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Penalisation techniques for one-dimensional reflected rough differential equations

Alexandre Richard* Etienne Tanré † Soledad Torres ‡

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Abstract

In this paper we solve real-valued rough differential equations (RDEs) reflected on a rough boundary. The solution Y is constructed as the limit of a sequence $(Y^n)_{n \in \mathbb{N}}$ of solutions to RDEs with unbounded drifts $(\psi_n)_{n \in \mathbb{N}}$. The penalisation ψ_n increases with n . Along the way, we thus also provide an existence theorem and a Doss-Sussmann representation for RDEs with a drift growing at most linearly. In addition, a speed of convergence of the sequence of penalised paths to the reflected solution is obtained.

We finally use the penalisation method to prove that under some conditions, the law of a reflected Gaussian RDE at time $t > 0$ is absolutely continuous with respect to the Lebesgue measure.

Key words: Reflected rough differential equation; Penalisation; Gaussian noise; Skorokhod problem.

MSC2010 Subject Classification: 34F05, 60G15, 60H10.

1 Introduction

Solving (stochastic) differential equations with a reflecting boundary condition is by now a classical problem. For a domain $D \subseteq \mathbb{R}^e$, a mapping $\sigma : \mathbb{R}^e \rightarrow \mathbb{R}^{e \times d}$, an initial value $y_0 \in \overline{D}$ and an \mathbb{R}^d -valued path $X = \{X_t\}_{t \in [0, T]}$ sometimes referred as the noise, this problem consists formally in finding \mathbb{R}^e -valued paths $\{Y_t\}_{t \in [0, T]}$ and $\{K_t\}_{t \in [0, T]}$ such that $\forall t \in [0, T]$,

$$\begin{aligned} Y_t &= y_0 + \int_0^t \sigma(Y_s) dX_s + K_t, \\ Y_t &\in \overline{D}, \quad |K|_T < \infty, \\ |K|_t &= \int_0^t \mathbf{1}_{\{X_s \in \partial D\}} d|K|_s \quad \text{and} \quad K_t = \int_0^t n(X_s) d|K|_s, \end{aligned}$$

where $|K|_t$ is the finite variation of K on $[0, t]$ and $n(x)$ is the unit inward normal of ∂D at x . If X is a Brownian motion and the integral is in the sense of Itô, this problem was first studied by Skorokhod [28], and then by McKean [24], El Karoui [10], Lions and Sznitman [22], to name but a few. For this reason, it is called the Skorokhod problem associated to X , σ and D (see Definition 2.9).

In the last few years, this problem has attracted a lot of attention when the driver X is a β -Hölder continuous path: in the “regular” case $\beta \in (\frac{1}{2}, 1)$, existence of a solution has been established in a multidimensional setting by Ferrante and Rovira [13] and uniqueness was then obtained by Falkowski and Słomiński [12]. In that case, the integral can be constructed by a Riemann sum approximation and is known as a Young integral [32]. Extensions of these results to the “irregular” case $\beta < \frac{1}{2}$ can be handled with rough paths. We recall that this theory was initiated by Lyons [23] and for a (multidimensional) β -Hölder continuous path X and σ a bounded vector field, it provides a way to solve the equation $dY_t = \sigma(Y_t) d\mathbf{X}_t$, where $\mathbf{X} = (X, \mathbb{X})$ is the path X with a supplementary two-parameters path \mathbb{X} (in fact higher order correction terms such as \mathbb{X} are needed

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if $\beta \leq \frac{1}{3}$, but we shall assume $\beta > \frac{1}{3}$ for simplicity). Solutions can be understood either as a limit of ODEs driven by a smooth driver X^k which converges to X ([17, Chap. 10]), or directly as an equality between Y_t and $\int_0^t \sigma(Y_s) d\mathbf{X}_s$ when this integral is defined in the sense of controlled rough paths [15, 18] (alternative approaches include the original definition of Lyons [23] and the one of Davie [6]). In this paper both notions will be useful and shown to coincide for the penalised RDEs. Existence of solutions of reflected RDEs with $\beta \in (\frac{1}{3}, \frac{1}{2})$ was proven by Aida [1] and Castaing, Marie, and Raynaud de Fitte [5] under slightly different conditions. While Deya, Gubinelli, Hofmanová, and Tindel [7] proved uniqueness for a one-dimensional path reflected on the horizontal line. In those works, the existence is obtained through Wong-Zakai or Euler-type approximations, assuming that the boundary is either a convex or sufficiently smooth set, or a hyperplane. On the other hand in the Brownian noise setting, the reflected solutions have often been constructed by a penalisation procedure (see in particular [11, 22, 30]). The present approach extends this classical penalisation technique to rough paths and covers the case of rough boundaries.

We focus on one-dimensional ($e = 1$) solutions to rough differential equations which are reflected on a moving boundary $L : [0, T] \rightarrow \mathbb{R}$, where the driver is a d -dimensional rough path \mathbf{X} with Hölder regularity $\beta \in (\frac{1}{3}, 1)$ (note that by a slight abuse of notations, we may use \mathbf{X} for X and the vocabulary of rough paths even in the smooth case). Following a classical method for reflected (stochastic) differential equations, we consider the following sequence of penalised RDEs with drift:

$$Y_t^n = y_0 + n \int_0^t (Y_s^n - L_s)_- ds + \int_0^t \sigma(Y_s^n) d\mathbf{X}_s. \quad (1.1)$$

For technical reasons, the drift function $n(\cdot)_-$ will be replaced by a smoother function ψ_n with at most linear growth, the interpretation remaining that of a stronger and stronger force pushing Y^n above L . But unlike classical ODEs and to some extent SDEs, solving RDEs with unbounded coefficients is known to be tricky [2, 20, 21]. However, in case only the drift is unbounded (smooth and at most linearly growing) and σ is smooth and bounded, Riedel and Scheutzow [27] proved the existence of a semiflow of solutions. We propose an alternative approach, without considering the flow but only the solution, based on an extension of a result of Friz and Oberhauser [14]. That is we prove that any RDE with drift having a bounded derivative has a unique global solution, and that it has a Doss-Sussmann-like representation [9, 29]. This last property turns to be extremely useful as it allows to transport the monotonicity of $\psi_n \leq \psi_{n+1}$ to the penalised solution, leading to $Y^n \leq Y^{n+1}$. We are then able to prove the uniform convergence of Y^n and $K^n := \int_0^\cdot \psi_n(Y_s^n - L_s) ds$ to Y and K , which are then identified as the solution to the Skorokhod problem described above. This reads (recall we assumed $e = 1$):

$$Y_t = y_0 + \int_0^t \sigma(Y_s) d\mathbf{X}_s + K_t \quad \text{and} \quad Y_t \geq L_t, \quad t \in [0, T], \quad (1.2)$$

and the non-decreasing path K increases only when Y hits L . Here, the reflection term also reads $K_t = \sup_{s \leq t} ((L_s - y_0 - \int_0^s \sigma(Y_u) d\mathbf{X}_u) \vee 0)$. Besides, when \mathbf{X} is a Gaussian rough path, the convergence of the sequence of penalised processes also happens uniformly in $L^\gamma(\Omega)$, $\gamma \geq 1$. The uniqueness of Y as the solution to the RDE with vector field σ and reflected on L follows from Deya, Gubinelli, Hofmanová, and Tindel [7, Th. 9], and the extension here to a non-constant boundary L bears no additional difficulty. Interestingly, it relies on a rough Gronwall lemma introduced in [8]. We provide a new application of this rough Gronwall lemma, obtaining a rate of convergence in the previous results. Namely, we obtain that the uniform distance between Y^n and Y is at most of order $n^{-\beta}$, where $\beta \in (\frac{1}{3}, 1)$ is the regularity of the driving signal. Up to a logarithmic factor, this result extends the optimal rate obtained in the Brownian framework by Słomiński [30].

The penalisation approach is a natural technique to solve reflected (ordinary, stochastic or rough) differential equations, and it also has fruitful applications to the study of the probabilistic properties of the solution. As an example, we prove that if σ is constant and if the noise is a fractional Brownian motion with Hurst parameter $H \in [\frac{1}{2}, 1)$, then at each time $t > 0$ the law of the solution Y_t restricted to $(0, \infty)$ is absolutely continuous with respect to the Lebesgue measure. We expect to carry further investigations in this direction to relax the assumption on σ and to get properties of the density.

Organisation of the paper. In Section 2, a brief overview of rough paths definitions and techniques is presented, followed by a set of precise assumptions and the statement of our main results. Then the existence

of a solution to the penalised equation is proven in Section 3, followed by some penalisation estimates. Most of the proofs that lead to the convergence of the penalised sequence to the reflected solution (Theorems 2.12 and 2.13) are contained in Section 4: first it is proven that Y^n and K^n converge uniformly (we show monotone convergence of Y^n towards a continuous limit), then that Y is controlled by X in the rough paths sense, which permits to use rough paths continuity theorems to show that Y and K solve the Skorokhod problem. In Section 5, we prove Theorem 2.14 which gives a rate of convergence of the sequence of penalised paths to the reflected solution, as well as Theorem 2.15 which gives a probabilistic estimate of the aforementioned rate. In Section 6, after recalling a few facts concerning Malliavin calculus and fractional Brownian motion, we prove that the reflected process with constant diffusion coefficient and driven by fractional noise admits a density at each time $t > 0$ (Theorem 2.16). Eventually, the proof of existence of solutions for RDEs with unbounded drift (Proposition 2.11) can be found in Appendix A.

Notations. C is a constant that may vary from line to line. For $k \in \mathbb{N}$ and $T > 0$, $\mathcal{C}_b^k([0, T]; F)$ (or simply \mathcal{C}_b^k) denotes the space of bounded functions which are k times continuously differentiable with bounded derivatives, with values in some linear space F . If E and F are two Banach spaces, $\mathcal{L}(E, F)$ denotes the space of continuous linear mappings from E to F . In the special case $E = \mathbb{R}^d$ and $F = \mathbb{R}$, we also write $(\mathbb{R}^d)'$ to denote the space of linear forms on \mathbb{R}^d . By a slight abuse of notations, we may consider row vectors as linear forms and vice versa. In this case, if $x \in \mathbb{R}^d$, the notation x^T will be used for the transpose operation. The tensor product of two finite-dimensional vector spaces E and F is denoted by $E \otimes F$. In particular, $\mathbb{R}^d \otimes \mathbb{R}^e \simeq \mathbb{R}^{d \times e} \simeq \mathcal{M}^{d,e}(\mathbb{R})$ is the space of real matrices of size $d \times e$. Let f be a function of one variable, and define

$$\delta f_{s,t} := f_t - f_s. \quad (1.3)$$

The 2-parameter functions are indexed by the simplex $\mathcal{S}_{[0,T]} = \{(s, t) \in [0, T]^2 : s \leq t\}$ rather than $[0, T]^2$. If I is a sub-interval of $[0, T]$, then $S_I = \{(s, t) \in I^2 : s \leq t\}$. For $\beta \in (0, 1)$ and a function $g : \mathcal{S}_{[0,T]} \rightarrow F$, the Hölder semi-norm of g on a sub-interval $I \subseteq [0, T]$, denoted by $\|g\|_{\beta, I}$ (or simply $\|g\|_{\beta}$ if $I = [0, T]$), is given by

$$\|g\|_{\beta, I} = \sup_{\substack{(s,t) \in \mathcal{S}_I \\ s \neq t}} \frac{|g_{s,t}|}{|t - s|^{\beta}}.$$

The β -Hölder space $\mathcal{C}_2^{\beta}([0, T]; F)$ is the space of functions $g : \mathcal{S}_{[0,T]} \rightarrow F$ such that $\|g\|_{\beta} < \infty$. The β -Hölder space $\mathcal{C}^{\beta}([0, T]; F)$ is the space of functions $f : [0, T] \rightarrow F$ such that $\|\delta f\|_{\beta} < \infty$ (hereafter $\|\delta f\|_{\beta}$ will simply be denoted by $\|f\|_{\beta}$). With a slight abuse of notations, we may write $g \in \mathcal{C}^{\beta}([0, T]; F)$ even for a 2-parameter function, and if the context is clear, we may just write $g \in \mathcal{C}^{\beta}$.

Similarly, we also remind the definitions of the p -variation semi-norm and space. For $p \geq 1$, a sub-interval $I \subseteq [0, T]$ and $g : \mathcal{S}_{[0,T]} \rightarrow F$, denote by $\|g\|_{p, I}$ (or simply $\|g\|_p$ if $I = [0, T]$) the semi-norm defined by

$$\|g\|_{p, I}^p = \sup_{\pi} \sum_{i=0}^{m-1} |g_{t_i, t_{i+1}}|^p,$$

where the supremum is taken over all finite subdivisions $\pi = (t_0, \dots, t_m)$ of I with $t_0 < t_1 < \dots < t_m \in I$, $\forall m \in \mathbb{N}$. We define \mathcal{V}_2^p the set of *continuous* 2-parameter paths g with finite p -variation, and \mathcal{V}^p the set of *continuous* paths $f : [0, T] \rightarrow F$ such that $\|\delta f\|_p < \infty$ (with the same abuse of notations, $\|\delta f\|_p$ will simply be denoted by $\|f\|_p$).

Note that we shall use roman letters (p, q, \dots) for the variation semi-norms and greek letters (α, β, \dots) for Hölder semi-norms in order not to confuse $\|\cdot\|_p$ and $\|\cdot\|_{\alpha}$. In case there might be a confusion, we shall write $\|\cdot\|_{p\text{-var}}$ or $\|\cdot\|_{\alpha\text{-Höl}}$, for instance $\|f\|_{1\text{-var}}$.

Remark 1.1. *The space \mathcal{C}^{β} (resp. \mathcal{V}^p) is Banach when equipped with the norm $f \mapsto |f_0| + \|f\|_{\beta}$. (resp. $|f_0| + \|f\|_p$). When this property will be needed, the paths will start from the same initial conditions, thus we may forget about the first term and consider $\|\cdot\|_{\beta}$ (resp. $\|\cdot\|_p$) as a norm.*

Lastly, the mapping $\phi_p(x) = x \vee x^p$, $x \geq 0$ will frequently appear in upper bounds of control functions that are used to control the p -variations of penalised and reflected solutions.

2 Preliminaries on rough paths and the Skorokhod problem

In this section, we briefly review the definitions of rough paths and rough differential equations, gathered mostly from Friz and Victoir [17] and Friz and Hairer [15]. We also make precise the meaning of the Skorokhod problem written in Equation (1.2).

2.1 Geometric rough paths

Definition 2.1 (Rough path). • Let $\beta \in (\frac{1}{3}, \frac{1}{2}]$ (resp. $p \in [2, 3)$). A β -Hölder rough path (resp. p -rough path) \mathbf{X} is a couple $\mathbf{X} = ((X_t)_{t \in [0, T]}, (\mathbb{X}_{s,t})_{s,t \in [0, T]}) \in \mathcal{C}^\beta([0, T]; \mathbb{R}^d) \times \mathcal{C}^{2\beta}([0, T]; \mathbb{R}^d \otimes \mathbb{R}^d)$ (resp. in $\mathcal{V}^p([0, T]; \mathbb{R}^d) \times \mathcal{V}^{\frac{p}{2}}([0, T]; \mathbb{R}^d \otimes \mathbb{R}^d)$) such that Chen's relation is satisfied:

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = \delta X_{s,u} \otimes \delta X_{u,t},$$

for any $s \leq u \leq t \in [0, T]^3$. The space of such paths is denoted by $\mathcal{C}^\beta([0, T]; \mathbb{R}^d)$, or simply \mathcal{C}^β (resp. $\mathcal{V}^p([0, T]; \mathbb{R}^d)$ and \mathcal{V}^p). For $\mathbf{X} \in \mathcal{C}^\beta$ (resp. in \mathcal{V}^p), we will need the following homogeneous rough path "norm"

$$\|\mathbf{X}\|_\beta = \|X\|_\beta + \sqrt{\|\mathbb{X}\|_{2\beta}} \quad (\text{resp. } \|\mathbf{X}\|_p^p = \|X\|_p^p + \|\mathbb{X}\|_{\frac{p}{2}}^{\frac{p}{2}}).$$

• $\mathbf{X} \in \mathcal{C}^\beta([0, T]; \mathbb{R}^d)$ (resp. in \mathcal{V}^p) is a geometric rough path if the symmetric part of \mathbb{X} , $\text{sym}(\mathbb{X}) = (\mathbb{X}^{ij} + \mathbb{X}^{ji})_{i,j=1\dots d}$, satisfies

$$\text{sym}(\mathbb{X})_{s,t} = \frac{1}{2} \delta X_{s,t} \otimes \delta X_{s,t}.$$

Intuitively, this relation implies that geometric rough paths admit a first order chain rule, as for smooth paths or Stratonovich calculus. The space of geometric β -Hölder rough paths (resp. p -rough paths) is denoted by $\mathcal{C}_g^\beta([0, T], \mathbb{R}^d)$ (resp. $\mathcal{V}_g^p([0, T], \mathbb{R}^d)$).

Although our main results are expressed in Hölder spaces only, the p -variations play an important role in the proofs, due to the nature of the compensator process K (which is non-decreasing and thus in \mathcal{V}^1).

For the following definition, we follow [7].

Definition 2.2 (Control function). Let I be an interval and recall that \mathcal{S}_I denotes the simplex on I . A control function is a map $w : \mathcal{S}_I \rightarrow \mathbb{R}_+$ which is super-additive, i.e. $w(s, t) + w(t, u) \leq w(s, u)$ for all $s \leq t \leq u \in I$. A control function is regular if $\lim_{|t-s| \rightarrow 0} w(s, t) = 0$.

For instance, if $\|X\|_{p,I}^p < \infty$ for some interval $I \subseteq [0, 1]$, then $w_X(s, t) = \|X\|_{p,[s,t]}^p$ is a control function on \mathcal{S}_I . If $X \in \mathcal{V}^p(I)$, then w_X is a regular control function.

2.2 Rough differential equations with drift

For a geometric rough path $\mathbf{X} \in \mathcal{C}_g^\beta([0, T], \mathbb{R}^d)$, we would like to give a meaning to the following formal equation:

$$dY_t = b(Y_t)dt + \sigma(Y_t)d\mathbf{X}_t. \quad (2.1)$$

We adopt the definition of solution given in [17, Definition 12.1] (see also [14, Definition 3]), which we recall for the reader's convenience. Note that this definition gives a meaning to (1.1), but not directly to $\int \sigma(Y_s^n) d\mathbf{X}_s$.

Definition 2.3 (RDE with drift). Let $\mathbf{X} \in \mathcal{C}_g^\beta([0, T], \mathbb{R}^d)$, with $\beta \in (\frac{1}{3}, \frac{1}{2}]$. We call $Y \in \mathcal{C}^0([0, T], \mathbb{R}^e)$ a solution to the RDE with drift (2.1) started at $y_0 \in \mathbb{R}^e$ if there exists a sequence $(X^k)_{k \in \mathbb{N}}$ of \mathbb{R}^d -valued Lipschitz paths such that

- $\sup_{k \in \mathbb{N}} \|\mathbf{X}^k\|_\beta < \infty$, where $\mathbf{X}^k = (X^k, \mathbb{X}^k)$ and $\mathbb{X}_{s,t}^k = \int_s^t (X_u^k - X_s^k) dX_u^k$;
- \mathbf{X}^k converges pointwise to \mathbf{X} ;
- for all k , the ODE $dY_t^k = b(Y_t^k)dt + \sigma(Y_t^k)dX_t^k$ has a solution and $\lim_{k \rightarrow \infty} \|Y^k - Y\|_{\infty, [0, T]} = 0$.

