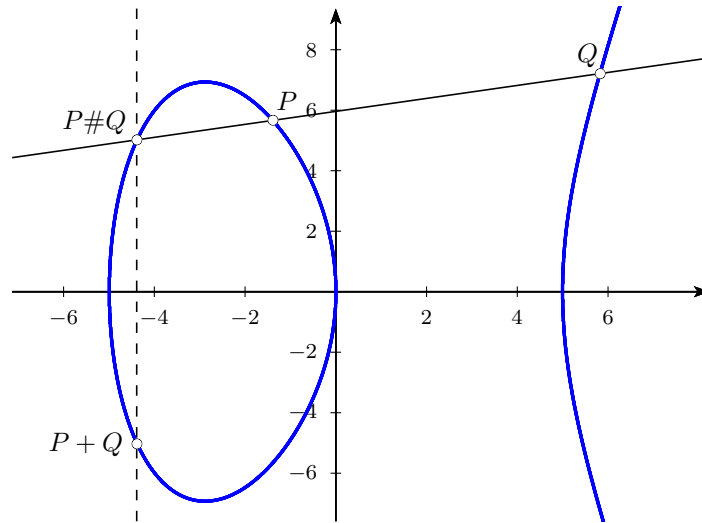
It is convenient to consider the curve  $C_A$  in the projective plane  $\mathbb{R}P^2$ , where the curve is given by

$$C_A : y^2z = x^3 - A^2xz^2.$$

On the points of  $C_A$ , one can define a commutative, binary, associative operation “+”, where  $\mathcal{O}$ , the neutral element of the operation, is the projective point  $(0, 1, 0)$  at infinity. More formally, if  $P$  and  $Q$  are two points on  $C_A$ , then let  $P\#Q$  be the third intersection point of the line through  $P$  and  $Q$  with the curve  $C_A$ . If  $P = Q$ , the line through  $P$  and  $Q$  is replaced by the tangent in  $P$ . Then  $P + Q$  is defined by stipulating

$$P + Q := \mathcal{O}\#(P\#Q),$$

where for a point  $R$  on  $C_A$ ,  $\mathcal{O}\#R$  is the point reflected across the  $x$ -axis. The following figure shows the congruent number curve  $C_A$  for  $A = 5$ , together with two points  $P$  and  $Q$  and their sum  $P + Q$ .



More formally, for two points  $P = (x_0, y_0)$  and  $Q = (x_1, y_1)$  on a congruent number curve  $C_A$ , the point  $P + Q = (x_2, y_2)$  is given by the following formulas:

- If  $x_0 \neq x_1$ , then

$$x_2 = \lambda^2 - x_0 - x_1, \quad y_2 = \lambda(x_0 - x_2) - y_0,$$

where

$$\lambda := \frac{y_1 - y_0}{x_1 - x_0}.$$

- If  $P = Q$ , i.e.,  $x_0 = x_1$  and  $y_0 = y_1$ , then

$$x_2 = \lambda^2 - 2x_0, \quad y_2 = 3x_0\lambda - \lambda^3 - y_0, \quad (1.3)$$

where

$$\lambda := \frac{3x_0^2 - A^2}{2y_0}. \quad (1.4)$$

Below we shall write  $2 * P$  instead of  $P + P$ .

- If  $x_0 = x_1$  and  $y_0 = -y_1$ , then  $P + Q := \mathcal{O}$ . In particular,  $(0, 0) + (0, 0) = (A, 0) + (A, 0) = (-A, 0) + (-A, 0) = \mathcal{O}$ .
- Finally, we define  $\mathcal{O} + P := P$  and  $P + \mathcal{O} := P$  for any point  $P$ , in particular,  $\mathcal{O} + \mathcal{O} = \mathcal{O}$ .