The classical Doss-Sussmann representation (see Doss [9] and Sussmann [29]) provides a way to write the solution of a stochastic differential equation as the composition of the flow of σ with the solution of a random ODE. It works for one-dimensional noises, even in some rough cases. However its multidimensional generalization requires strong geometric assumptions on σ (see [9]). Instead we recall a less explicit formulation borrowed from Friz and Oberhauser [14], which requires no additional assumption on σ and shall be enough for our needs.

For some $\sigma : \mathbb{R}^e \rightarrow \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$, consider the RDE

$$d\tilde{Y}_t = \sigma(\tilde{Y}_t) d\mathbf{X}_t, \quad (2.2)$$

and if they exist, denote by $y_0 \mapsto U_{t \leftarrow 0}^{\mathbf{X}; y_0}$ the flow of the solution (i.e. $\tilde{Y}_t = U_{t \leftarrow 0}^{\mathbf{X}; y_0}$ when $\tilde{Y}_0 = y_0$), by $J_{t \leftarrow 0}^{\mathbf{X}; y_0}$ its Jacobian and by $J_{0 \leftarrow t}^{\mathbf{X}; y_0}$ the inverse of the Jacobian.

Proposition 2.4 ([14]). *Assume that $b \in \mathcal{C}_b^1(\mathbb{R}^e; \mathbb{R}^e)$, $\sigma \in \mathcal{C}_b^4(\mathbb{R}^e; \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e))$ and that $\mathbf{X} \in \mathcal{C}_g^\beta([0, T]; \mathbb{R}^d)$, with $\beta \in (\frac{1}{3}, \frac{1}{2})$. Then for any $y_0 \in \mathbb{R}^e$, there exists a unique solution Y to the RDE with drift (2.1) started from y_0 . Moreover, this solution has the following Doss-Sussmann representation:*

$$\begin{cases} Y_t &= U_{t \leftarrow 0}^{\mathbf{X}; Z_t} \\ Z_t &= y_0 + \int_0^t W(s, Z_s) ds \end{cases}, \quad t \in [0, T],$$

where

$$W(t, z) = J_{0 \leftarrow t}^{\mathbf{X}; z} b \left(U_{t \leftarrow 0}^{\mathbf{X}; z} \right), \quad (t, z) \in [0, T] \times \mathbb{R}^e. \quad (2.3)$$

2.3 Assumptions

We shall assume throughout the paper that $e = 1$ (except in the more general Proposition 2.11) and

$$\sigma \in \mathcal{C}_b^4(\mathbb{R}, (\mathbb{R}^d)'). \quad (2.4)$$

Since the penalisation term $n(\cdot)_-$ in (1.1) is not differentiable, we approximate it by a smooth non-increasing function ψ_n such that

$$\forall y \in \mathbb{R}, \quad \psi_n(y) = \begin{cases} 0 & \text{if } y > 0, \\ -\frac{1}{2} - ny & \text{if } y \leq -\frac{1}{n}, \\ \text{convex interpolation} & \text{if } -\frac{1}{n} < y \leq 0. \end{cases} \quad (2.5)$$

In fact, for any $n \in \mathbb{N}$ we can choose ψ_n as above and which also satisfies:

$$\begin{cases} \psi_n \in \mathcal{C}^\infty \text{ and } \psi_n' \in \mathcal{C}_b^\infty; \\ \forall y \in \mathbb{R}, \quad \psi_n(y) \leq \psi_{n+1}(y) \text{ and } -\frac{1}{2} + ny_- \leq \psi_n(y) \leq ny_- \end{cases} \quad (2.6)$$

We assume that the driving signal is a geometric β -Hölder rough path, for some $\beta \in (\frac{1}{3}, \frac{1}{2}]$, i.e. $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{C}_g^\beta([0, T]; \mathbb{R}^d)$. The boundary process L is assumed to have at least the same Hölder regularity as X , and further that

$$\widehat{X} := (X, L) \text{ can be enhanced into a geometric } \beta\text{-Hölder rough path } \widehat{\mathbf{X}} = (\widehat{X}, \widehat{\mathbb{X}}) \in \mathcal{C}_g^\beta([0, T]; \mathbb{R}^{d+1}). \quad (2.7)$$

In that case, we still denote by \mathbb{X} the projection of $\widehat{\mathbb{X}}$ on the X component, and by $\mathbf{X} = (X, \mathbb{X})$ the associated rough path. Note that the previous assumption is not trivial in general because of the roughness $\beta \leq \frac{1}{2}$. In fact since we consider RDEs with drifts, we will also need (X, L, t) to be lifted into a geometric rough path. In that case, since the identity function of \mathbb{R} is smooth, it is always possible to realise this lift, in such a way that the projection on (X, L) coincides with $\widehat{\mathbf{X}}$ (see Young pairings [17, Section 9.4]). Observe that Young pairings can also be used to obtain (2.7), but then one has to assume more regularity on L , namely that $L \in \mathcal{V}^q$, with $q \geq 1$ such that $\frac{1}{q} + \frac{1}{p} > 1$ ($p = \beta^{-1}$).

With these notations and assumptions, we consider

$$Y_t^n = y_0 + \int_0^t \psi_n(Y_s^n - L_s) ds + \int_0^t \sigma(Y_s^n) d\mathbf{X}_s, \quad t \in [0, T]. \quad (2.8)$$

For each $n \in \mathbb{N}$, Proposition 3.1 ensures that there is a unique solution to (2.8).

2.4 Gaussian rough paths

In case X is a Gaussian process, several papers give conditions (see in particular Cass, Hairer, Litterer, and Tindel [4]) for X to be enhanced into a geometric rough path. Cass, Litterer, and Lyons [3] also proved that such conditions yield that the Jacobian of the flow has finite moments of all order (see also [4] with a bounded drift).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and let $X = (X^1, \dots, X^d)$ be a continuous, centred Gaussian process with independent and identically distributed components and let $R(s, t) = \mathbb{E}(X_s^1 X_t^1)$ denote the covariance function of X^1 . Following Cass et al. [3], let

$$R \begin{pmatrix} s, t \\ u, v \end{pmatrix} = \mathbb{E} [(X_t^1 - X_s^1)(X_v^1 - X_u^1)]$$

be the rectangular increments of R . Then for $r \in [1, \frac{3}{2})$, we might assume that R has finite second-order r -variation in the sense

$$\|R\|_{r; [0, T]^2} := \left(\sup_{\substack{\pi=(t_i) \\ \pi'=(t'_j)}} \sum_{i,j} R \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right)^{\frac{1}{r}} < \infty. \quad (\mathbf{H}_{\text{Cov}})$$

Under this assumption, X can almost surely be enhanced into a geometric rough path $\mathbf{X} = (X, \mathbb{X})$ and for any $\alpha \in (\frac{1}{3}, \frac{1}{2r})$, $\mathbf{X} \in \mathcal{C}_g^\alpha$. Moreover, this assumption permits to obtain upper bounds on the Jacobian of the flow of a Gaussian RDE, which shall help us obtain convergence results in $L^\gamma(\Omega)$ (Theorem 2.13).

Remark 2.5. *A typical example of process satisfying $(\mathbf{H}_{\text{Cov}})$ is the fractional Brownian motion $(B_t^H)_{t \geq 0}$. We recall that for any Hurst parameter $H \in (0, 1)$, $(B_t^H)_{t \geq 0}$ is the centred Gaussian process with covariance*

$$\mathbb{E}(B_t^H B_s^H) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}), \quad \forall t, s \geq 0.$$

Such a process is statistically H -self-similar and increment stationary (e.g. for $H = \frac{1}{2}$, this is a standard Brownian motion). Most importantly regarding the theory of rough paths, if $H \in (\frac{1}{3}, \frac{1}{2})$, its covariance satisfies $(\mathbf{H}_{\text{Cov}})$ with $r = \frac{1}{2H}$, so that it can be enhanced into a geometric rough paths. If $H \in (\frac{1}{2}, 1)$, then one can solve differential equations driven by B^H in the Young sense (i.e. without needing to enhance B^H). Besides, its sample paths are almost surely β -Hölder continuous, for any $\beta < H$.

2.5 Controlled rough paths

We choose to define controlled rough paths with respect to the p -variation topology. This is because the compensator K and its approximations K^n are clearly in \mathcal{V}^1 while it seems more difficult to prove that they have some Hölder regularity. It then becomes possible to use rough paths continuity results such as Theorem 2.7.

Definition 2.6 (Controlled rough path). *Let $p \in [2, 3)$ and $X \in \mathcal{V}^p([0, T]; \mathbb{R}^d)$. A path $Y \in \mathcal{V}^p([0, T]; E)$ is controlled by X if there exist a path $Y' \in \mathcal{V}^p([0, T]; \mathcal{L}(\mathbb{R}^d, E))$ and a map $R^Y \in \mathcal{V}^{\frac{p}{2}}([0, T]; E)$ such that*

$$\forall s \leq t \in [0, T], \quad \delta Y_{s,t} = Y'_s \delta X_{s,t} + R_{s,t}^Y.$$

The path Y' is called the Gubinelli derivative of Y (although it might not be unique), and R^Y is a remainder term. The space of such couples of paths (Y, Y') controlled by X is denoted by $\mathcal{V}_X^p(E)$ ($\mathcal{C}_X^\beta(E)$ for a corresponding definition in β -Hölder norm, see [15, Definition 4.6]).

Now if $\mathbf{X} \in \mathcal{V}^p([0, T]; \mathbb{R}^d)$ and $(Y, Y') \in \mathcal{V}_X^p(\mathcal{L}(\mathbb{R}^d, \mathbb{R}^e))$, then the rough integral of Y against \mathbf{X} is defined by

$$\int_0^T Y_s d\mathbf{X}_s = \lim_{m \rightarrow \infty} \sum_{\pi_m=(t_i^m)} Y_{t_i^m} \delta X_{t_i^m, t_{i+1}^m} + Y'_{t_i^m} \mathbb{X}_{t_i^m, t_{i+1}^m}, \quad (2.9)$$

where $(\pi_m)_{m \in \mathbb{N}}$ is an increasing sequence of subdivisions of $[0, T]$ such that $\lim_{m \rightarrow \infty} \max_i (t_{i+1}^m - t_i^m) = 0$ and $t_0^m = 0, t_m^m = T$.

The existence of this integral has been established by Gubinelli [18] for the Hölder topology (see also [15, Proposition 4.10]). In the p -variation topology, we refer to Friz and Shekhar [16, Theorem 31]:

Theorem 2.7. *Let $p \in [2, 3)$. If $\mathbf{X} \in \mathcal{V}^p([0, T]; \mathbb{R}^d)$ and $(Y, Y') \in \mathcal{V}_X^p(\mathcal{L}(\mathbb{R}^d, \mathbb{R}^e))$, then the rough integral of Y against \mathbf{X} exists (and the limit in (2.9) does not depend on the choice of a sequence of subdivisions). Moreover, for any $s, t \in [0, T]$,*

$$\left| \int_s^t Y_u d\mathbf{X}_u - Y_s \delta X_{s,t} - Y'_s \mathbb{X}_{s,t} \right| \leq C_p \left(\|X\|_{p,[s,t]} \|R^Y\|_{\frac{p}{2},[s,t]} + \|\mathbb{X}\|_{\frac{p}{2},[s,t]} \|Y'\|_{p,[s,t]} \right). \quad (2.10)$$

Let us finally recall Proposition 2.12 of [5].

Proposition 2.8. *Let $p \in [2, 3)$. Let $\mathbf{X} \in \mathcal{V}^p([0, T]; \mathbb{R}^d)$ and assume that $(Y^n, (Y^n)')_{n \in \mathbb{N}} \subset \mathcal{V}_X^p(\mathcal{L}(\mathbb{R}^d, \mathbb{R}^e))$ is a sequence such that:*

$$(Y^n)' \text{ and } R^{Y^n} \text{ converge in the uniform topology on } [0, T] \text{ (resp. } [0, T]^2),$$

and

$$\sup_{n \in \mathbb{N}} \left(\|(Y^n)'\|_{p,[0,T]} + \|R^{Y^n}\|_{\frac{p}{2},[0,T]} \right) < \infty,$$

then $(Y^n, (Y^n)')$ converges uniformly to some $(Y, Y') \in \mathcal{V}_X^p$ and

$$\lim_{n \rightarrow \infty} \left\| \int_0^\cdot Y_s^n d\mathbf{X}_s - \int_0^\cdot Y_s d\mathbf{X}_s \right\|_{\infty, [0, T]} = 0.$$

2.6 The Skorokhod problem

Having at our disposal a rough integral in the sense of Equation (2.9), we can give a meaning to Equation (1.2), also referred to as Skorokhod problem associated to σ and L , denoted by $SP(\sigma, L)$.

Definition 2.9. *Let $\mathbf{X} \in \mathcal{V}^p([0, T]; \mathbb{R}^d)$. We say that (Y, K) solves $SP(\sigma, L)$, or that it is a solution to the reflected RDE with diffusion coefficient σ started from $y_0 \geq L_0$ and reflected on the path L , if*

(i) $(Y, \sigma(Y)) \in \mathcal{V}_X^p$ and (Y, K) satisfies Equation (1.2), in the sense that both sides are equal, where the integral $\int_0^\cdot \sigma(Y_s) d\mathbf{X}_s$ is understood in the sense of (2.9);

(ii) $\forall t \in [0, T], Y_t \geq L_t$;

(iii) K is nondecreasing;

(iv) $\forall t \in [0, T], \int_0^t (Y_s - L_s) dK_s = 0$, or equivalently, $\int_0^t \mathbf{1}_{\{Y_s \neq L_s\}} dK_s = 0$.

Remark 2.10. *In item (i), it is also possible to define solutions to reflected RDEs in the sense of Davie as in Deya et al. [7]. For RDEs with bounded coefficients (without reflection), Davie's solution and the solution in the sense of controlled rough paths coincide ([15, Proposition 8.8]).*

2.7 Main results

Hereafter, we use the notation $\mathbf{X} \in \mathcal{C}_g^\beta$ even if $\beta > \frac{1}{2}$, although the iterated integral \mathbb{X} is irrelevant in this case. This notation permits to present our results in a unified form.

Our first result states the global existence and uniqueness of solutions for RDEs with an unbounded drift which has at most linear growth. It is generally a difficult task to obtain global existence for RDEs when the vector fields are unbounded (which is the case of ψ_n), and known counter-examples show that global solutions may not exist in general. Nevertheless, for an RDE with coefficient $V = (V_1, \dots, V_d)$ on \mathbb{R}^e , where each V_i has components V_i^k , there are several results in this direction ([20, 21], [17, Exercise 10.56] and [2])

which ask roughly for $V_i^k \nabla V_j^l$ to be bounded and Hölder continuous for all i, j, k, l . Observe that in our case (assuming $L \equiv 0$ for simplicity), the vector field V would be $V(y, t) = (\psi_n(y), \sigma(y))$ but that $\psi_n \sigma'$ is not bounded. However this general approach neglects the special nature of the drift term and its smooth driver “dt” by considering it as any other component of the rough driver. On the other hand, it has been proven recently by Riedel and Scheutzow [27] that under a linear growth assumption of b , the RDE with drift (2.1) has a unique solution (there exists in fact a semiflow of solutions). Under similar assumptions, we provide here a Doss-Sussmann representation of the solution.

Proposition 2.11. *Let $\sigma \in \mathcal{C}_b^4(\mathbb{R}^e, \mathcal{L}(\mathbb{R}^d; \mathbb{R}^e))$, $n \in \mathbb{N}$ and assume that*

$$b \in \mathcal{C}^1(\mathbb{R}^e, \mathbb{R}^e) \quad \text{and} \quad \nabla b \in \mathcal{C}_b(\mathbb{R}^e, \mathbb{R}^{e \times e}).$$

Let $\beta \in (\frac{1}{3}, \frac{1}{2})$, and let \mathbf{X} be a d -dimensional β -Hölder geometric rough path. Then for any initial condition y_0 , there exists a unique solution Y to the drifted RDE on $[0, T]$. Moreover, this solution is a path $Y \in \mathcal{C}^\beta$ which also solves:

$$\begin{cases} Y_t &= U_{t \leftarrow 0}^{\mathbf{X}; Z_t} \\ Z_t &= y_0 + \int_0^t W(s, Z_s) ds \end{cases}, \quad t \in [0, T],$$

where

$$W(t, z) = J_{0 \leftarrow t}^{\mathbf{X}; z} b \left(U_{t \leftarrow 0}^{\mathbf{X}; z} \right), \quad (t, z) \in [0, T] \times \mathbb{R}^e.$$

Our proof being inspired by the one of Friz and Oberhauser [14] (the difference is that b is bounded in [14]), it is postponed to the Appendix. The idea is to derive first the local existence and a Doss-Sussmann representation on a small time interval where the existence of the solution is known. Global existence is then achieved by stability of the ODE in the Doss-Sussmann representation.

Besides enabling us to prove the previous proposition, the Doss-Sussmann representation also yields a monotonicity property that will be very useful for the penalisation procedure. In particular, we will be able to deduce that there exists a path Y which is the limit of the non-decreasing sequence $(Y^n)_{n \in \mathbb{N}}$ and that this path is controlled by \mathbf{X} (Proposition 4.9).

We are now in a position to state our first main result.

Theorem 2.12. *Let $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{C}_g^\beta$ be a geometric β -Hölder rough path, $\beta \in (\frac{1}{3}, 1) \setminus \{\frac{1}{2}\}$. Assume that $\{\psi_n\}_{n \in \mathbb{N}}$, σ and L satisfy conditions (2.4)-(2.7), and that $y_0 \geq L_0$.*

- (i) *Then the sequence $(Y^n, \int_0^\cdot \psi_n(Y_s^n - L_s) ds)_{n \in \mathbb{N}}$, defined as the solution to (2.8), converges uniformly on $[0, T]$ to some path $(Y, K) \in \mathcal{C}_X^\beta \times \mathcal{V}^1$.*
- (ii) *Besides, (Y, K) is the unique solution to the reflected RDE (1.2) (in the sense of Definition 2.9), i.e. it is the solution to the Skorokhod problem $SP(\sigma, L)$.*

So far, the result only involved deterministic rough paths. Using some recent results on Gaussian rough paths leads to the following theorem.

Theorem 2.13. *Let σ and $\{\psi_n\}_{n \in \mathbb{N}}$ satisfy conditions (2.4)-(2.6) and let $y_0 \geq L_0$ almost surely. Let $X = (X^1, \dots, X^d)$ be an a.s. continuous, centred Gaussian process with independent and identically distributed components, and let R be its covariance function. Assume that either $X \in \mathcal{C}^\beta([0, T]; \mathbb{R}^d)$ a.s. for some $\beta \in (\frac{1}{2}, 1)$, or that:*

- *R has finite second-order r -variations for some $r \in [1, \frac{3}{2})$, as in $(\mathbf{H}_{\text{Cov}})$;*
- *L satisfies almost surely condition (2.7) for any $\beta < \frac{1}{2r}$ and that $\mathbb{E} \left[\|L\|_\beta^\gamma \right] < \infty$, for any $\gamma \geq 1$.*

Then the conclusions of Theorem 2.12 hold in the almost sure sense and moreover, the convergence holds in the following sense: $\forall \gamma \geq 1$,

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[\sup_{t \in [0, T]} |Y_t - Y_t^n|^\gamma \right] = 0. \quad (2.11)$$

Furthermore, we obtain a rate of convergence of the sequence of penalised processes to the reflected solution.

Theorem 2.14. *Assume that the hypotheses of Theorem 2.12 hold. In particular, X is a β -Hölder path, with $\beta \in (\frac{1}{3}, 1) \setminus \{\frac{1}{2}\}$. Then the penalised solution Y^n converges to Y with the following rate: there exists $C > 0$ such that*

$$\forall n \in \mathbb{N}^*, \quad \sup_{t \in [0, T]} |Y_t^n - Y_t| \leq C n^{-\beta}.$$

Compared with the Theorem 4.1 of Słomiński [30], we see that our result matches the optimal rate, up to a logarithmic correction. However the result of Słomiński [30] is in $L^p(\Omega)$ whereas the previous theorem is only *a.s.*. We will be able to close this gap partially in the next result. But let us observe first that Theorem 2.14 is proven through a Gronwall argument and the constant C appearing there is thus of exponential form. Besides, the p -variation norm of $J^{\mathbf{X}}$ (the Jacobian of the flow of the RDE) appears in this exponential, and $J^{\mathbf{X}}$ is known to have only sub-exponential moments ([3, Theorem 6.5]). This explains the log appearing in the following result.

Theorem 2.15. *Assume that the hypotheses of Theorem 2.13 hold. In particular X is a Gaussian process, and either X has β -Hölder paths *a.s.* with $\beta \in (\frac{1}{2}, 1)$, or its covariance has finite second-order r -variations, and then X has β -Hölder paths *a.s.* with $\beta \in (\frac{1}{3}, \frac{1}{2r})$. Then for any $\gamma \geq 1$,*

$$\mathbb{E} \left[\left| \sup_{n \in \mathbb{N}} \log (n^\beta \|\delta_n Y\|_{\infty, [0, T]}) \right|^\gamma \right] < \infty.$$

The last result of this paper is a nice application of the previous penalisation technique and results, which are used to prove the existence of a density for the reflected process when the noise is a fractional Brownian motion. It is presented under simplified assumptions as the general case would be out of the scope of the present paper and will be further investigated in a separate work.

Theorem 2.16. *Let B be a one-dimensional fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$, and let $b \in C_b^1$. Let (Y, K) be the solution to the Skorokhod problem reflected on the horizontal axis*

$$\forall t \geq 0, \quad Y_t = y_0 + \int_0^t b(Y_s) ds + K_t + B_t.$$

Then for any $t > 0$, the restriction of the law of Y_t to $(0, \infty)$, i.e. the measure $[\mathbf{1}_{\{Y_t > 0\}} \mathbb{P}] \circ Y_t^{-1}$, admits a density with respect to the Lebesgue measure.

Note that unless otherwise stated (mostly in Section 6), we will only consider the case $\beta \in (\frac{1}{3}, \frac{1}{2})$. Indeed if $\beta \in (\frac{1}{2}, 1)$, Young integrals can be used, which makes proofs easier.

3 Penalisation for RDEs

3.1 Flow of an RDE

In this paragraph, we gather several useful properties of the flow of the solution of an RDE, and of its Jacobian. Let $\beta \in (\frac{1}{3}, \frac{1}{2})$ and $p = \frac{1}{\beta} \in (2, 3)$. Hereafter, $\left\{ U_{t \leftarrow 0}^{\mathbf{X}; y_0}, t \in [0, T] \right\}$ denotes the solution to the RDE (2.2) started from y_0 , with $\mathbf{X} \in \mathcal{C}_g^\beta([0, T]; \mathbb{R}^d)$ and $\sigma \in \mathcal{C}_b^4(\mathbb{R}^e; \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e))$ (note that \mathcal{C}_b^3 is enough for existence and uniqueness in (2.2)).

First, we know that the smoothness of the flow depends on the smoothness of σ : for any t , $y_0 \mapsto U_{t \leftarrow 0}^{\mathbf{X}; y_0}$ is Lipschitz continuous and twice differentiable (see for instance [14, Proposition 3]). Denote by $J_{t \leftarrow 0}^{\mathbf{X}; y_0}$ its Jacobian matrix, which according to [3, Corollary 4.6] is uniformly (in $(t, y_0) \in [0, T] \times \mathbb{R}$) bounded by a quantity depending only on p , $\|\mathbf{X}\|_{p, [0, T]}$ and the so-called α -local p -variation of \mathbf{X} (see [3, Definition 4.3]). We denote this upper bound by $C_J^{\mathbf{X}}$.

Denote by $J_{0 \leftarrow t}^{\mathbf{X}; \cdot} := (J_{t \leftarrow 0}^{\mathbf{X}; \cdot})^{-1}$ its inverse matrix, which can be interpreted as the Jacobian of the flow of the same RDE with X evolving backward. Hence as noticed in the proof of [4, Theorem 7.2], $J_{0 \leftarrow t}^{\mathbf{X}; \cdot}$ is also bounded by $C_J^{\mathbf{X}}$, so that altogether the following inequality is fulfilled:

$$\sup_{y_0 \in \mathbb{R}} \max \left(\|J_{\cdot \leftarrow 0}^{\mathbf{X}; y_0}\|_{\infty, [0, T]}, \|J_{0 \leftarrow \cdot}^{\mathbf{X}; y_0}\|_{\infty, [0, T]} \right) \leq C_J^{\mathbf{X}} < \infty. \quad (3.1)$$

Note also that with $\sigma \in C_b^4$, $J_{0\leftarrow t}^{\mathbf{X};\cdot}$ and $J_{t\leftarrow 0}^{\mathbf{X};\cdot}$ are Lipschitz continuous, uniformly in t . Besides, when X is Gaussian with *i.i.d.* components and satisfies $(\mathbf{H}_{\text{Cov}})$, $C_{\mathcal{J}}^{\mathbf{X}}$ has finite moments of all orders ([3, Theorem 6.5] and [4, Theorem 7.2]). As observed in [4, Section 7], $J_{0\leftarrow t}^{\mathbf{X};z}$ satisfies the following linear RDE, for any fixed z :

$$dJ_{0\leftarrow t}^{\mathbf{X};z} = d\mathbf{M}_t J_{0\leftarrow t}^{\mathbf{X};z},$$

where \mathbf{M} depends on the flow $U_{t\leftarrow 0}^{\mathbf{X};z}$. If $e = 1$ (recall e is the dimension of the space in which y lives), it is thus a consequence of the fact that $J_{0\leftarrow 0}^{\mathbf{X};z} = 1$ and of the uniqueness in the previous equation that $J_{0\leftarrow t}^{\mathbf{X};z} > 0$ for any $z \in \mathbb{R}$ and any $t \geq 0$. Hence it follows from (3.1) that

$$\forall z \in \mathbb{R}, \quad J_{0\leftarrow t}^{\mathbf{X};z} \geq (C_{\mathcal{J}}^{\mathbf{X}})^{-1} (> 0). \quad (3.2)$$

It will be important to keep in mind that all the above properties are independent of the choice of a drift function b .

Finally, the mapping $W(t, z)$ defined in (2.3) is continuous in t , Lipschitz continuous in z uniformly in t if b is bounded (this is however not true anymore if b is unbounded). This ensures that there is a unique solution to

$$z'_t = W(t, z_t), \quad z_0 = y_0.$$

3.2 Existence of a global solution to (2.8)

The result below states the global existence of a solution to the rough differential equation (2.8). Due to the boundary term L in (2.8), we cannot apply directly Proposition 2.11. However, provided that (2.8) can be cast into a proper RDE with drift using Assumption (2.7), then the result will hold.

Proposition 3.1. *Let $\sigma \in C_b^4(\mathbb{R}; (\mathbb{R}^d)')$, $n \in \mathbb{N}$ and ψ_n satisfying (2.5)-(2.6). Let $\beta \in (\frac{1}{3}, \frac{1}{2})$, let \mathbf{X} be a β -Hölder geometric rough path and let $\{L_t\}_{t \in [0, T]}$ be a barrier process satisfying (2.7). Then for any initial condition y_0 such that $y_0 \geq L_0$, there exists a unique solution to (2.8). Moreover, this solution is a path $\{Y_t^n\}_{t \in [0, T]} \in \mathcal{C}^\beta$ which also solves*

$$\begin{cases} Y_t^n &= U_{t\leftarrow 0}^{\mathbf{X}; Z_t^n} \\ Z_t^n &= y_0 + \int_0^t W^n(s, Z_s^n) ds \end{cases}, \quad t \in [0, T], \quad (3.3)$$

where

$$W^n(t, z) = J_{0\leftarrow t}^{\mathbf{X};z} \psi_n \left(U_{t\leftarrow 0}^{\mathbf{X};z} - L_t \right), \quad (t, z) \in [0, T] \times \mathbb{R}.$$

Remark 3.2. • For $\beta > \frac{1}{2}$, our assumptions on the coefficients meet those from [19] and thus there exists a unique solution to (2.8). Moreover, the previous Doss-Sussmann representation holds also true by a simple application of the usual chain rule.

• If the dimension of the noise d equals 1, then the usual Doss-Sussmann representation [9] can be used.

Proof. For $y \in \mathbb{R}^2$, define $\widehat{b}_n(y) = (\psi_n(y^1 - y^2), 0)^T$, where we used the notation $y = (y^1, y^2)^T \in \mathbb{R}^2$. In the same way, define $\widehat{\sigma}(y) = \begin{pmatrix} \sigma(y^1) & 0 \\ 0 & 1 \end{pmatrix}$, so that $\widehat{\sigma} \in C_b^4(\mathbb{R}^2; \mathcal{L}(\mathbb{R}^{d+1}, \mathbb{R}^2))$. Finally, let $\widehat{\mathbf{X}} \in \mathcal{C}_g^\beta$ be the rough path above (X, L) , as in (2.7). Proposition 2.11 ensures that there exists a unique solution $\widehat{Y}^n \in \mathcal{C}^\beta([0, T]; \mathbb{R}^2)$ to the following RDE with drift

$$d\widehat{Y}_t^n = \widehat{b}_n(\widehat{Y}_t^n)dt + \widehat{\sigma}(\widehat{Y}_t^n)d\widehat{\mathbf{X}}_t.$$

Since Y^n corresponds to the first component of \widehat{Y}^n , the result follows. \square

3.3 Penalisation estimates

In the sequel, we will use several times the following result, which gives uniform estimates for solutions of integral equations with drift coefficient ψ_n .

Lemma 3.3. *Let $\Psi > 0$, $\ell, \{g^n\}_{n \in \mathbb{N}}$ be continuous functions such that $g_0^n = 0$, and assume that for each $n \in \mathbb{N}$, f^n is a solution to:*

$$\begin{cases} f_t^n = f_0^n + g_t^n + \Psi \int_0^t \psi_n(f_u^n - \ell_u) du, & \forall t \in [0, T], \\ f_0^n = f_0 \geq \ell_0. \end{cases}$$

Then,

(i) For all $t \in [0, T]$,

$$\forall n \in \mathbb{N}, |\delta f_{0,t}^n - \delta \ell_{0,t}| \leq \sqrt{26} \|g^n - \delta \ell_{0,\cdot}\|_{\infty, [0,t]};$$

(ii) Let $\beta \in (0, 1)$. If $\ell, \{g^n\}_{n \in \mathbb{N}} \in \mathcal{C}^\beta([0, T], \mathbb{R})$ and $f_0^n \geq \ell_0$, then

$$\forall t \in [0, T], \forall n \in \mathbb{N}, \quad \psi_n(f_t^n - \ell_t) \leq \bar{\Psi}_n (\Psi^{-\beta} + \Psi^{1-\beta}) n^{1-\beta},$$

where $\bar{\Psi}_n = C(\|\ell\|_\beta + \|g^n\|_\beta + \frac{1}{2}\Psi T^{1-\beta})$.

Proof. (i) Denoting $k_t^n := \Psi \int_0^t \psi_n(f_u^n - \ell_u) du$, let \bar{f}^n and \bar{g}^n be defined as follows:

$$\begin{aligned} \bar{f}_t^n &:= \delta f_{0,t}^n - \delta \ell_{0,t} = -\delta \ell_{0,t} + g_t^n + \Psi \int_0^t \psi_n(f_u^n - \ell_u) du \\ &=: \bar{g}_t^n + k_t^n. \end{aligned}$$

Observe that

$$\begin{aligned} (\bar{f}_t^n)^2 &= (\bar{g}_t^n)^2 + (k_t^n)^2 + 2 \int_0^t \bar{g}_t^n dk_u^n = (\bar{g}_t^n)^2 + 2 \int_0^t (k_u^n + \bar{g}_t^n) dk_u^n \\ &\leq (\bar{g}_t^n)^2 + 2 \int_0^t (\bar{g}_t^n - \bar{g}_u^n) dk_u^n \end{aligned}$$

where we used the inequality $\bar{f}_u^n \psi_n(f_u^n - \ell_u) \leq (f_u^n - \ell_u) \psi_n(f_u^n - \ell_u) \leq 0$. It follows that

$$\begin{aligned} (\bar{f}_t^n)^2 &\leq (\bar{g}_t^n)^2 + 2k_t^n \|\bar{g}_t^n - \bar{g}^n\|_{\infty, [0,t]} \leq (\bar{g}_t^n)^2 + 2(|\bar{f}_t^n| + |\bar{g}_t^n|) \|\bar{g}_t^n - \bar{g}^n\|_{\infty, [0,t]} \\ &\leq 5\|\bar{g}^n\|_{\infty, [0,t]}^2 + 4|\bar{f}_t^n| \|\bar{g}^n\|_{\infty, [0,t]} \\ &\leq 5\|\bar{g}^n\|_{\infty, [0,t]}^2 + \frac{1}{2} \left(|\bar{f}_t^n|^2 + 16\|\bar{g}^n\|_{\infty, [0,t]}^2 \right), \end{aligned}$$

which implies the result.

(ii) The inequality $\psi_n(x) \geq -\frac{1}{2} - nx$ yields

$$f_t^n - \ell_t \geq f_0^n - \ell_0 + \bar{g}_t^n - \frac{1}{2}\Psi t - n\Psi \int_0^t (f_u^n - \ell_u) du.$$

Denote $\tilde{g}_t^n := \bar{g}_t^n - \frac{1}{2}\Psi t$ and \tilde{f}^n the solution to

$$\tilde{f}_t^n - \ell_t = f_0^n - \ell_0 + \tilde{g}_t^n - n\Psi \int_0^t (\tilde{f}_u^n - \ell_u) du. \quad (3.4)$$

It follows from the comparison principle of ODEs that for any $t \in [0, T]$, $f_t^n - \ell_t \geq \tilde{f}_t^n - \ell_t$. Solving (3.4) yields

$$\begin{aligned} f_t^n - \ell_t &\geq (f_0^n - \ell_0)e^{-n\Psi t} - \int_0^t e^{-n\Psi(t-u)} d\tilde{g}_u^n \\ &\geq (f_0^n - \ell_0)e^{-n\Psi t} + \tilde{g}_t^n e^{-n\Psi t} + n\Psi \int_0^t e^{-n\Psi(t-u)} (\tilde{g}_u^n - \tilde{g}_t^n) du, \quad t \in [0, T]. \end{aligned} \quad (3.5)$$

Since $\psi_n(x) \leq nx_-$, we now obtain from (3.5) that

$$\begin{aligned} \psi_n(f_t^n - \ell_t) &\leq n \left(\tilde{g}_t^n e^{-n\Psi t} + n\Psi \int_0^t e^{-n\Psi(t-u)} (\tilde{g}_u^n - \tilde{g}_t^n) du \right)_- \\ &\leq n \|\tilde{g}^n\|_\beta t^\beta e^{-n\Psi t} + \|\tilde{g}^n\|_\beta n^2 \Psi \int_0^t e^{-n\Psi(t-u)} (t-u)^\beta du, \quad t \in [0, T]. \end{aligned} \quad (3.6)$$

It is clear that $n \|\tilde{g}^n\|_\beta t^\beta e^{-n\Psi t} \leq \|\tilde{g}^n\|_\beta \Psi^{-\beta} n^{1-\beta}$. Thus one focuses now on the second term: an integration-by-parts and the change of variables $v = n\Psi u$ yield

$$\begin{aligned} n^2 \Psi \int_0^t e^{-n\Psi(t-u)} (t-u)^\beta du &= -nt^\beta e^{-n\Psi t} + \beta n \int_0^t e^{-n\Psi u} u^{\beta-1} du \\ &= -nt^\beta e^{-n\Psi t} + \beta n^{1-\beta} \Psi^{1-\beta} \int_0^{n\Psi t} v^{\beta-1} e^{-v} dv \\ &\leq C n^{1-\beta} \Psi^{1-\beta}. \end{aligned}$$

Plugging the last inequality in (3.6) gives the desired result. \square

4 Existence of a solution to the Skorokhod problem

4.1 Existence of the limit process

We use first comparison theorems, the Doss-Sussmann representation (3.3) and Lemma 3.3 to get the following result, which implies the existence of paths Z and Y as pointwise limits of (Z^n) and (Y^n) .

Proposition 4.1. *(i) Let the notations and assumptions of Theorem 2.12 be in force. Then the sequences of paths $(Z^n)_{n \in \mathbb{N}}$ and $(Y^n)_{n \in \mathbb{N}}$ defined in (3.3) are nondecreasing with n . Besides, the following inequalities are satisfied*

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} |Z_t^n| < +\infty \quad \text{and} \quad \sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} |Y_t^n| < +\infty.$$

(ii) Now let the assumptions of Theorem 2.13 be in force. Then the previous conclusions hold in the almost sure sense and moreover, for any $\gamma \geq 1$,

$$\mathbb{E} \left[\sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} |Z_t^n|^\gamma \right] < +\infty \quad \text{and} \quad \mathbb{E} \left[\sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} |Y_t^n|^\gamma \right] < +\infty.$$

Proof. (i) For each $n \in \mathbb{N}$, recall from (3.3) that Z^n is the solution of an ODE with coefficient $W^n(t, z) = J_{0 \leftarrow t}^{\mathbf{X}; z} \psi_n(U_{t \leftarrow 0}^{\mathbf{X}; z} - L_t)$. In view of (3.2) and the fact that $\psi_n \leq \psi_{n+1}$, it follows from the comparison theorem for ODEs that $Z^n \leq Z^{n+1}$. Besides, the mapping $z \mapsto U_{t \leftarrow 0}^{\mathbf{X}; z}$ is increasing since its derivative is $J_{t \leftarrow 0}^{\mathbf{X}; z}$ which, similarly to (3.2), is positive. Hence $Y^n \leq Y^{n+1}$.