With the operation “+”,  $(C_A, +)$  is an abelian group with neutral element  $\mathcal{O}$ . Let  $C_A(\mathbb{Q})$  be the set of rational points on  $C_A$  together with  $\mathcal{O}$ . It is easy to see that  $(C_A(\mathbb{Q}), +)$  is a subgroup of  $(C_A, +)$ . Moreover, it is well known that the group  $(C_A(\mathbb{Q}), +)$  is finitely generated. One can readily check that the three points  $(0, 0)$  and  $(\pm A, 0)$  are the only points on  $C_A$  of order 2, and one easily finds other points of finite order on  $C_A$ . But do we find also rational points of finite order on  $C_A$ ? This question is answered by the following

**Theorem 1.** *If  $A$  is a congruent number and  $(x_0, y_0)$  is a rational point on  $C_A$  with  $y_0 \neq 0$ , then the order of  $(x_0, y_0)$  is infinite. In particular, if there exists one rational Pythagorean  $A$ -triple, then there exist infinitely many such triples.*

The usual proofs of Theorem 1 are quite involved. For example, Koblitz [Kob93, Ch.I, §9, Prop. 17] gives a proof using Dirichlet’s theorem on primes in an arithmetic progression, and in Chahal [Cha06, Thm. 3], a proof is given using the Lutz-Nagell theorem, which states that rational points of finite order are integral. However, both results, Dirichlet’s theorem and the Lutz-Nagell theorem, are quite deep results, and the aim of this article is to provide a simple proof of Theorem 1 which relies on an elementary theorem of Fermat.

## 2. A Theorem of Fermat

In [Fer1670], Fermat gives an algorithm to construct different right-angled triangles with three rational sides having the same area (see also Hungerbühler [Hun96]). Moreover, Fermat claims that his algorithm yields infinitely many distinct such right-angled triangles. However, he did not provide a proof for this claim. In this section, we first present Fermat’s algorithm and then we show that this algorithm delivers infinitely many pairwise distinct rational right-angled triangles of the same area.

**Fermat’s Algorithm 2.** *Assume that  $A$  is a congruent number, and that  $(a_0, b_0, c_0)$  is a rational Pythagorean  $A$ -triple, i.e.,  $A = \left| \frac{a_0 b_0}{2} \right|$ . Then*

$$a_1 := \frac{4c_0^2 a_0 b_0}{2c_0(a_0^2 - b_0^2)}, \quad b_1 := \frac{c_0^4 - 4a_0^2 b_0^2}{2c_0(a_0^2 - b_0^2)}, \quad c_1 := \frac{c_0^4 + 4a_0^2 b_0^2}{2c_0(a_0^2 - b_0^2)}, \quad (2.5)$$

*is also a rational Pythagorean  $A$ -triple. Moreover,  $a_0 b_0 = a_1 b_1$ , i.e., if  $(a_0, b_0, c_0, A) \in P(\mathbb{Q})$ , then  $(a_1, b_1, c_1, A) \in P(\mathbb{Q})$ .*

*Proof.* Let  $m := c_0^2$ , let  $n := 2a_0 b_0$ , and let

$$X := 2mn, \quad Y := m^2 - n^2, \quad Z := m^2 + n^2,$$

in other words,

$$X = 4c_0^2 a_0 b_0, \quad Y = c_0^4 - 4a_0^2 b_0^2, \quad Z = c_0^4 + 4a_0^2 b_0^2.$$

Then obviously,  $X^2 + Y^2 = Z^2$ , and since  $a_0, b_0, c_0 \in \mathbb{Q}$ ,  $(|X|, |Y|, |Z|)$  is a rational Pythagorean triple, where the area of the corresponding right-angled triangle is

$$\tilde{A} = \left| \frac{XY}{2} \right| = |2a_0 b_0 c_0^2 (c_0^4 - 4a_0^2 b_0^2)|.$$

Since  $a_0^2 + b_0^2 = c_0^2$ , we get  $c_0^4 = (a_0^2 + b_0^2)^2 = a_0^4 + 2a_0^2 b_0^2 + b_0^4$  and therefore

$$c_0^4 - 4a_0^2 b_0^2 = a_0^4 - 2a_0^2 b_0^2 + b_0^4 = (a_0^2 - b_0^2)^2 > 0.$$

So, for

$$a_1 = \frac{X}{2c_0(a_0^2 - b_0^2)}, \quad b_1 = \frac{Y}{2c_0(a_0^2 - b_0^2)}, \quad c_1 = \frac{Z}{2c_0(a_0^2 - b_0^2)},$$

we have  $a_1^2 + b_1^2 = c_1^2$  and

$$\frac{a_1 b_1}{2} = \frac{XY}{2 \cdot 4c_0^2(a_0^2 - b_0^2)^2} = \frac{2a_0 b_0 c_0^2(c_0^4 - 4a_0^2 b_0^2)}{4c_0^2(a_0^2 - b_0^2)^2} = \frac{2a_0 b_0 c_0^2(a_0^2 - b_0^2)^2}{4c_0^2(a_0^2 - b_0^2)^2} = \frac{a_0 b_0}{2}.$$