To prove the boundedness of Z^n and Y^n , define \tilde{Z}^n as the solution of the following ODE:

$$\tilde{Z}_t^n = y_0 + C_J^{\mathbf{X}} \int_0^t \psi_n \left(U_{s \leftarrow 0}^{\mathbf{X}; \tilde{Z}_s^n} - L_s \right) ds, \quad t \in [0, T],$$

which in view of the bound (3.1) and the comparison principle yields $\tilde{Z}_t^n \geq Z_t^n$. Observing that

$$\begin{aligned} U_{s \leftarrow 0}^{\mathbf{X}; \tilde{Z}_s^n} &= U_{s \leftarrow 0}^{\mathbf{X}; y_0} + U_{s \leftarrow 0}^{\mathbf{X}; \tilde{Z}_s^n} - U_{s \leftarrow 0}^{\mathbf{X}; y_0} \\ &= U_{s \leftarrow 0}^{\mathbf{X}; y_0} + \int_{y_0}^{\tilde{Z}_s^n} J_{s \leftarrow 0}^{\mathbf{X}; z} dz \\ &\geq U_{s \leftarrow 0}^{\mathbf{X}; y_0} + (C_J^{\mathbf{X}})^{-1}(\tilde{Z}_s^n - y_0), \end{aligned}$$

where the last inequality follows from (3.2), it comes that

$$\tilde{Z}_t^n \leq y_0 + C_J^{\mathbf{X}} \int_0^t \psi_n \left(U_{s \leftarrow 0}^{\mathbf{X}; y_0} + (C_J^{\mathbf{X}})^{-1}(\tilde{Z}_s^n - y_0) - L_s \right) ds, \quad t \in [0, T].$$

Note that as the solution of an RDE, $U_{s \leftarrow 0}^{\mathbf{X}; y_0}$ satisfies (see [15, Proposition 8.3]):

$$\|U_{\cdot \leftarrow 0}^{\mathbf{X}; y_0}\|_{\beta, [0, T]} \leq C \left\{ \left(\|\sigma\|_{C_b^2} \|\mathbf{X}\|_{\beta, [0, T]} \right) \vee \left(\|\sigma\|_{C_b^2} \|\mathbf{X}\|_{\beta, [0, T]} \right)^{\frac{1}{\beta}} \right\}, \quad (4.1)$$

where C depends only on β . Hence, denoting temporarily by $C_{\sigma, \mathbf{X}, \beta}$ the right-hand side of the previous inequality, and since $y_0 \geq L_0$,

$$\tilde{Z}_t^n \leq y_0 + C_J^{\mathbf{X}} \int_0^t \psi_n \left(-C_{\sigma, \mathbf{X}, \beta} s^\beta + (C_J^{\mathbf{X}})^{-1}(\tilde{Z}_s^n - y_0) - (L_s - L_0) \right) ds, \quad t \in [0, T], \quad (4.2)$$

so that the process \bar{Z}^n which is the solution to the ODE

$$\bar{Z}_t^n = -C_{\sigma, \mathbf{X}, \beta} t^\beta + \int_0^t \psi_n \left(\bar{Z}_s^n - (L_s - L_0) \right) ds, \quad t \in [0, T]$$

satisfies $-C_{\sigma, \mathbf{X}, \beta} t^\beta + (C_J^{\mathbf{X}})^{-1}(\tilde{Z}_t^n - y_0) \leq \bar{Z}_t^n$, $\forall t \in [0, T]$ (by the comparison principle of ODEs). By Lemma 3.3, \bar{Z}^n satisfies:

$$|\bar{Z}_t^n - (L_t - L_0)| \leq \sqrt{26} \left(C_{\sigma, \mathbf{X}, \beta} t^\beta + \sup_{s \in [0, t]} |L_s - L_0| \right),$$

which then leads to the following bound: there exists $C > 0$ which depends only on σ, β, T such that

$$Z_t^n \leq y_0 + C C_J^{\mathbf{X}} \left((\|\mathbf{X}\|_{\beta, [0, T]} \vee \|\mathbf{X}\|_{\beta, [0, T]}^{\frac{1}{\beta}}) t^\beta + \sup_{s \in [0, t]} |L_s - L_0| \right). \quad (4.3)$$

Moreover, for any $t \geq 0$, $Z_t^n \geq y_0$, hence $\sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} |Z_t^n| < +\infty$. To prove the second part of claim (i), observe that

$$\begin{aligned} |Y_t^n| &= |U_{t \leftarrow 0}^{\mathbf{X}; Z_t^n}| = |U_{t \leftarrow 0}^{\mathbf{X}; y_0} + U_{t \leftarrow 0}^{\mathbf{X}; Z_t^n} - U_{t \leftarrow 0}^{\mathbf{X}; y_0}| \\ &\leq |y_0| + \|U_{\cdot \leftarrow 0}^{\mathbf{X}; y_0}\|_{\beta, [0, t]} t^\beta + C_J^{\mathbf{X}} |Z_t^n - y_0|. \end{aligned} \quad (4.4)$$

Claim (i) then follows from (4.1) and (4.3).

(ii) Now if X is a Gaussian process satisfying the assumptions of Theorem 2.13, it suffices to use the deterministic estimates (4.1), (4.3) and (4.4), as well as the following probabilistic estimates: for any $\gamma \geq 1$,

$$\mathbb{E} \left[\|\mathbf{X}\|_{\beta, [0, T]}^\gamma \right] < \infty, \quad \mathbb{E} \left[(C_J^{\mathbf{X}})^\gamma \right] < \infty \quad \text{and} \quad \mathbb{E} \left[\|L\|_{\beta, [0, T]}^\gamma \right] < \infty \quad (4.5)$$

where the first bound is a classical consequence of Kolmogorov's continuity theorem (which follows from $(\mathbf{H}_{\text{Cov}})$ for any $\beta < \frac{1}{2r}$), the second one is [3, Theorem 6.5] and the third one was an assumption in Theorem 2.13. Then Claim (ii) holds true. \square

Remark 4.2. Observe that in the previous proof, we carefully avoided to estimate directly the Hölder regularity of $t \mapsto \int_0^t \sigma(U_{s \leftarrow 0}^{\mathbf{X}; Z_s^n}) d\mathbf{X}_s$, since any basic a priori estimate would have depended on n . However, we will be able to treat such questions in the next section.

4.2 Uniform (in n) continuity of the sequence of penalised processes

So far we only obtained pointwise convergence of the sequences of paths. Now, we obtain uniform convergence and derive Hölder continuity of the limiting path. This section is organised as follows: Lemmas 4.3 to 4.5 are technical results which will permit to overcome the main difficulty, namely that the negative part of $Y^n - L$ converges to 0 as $n \rightarrow \infty$ (Proposition 4.7). Finally, we prove that this implies the desired uniform convergence of Y^n and Z^n (Proposition 4.8).

For any $p \geq 1$ and any $x \geq 0$, recall that $\phi_p(x) = x \vee x^p$, and define the control functions

$$\forall (s, t) \in \mathcal{S}_{[0, T]}, \quad \kappa_{\mathbf{X}}(s, t) := \phi_p(\|\mathbf{X}\|_{p, [s, t]})^p$$

(note the implicit dependence in p) and

$$\begin{aligned} \forall (s, t) \in \mathcal{S}_{[0, T]}, \quad \kappa_{\mathbf{X}, Z}(s, t) &:= \kappa_{\mathbf{X}}(s, t) + (C_J^{\mathbf{X}}(Z_t - Z_s))^p, \\ \kappa_{\mathbf{X}, Z^n}(s, t) &:= \kappa_{\mathbf{X}}(s, t) + (C_J^{\mathbf{X}}(Z_t^n - Z_s^n))^p. \end{aligned}$$

Furthermore, let us denote by K^n the penalisation term in (2.8),

$$K_t^n := \int_0^t \psi_n(Y_s^n - L_s) ds, \quad t \in [0, T], \quad (4.6)$$

and another control function $\kappa_{\mathbf{X}, K^n}$ related to the variations of K^n :

$$\forall (s, t) \in \mathcal{S}_{[0, T]}, \quad \kappa_{\mathbf{X}, K^n}(s, t) := \kappa_{\mathbf{X}}(s, t) + (\delta K_{s, t}^n)^p. \quad (4.7)$$

The p -variations of K^n are controlled by those of Z^n and reciprocally:

Lemma 4.3. *Consider the continuous process K^n defined in (4.6). Then for any $n \in \mathbb{N}$ and any $(s, t) \in \mathcal{S}_{[0, T]}$,*

$$\delta K_{s, t}^n \leq C_J^{\mathbf{X}} \delta Z_{s, t}^n \quad \text{and} \quad \delta Z_{s, t}^n \leq C_J^{\mathbf{X}} \delta K_{s, t}^n.$$

This implies that for any $q \geq 1$, $\|K^n\|_{q, [s, t]} \leq C_J^{\mathbf{X}} \|Z^n\|_{q, [s, t]}$.

Proof. Using the definition (3.3) of Z^n and the bound (3.1) on J (recall also that J is positive), one has

$$\begin{aligned} K_t^n - K_s^n &= \int_s^t J_{u \leftarrow 0}^{\mathbf{X}; Z_u^n} J_{0 \leftarrow u}^{\mathbf{X}; Z_u^n} \psi_n(Y_u^n - L_u) du \\ &\leq C_J^{\mathbf{X}} \int_s^t J_{0 \leftarrow u}^{\mathbf{X}; Z_u^n} \psi_n(Y_u^n - L_u) du = C_J^{\mathbf{X}} (Z_t^n - Z_s^n). \end{aligned}$$

The converse statement is obtained similarly. \square

In the next two lemmas, it is proven that Y^n and $\sigma(Y^n)$ are controlled by X . In particular, let us set for any $(s, t) \in \mathcal{S}_{[0, T]}$,

$$R_{s, t}^{Y^n} := \delta Y_{s, t}^n - \sigma(Y_s^n) \delta X_{s, t}, \quad (4.8)$$

and

$$R_{s, t}^{\sigma(Y^n)} := \delta \sigma(Y^n)_{s, t} - \sigma'(Y_s^n) \sigma(Y_s^n) \delta X_{s, t}. \quad (4.9)$$

The variations of R^{Y^n} and $R^{\sigma(Y^n)}$, which are the remainder terms in the sense of Definition 2.6, are shown to be bounded.

Lemma 4.4. *The path $\sigma(Y^n)$ is controlled by X and its Gubinelli derivative is $\sigma'(Y^n)\sigma(Y^n)$ ($\sigma'(y)$ is an element of $\mathcal{L}(\mathbb{R}, \mathcal{L}(\mathbb{R}^d, \mathbb{R}))$). In particular, $R^{Y^n} \in \mathcal{C}^{2\beta}([0, T], \mathbb{R})$ and $R^{\sigma(Y^n)} \in \mathcal{C}^{2\beta}([0, T], (\mathbb{R}^d)')$.*

Proof. Recall that Y^n is a solution obtained by approximation, i.e. in the sense of Definition 2.3. The first goal of this proof is to show that Y^n can also be understood as a solution in the sense of controlled rough paths.

We know from Proposition 3.1 that Y^n does not blow up in finite time. Denote $M = \sup_{t \in [0, T]} |Y_t^n|$ and consider a bounded smooth function $\psi_n^{(M)}$ equal to ψ_n on the interval $(-2M, 2M)$. We denote by $Y^{n, M}$ the solution to (2.8) with $\psi_n^{(M)}$ instead of ψ_n . We have $Y_t^{n, M} = Y_t^n$ for all $t \in [0, T]$.

Now consider the following RDE in the augmented form (i.e. without drift):

$$d\widehat{Y}_t^{n, M} = \widehat{\sigma}(\widehat{Y}_t^{n, M})d\widehat{\mathbf{X}}_t, \quad (4.10)$$

where:

- for any $(y, l) \in \mathbb{R}^2$,

$$\widehat{\sigma} \left(\begin{pmatrix} y \\ l \end{pmatrix} \right) = \begin{pmatrix} \sigma(y) & \psi_n^{(M)}(y-l) & 0 \\ 0 \dots 0 & 0 & 1 \end{pmatrix};$$

- $\widehat{\mathbf{X}}$ is the canonical rough path above $\widehat{X}_t = (X_t^T, t, L_t)^T$;
- $\widehat{Y}_t^{n, M} = (Y_t^{n, M}, L_t)^T$ is the solution to the RDE (4.10) in the sense of Definition 2.3.

Since $\widehat{\sigma}$ is smooth and bounded, Equation (4.10) can also be solved in the sense of controlled rough paths (see [15, Theorem 8.4]), and the solution that we denote by $\overline{Y}^{n, M}$ is controlled by \widehat{X} (with Gubinelli derivative $\widehat{\sigma}(\overline{Y}^{n, M})$). Considering that $\widehat{\mathbf{X}}$ can be approximated by a sequence $((\widehat{X}^k, \widehat{\mathbf{X}}^k))_{k \in \mathbb{N}}$ of smooth paths in the α -Hölder rough path topology, with $\alpha < \beta$ and $\widehat{\mathbf{X}}^k$ the Riemann-Stieltjes iterated integral of \widehat{X}^k (see [15, Proposition 2.5]), we can associate a unique solution $\widehat{Y}^{n, M, k}$ to the equation $d\widehat{Y}_t^{n, M, k} = \widehat{\sigma}(\widehat{Y}_t^{n, M, k})d\widehat{X}_t^k$, for each $k \in \mathbb{N}$. In view of the continuity of the Itô-Lyons map $\mathbf{X} \in \mathcal{C}^\beta \mapsto Y \in \mathcal{C}_X^\alpha$, where Y is the solution in the controlled rough paths sense ([15, Theorem 8.5]), we obtain that $\widehat{Y}^{n, M, k}$ converges in α -Hölder norm to $\overline{Y}^{n, M}$. Since $\widehat{Y}^{n, M, k}$ is in fact a solution in the usual sense of ODEs, it also converges in the uniform topology to $\widehat{Y}^{n, M}$. Hence $\widehat{Y}^{n, M} = \overline{Y}^{n, M}$, and as noticed in the first paragraph, $Y^{n, M} = Y^n$, so that the two notions of solution coincide.

In particular, $(\widehat{Y}^{n, M}, \widehat{\sigma}(\widehat{Y}^{n, M})) \in \mathcal{C}_{\widehat{X}}^\beta$. This immediately yields that $(Y^n, (\sigma(Y^n), \psi_n(Y^n - L)))^T$ is controlled by the paths $X_t^a := (X_t^T, t)^T$, which in other words states that the mapping $Q_{s, t}^n := \delta Y_{s, t}^n - (\sigma(Y_s^n), \psi_n(Y_s^n - L_s))\delta X_{s, t}^a$ satisfies $\|Q^n\|_{2\beta, [0, T]} < \infty$. Hence one deduces that $(Y^n, \sigma(Y^n)) \in \mathcal{C}_X^\beta$: Indeed,

$$\begin{aligned} |R_{s, t}^{Y^n}| &= |Q_{s, t}^n + \psi_n(Y_s^n - L_s)(t - s)| \\ &\leq |Q_{s, t}^n| + \|\psi_n(Y^n - L)\|_{\infty, [0, T]}(t - s) \leq |Q_{s, t}^n| + n\|Y^n - L\|_{\infty, [0, T]}(t - s). \end{aligned}$$

Thus it follows from the above and Proposition 4.1 that

$$\|R^{Y^n}\|_{2\beta, [0, T]} \leq C (\|Q^n\|_{2\beta, [0, T]} + n\|Y^n - L\|_{\infty, [0, T]}T^{1-\beta}) < \infty,$$

and in particular the Gubinelli derivative of Y^n is $\sigma(Y^n) \in \mathcal{C}^\beta([0, T]; (\mathbb{R}^d)')$.

Finally, $\sigma(Y^n)$ is controlled by X with Gubinelli derivative $\sigma'(Y^n)\sigma(Y^n)$ (see [15, Lemma 7.3]). \square

The previous lemma is now refined to get estimates on Y^n and $R^{\sigma(Y^n)}$ which are uniform in n . These will be crucial in the limiting procedure that leads to the proof of Theorem 2.12.

Lemma 4.5. (i) *Under the assumptions of Theorem 2.12, one has the following inequality:*

$$\Theta := \sup_{n \in \mathbb{N}} \left(\|\sigma'(Y^n)\sigma(Y^n)\|_{p, [0, T]} + \|R^{\sigma(Y^n)}\|_{\frac{p}{2}, [0, T]} \right) < \infty. \quad (4.11)$$

(ii) *If in addition, the assumptions of Theorem 2.13 hold, then $\mathbb{E}(\Theta^\gamma) < \infty$, for any $\gamma \geq 1$.*

Proof. **1st step.** By a Taylor expansion,

$$\delta\sigma(Y^n)_{s,t} = \sigma'(Y_s^n)\delta Y_{s,t}^n + \int_{Y_s^n}^{Y_t^n} \sigma''(y)(Y_t^n - y)dy. \quad (4.12)$$

Hence the combination of (4.8), (4.9) and (4.12) yields

$$R_{s,t}^{\sigma(Y^n)} = \sigma'(Y_s^n)R_{s,t}^{Y^n} + \int_{Y_s^n}^{Y_t^n} \sigma''(y)(Y_t^n - y)dy \quad (4.13)$$

and

$$|R_{s,t}^{\sigma(Y^n)}| \leq \|\sigma'\|_\infty |R_{s,t}^{Y^n}| + \frac{1}{2}\|\sigma''\|_\infty (Y_t^n - Y_s^n)^2 \leq C \left(|R_{s,t}^{Y^n}| + (Y_t^n - Y_s^n)^2 \right). \quad (4.14)$$

2nd step. From Lemma 4.4, the Definition (4.8) of R^{Y^n} , and inequality (2.10) applied to $|\int_s^t \sigma(Y_u^n)d\mathbf{X}_u - \sigma(Y_s^n)\delta X_{s,t}|$, one gets

$$|R_{s,t}^{Y^n}| \leq \delta K_{s,t}^n + |\sigma'(Y_s^n)\sigma(Y_s^n)|\|\mathbb{X}_{s,t}\| + C_p \left(\|X\|_{p,[s,t]}\|R^{\sigma(Y^n)}\|_{\frac{p}{2},[s,t]} + \|\sigma'(Y^n)\sigma(Y^n)\|_{p,[s,t]}\|\mathbb{X}\|_{\frac{p}{2},[s,t]} \right).$$

Hence

$$|R_{s,t}^{Y^n}| \leq \delta K_{s,t}^n + M \left(\|\mathbb{X}\|_{\frac{p}{2},[s,t]} + \|X\|_{p,[s,t]}(\|R^{Y^n}\|_{\frac{p}{2},[s,t]} + \|Y^n\|_{p,[s,t]}^2) + \|Y^n\|_{p,[s,t]}\|\mathbb{X}\|_{\frac{p}{2},[s,t]} \right), \quad (4.15)$$

for some M that depends only on p , the uniform norms of σ and its derivatives. Thus for any $(s, t) \in \mathcal{S}_{[0,T]}$ such that $|t - s| \leq \delta_{\mathbf{X}}$, where

$$\delta_{\mathbf{X}} := T \wedge \sup \left\{ \delta > 0 : \kappa_{\mathbf{X}}(s, t)^{\frac{1}{p}} \leq \frac{1}{2}M^{-1}, \forall s, t \in [0, T] \text{ s.t. } |t - s| \leq \delta \right\}, \quad (4.16)$$

one gets

$$\begin{aligned} \|R^{Y^n}\|_{\frac{p}{2},[s,t]} &\leq 2\delta K_{s,t}^n + 2M\|\mathbb{X}\|_{\frac{p}{2},[s,t]} + \|Y^n\|_{p,[s,t]}^2 + \|Y^n\|_{p,[s,t]}\|\mathbb{X}\|_{\frac{p}{2},[s,t]} \\ &\leq 2\delta K_{s,t}^n + (2M + 1)\|\mathbb{X}\|_{\frac{p}{2},[s,t]} + 2\|Y^n\|_{p,[s,t]}^2. \end{aligned} \quad (4.17)$$

3rd step. We will now be able to bound $\|Y^n\|_p$. By the definition (4.8) of R^{Y^n} , and using estimate (4.17),

$$\begin{aligned} |\delta Y_{s,t}^n| &\leq |R_{s,t}^{Y^n}| + |\sigma(Y_s^n)|\|\delta X_{s,t}\| \\ &\leq 2\delta K_{s,t}^n + (2M + 1)\|\mathbb{X}\|_{\frac{p}{2},[s,t]} + \|\sigma\|_\infty \|X\|_{p,[s,t]} + 2\|Y^n\|_{p,[s,t]}^2. \end{aligned}$$

Multiplying the equation by 2 and in view of the definition (4.7) of $\kappa_{\mathbf{X},K^n}$, the previous equation reads

$$2\|Y^n\|_{p,[s,t]} \leq \widetilde{M}\kappa_{\mathbf{X},K^n}(s, t)^{\frac{1}{p}} + 4\|Y^n\|_{p,[s,t]}^2,$$

for some \widetilde{M} which depends only on p and σ . Arguing as in [15, p.111-112] (with the notations of [15], $\psi_h \equiv 2\|Y^n\|_{p,[s,t]}$ and $\lambda_h \equiv \widetilde{M}\kappa_{\mathbf{X},K^n}(s, t)^{\frac{1}{p}}$) one obtains that for any $(s, t) \in \mathcal{S}_{[0,T]}$ such that $\widetilde{M}\kappa_{\mathbf{X},K^n}(s, t)^{\frac{1}{p}} < \frac{1}{4}$ and

$$\frac{1}{2} - \sqrt{\frac{1}{4} - \widetilde{M}\kappa_{\mathbf{X},K^n}(s, t)^{\frac{1}{p}}} < \frac{1}{6}, \quad (4.18)$$

then for any such s and t ,

$$\|Y^n\|_{p,[s,t]} \leq \widetilde{M}\kappa_{\mathbf{X},K^n}(s, t)^{\frac{1}{p}}. \quad (4.19)$$

Observe that (4.18) is fulfilled if and only if $\kappa_{\mathbf{X},K^n}(s,t) < \left(\frac{5}{36}\widetilde{M}^{-1}\right)^p$. Hence by a classical argument, one gets that for any $(s,t) \in \mathcal{S}_{[0,T]}$,

$$\begin{aligned} \|Y^n\|_{p,[s,t]}^p &\leq \widetilde{M} \left\{ 1 \vee \left(2 \left(\frac{36}{5}\widetilde{M} \right)^p \kappa_{\mathbf{X},K^n}(s,t) \right)^{p-1} \right\} \kappa_{\mathbf{X},K^n}(s,t) \\ &\leq C (\kappa_{\mathbf{X},K^n}(s,t) \vee \kappa_{\mathbf{X},K^n}(s,t)^p), \end{aligned} \quad (4.20)$$

for some $C > 0$ that depends only on p and σ .

Note that using the Doss-Sussmann representation of Y^n , we could have obtained more easily that

$$\forall (s,t) \in \mathcal{S}_{[0,T]}, \quad \|Y^n\|_{p,[s,t]} \leq C \phi_p \left(\|\mathbf{X}\|_{p,[s,t]} \right) + C_J^{\mathbf{X}} (Z_t^n - Z_s^n).$$

However, this bound is weaker than (4.20) since it can be deduced from it using Lemma 4.3. Most importantly, it involves $C_J^{\mathbf{X}}$ which does not have exponential moments (when \mathbf{X} is a Gaussian rough path satisfying $(\mathbf{H}_{\text{Cov}})$) and this would bring issues in Section 5 when Gronwall arguments are to be used.

4th Step. In view of (4.20), it comes from (4.17) that for any $(s,t) \in \mathcal{S}_{[0,T]}$ such that $|t-s| \leq \delta_{\mathbf{X}}$,

$$\|R^{Y^n}\|_{\frac{p}{2},[s,t]}^{\frac{p}{2}} \leq C (\kappa_{\mathbf{X},K^n}(s,t) \vee \kappa_{\mathbf{X},K^n}(s,t)^p). \quad (4.21)$$

Thus (4.14) implies that for any $(s,t) \in \mathcal{S}_{[0,T]}$ such that $|t-s| \leq \delta_{\mathbf{X}}$,

$$|R_{s,t}^{\sigma(Y^n)}| \leq C (\kappa_{\mathbf{X},K^n}(s,t) \vee \kappa_{\mathbf{X},K^n}(s,t)^p)^{\frac{2}{p}}. \quad (4.22)$$

Then for any $(s,t) \in \mathcal{S}_{[0,T]}$,

$$\begin{aligned} \|R^{\sigma(Y^n)}\|_{\frac{p}{2},[s,t]}^{\frac{p}{2}} &= \sup_{\pi=(t_i) \subset [s,t]} \left(\sum_{t_{i+1}-t_i \leq \delta_{\mathbf{X}}} |R_{t_i,t_{i+1}}^{\sigma(Y^n)}|^{\frac{p}{2}} + \sum_{t_{i+1}-t_i > \delta_{\mathbf{X}}} |R_{t_i,t_{i+1}}^{\sigma(Y^n)}|^{\frac{p}{2}} \right) \\ &\leq C (\kappa_{\mathbf{X},K^n}(s,t) \vee \kappa_{\mathbf{X},K^n}(s,t)^p) + \sup_{\pi=(t_i) \subset [s,t]} \sum_{t_{i+1}-t_i > \delta_{\mathbf{X}}} |R_{t_i,t_{i+1}}^{\sigma(Y^n)}|^{\frac{p}{2}}. \end{aligned} \quad (4.23)$$

Hereafter, we assume that $t-s > \delta_{\mathbf{X}}$. By a simple induction, one can verify that for any $s_0 < s_1 < \dots < s_N$,

$$R_{s_0,s_N}^{\sigma(Y^n)} = \sum_{k=0}^{N-1} R_{s_k,s_{k+1}}^{\sigma(Y^n)} + \sum_{k=1}^{N-1} \delta R_{s_0,s_k,s_{k+1}}^{\sigma(Y^n)},$$

where $\delta R_{s_0,s_k,s_{k+1}}^{\sigma(Y^n)} = R_{s_0,s_{k+1}}^{\sigma(Y^n)} - R_{s_0,s_k}^{\sigma(Y^n)} - R_{s_k,s_{k+1}}^{\sigma(Y^n)}$. In view of (4.9), there is $\delta R_{s_0,s_k,s_{k+1}}^{\sigma(Y^n)} = -\delta(\sigma'\sigma(Y^n))_{s_0,s_k} \delta X_{s_k,s_{k+1}}$. Hence

$$\begin{aligned} |\delta R_{s_0,s_k,s_{k+1}}^{\sigma(Y^n)}| &\leq C |\delta Y_{s_0,s_k}^n| |\delta X_{s_k,s_{k+1}}| \\ &\leq C |\delta X_{s_k,s_{k+1}}| \sum_{j=0}^{k-1} |\delta Y_{s_j,s_{j+1}}^n|. \end{aligned} \quad (4.24)$$

For any index i in (4.23) such that $t_{i+1} - t_i > \delta_{\mathbf{X}}$, denote $K_i = \lfloor \frac{t_{i+1}-t_i}{\delta_{\mathbf{X}}} \rfloor$ ($\leq \frac{T}{\delta_{\mathbf{X}}}$), $t_{i,k} = t_i + k\delta_{\mathbf{X}}$ for any $k = 0 \dots K_i$ and $t_{i,K_i+1} = t_{i+1}$. We obtain

$$\begin{aligned} \sum_{t_{i+1}-t_i > \delta_{\mathbf{X}}} |R_{t_i,t_{i+1}}^{\sigma(Y^n)}|^{\frac{p}{2}} &= \sum_{t_{i+1}-t_i > \delta_{\mathbf{X}}} \left| \sum_{k=0}^{K_i} R_{t_i,k,t_{i,k+1}}^{\sigma(Y^n)} + \delta R_{t_i,t_i,t_{i,k+1}}^{\sigma(Y^n)} \right|^{\frac{p}{2}} \\ &\leq \sum_{t_{i+1}-t_i > \delta_{\mathbf{X}}} \left(2 \frac{t-s}{\delta_{\mathbf{X}}} \right)^{\frac{p}{2}-1} \sum_{k=0}^{K_i} |R_{t_i,k,t_{i,k+1}}^{\sigma(Y^n)}|^{\frac{p}{2}} + |\delta R_{t_i,t_i,t_{i,k+1}}^{\sigma(Y^n)}|^{\frac{p}{2}} \\ &\leq \left(2 \frac{t-s}{\delta_{\mathbf{X}}} \right)^{\frac{p}{2}-1} \sum_{t_{i+1}-t_i > \delta_{\mathbf{X}}} \left\{ \sum_{k=0}^{K_i} |R_{t_i,k,t_{i,k+1}}^{\sigma(Y^n)}|^{\frac{p}{2}} + C \left(\sum_{k=0}^{K_i} |\delta Y_{t_i,k,t_{i,k+1}}^n| \right)^{\frac{p}{2}} \sum_{k=0}^{K_i} |\delta X_{t_i,k,t_{i,k+1}}|^{\frac{p}{2}} \right\}, \end{aligned}$$

using (4.24) in the last inequality. Hence

$$\begin{aligned}
& \sum_{t_{i+1}-t_i > \delta_{\mathbf{X}}} |R_{t_i, t_{i+1}}^{\sigma(Y^n)}|^{\frac{p}{2}} \\
& \leq C \left(\frac{t-s}{\delta_{\mathbf{X}}} \right)^{\frac{p}{2}-1} \sum_{t_{i+1}-t_i > \delta_{\mathbf{X}}} \left\{ \sum_{k=0}^{K_i} |R_{t_i, k, t_i, k+1}^{\sigma(Y^n)}|^{\frac{p}{2}} + \left(\frac{t-s}{\delta_{\mathbf{X}}} \right)^{p-1} \sum_{k=0}^{K_i} |\delta Y_{t_i, k, t_i, k+1}^n|^p + \frac{t-s}{\delta_{\mathbf{X}}} \sum_{k=0}^{K_i} |\delta X_{t_i, k, t_i, k+1}|^p \right\} \\
& \leq C \left(\frac{t-s}{\delta_{\mathbf{X}}} \right)^{\frac{p}{2}-1} \sum_{t_{i+1}-t_i > \delta_{\mathbf{X}}} \sum_{k=0}^{K_i} |R_{t_i, k, t_i, k+1}^{\sigma(Y^n)}|^{\frac{p}{2}} + C \left(\frac{t-s}{\delta_{\mathbf{X}}} \right)^{\frac{3p}{2}-2} \left(\|Y^n\|_{p, [s, t]}^p + \|X\|_{p, [s, t]}^p \right),
\end{aligned}$$

Now, since $t_{i, k+1} - t_{i, k} \leq \delta_{\mathbf{X}}$, we can use (4.22) and the super-additivity of $(s, t) \mapsto \kappa_{\mathbf{X}, K^n}(s, t) \vee \kappa_{\mathbf{X}, K^n}(s, t)^p$ to get that

$$\sum_{t_{i+1}-t_i > \delta_{\mathbf{X}}} \sum_{k=0}^{K_i} |R_{t_i, k, t_i, k+1}^{\sigma(Y^n)}|^{\frac{p}{2}} \leq C (\kappa_{\mathbf{X}, K^n}(s, t) \vee \kappa_{\mathbf{X}, K^n}(s, t)^p).$$

Eventually, one gets that

$$\sum_{t_{i+1}-t_i > \delta_{\mathbf{X}}} |R_{t_i, t_{i+1}}^{\sigma(Y^n)}|^{\frac{p}{2}} \leq C \left(\frac{t-s}{\delta_{\mathbf{X}}} \right)^{\frac{3p}{2}-2} \left(\kappa_{\mathbf{X}, K^n}(s, t) \vee \kappa_{\mathbf{X}, K^n}(s, t)^p + \|Y^n\|_{p, [s, t]}^p + \|X\|_{p, [s, t]}^p \right),$$

so that using again (4.20),

$$\|R^{\sigma(Y^n)}\|_{\frac{p}{2}, [s, t]}^{\frac{p}{2}} \leq C \left(\frac{t-s}{\delta_{\mathbf{X}}} \right)^{\frac{3p}{2}-2} (\kappa_{\mathbf{X}, K^n}(s, t) \vee \kappa_{\mathbf{X}, K^n}(s, t)^p). \quad (4.25)$$

In view of Lemma 4.3, $\kappa_{\mathbf{X}, K^n}(s, t) \leq (C_J^{\mathbf{X}} \delta Z_{s, t}^n)^p + \|\mathbb{X}\|_{\frac{p}{2}, [s, t]}^p + \|X\|_{p, [s, t]}^p$ and since $Z^n \leq Z^{n+1}$ (and $Z_0^n = Z_0^{n+1} = y_0$),

$$\|R^{\sigma(Y^n)}\|_{\frac{p}{2}, [0, T]}^{\frac{p}{2}} \leq C (\delta_{\mathbf{X}})^{2-\frac{3p}{2}} \left((C_J^{\mathbf{X}} Z_T)^p \vee (C_J^{\mathbf{X}} Z_T)^{p^2} + \phi_p \left(\|\mathbb{X}\|_{p, [0, T]} \right)^p \right). \quad (4.26)$$

Finally, since we assumed that $\|X\|_{\beta} < \infty$, it follows that $\|X\|_{p, [s, t]} \leq \|X\|_{\beta} |t-s|^{\beta}$ and then (since $\beta = p^{-1}$) that $\delta_{\mathbf{X}} \geq \frac{1}{2} (\|X\|_{\beta} C)^{-p} > 0$. This proves (i).

(ii) Based on (4.20) and the observation of the previous paragraph,

$$\|Y^n\|_{p, [s, t]} \leq C \left((C_J^{\mathbf{X}} Z_T)^p \vee (C_J^{\mathbf{X}} Z_T)^{p^2} + \phi_p \left(\|\mathbb{X}\|_{p, [0, T]} \right)^p \right).$$

Besides, from the combination of inequalities (4.3) and (4.5), one gets that $\mathbb{E} \left[\sup_{n \in \mathbb{N}} \|Y^n\|_{p, [0, T]}^{\gamma} \right] < \infty$, $\forall \gamma \geq 1$. Using the regularity of σ , we deduce that $\mathbb{E} \left[\sup_{n \in \mathbb{N}} \|\sigma'(Y^n) \sigma(Y^n)\|_{p, [0, T]}^{\gamma} \right] < \infty$.

One can verify that $\delta_{\mathbf{X}} \geq T(2M)^{-\frac{1}{p}} \kappa_{\mathbf{X}}(0, T)^{-1}$, so that for any $\gamma \geq 1$,

$$\mathbb{E} [\delta_{\mathbf{X}}^{(2-\frac{3p}{2})\gamma}] \leq C \mathbb{E} [\kappa_{\mathbf{X}}(s, t)^{(\frac{3p}{2}-2)\gamma}] < \infty. \quad (4.27)$$

In view of (4.26) and the previous inequality, it follows that $\mathbb{E} \left[\sup_{n \in \mathbb{N}} \|R^{\sigma(Y^n)}\|_{p, [0, T]}^{\gamma} \right] < \infty$. \square

Corollary 4.6. *Recall that $\beta = \frac{1}{p}$. The rough integral in (2.8) is β -Hölder continuous on $[0, T]$, uniformly in n : there exists $C > 0$ depending only on β , $\|\sigma\|_{\infty}$ and $\|\sigma'\|_{\infty}$ such that*

$$\forall n \in \mathbb{N}, \forall s, t \in [0, T], \quad \left| \int_s^t \sigma(Y_u^n) d\mathbf{X}_u \right| \leq C(1 + \Theta) \|\mathbb{X}\|_{\beta} |t-s|^{\beta},$$

where Θ was defined (independently of n) in (4.11).

Proof. In view of Lemmas 4.4 and 4.5, Theorem 2.7 implies that $\|\int_0^\cdot \sigma(Y_u^n) d\mathbf{X}_u\|_{p,[s,t]}$ is bounded from above by some control function which is independent of n :

$$\begin{aligned} \left\| \int_0^\cdot \sigma(Y_u^n) d\mathbf{X}_u \right\|_{p,[s,t]} &\leq \|\sigma(Y^n)\|_{\infty,[s,t]} \|X\|_{p,[s,t]} + \|\sigma'(Y^n)\sigma(Y^n)\|_{\infty,[s,t]} \|\mathbb{X}\|_{\frac{p}{2},[s,t]} \\ &\quad + C_p \left(\|X\|_{p,[s,t]} \|R^{\sigma(Y^n)}\|_{\frac{p}{2},[s,t]} + \|\mathbb{X}\|_{\frac{p}{2},[s,t]} \|\sigma'(Y^n)\sigma(Y^n)\|_{p,[s,t]} \right) \\ &\leq C(1 + \Theta) \left(\|X\|_{p,[s,t]} + \|\mathbb{X}\|_{\frac{p}{2},[s,t]} \right), \end{aligned}$$

where Θ was defined (independently of n) in (4.11). To conclude the proof, it remains to notice that since $(X, \mathbb{X}) \in \mathcal{C}^\beta$, $\|X\|_{p,[s,t]} \leq \|X\|_\beta |t-s|^\beta$ and $\|\mathbb{X}\|_{\frac{p}{2},[s,t]} \leq \|\mathbb{X}\|_{2\beta} |t-s|^{2\beta}$. \square

Proposition 4.7. (i) Under the assumptions of Theorem 2.12, there exists $C > 0$ which depends only on p and T , such that for any $n \in \mathbb{N}^*$,

$$\sup_{s \in [0, T]} (Y_s^n - L_s)_- \leq C \left(1 + (1 + \Theta) \|\mathbf{X}\|_\beta + \|L\|_\beta \right) n^{-\beta},$$

where Θ was defined in (4.11). In particular, $\lim_{n \rightarrow \infty} \sup_{s \in [0, T]} (Y_s^n - L_s)_- = 0$.

(ii) If in addition, the assumptions of Theorem 2.13 hold, then $\lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{s \in [0, T]} |(Y_s^n - L_s)_-|^\gamma \right] = 0$, for any $\gamma \geq 1$.

Proof. (i) Recall that $(x)_- \leq \frac{1}{n} \psi_n(x) + \frac{1}{2n}$. Applying Lemma 3.3(ii) and Corollary 4.6, one gets that

$$\begin{aligned} \forall n, \quad \sup_{s \in [0, T]} (Y_s^n - L_s)_- &\leq C \left(1 + \|L\|_\beta + \left\| \int_0^\cdot \sigma(Y_u^n) d\mathbf{X}_u \right\|_\beta \right) n^{-\beta} + \frac{1}{2n} \\ &\leq C \left(1 + \|L\|_\beta + (1 + \Theta) \|\mathbf{X}\|_\beta \right) n^{-\beta}. \end{aligned} \quad (4.28)$$

which is the desired result.

(ii) Using (4.5) and Lemma 4.5 (ii), one gets that $\mathbb{E}[(1 + \Theta) \|\mathbf{X}\|_\beta]^\gamma < \infty$, $\forall \gamma \geq 1$, so the result follows from (4.28). \square

Proposition 4.8. (i) Under the assumptions of Theorem 2.12, $(Y^n)_{n \in \mathbb{N}}$ converges uniformly to some process $\{Y_t\}_{t \in [0, T]} \in \mathcal{C}^\beta$.

(ii) If in addition, the driving noise X is Gaussian and the assumptions of Theorem 2.13 are satisfied, then the convergence happens in $L^\gamma(\Omega; (\mathcal{C}^0, \|\cdot\|_{\infty, [0, T]})$) (i.e. as in (2.11)), $\forall \gamma \geq 1$. Besides, Y has a β -Hölder continuous modification and $\mathbb{E}[\|Y\|_{\beta, [0, T]}^\gamma] < \infty$, $\forall \gamma \geq 1$.

Proof. (i) This proof uses estimates which are similar to those in the proof of Proposition 4.1, but this time a bound on the increment $Z_t^n - Z_s^n$ is needed.