□

**Theorem 3.** *Assume that  $A$  is a congruent number, that  $(a_0, b_0, c_0)$  is a rational Pythagorean  $A$ -triple, and for positive integers  $n$ , let  $(a_n, b_n, c_n)$  be the rational Pythagorean  $A$ -triple we obtain by Fermat's Algorithm from  $(a_{n-1}, b_{n-1}, c_{n-1})$ . Then for any distinct non-negative integers  $n, n'$ , we have  $|c_n| \neq |c_{n'}|$ .*

*Proof.* Let  $n$  be an arbitrary but fixed non-negative integer. Since  $A = \left| \frac{a_n b_n}{2} \right|$ , we have  $2A = |a_n b_n|$ , and consequently

$$a_n^2 b_n^2 = 4A^2. \quad (2.6)$$

Furthermore, since  $a_n^2 + b_n^2 = c_n^2$ , we have

$$(a_n^2 + b_n^2)^2 = a_n^4 + 2a_n^2 b_n^2 + b_n^4 = a_n^4 + 8A^2 + b_n^4 = c_n^4,$$

and consequently we get

$$c_n^4 - 16A^2 = a_n^4 - 8A^2 + b_n^4 = a_n^4 - 2a_n^2 b_n^2 + b_n^4 = (a_n^2 - b_n^2)^2 > 0.$$

Therefore,

$$\sqrt{(a_n^2 - b_n^2)^2} = |a_n^2 - b_n^2| = \sqrt{c_n^4 - 16A^2},$$

and with (2.5) and (2.6) we finally have

$$|c_{n+1}| = \frac{c_n^4 + 16A^2}{2c_n \sqrt{c_n^4 - 16A^2}}.$$

Now, assume that  $c_n = \frac{u}{v}$  where  $u$  and  $v$  are in lowest terms. We consider the following two cases:

*$u$  is odd:* First, we write  $v = 2^k \cdot \tilde{v}$ , where  $k \geq 0$  and  $\tilde{v}$  is odd. In particular,  $c_n = \frac{u}{2^k \cdot \tilde{v}}$ . Since  $c_{n+1}$  is rational,  $\sqrt{c_n^4 - 16A^2} \in \mathbb{Q}$ . So,

$$\sqrt{c_n^4 - 16A^2} = \sqrt{\frac{u^4 - 16A^2 v^4}{v^4}} = \frac{\tilde{u}}{v^2}$$

for a positive odd integer  $\tilde{u}$ . Then

$$|c_{n+1}| = \frac{\frac{u^4 + 16A^2 v^4}{v^4}}{\frac{2u\tilde{u}}{v^3}} = \frac{\bar{u}}{2u\tilde{u}v} = \frac{\bar{u}}{2u\tilde{u}2^k \tilde{v}} = \frac{\bar{u}}{2^{k+1} u \tilde{u} \tilde{v}} = \frac{u'}{2^{k+1} \cdot v'}$$

where  $\bar{u}, u', v'$  are odd integers and  $\gcd(u', v') = 1$ . This shows that

$$c_n = \frac{u}{2^k \cdot \tilde{v}} \quad \Rightarrow \quad |c_{n+1}| = \frac{u'}{2^{k+1} \cdot v'}$$

where  $u, \tilde{v}, u', v'$  are odd.

*u is even:* First, we write  $u = 2^k \cdot \tilde{u}$ , where  $k \geq 1$  and  $\tilde{u}$  is odd. In particular,  $c_n = \frac{2^k \cdot \tilde{u}}{v}$ , where  $v$  is odd. Similarly,  $A = 2^l \cdot \tilde{A}$ , where  $l \geq 0$  and  $\tilde{A}$  is odd. Then

$$c_n^4 \pm 16A^2 = \frac{2^{4k} \cdot \tilde{u}^4 \pm 2^{4+2l} \tilde{A}^2 v^4}{v^4},$$

where both numbers are of the form

$$\frac{2^{2m} \bar{u}}{v^4},$$

where  $\bar{u}$  is odd and  $4 \leq 2m \leq 4k$ , i.e.,  $2 \leq m \leq 2k$ . Therefore,

$$|c_{n+1}| = \frac{2^{2m} u_0 \cdot v^3}{2 \cdot 2^k \tilde{u} \cdot v^4 \cdot 2^m u_1} = \frac{2^{m-k-1} \cdot u'}{v'},$$

where  $u_0, u_1, u', v'$  are odd. Since  $m < 2k + 1$ , we have  $m - k - 1 < k$ , and therefore we obtain

$$c_n = \frac{2^k \cdot \tilde{u}}{v} \quad \Rightarrow \quad |c_{n+1}| = \frac{2^{k'} \cdot u'}{v'}$$

where  $\tilde{u}, v, u', v'$  are odd and  $0 \leq k' < k$ .

Both cases together show that whenever  $c_n = 2^k \cdot \frac{u}{v}$ , where  $k \in \mathbb{Z}$  and  $u, v$  are odd, then  $|c_{n+1}| = 2^{k'} \cdot \frac{u'}{v'}$ , where  $u', v'$  are odd and  $k' < k$ . So, for any distinct non-negative integers  $n$  and  $n'$ ,  $|c_n| \neq |c_{n+1}|$ .  $\square$

The proof of Theorem 3 gives us the following reformulation of Fermat's Algorithm:

**Corollary 4.** *Assume that  $A$  is a congruent number, and that  $(a_0, b_0, c_0)$  is a rational Pythagorean  $A$ -triple, i.e.,  $A = \left| \frac{a_0 b_0}{2} \right|$ . Then*

$$a_1 = \frac{4Ac_0}{\sqrt{c_0^4 - 16A^2}}, \quad b_1 = \frac{\sqrt{c_0^4 - 16A^2}}{2c_0}, \quad c_1 = \frac{c_0^4 + 16A^2}{2c_0 \sqrt{c_0^4 - 16A^2}},$$

*is also a rational Pythagorean  $A$ -triple.*

*Proof.* Notice that  $c_0^4 - 4a_0^2 b_0^2 = c_0^4 - 16A^2$  and recall that  $|a_0^2 - b_0^2| = \sqrt{c_0^4 - 16A^2}$ .  $\square$

### 3. Doubling points with Fermat's Algorithm

Before we prove Theorem 1 (i.e., that congruent number curves do not contain rational points of finite order), we first prove that Fermat's Algorithm 2 is essentially doubling points on congruent number curves.

**Lemma 5.** *Let  $A$  be a congruent number, let  $(a_0, b_0, c_0)$  be a rational Pythagorean  $A$ -triple, and let  $(a_1, b_1, c_1)$  be the rational Pythagorean  $A$ -triple obtained by Fermat's Algorithm from  $(a_0, b_0, c_0)$ . Furthermore, let  $(x_0, y_0)$  and  $(x_1, y_1)$  be the rational points on the curve  $C_A$  which correspond to  $(a_0, b_0, c_0)$  and  $(a_1, b_1, c_1)$ , respectively. Then we have*

$$2 * (x_0, y_0) = (x_1, -y_1).$$

*Proof.* Let  $(a_0, b_0, c_0)$  be a rational Pythagorean  $A$ -triple. Then, according to (2.5), the rational Pythagorean  $A$ -triple  $(a_1, b_1, c_1)$  which we obtain by Fermat's Algorithm is given by

$$a_1 := \frac{4c_0^2 a_0 b_0}{2c_0(a_0^2 - b_0^2)}, \quad b_1 := \frac{c_0^4 - 4a_0^2 b_0^2}{2c_0(a_0^2 - b_0^2)}, \quad c_1 := \frac{c_0^4 + 4a_0^2 b_0^2}{2c_0(a_0^2 - b_0^2)}.$$

Now, by (1.1), the coordinates of the rational point  $(x_1, y_1)$  on  $C_A$  which corresponds to the rational Pythagorean  $A$ -triple  $(a_1, b_1, c_1)$  are given by

$$\begin{aligned} x_1 &= \frac{a_0 b_0 \cdot (b_1 + c_1)}{2 \cdot a_1} = \frac{a_0 b_0 \cdot 2c_0^4}{2 \cdot 4c_0^2 a_0 b_0} = \frac{c_0^2}{4}, \\ y_1 &= \frac{2\left(\frac{a_0 b_0}{2}\right)^2 (b_1 + c_1)}{a_1^2} = \frac{1}{8}(a_0^2 - b_0^2)c_0. \end{aligned}$$