For any $s \in [0, T]$, define $\{\tilde{Z}_{t \leftarrow s}^n\}_{t \in [s, T]}$ as the solution of the following (random) ODE:

$$\tilde{Z}_{t \leftarrow s}^n = Z_s^n + C_J^{\mathbf{X}} \int_s^t \psi_n \left(U_{u \leftarrow 0}^{\mathbf{X}; \tilde{Z}_{u \leftarrow s}^n} - L_u \right) du, \quad t \in [s, T],$$

Once again, there is $\tilde{Z}_{t \leftarrow s}^n \geq Z_t^n$ and for $C_{\sigma, \mathbf{X}, \beta}$ as in (4.2),

$$\tilde{Z}_{t \leftarrow s}^n \leq Z_s^n + C_J^{\mathbf{X}} \int_s^t \psi_n \left(Y_s^n - C_{\sigma, \mathbf{X}, \beta} (u-s)^\beta + (C_J^{\mathbf{X}})^{-1} (\tilde{Z}_{u \leftarrow s}^n - \tilde{Z}_{s \leftarrow s}^n) - L_u \right) du, \quad t \in [s, T],$$

but unlike in (4.2), it is no longer true that the starting point Y_s^n is larger than L_s . Thus we only get that

$$\tilde{Z}_{t \leftarrow s}^n \leq Z_s^n + C_J^{\mathbf{X}} \int_s^t \psi_n \left(-(Y_s^n - L_s)_- - C_{\sigma, \mathbf{X}, \beta} (u-s)^\beta + (C_J^{\mathbf{X}})^{-1} (\tilde{Z}_{u \leftarrow s}^n - \tilde{Z}_{s \leftarrow s}^n) - (L_u - L_s) \right) ds, \quad t \in [s, T].$$

As in (4.3), one can then verify that the previous bound leads to

$$\forall t \in [s, T], \quad Z_t^n \leq \tilde{Z}_{t \leftarrow s}^n \leq Z_s^n + C C_J^{\mathbf{X}} \left((\|\mathbf{X}\|_{\beta, [0, T]} \vee \|\mathbf{X}\|_{\beta, [0, T]}^{\frac{1}{\beta}}) (t-s)^\beta + \sup_{u \in [s, t]} |L_u - L_s| + (Y_s^n - L_s)_- \right),$$

where C depends only on σ, β, T (an in particular not in n or s). Now as in (4.4),

$$\begin{aligned} |Y_t^n - Y_s^n| &\leq \sup_{y \in \mathbb{R}} \|U_{\cdot \leftarrow 0}^{\mathbf{X}; y}\|_{\beta, [0, T]} (t-s)^\beta + C_J^{\mathbf{X}} |Z_t^n - Z_s^n| \\ &\leq C (C_J^{\mathbf{X}})^2 \left((\|\mathbf{X}\|_{\beta, [0, T]} \vee \|\mathbf{X}\|_{\beta, [0, T]}^{\frac{1}{\beta}}) (t-s)^\beta + \sup_{u \in [s, t]} |L_u - L_s| + (Y_s^n - L_s)_- \right). \end{aligned} \quad (4.29)$$

Using Proposition 4.7, we can now take the (pointwise) limit as $n \rightarrow \infty$ in the two previous inequalities to get that for any $t \in [s, T]$,

$$\begin{aligned} Z_t &\leq Z_s + C C_J^{\mathbf{X}} \left((\|\mathbf{X}\|_{\beta, [0, T]} \vee \|\mathbf{X}\|_{\beta, [0, T]}^{\frac{1}{\beta}}) (t-s)^\beta + \sup_{u \in [s, t]} |L_u - L_s| \right) \\ \text{and } |Y_t - Y_s| &\leq C (C_J^{\mathbf{X}})^2 \left((\|\mathbf{X}\|_{\beta, [0, T]} \vee \|\mathbf{X}\|_{\beta, [0, T]}^{\frac{1}{\beta}}) (t-s)^\beta + \sup_{u \in [s, t]} |L_u - L_s| \right). \end{aligned}$$

Hence Z and Y are (Hölder-)continuous, so arguing with Dini's Theorem, we are now able to conclude that the convergences are uniform.

(ii) Under the assumptions of Theorem 2.13, one deduces from the previous point that $\lim_{n \rightarrow \infty} \|Y^n - Y\|_{\infty, [0, T]} = 0$ almost surely. Moreover, Proposition 4.1 states that $(Y^n)_{n \in \mathbb{N}}$ is a nondecreasing sequence. Thus $\|Y^n - Y\|_{\infty, [0, T]} \leq 2\|Y\|_{\infty, [0, T]}$, and since $\mathbb{E}[\|Y\|_{\infty, [0, T]}^\gamma] < \infty$ (by Proposition 4.1 (ii)), the convergence result is obtained by using Lebesgue's theorem.

The Hölder continuity of Y is a consequence of (4.29) and Proposition 4.7 (ii). \square

4.3 Proof of Theorems 2.12 and 2.13: Identification of the limit process Y

To achieve the proof of Theorem 2.12, we will show that $(Y, \sigma(Y)) \in \mathcal{C}_X^\beta$ and deduce that $\int_0^\cdot \sigma(Y_s) d\mathbf{X}_s$ is the uniform limit of the sequence $\{\int_0^\cdot \sigma(Y_s^n) d\mathbf{X}_s\}_{n \in \mathbb{N}}$ and that K^n converges uniformly to a non-decreasing path K . Then, by checking the properties (i), (ii), (iii) and (iv) of Definition 2.9, we will be able to prove that the couple (Y, K) so constructed is indeed solution to the Skorokhod problem $SP(\sigma, L)$.

Step 1: convergence of the rough integral.

Proposition 4.9. *The path $\sigma(Y)$ is controlled by X , i.e. $(\sigma(Y), \sigma'(Y)\sigma(Y)) \in \mathcal{C}_X^\beta$, and the following convergence happens in $\mathcal{C}^0([0, T], \mathbb{R})$:*

$$\lim_{n \rightarrow \infty} \left\| \int_0^\cdot \sigma(Y_s^n) d\mathbf{X}_s - \int_0^\cdot \sigma(Y_s) d\mathbf{X}_s \right\|_{\infty, [0, T]} = 0.$$

Proof. Our aim is to check that Proposition 2.8 can be applied. First recall that $\sigma(Y^n)$ is controlled by X and that its Gubinelli derivative is $\sigma'(Y^n)\sigma(Y^n)$ (Lemma 4.4). By Proposition 4.8, $\sigma'(Y^n)\sigma(Y^n)$ converges uniformly to $\sigma'(Y)\sigma(Y)$. Similarly, $R_{s,t}^{\sigma(Y^n)}$ converges uniformly a.s. to $R_{s,t}^{\sigma(Y)} := \delta\sigma(Y)_{s,t} - \sigma'(Y_s)\sigma(Y_s)\delta X_{s,t}$. Hence in view of Lemma 4.5, the assumptions of Proposition 2.8 are matched and the desired result follows. \square

A direct consequence of the previous proposition and of Proposition 4.8 is that K^n converges (uniformly) to a limit path K so that

$$\forall t \in [0, T], \quad Y_t = y_0 + \int_0^t \sigma(Y_s) d\mathbf{X}_s + K_t.$$

As a limit of non-decreasing paths, K is non-decreasing. Hence the properties (i) and (iii) of Definition 2.9 are verified.

Step 2: $Y \geq L$. This is the result of Proposition 4.7. Thus property (ii) of Definition 2.9 is satisfied.

Step 3: points of increase of K . By the uniform convergence of K^n and the non-decreasing property of K^n and K , it follows that dK^n weakly converges towards dK and since Y^n converges uniformly to Y ,

$$0 \geq \int_0^t (Y_s^n - L_s) \psi_n(Y_s^n - L_s) ds = \int_0^t (Y_s^n - L_s) dK_s^n \rightarrow \int_0^t (Y_s - L_s) dK_s,$$

where the last integral exists in the sense of Lebesgue-Stieltjes integrals, since K is a non-decreasing path. Since $Y_s - L_s \geq 0$ (by the previous step) and K is non-decreasing, it follows that $\int_0^t (Y_s - L_s) dK_s \geq 0$. Hence for any $t \in [0, T]$, $\int_0^t (Y_s - L_s) dK_s = 0$, which proves that the point (iv) is satisfied.

Remark 4.10. *In view of the previous Steps 1 to 3, we conclude that (Y, K) is a solution to $SP(\sigma, L)$, which achieves the proof of the existence part in Theorem 2.12. In addition, if X is a Gaussian process satisfying Assumption (H_{Cov}) , we obtained all along Section 4 the probabilistic estimates to ensure that Theorem 2.13 holds.*

4.4 Proof of Theorems 2.12 and 2.13: Uniqueness

Uniqueness is in general the most tricky part in reflection problems. Here we rely on earlier results in the literature. In the case $\beta > \frac{1}{2}$, the uniqueness of the reflected solution is due to Falkowski and Słomiński [12].

In the case $\beta \leq \frac{1}{2}$, the uniqueness of the reflected RDE has been proven recently by Deya, Gubinelli, Hofmanová, and Tindel [7]. The difference between our work and [7] is that they have a fixed boundary process $L \equiv 0$. But their proof of uniqueness can adapt to a moving boundary, as will become clear in Section 5, where we implement their method to get new results.

5 Rate of convergence of the sequence of penalised processes

5.1 A priori estimate

For each $n \in \mathbb{N}$, we introduce the operator δ_n which acts on functionals of Y and Y^n as follows: for any $\Phi : \mathcal{C}([0, T]; \mathbb{R}^d) \rightarrow \mathcal{C}([0, T]; \mathbb{R}^d)$,

$$\delta_n \Phi(Y) = \Phi(Y) - \Phi(Y^n).$$

For instance, we shall write $\delta_n Y_s = Y_s - Y_s^n$, $\delta_n \sigma(Y)_s = \sigma(Y_s) - \sigma(Y_s^n)$ and also $\delta(\delta_n Y)_{s,t} = \delta_n(\delta Y_{s,t}) = Y_t - Y_s - Y_t^n + Y_s^n$, etc.

By linearity, $\delta_n Y$ has Gubinelli derivative $\delta_n \sigma(Y)$, i.e. $(\delta_n Y, \delta_n \sigma(Y)) \in \mathcal{V}_X^p$, and $R_{s,t}^{\delta_n Y} := \delta(\delta_n Y)_{s,t} - \delta_n \sigma(Y)_s \delta X_{s,t} \in \mathcal{V}^{\frac{p}{2}}$. Similarly, $\delta_n \sigma(Y)$ has Gubinelli derivative $\delta_n \sigma' \sigma(Y)$, i.e. $(\delta_n \sigma(Y), \delta_n \sigma' \sigma(Y)) \in \mathcal{V}_X^p$, and $R_{s,t}^{\delta_n \sigma(Y)} := \delta(\delta_n \sigma(Y))_{s,t} - \delta_n \sigma' \sigma(Y)_s \delta X_{s,t} \in \mathcal{V}^{\frac{p}{2}}$. Note that $R^{\delta_n Y} = \delta_n R^Y$ and that $R^{\delta_n \sigma(Y)} = \delta_n R^{\sigma(Y)}$.

We also need the control function associated to (K, K^n) . Thus for each $n \in \mathbb{N}$, define $w_n : \mathcal{S}_{[0, T]} \rightarrow \mathbb{R}_+$ as follows:

$$w_n(s, t) = \|(K, K^n)\|_{1\text{-var}, [s, t]} = \delta K_{s,t} + \delta K_{s,t}^n.$$

It will be enough in the sequel to get bounds on the remainder $R^{\sigma(Y^n)}$ on the “small” set $\mathcal{S}_{\mathbf{X}}$ constructed from the quantity $\delta_{\mathbf{X}}$ defined previously (see Equation (4.16)):

$$\mathcal{S}_{\mathbf{X}} := \{(u, v) \in \mathcal{S}_{[0, T]} : v - u \leq \delta_{\mathbf{X}}\}. \quad (5.1)$$

Finally, define the control functions $\kappa_{\mathbf{X},w_n}$ and $\tilde{\kappa}_{\mathbf{X},w_n}$ by

$$\begin{aligned} \forall (s, t) \in \mathcal{S}_{[0,T]}, \quad \kappa_{\mathbf{X},w_n}(s, t) &:= \phi_p \left(\|\mathbf{X}\|_{p,[s,t]} \right)^p + \phi_p \left(w_n(s, t) \right)^p \\ \tilde{\kappa}_{\mathbf{X},w_n}(s, t) &:= \sum_{k=1}^4 \kappa_{\mathbf{X},w_n}(s, t)^k. \end{aligned} \quad (5.2)$$

Lemma 5.1. *Let the assumptions of Theorem 2.12 hold. Then for any $(s, t) \in \mathcal{S}_{\mathbf{X}}$,*

$$\|R^{\delta_n \sigma(Y)}\|_{\frac{p}{2},[s,t]} \leq C \left\{ w_n(s, t) + (w_n(s, t) + \|\delta_n Y\|_{\infty,[s,t]}) \left(\tilde{\kappa}_{\mathbf{X},w_n}(s, t)^{\frac{1}{p}} \vee \tilde{\kappa}_{\mathbf{X},w_n}(s, t) \right) \right\}.$$

Proof. **1st step.** Having in mind the previous definition of $R^{\delta_n \sigma(Y)}$ and $R^{\delta_n Y}$, we deduce from (4.13) that

$$\begin{aligned} R_{s,t}^{\delta_n \sigma(Y)} &= \sigma'(Y_s) R_{s,t}^Y + (\delta Y_{s,t})^2 \int_0^1 \sigma''(Y_t - (\delta Y_{s,t})u) u \, du - \sigma'(Y_s^n) R_{s,t}^{Y^n} - (\delta Y_{s,t}^n)^2 \int_0^1 \sigma''(Y_t^n - (\delta Y_{s,t}^n)u) u \, du \\ &= \sigma'(Y_s) R_{s,t}^{\delta_n Y} + \delta_n \sigma'(Y)_s R_{s,t}^{Y^n} \\ &\quad + \delta_n (\delta Y_{s,t})^2 \int_0^1 \sigma''(Y_t - (\delta Y_{s,t})u) u \, du + (\delta Y_{s,t}^n)^2 \int_0^1 \delta_n \sigma''(Y_t - (\delta Y_{s,t})u) u \, du. \end{aligned}$$

Hence

$$|R_{s,t}^{\delta_n \sigma(Y)}| \leq C \left\{ |R_{s,t}^{\delta_n Y}| + |\delta_n Y_s| |R_{s,t}^{Y^n}| + |\delta_n (\delta Y_{s,t})^2| + \|\delta_n Y\|_{\infty,[s,t]} (\delta Y_{s,t}^n)^2 \right\}. \quad (5.3)$$

2nd step. We provide now an upper bound on $|R_{s,t}^{\delta_n Y}|$ in terms of $R_{s,t}^{\delta_n \sigma(Y)}$:

$$\begin{aligned} |R_{s,t}^{\delta_n Y}| &= |\delta(\delta_n Y)_{s,t} - \delta_n \sigma(Y)_s \delta X_{s,t}| \\ &= |\delta(\delta_n K)_{s,t} + \int_s^t \delta_n \sigma(Y)_u \, d\mathbf{X}_u - \delta_n \sigma(Y)_s \delta X_{s,t}| \\ &\leq w_n(s, t) + |\delta_n \sigma'(Y)_s \mathbb{X}_{s,t}| + C_p \left(\|R^{\delta_n \sigma(Y)}\|_{\frac{p}{2},[s,t]} \|X\|_{p,[s,t]} + \|\delta_n \sigma'(Y)\|_{p,[s,t]} \|\mathbb{X}\|_{\frac{p}{2},[s,t]} \right). \end{aligned} \quad (5.4)$$

Thus plugging (5.4) into (5.3) yields

$$\begin{aligned} |R_{s,t}^{\delta_n \sigma(Y)}| &\leq C_p \|R^{\delta_n \sigma(Y)}\|_{\frac{p}{2},[s,t]} \|X\|_{p,[s,t]} + C \left\{ w_n(s, t) + |\delta_n Y_s| (|R_{s,t}^{Y^n}| + |\mathbb{X}_{s,t}|) + |\delta_n (\delta Y_{s,t})^2| \right. \\ &\quad \left. + \|\delta_n Y\|_{\infty,[s,t]} (\delta Y_{s,t}^n)^2 + \|\delta_n \sigma'(Y)\|_{p,[s,t]} \|\mathbb{X}\|_{\frac{p}{2},[s,t]} \right\}. \end{aligned} \quad (5.5)$$

3rd step. Now we provide bounds for $|\delta_n (\delta Y_{s,t})^2|$ and $\|\delta_n \sigma'(Y)\|_{p,[s,t]}$. Recall from (4.20) that

$$\|Y^n\|_{p,[s,t]} \leq C \left(\phi_p(\|\mathbf{X}\|_{p,[s,t]}) + \delta K_{s,t}^n \vee (\delta K_{s,t}^n)^p \right) = C \left(\phi_p(\|\mathbf{X}\|_{p,[s,t]}) + \phi_p(\delta K_{s,t}^n) \right). \quad (5.6)$$

Similarly, one gets $\|Y\|_{p,[s,t]} \leq C \left(\phi_p(\|\mathbf{X}\|_{p,[s,t]}) + \phi_p(\delta K_{s,t}) \right)$. Recall also that from (4.21) that for any $s < t \in [0, T]$ such that $|t - s| \leq \delta_{\mathbf{X}}$,

$$|R_{s,t}^{Y^n}| \leq C \left(\phi_p(\|\mathbf{X}\|_{p,[s,t]})^p + \phi_p(\delta K_{s,t}^n)^p \right)^{\frac{2}{p}}. \quad (5.7)$$

Now, observing that $|\delta_n (\delta Y_{s,t})^2| = |\delta(\delta_n Y)_{s,t}| |\delta Y_{s,t} + \delta Y_{s,t}^n|$ and using the Cauchy-Schwarz inequality, one gets that for any subdivision $\pi = (t_i)$ of $[s, t]$,

$$\begin{aligned} \sum_i |\delta_n (\delta Y_{t_i, t_{i+1}})|^{\frac{2}{p}} &\leq \left(\sum_i |\delta(\delta_n Y)_{t_i, t_{i+1}}|^p \times \sum_i |\delta Y_{s,t} + \delta Y_{s,t}^n|^p \right)^{\frac{1}{2}} \\ &\leq \|\delta_n Y\|_{p,[s,t]}^{\frac{2}{p}} \|Y + Y^n\|_{p,[s,t]}^{\frac{2}{p}}. \end{aligned}$$

Thus from (5.6),

$$\begin{aligned} \sup_{\pi} \sum_i |\delta_n(\delta Y_{t_i, t_{i+1}})|^{\frac{p}{2}} &\leq C \|\delta_n Y\|_{p, [s, t]}^{\frac{p}{2}} \left(\phi_p(\|\mathbf{X}\|_{p, [s, t]}) + \phi_p(\delta K_{s, t}^n + \delta K_{s, t}) \right)^{\frac{p}{2}} \\ &\leq C \|\delta_n Y\|_{p, [s, t]}^{\frac{p}{2}} \kappa_{\mathbf{X}, w_n}(s, t)^{\frac{1}{2}}, \end{aligned} \quad (5.8)$$

for some $C > 0$ which depends only on p and σ . Eventually, using the smoothness of σ , observe that

$$\begin{aligned} |\delta(\delta_n \sigma'(Y))_{s, t}| &= |\delta_n Y_t \int_0^1 (\sigma' \sigma)'(Y_t^n + \delta_n Y_t u) du - \delta_n Y_s \int_0^1 (\sigma' \sigma)'(Y_s^n + \delta_n Y_s u) du| \\ &\leq |\delta(\delta_n Y)_{s, t} \int_0^1 (\sigma' \sigma)'(Y_t^n + \delta_n Y_t u) du| + |\delta_n Y_s| \int_0^1 |(\sigma' \sigma)'(Y_t^n + \delta_n Y_t u) - (\sigma' \sigma)'(Y_s^n + \delta_n Y_s u)| du \\ &\leq C (|\delta(\delta_n Y)_{s, t}| + |\delta_n Y_s| (|\delta Y_{s, t}^n| + |\delta(\delta_n Y)_{s, t}|)) \\ &\leq C (|\delta(\delta_n Y)_{s, t}| + |\delta_n Y_s| (|\delta Y_{s, t}^n| + |\delta Y_{s, t}|)). \end{aligned}$$

Therefore, using again (5.6),

$$\|\delta_n \sigma'(Y)\|_{p, [s, t]} \leq C \left(\|\delta_n Y\|_{p, [s, t]} + \|\delta_n Y\|_{\infty, [s, t]} \left(\phi_p(\|\mathbf{X}\|_{p, [s, t]}) + \phi_p(w_n(s, t)) \right) \right). \quad (5.9)$$

4th step. The inequalities (5.6), (5.7), (5.8) and (5.9) plugged into (5.5) now provide that for any $(s, t) \in \mathcal{S}_{\mathbf{X}}$,

$$\begin{aligned} \|R^{\delta_n \sigma(Y)}\|_{\frac{p}{2}, [s, t]} &\leq C_p \|R^{\delta_n \sigma(Y)}\|_{\frac{p}{2}, [s, t]} \|X\|_{p, [s, t]} \\ &\quad + C \left\{ w_n(s, t) + \|\delta_n Y\|_{\infty, [s, t]} \left(\left(\phi_p(\|\mathbf{X}\|_{p, [s, t]})^p + \phi_p(\delta K_{s, t}^n)^p \right)^{\frac{2}{p}} + \|\mathbb{X}\|_{\frac{p}{2}, [s, t]} \right) \right. \\ &\quad + \|\delta_n Y\|_{p, [s, t]} \kappa_{\mathbf{X}, w_n}(s, t)^{\frac{1}{p}} + \|\delta_n Y\|_{\infty, [s, t]} \left(\phi_p(\|\mathbf{X}\|_{p, [s, t]}) + \phi_p(\delta K_{s, t}^n) \right)^2 \\ &\quad \left. + \left(\|\delta_n Y\|_{p, [s, t]} + \|\delta_n Y\|_{\infty, [s, t]} \left(\phi_p(\|\mathbf{X}\|_{p, [s, t]}) + \phi_p(w_n(s, t)) \right) \right) \|\mathbb{X}\|_{\frac{p}{2}, [s, t]} \right\}. \end{aligned}$$

Recall that $\kappa_{\mathbf{X}, w_n}$ was defined in (5.2), and that $\delta K_{s, t}^n \leq w_n(s, t)$. Besides, since

$$\left(\phi_p(\|\mathbf{X}\|_{p, [s, t]}) + \phi_p(w_n(s, t)) \right) \|\mathbb{X}\|_{\frac{p}{2}, [s, t]} \leq \kappa_{\mathbf{X}, w_n}(s, t)^{\frac{2}{p}},$$

we get that for any $(s, t) \in \mathcal{S}_{\mathbf{X}}$,

$$\begin{aligned} \|R^{\delta_n \sigma(Y)}\|_{\frac{p}{2}, [s, t]} &\leq C_p \|R^{\delta_n \sigma(Y)}\|_{\frac{p}{2}, [s, t]} \|X\|_{p, [s, t]} \\ &\quad + C \left\{ w_n(s, t) + \|\delta_n Y\|_{\infty, [s, t]} \kappa_{\mathbf{X}, w_n}(s, t)^{\frac{2}{p}} + \|\delta_n Y\|_{p, [s, t]} \kappa_{\mathbf{X}, w_n}(s, t)^{\frac{1}{p}} \right\}. \end{aligned}$$

Consider the set $\mathcal{S}_X := \{(u, v) \in \mathcal{S}_{[0, T]} : v - u \leq \delta_X\}$, where

$$\delta_X = T \wedge \sup\{\delta > 0 : \forall (u, v) \in \mathcal{S}_{[0, T]}, v - u \leq \delta \Rightarrow \|X\|_{p, [u, v]} \leq (2C_p)^{-1}\}.$$

Up to redefining $\delta_{\mathbf{X}}$ into $\delta_X \wedge \delta_{\mathbf{X}}$, we can assume without loss of generality that $\mathcal{S}_{\mathbf{X}} \subseteq \mathcal{S}_X$. Thus for any $(s, t) \in \mathcal{S}_{\mathbf{X}}$, one gets that

$$\|R^{\delta_n \sigma(Y)}\|_{\frac{p}{2}, [s, t]} \leq 2C \left\{ w_n(s, t) + \|\delta_n Y\|_{\infty, [s, t]} \kappa_{\mathbf{X}, w_n}(s, t)^{\frac{2}{p}} + \|\delta_n Y\|_{p, [s, t]} \kappa_{\mathbf{X}, w_n}(s, t)^{\frac{1}{p}} \right\}. \quad (5.10)$$

5th step. Back to the definition of $\delta_n Y$, we obtain

$$\begin{aligned} |\delta(\delta_n Y)_{s, t}| &= |\delta(\delta_n K)_{s, t} + \int_s^t \delta_n \sigma(Y)_u d\mathbf{X}_u| \\ &\leq w_n(s, t) + |\delta_n \sigma(Y)_s| |\delta X_{s, t}| + |\delta_n \sigma'(Y)_s| \|\mathbb{X}_{s, t}\| \\ &\quad + C_p \left(\|R^{\delta_n \sigma(Y)}\|_{\frac{p}{2}, [s, t]} \|X\|_{p, [s, t]} + \|\delta_n \sigma'(Y)\|_{p, [s, t]} \|\mathbb{X}\|_{\frac{p}{2}, [s, t]} \right). \end{aligned}$$

The previous bounds (5.9) and (5.10) can now be used as follows: $\forall(s, t) \in \mathcal{S}_{\mathbf{X}}$,

$$\begin{aligned}
\|\delta_n Y\|_{p,[s,t]} &\leq C \left\{ w_n(s, t) + \|\delta_n Y\|_{\infty,[s,t]} \|X\|_{p,[s,t]} + \|\delta_n Y\|_{\infty,[s,t]} \|\mathbb{X}\|_{\frac{p}{2},[s,t]} \right. \\
&\quad + \|X\|_{p,[s,t]} \left(w_n(s, t) + \|\delta_n Y\|_{\infty,[s,t]} \mathcal{K}_{\mathbf{X},w_n}(s, t)^{\frac{2}{p}} + \|\delta_n Y\|_{p,[s,t]} \mathcal{K}_{\mathbf{X},w_n}(s, t)^{\frac{1}{p}} \right) \\
&\quad \left. + \|\mathbb{X}\|_{\frac{p}{2},[s,t]} \left(\|\delta_n Y\|_{p,[s,t]} + \|\delta_n Y\|_{\infty,[s,t]} \mathcal{K}_{\mathbf{X},w_n}(s, t)^{\frac{1}{p}} \right) \right\} \\
&\leq \frac{1}{2} C_Y \|\delta_n Y\|_{p,[s,t]} \left(\|X\|_{p,[s,t]} \mathcal{K}_{\mathbf{X},w_n}(s, t)^{\frac{1}{p}} + \|\mathbb{X}\|_{\frac{p}{2},[s,t]} \right) + C w_n(s, t) (1 + \|X\|_{p,[s,t]}) \\
&\quad + C \|\delta_n Y\|_{\infty,[s,t]} \left(\mathcal{K}_{\mathbf{X},w_n}(s, t)^{\frac{1}{p}} + \mathcal{K}_{\mathbf{X},w_n}(s, t)^{\frac{3}{p}} \right) \\
&\leq C_Y \|\delta_n Y\|_{p,[s,t]} \mathcal{K}_{\mathbf{X},w_n}(s, t)^{\frac{2}{p}} + C (w_n(s, t) + \|\delta_n Y\|_{\infty,[s,t]}) \left(1 + \mathcal{K}_{\mathbf{X},w_n}(s, t)^{\frac{1}{p}} + \mathcal{K}_{\mathbf{X},w_n}(s, t)^{\frac{3}{p}} \right),
\end{aligned}$$

where $C_Y > 0$ depends only on p and σ . Since

$$(s, t) \mapsto C (w_n(s, t) + \|\delta_n Y\|_{\infty,[s,t]})^p \left(1 + \mathcal{K}_{\mathbf{X},w_n}(s, t)^{\frac{1}{p}} + \mathcal{K}_{\mathbf{X},w_n}(s, t)^{\frac{3}{p}} \right)^p$$

is super-additive, we obtain by a classical argument that $\forall(s, t) \in \mathcal{S}_{[0,T]}$,

$$\|\delta_n Y\|_{p,[s,t]} \leq \left(2(2C_Y)^{\frac{p}{2}} \right)^{\frac{p-1}{p}} \left(1 \vee \mathcal{K}_{\mathbf{X},w_n}(s, t)^{\frac{p-1}{p}} \right) \times C (w_n(s, t) + \|\delta_n Y\|_{\infty,[s,t]}) \sum_{\substack{k=0 \\ k \neq 2}}^3 \mathcal{K}_{\mathbf{X},w_n}(s, t)^{\frac{k}{p}},$$

so that $\forall(s, t) \in \mathcal{S}_{[0,T]}$,

$$\begin{aligned}
\|\delta_n Y\|_{p,[s,t]} \mathcal{K}_{\mathbf{X},w_n}(s, t)^{\frac{1}{p}} &\leq C (w_n(s, t) + \|\delta_n Y\|_{\infty,[s,t]}) \left(\mathcal{K}_{\mathbf{X},w_n}(s, t)^{\frac{1}{p}} \vee \mathcal{K}_{\mathbf{X},w_n}(s, t) \right) \sum_{\substack{k=0 \\ k \neq 2}}^3 \mathcal{K}_{\mathbf{X},w_n}(s, t)^{\frac{k}{p}} \\
&\leq C (w_n(s, t) + \|\delta_n Y\|_{\infty,[s,t]}) \left(\tilde{\mathcal{K}}_{\mathbf{X},w_n}(s, t)^{\frac{1}{p}} \vee \tilde{\mathcal{K}}_{\mathbf{X},w_n}(s, t) \right).
\end{aligned}$$

By plugging the above bound into the right-hand side of (5.10), one obtains the desired result. For further use, note also that the previous bound can be used in (5.9) to get that $\forall(s, t) \in \mathcal{S}_{[0,T]}$,

$$\begin{aligned}
\|\delta_n \sigma' \sigma(Y)\|_{p,[s,t]} \|\mathbb{X}\|_{\frac{p}{2},[s,t]} &\leq C \left(\|\delta_n Y\|_{p,[s,t]} \mathcal{K}_{\mathbf{X},w_n}(s, t)^{\frac{1}{p}} + \|\delta_n Y\|_{\infty,[s,t]} \mathcal{K}_{\mathbf{X},K^n}(s, t)^{\frac{2}{p}} \right) \\
&\leq C (w_n(s, t) + \|\delta_n Y\|_{\infty,[s,t]}) \left(\tilde{\mathcal{K}}_{\mathbf{X},w_n}(s, t)^{\frac{1}{p}} \vee \tilde{\mathcal{K}}_{\mathbf{X},w_n}(s, t) \right). \tag{5.11}
\end{aligned}$$

□

5.2 Proof of Theorems 2.14 and 2.15

In this section, denote by $\Pi_t(y)$ the projection on the epigraph of L , i.e. $\Pi_t(y) = y \vee L_t$. Consider the control function defined by: $\forall(s, t) \in \mathcal{S}_{[0,T]}$,

$$\bar{\mathcal{K}}_{\mathbf{X},w_n}(s, t) := (1 + \|\mathbb{X}\|_{\frac{p}{2},[s,t]})^p \left(\tilde{\mathcal{K}}_{\mathbf{X},w_n}(s, t) \vee \tilde{\mathcal{K}}_{\mathbf{X},w_n}(s, t)^p \right). \tag{5.12}$$

Lemma 5.2. *Set $c = (2e^2)^{\frac{1}{p}}$ and denote by $\bar{\Theta}$ the quantity $C \left(1 + (1 + \Theta) \|\mathbf{X}\|_{\beta} + \|L\|_{\beta} \right)$ that appears in the upper bound of $\sup_{s \in [0,T]} (Y_s^n - L_s)_-$ in Proposition 4.7. For any $n \in \mathbb{N}^*$, one has*

$$\forall(s, t) \in \mathcal{S}_{\mathbf{X}}, \quad \|Y - \Pi.(Y^n)\|_{\infty,[s,t]} \leq 2e^{1 \vee (c \bar{\mathcal{K}}_{\mathbf{X},w_n}(s,t))} (Y_s - \Pi_s(Y_s^n) + \bar{\Theta} n^{-\beta} + 2\delta K_{s,t}).$$

Proof. Observe that

$$\begin{aligned}\delta(\delta_n Y)_{s,t} + w_n(s,t) &= \int_s^t \delta_n \sigma(Y)_u d\mathbf{X}_u + \delta(\delta_n K)_{s,t} + w_n(s,t) \\ &= \int_s^t \delta_n \sigma(Y)_u d\mathbf{X}_u + 2\delta K_{s,t}.\end{aligned}\tag{5.13}$$

In view of the following classical inequality

$$\left| \int_s^t \delta_n \sigma(Y)_u d\mathbf{X}_u \right| \leq C \left\{ \delta_n Y_s |\delta X_{s,t}| + \delta_n Y_s \|\mathbb{X}_{s,t}\| + \|R^{\delta_n \sigma(Y)}\|_{\frac{p}{2},[s,t]} \|X\|_{p,[s,t]} + \|\delta_n \sigma'(Y)\|_{p,[s,t]} \|\mathbb{X}\|_{\frac{p}{2},[s,t]} \right\},$$

one can apply Lemma 5.1 and Inequality (5.11) to get that for any $(s,t) \in \mathcal{S}_{\mathbf{X}}$,

$$\left| \int_s^t \delta_n \sigma(Y)_u d\mathbf{X}_u \right| \leq C \left(\|\delta_n Y\|_{\infty,[s,t]} + w_n(s,t) \right) \left\{ (1 + \|\mathbb{X}\|_{\frac{p}{2},[s,t]}) \left(\tilde{\mathcal{K}}_{\mathbf{X},w_n}(s,t)^{\frac{1}{p}} \vee \tilde{\mathcal{K}}_{\mathbf{X},w_n}(s,t) \right) \right\}.\tag{5.14}$$

Now we get from (5.13) and (5.14) that

$$\delta(\delta_n Y)_{s,t} + w_n(s,t) \leq \left(\|\delta_n Y\|_{\infty,[s,t]} + w_n(s,t) \right) \bar{\mathcal{K}}_{\mathbf{X},w_n}(s,t)^{\frac{1}{p}} + 2\delta K_{s,t}.$$

Hence we are now in a position to apply the rough Gronwall lemma of Deya et al. [8, Lemma 2.11] which reads

$$\forall (s,t) \in \mathcal{S}_{\mathbf{X}}, \quad \|\delta_n Y\|_{\infty,[s,t]} + w_n(s,t) \leq 2 \exp \left\{ 1 \vee (c\bar{\mathcal{K}}_{\mathbf{X},w_n}(s,t)) \right\} (\delta_n Y_s + 2\delta K_{s,t}).$$

With the current notation, Proposition 4.7 yields $\Pi_t(Y_t^n) - Y_t^n \leq \bar{\Theta} n^{-\beta}$, $\forall t \in [0, T]$. Since $Y - \Pi.(Y^n) \leq \delta_n Y$, we obtain

$$\forall (s,t) \in \mathcal{S}_{\mathbf{X}}, \quad \|Y - \Pi.(Y^n)\|_{\infty,[s,t]} \leq 2e^{1 \vee (c\bar{\mathcal{K}}_{\mathbf{X},w_n}(s,t))} (Y_s - \Pi_s(Y_s^n) + \Pi_s(Y_s^n) - Y_s^n + 2\delta K_{s,t})$$

which gives the expected result. \square

We now have all the ingredients to carry out the proof of the Theorems.

Proof of Theorem 2.14. We will use Lemma 5.2 to obtain

$$\|Y - \Pi.(Y^n)\|_{\infty,[0,T]} \leq \exp \left\{ 2T\delta_{\mathbf{X}}^{-1} + (c\bar{\mathcal{K}}_{\mathbf{X},w_n}(0,T)) \right\} (2 + T\delta_{\mathbf{X}}^{-1}\bar{\Theta})n^{-\beta}.\tag{5.15}$$

It is clear that the quantity $\bar{\mathcal{K}}_{\mathbf{X},K}(0,T) := \sup_{n \in \mathbb{N}} \bar{\mathcal{K}}_{\mathbf{X},w_n}(0,T)$ is finite (we give more details in the next proof, where \mathbf{X} is a Gaussian rough paths). Hence the Inequality (5.15) yields the desired result since

$$\begin{aligned}\|\delta_n Y\|_{\infty,[0,T]} &\leq \|Y - \Pi.(Y^n)\|_{\infty,[0,T]} + \|\Pi.(Y^n) - Y^n\|_{\infty,[0,T]} \\ &\leq \exp \left\{ 2T\delta_{\mathbf{X}}^{-1} + (c\bar{\mathcal{K}}_{\mathbf{X},K}(0,T)) \right\} (2 + \bar{\Theta} + T\delta_{\mathbf{X}}^{-1}\bar{\Theta})n^{-\beta},\end{aligned}\tag{5.16}$$

using Proposition 4.7 in the last inequality.