Let still  $(a_0, b_0, c_0)$  be a rational Pythagorean  $A$ -triple. Then, again by (1.1), the corresponding rational point  $(x_0, y_0)$  on  $C_A$  is given by

$$x_0 = \frac{b_0(b_0 + c_0)}{2}, \quad y_0 = \frac{b_0^2(b_0 + c_0)}{2}.$$

Now, as we have seen in (1.3) and (1.4), the coordinates of the point  $(x'_1, y'_1) := 2 * (x_0, y_0)$  are given by  $x'_1 = \lambda^2 - 2x_0$ ,  $y'_1 = 3x_0\lambda - \lambda^3 - y_0$ , where

$$\begin{aligned} \lambda &= \frac{3x_0^2 - \left(\frac{a_0 b_0}{2}\right)^2}{2y_0} = \frac{\frac{3b_0^2(b_0+c_0)^2 - a_0^2 b_0^2}{4}}{b_0^2(b_0+c_0)} = \frac{3(b_0+c_0)^2 - a_0^2}{4(b_0+c_0)} = \frac{3(b_0+c_0)^2 + (b_0^2 - c_0^2)}{4(b_0+c_0)} = \\ &= \frac{(3b_0^2 + 6b_0c_0 + 3c_0^2) + (b_0^2 - c_0^2)}{4(b_0+c_0)} = \frac{4b_0^2 + 6b_0c_0 + 2c_0^2}{4(b_0+c_0)} = \frac{2b_0^2 + 3b_0c_0 + c_0^2}{2(b_0+c_0)} = \\ &= \frac{(2b_0+c_0)(b_0+c_0)}{2(b_0+c_0)} = \frac{(2b_0+c_0)}{2}. \end{aligned}$$

Hence,

$$x'_1 = \lambda^2 - 2x_0 = \frac{(2b_0+c_0)^2}{4} - b_0(b_0+c_0) = \frac{(4b_0^2 + 4b_0c_0 + c_0^2) - (4b_0^2 + 4b_0c_0)}{4} = \frac{c_0^2}{4}$$

and

$$y'_1 = 3x_0\lambda - \lambda^3 - y_0 = \frac{1}{8}(2b_0^2c_0 - c_0^3) = \frac{1}{8}(b_0^2 - a_0^2)c_0,$$

i.e.,  $x_1 = x'_1$  and  $y_1 = -y'_1$ , as claimed.  $\square$

With Lemma 5, we are now able to prove Theorem 1, which states that for a congruent number  $A$ , the curve  $C_A : y^2 = x^3 - A^2x$  does not have rational points of finite order other than  $(0, 0)$  and  $(\pm A, 0)$ .

*Proof of Theorem 1.* Assume that  $A$  is a congruent number, let  $(x_0, y_0)$  be a rational point on  $C_A$  which  $y_0 \neq 0$ , and let  $(a_0, b_0, c_0)$  be the rational Pythagorean  $A$ -triple which corresponds to  $(x_0, y_0)$  by (1.2). Furthermore, for positive integers  $n$ , let  $(a_n, b_n, c_n)$  be the rational Pythagorean  $A$ -triple we obtain by Fermat's Algorithm from  $(a_{n-1}, b_{n-1}, c_{n-1})$ , and let  $(x_n, y_n)$  be the rational point on  $C_A$  which corresponds to the rational Pythagorean  $A$ -triple  $(a_n, b_n, c_n)$  by (1.1).

By the proof of Lemma 5 we know that the  $x$ -coordinate of  $2 * (x_n, y_n)$  is equal to  $\frac{c_n^2}{4}$ , and by Theorem 3 we have that for any distinct non-negative integers  $n, n'$ ,  $|c_n| \neq |c_{n'}|$ . Hence, for all distinct non-negative integers  $n, n'$  we have

$$(x_n, y_n) \neq (x_{n'}, y_{n'}),$$

which shows that the order of  $(x_0, y_0)$  is infinite.  $\square$

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