To see that (5.15) holds, consider first the case $s = 0$ and $t > 0$ such that $(0,t) \in \mathcal{S}_{\mathbf{X}}$: since $Y_0 = \Pi_0(Y_0^n) = y_0$, one can define

$$t_0^n := T \wedge \inf \{ t > 0 : Y_t - \Pi_t(Y_t^n) = 2n^{-\beta} \}.$$

Of course, if $t_0^n = T$ then the proof is over. So let us assume that $t_0^n < T$ and define

$$t_1^n := T \wedge \inf \{ t > t_0^n : Y_t - \Pi_t(Y_t^n) = n^{-\beta} \}$$

and the mapping $\vartheta : [0, T] \rightarrow [0, T]$ associated to $\mathcal{S}_{\mathbf{X}}$ as follows:

$$\forall t \in [0, T], \quad \vartheta(t) = \sup \{ t' > t : (t, t') \in \mathcal{S}_{\mathbf{X}} \}.$$

Notice that for any $u \in [t_0^n, t_1^n]$, Y_u lies strictly above the boundary L_u since $Y_u \geq n^{-\beta} + \Pi_u(Y_u^n) > L_u$. Hence for any $u \in [t_0^n, t_1^n]$, $\delta K_{s,t} = 0$. It follows that for any $u \in [t_0^n, t_1^n \wedge \vartheta(t_0^n)]$,

$$\|Y - \Pi.(Y^n)\|_{\infty, [t_0^n, u]} \leq 2e^{1\vee(c\bar{K}_{\mathbf{X}, w_n}(t_0^n, t_1^n \wedge \vartheta(t_0^n)))} (2n^{-\beta} + \bar{\Theta}n^{-\beta} + 0). \quad (5.17)$$

Then, we have one of the following possibilities:

1. if $t_1^n \wedge \vartheta(t_0^n) = T$, then rephrasing (5.17), we get $\|Y - \Pi.(Y^n)\|_{\infty, [t_0^n, T]} \leq 2e^{1\vee(c\bar{K}_{\mathbf{X}, w_n}(t_0^n, T))} (2 + \bar{\Theta})n^{-\beta}$ and (5.15) is proven (recall that by definition, $\|Y - \Pi.(Y^n)\|_{\infty, [0, t_0^n]} \leq 2n^{-\beta}$);
2. if $t_1^n \wedge \vartheta(t_0^n) = t_1^n < T$, then (5.17) now reads

$$\|Y - \Pi.(Y^n)\|_{\infty, [t_0^n, t_1^n]} \leq 2e^{1\vee(c\bar{K}_{\mathbf{X}, w_n}(t_0^n, t_1^n))} (2 + \bar{\Theta})n^{-\beta},$$

which is smaller than the right-hand side of inequality (5.15). Then, since $Y_{t_1^n} - \Pi_{t_1^n}(Y_{t_1^n}^n) = n^{-\beta}$, we can define

$$t_2^n := T \wedge \inf \{t > t_1^n : Y_t - \Pi_t(Y_t^n) = 2n^{-\beta}\}$$

and if $t_2^n < T$, define also

$$t_3^n := T \wedge \inf \{t > t_2^n : Y_t - \Pi_t(Y_t^n) = n^{-\beta}\}$$

and move to the next step (iterate);

3. if $t_1^n \wedge \vartheta(t_0^n) = \vartheta(t_0^n) < T$, then as in (5.17), we get $\|Y - \Pi.(Y^n)\|_{\infty, [t_0^n, \vartheta(t_0^n)]} \leq 2e^{1\vee(c\bar{K}_{\mathbf{X}, w_n}(t_0^n, \vartheta(t_0^n)))} (2 + \bar{\Theta})n^{-\beta}$. But now, we need to consider the times $\vartheta \circ \vartheta(t_0^n) \equiv \vartheta^2(t_0^n)$, $\vartheta^3(t_0^n)$, \dots , $\vartheta^J(t_0^n)$, as long as $\vartheta^J(t_0^n) < T \wedge t_1^n$. By an immediate induction, we obtain that for such J ,

$$\|Y - \Pi.(Y^n)\|_{\infty, [t_0^n, \vartheta^J(t_0^n)]} \leq 2^J \exp \left\{ J \vee \left(c \sum_{j=0}^{J-1} \bar{K}_{\mathbf{X}, w_n}(\vartheta^j t_0^n, \vartheta^{j+1}(t_0^n)) \right) \right\} (2 + J\bar{\Theta})n^{-\beta},$$

where we used $\vartheta^0(t_0^n) = t_0^n$. Note however that $\vartheta^J(t_0^n) = J\delta_{\mathbf{X}}$, so that J must be smaller than $T\delta_{\mathbf{X}}^{-1}$. Besides, by the super-additivity property of $\bar{K}_{\mathbf{X}, w_n}$, one gets

$$\|Y - \Pi.(Y^n)\|_{\infty, [t_0^n, \vartheta^J(t_0^n)]} \leq 2^{T\delta_{\mathbf{X}}^{-1}} \exp \{T\delta_{\mathbf{X}}^{-1} \vee (c\bar{K}_{\mathbf{X}, w_n}(t_0^n, T))\} (2 + T\delta_{\mathbf{X}}^{-1}\bar{\Theta})n^{-\beta},$$

which is smaller than the right-hand side of (5.15).

To conclude this step, observe that if $\vartheta^{J+1}(t_0^n) = T$, then we proved (5.15). While if $\vartheta^{J+1}(t_0^n) = t_1^n$, then one can move to point 2. and iterate.

Following this construction, there exists an increasing sequence $(t_k^n)_{k \in \mathbb{N}} \in [0, T]^{\mathbb{N}}$ (possibly taking only finitely many different values) such that $\lim_{k \rightarrow \infty} t_k^n = T$ and for any $k \geq 0$, $\|Y - \Pi.(Y^n)\|_{\infty, [t_{2k}^n, t_{2k+1}^n]}$ is bounded by the right-hand side of inequality (5.15). This achieves the proof of this theorem. \square

We will now conclude this section using the previous estimate (5.16) and the probabilistic bound on Θ .

Proof of Theorem 2.15. First, we provide a bound on K_T where (Y, K) is a solution of the Skorokhod problem associated to $y_0 + \int_0^\cdot \sigma(Y_u) d\mathbf{X}_u$ reflected on L . Similarly, observe that $(L, 0)$ is solution of the Skorokhod problem associated to L reflected on L . We call Skorokhod mapping the function that takes any continuous paths (z, l) and maps it to (y, k) , where $y = z + k$ is a path reflected on l . The Skorokhod mapping is Lipschitz continuous in the uniform topology (call C_S the Lipschitz constant), see for instance Equations (2.1)-(2.2) in [12]. Thus one gets that

$$\begin{aligned} K_T &= \|K\|_{\infty, [0, T]} \leq C_S \|y_0 + \int_0^\cdot \sigma(Y_u) d\mathbf{X}_u - L\|_{\infty, [0, T]} \\ &\leq C_S \left(C(1 + \Theta) \|\mathbf{X}\|_{\beta} + \|L\|_{\beta} \right) T^{\beta} \end{aligned} \quad (5.18)$$

where we used Corollary 4.6. But Θ depends linearly on $C_J^{\mathbf{X}}$, and as already mentioned, this quantity may not have exponential moments. Since an exponential of Θ appears in Inequality (5.16), this explains why we cannot get a simple upper bound for $\mathbb{E}(\|\delta_n Y\|_{\infty, [0, T]})$. From Lemma 4.5 and the definition of $\bar{\Theta}$ in Lemma 5.2, recall that $\forall \gamma \geq 1$, $\mathbb{E}(\bar{\Theta}^\gamma) < \infty$. Moreover, we know from (4.27) that $\mathbb{E}(\delta_{\mathbf{X}}^\gamma) < \infty$. Finally, since $K_T^n \leq K_T$, one gets from (5.2) and (5.12) that

$$\begin{aligned} \bar{K}_{\mathbf{X}, w_n}(0, T) &\leq C \left(1 + \|\mathbb{X}\|_{\frac{p}{2}, [0, T]}^p \right) \\ &\quad \times \left(\sum_{k=1}^4 \left(\phi_p \left(\|\mathbf{X}\|_{p, [0, T]} \right)^{kp} + \phi_p(K_T)^{kp} \right) \vee \sum_{k=1}^4 \left(\phi_p \left(\|\mathbf{X}\|_{p, [0, T]} \right)^{kp^2} + \phi_p(K_T)^{kp^2} \right) \right). \end{aligned}$$

Hence, in view of the bound (5.18) on K_T , there exists a random variable \bar{K} such that, for any $\gamma \geq 1$,

$$\mathbb{E} \left(\left(\sup_{n \in \mathbb{N}} \bar{K}_{\mathbf{X}, w_n}(0, T) \right)^\gamma \right) = \mathbb{E}(\bar{K}^\gamma) < \infty.$$

Using (5.16), it is now clear that for any $\gamma \geq 1$,

$$\mathbb{E} \left[\left| \sup_{n \in \mathbb{N}} \log \left(n^\beta \|\delta_n Y\|_{\infty, [0, T]} \right) \right|^\gamma \right] < \infty.$$

□

6 Application: existence of a density for the reflected process

In this last section, we aim at proving Theorem 2.16. Thus let us consider the following simplified problem (compared to (1.2)), with constant diffusion coefficient and one-dimensional fractional Brownian noise:

$$\forall t \geq 0, \quad Y_t = y_0 + \int_0^t b(Y_u) du + K_t + B_t,$$

where B is a fractional Brownian motion with Hurst parameter $H \in [\frac{1}{2}, 1)$, $y_0 \geq 0$, $b \in \mathcal{C}_b^1(\mathbb{R})$ and (Y, K) is the solution of the Skorokhod problem of reflection above the constant boundary 0. Note that we added a drift b compared to previous equations, which in the previous section could have been part of the vector field σ . However it is not necessary here to assume as much as a fourth order bounded derivative in this case, and one can check that $b \in \mathcal{C}_b^1$ is enough to get existence with the method of Sections 3 and 4.

As in the previous sections, Y is approximated by the non-decreasing sequence of processes $(Y^n)_{n \in \mathbb{N}}$:

$$\forall t \geq 0, \quad Y_t^n = y_0 + \int_0^t b(Y_u^n) du + \int_0^t \psi_n(Y_u^n) du + B_t. \quad (6.1)$$

6.1 Malliavin calculus and fractional Brownian motion

Let us briefly review some fundamental tools and results of Malliavin calculus that permit to prove that some random variables are absolutely continuous with respect to the Lebesgue measure. In a second paragraph, we shall give a brief account of Malliavin calculus for the fractional Brownian motion, in a manner that emphasises the applicability of the general results to the fBm framework.

Let D denote the usual Malliavin derivative on the Cameron-Martin space $\mathcal{H} = L^2([0, T])$. For any $p \geq 1$, let $\mathbb{D}^{1,p}$ be the Malliavin-Sobolev space associated to the derivative operator D . We aim at applying the following result of Bouleau and Hirsch to Y_t .

Theorem 6.1 ([25]). *Let $X \in \mathbb{D}^{1,2}$ be a real-valued random variable. If $A \in \mathcal{F}$ and $\|DX\|_{\mathcal{H}} > 0$ a.s. on A , then the restriction of the law of X to A , i.e. the measure $[\mathbf{1}_A \mathbb{P}] \circ X^{-1}$, admits a density with respect to the Lebesgue measure.*

Concerning the Malliavin calculus for fBm, we recall some definition and properties borrowed from [25, Chapter 5] (see also [26, Section 2] for a more detailed introduction than what we present here). One possible approach to this calculus is to consider the kernel K , defined for any $H \in (\frac{1}{2}, 1)$ by:

$$K(t, s) = \begin{cases} c_H s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}} du & \text{if } t > s > 0 \\ 0 & \text{if } t \leq s, \end{cases} \quad (6.2)$$

where c_H is a positive constant (see [25, Eq. (5.8)]). Note that for the standard Brownian motion ($H = \frac{1}{2}$), $K(t, s) = \mathbf{1}_{[0,t]}(s)$. Then if B is a fractional Brownian motion,

$$D_s B_t = K(t, s).$$

Furthermore, one can define an $L^2([0, T])$ -valued linear isometry K^* as follows: for any simple function φ ,

$$K^* \varphi(s) = \int_s^T \varphi(u) \frac{\partial K}{\partial u}(u, s) du = c_H \int_s^T \varphi(u) \left(\frac{u}{s}\right)^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}} du.$$

The domain of K^* is thus a Hilbert space, which we do not need to characterise here, but only recall that for any $H \in [\frac{1}{2}, 1)$, it contains $\mathcal{C}_b([0, T])$. Besides, if the support of φ is contained in $[0, t]$ for some $t > 0$, one can verify that

$$K^* \varphi(s) = \int_s^t \frac{\partial K}{\partial u}(u, s) \varphi(u) du. \quad (6.3)$$

Applying the Malliavin derivative on both sides of (6.1), we obtain

$$\forall s, t \geq 0, \quad D_s Y_t^n = \int_0^t D_s Y_u^n b'(Y_u^n) du + \int_0^t D_s Y_u^n \psi'_n(Y_u^n) du + K(t, s).$$

Since for any fixed $s \geq 0$, the previous equality is an ODE in t , solving it yields

$$\forall t \geq 0, \quad D_s Y_t^n = K^* \left[\mathbf{1}_{[0,t]}(\cdot) \exp \left\{ \int_{\cdot}^t (b'(Y_v^n) + \psi'_n(Y_v^n)) dv \right\} \right] (s). \quad (6.4)$$

For the ease of notations, let us define, for $\varphi = b' + \psi'_n$ or $\varphi = b'$, the process

$$\mathcal{E}_{s,t}[\varphi] = \mathbf{1}_{[0,t]}(s) \exp \left\{ \int_s^t \varphi(Y_v^n) dv \right\}.$$

6.2 Proof of existence of a density (Theorem 2.16)

Let $t > 0$. The scheme of proof is very similar to the one used by Tindel [31] for elliptic PDEs perturbed by an additive white noise. First, notice that the sequence $(DY_t^n)_{n \in \mathbb{N}}$ is bounded in $L^2(\Omega; \mathcal{H})$. Indeed, the mapping $\mathcal{E}_{\cdot,t}[b' + \psi'_n]$ is bounded uniformly in n and $s \in [0, T]$, since $\psi'_n \leq 0$. Hence it is clear that the same is true of $K^* Y_t^n$ (note that this is where things become difficult if one wants to consider the case $H < \frac{1}{2}$). Since Y^n converges to Y in L^2 (cf Theorem 2.13) and $\sup_{n \in \mathbb{N}} \mathbb{E}(\|DY_t^n\|_{\mathcal{H}}) < \infty$ in view of the preceding discussion, we deduce that $Y_t \in \mathbb{D}^{1,2}$ and that $(DY_t^n)_{n \in \mathbb{N}}$ converges weakly to DY_t in $L^2(\Omega; \mathcal{H})$ (see for instance Lemma 1.2.3 in [25]), i.e. that for any $\chi \in L^2(\Omega)$ and any $f \in \mathcal{H}$,

$$\lim_{n \rightarrow \infty} \mathbb{E}[\chi \langle DY_t^n, f \rangle] = \mathbb{E}[\chi \langle DY_t, f \rangle]. \quad (6.5)$$

In the sequel, we shall apply this convergence to any non-negative $\chi \in L^2(\Omega)$, and to any $f \in \mathcal{H}$ with a sufficiently small support.

Let Ω_0 be a measurable set of measure 1 on which Y^n converges (uniformly) (cf Theorem 2.13). In view of Theorem 6.1, it suffices to prove that for any $a > 0$, $\|DY_t\|_{\mathcal{H}} > 0$ *a.s.* on the event

$$\Omega_a = \{\omega \in \Omega_0 : Y_t \geq 3a\}.$$

As in [31], we notice that it suffices to prove that for any $k, j \in \mathbb{N}^*$, $\|DY_t\|_{\mathcal{H}} > 0$ a.s. on the following event

$$\Omega_{a,k,j} := \{\omega \in \Omega_a : Y_t^k \geq 2a \text{ and } Y_s^k \geq a, \text{ for any } s \text{ such that } |t-s| \leq j^{-1}\},$$

since $\bigcup_{k,j \in \mathbb{N}^*} \Omega_{a,k,j} = \Omega_a$.

Hence let now a, k, j be fixed. Since the sequence Y^n is non-decreasing, $Y_s^n \geq a$ a.s. on $\Omega_{a,k,j}$ for all $n \geq k$ and all $s \in [t-j^{-1}, t+j^{-1}]$, and thus $\psi'_n(Y_s^n) = 0$. Hence on $\Omega_{a,k,j}$ and for all $n \geq k$, (6.4) now reads

$$\forall s \in [t-j^{-1}, t], \quad D_s Y_t^n = K^* \mathcal{E}_{\cdot, t}[b'](s). \quad (6.6)$$

Based on the definitions of the previous section (in particular (6.3)), and for any non-negative $f \in L^2$ with support in $[t-j^{-1}, t]$,

$$\langle DY_t^n, f \rangle = c_H \int_{t-j^{-1}}^t f(s) \int_s^t \left(\frac{u}{s}\right)^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}} \mathcal{E}_{u,t}[b' + \psi'_n] \, dud s. \quad (6.7)$$

Thus using (6.6) and the boundedness of b' , it follows that $\mathbf{1}_{\Omega_{a,k,j}} \mathcal{E}_{u,t}[b' + \psi_n] \geq \mathbf{1}_{\Omega_{a,k,j}} e^{-(t-u)\|b'\|_\infty}$, hence one gets

$$\begin{aligned} \mathbf{1}_{\Omega_{a,k,j}} \langle DY_t^n, f \rangle &\geq \mathbf{1}_{\Omega_{a,k,j}} e^{-j^{-1}\|b'\|_\infty} \int_{t-j^{-1}}^t f(s) c_H \int_s^t \left(\frac{u}{s}\right)^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}} \, dud s \\ &= \mathbf{1}_{\Omega_{a,k,j}} e^{-j^{-1}\|b'\|_\infty} \int_{t-j^{-1}}^t f(s) K(t, s) \, ds. \end{aligned}$$

Hence the previous inequality and (6.5) yield that for any non-negative $\chi \in L^2(\Omega)$ and any non-negative $f \in L^2$ with support in $[t-j^{-1}, t]$

$$\mathbb{E}(\mathbf{1}_{\Omega_{a,k,j}} \chi \langle DY_t, f \rangle) \geq \mathbb{E}\left(\mathbf{1}_{\Omega_{a,k,j}} \chi e^{-j^{-1}\|b'\|_\infty} \langle K(t, \cdot), f \rangle\right).$$

It follows that almost surely, $\mathbf{1}_{\Omega_{a,k,j}} D_s Y_t \geq \mathbf{1}_{\Omega_{a,k,j}} e^{-j^{-1}\|b'\|_\infty} K(t, s)$ for almost all $s \in [t-j^{-1}, t]$. Hence the following inequality holds almost surely on $\Omega_{a,k,j}$:

$$\|DY_t\|_{\mathcal{H}} \geq e^{-j^{-1}\|b'\|_\infty} \|K(t, \cdot) \mathbf{1}_{[t-j^{-1}, t]}\|_{\mathcal{H}} > 0.$$

Applying Theorem 6.1 to DY_t concludes the proof of Theorem 2.16.

A Appendix

Proof of Proposition 2.11. The local existence of a solution of (2.8) comes from [20], Theorem 3 (see also [17, Theorem 10.21]). Uniqueness is also granted given the regularity of σ and b . In view of [20, Lemma 1] (see also [17, Theorem 10.21]), we know that either Y is a global solution on $[0, T]$, or that there is some time $\tau' \leq T$ such that for any $\tau \in [0, \tau')$, $\{Y_s\}_{s \in [0, \tau]}$ is a solution to (2.8) and that $\lim_{t \rightarrow \tau'} |Y_t| = \infty$. In the remaining of this proof, we shall prove that Y_t coincides on $[0, \tau')$ with the solution to (3.3). Since the latter does not explode, it will follow that Y is a global solution.

Let us turn to the Doss-Sussmann representation. Recall that according to (3.1), $J_{0 \leftarrow t}^{\mathbf{X}; \cdot}$ is Lipschitz uniformly in t but that due to the unboundedness of b , $W(t, \cdot)$ is only locally Lipschitz (uniformly in t). This suffices to prove existence and uniqueness of a solution to $\dot{z}_t = W(t, z_t)$ on a small enough time interval. In fact, $J_{0 \leftarrow t}^{\mathbf{X}; \cdot}$ is bounded (see (3.1)) and $C_U^{\mathbf{X}} := \sup_{t \in [0, T]} |U_{t \leftarrow 0}^{\mathbf{X}; 0}| < \infty$. Denote by $B(C_U^{\mathbf{X}})$ the ball of \mathbb{R}^e centred in 0 and with radius $C_U^{\mathbf{X}}$. Thus

$$\begin{aligned} |W(t, z)| &\leq |J_{0 \leftarrow t}^{\mathbf{X}; z} b(U_{t \leftarrow 0}^{\mathbf{X}; 0})| + |W(t, z) - J_{0 \leftarrow t}^{\mathbf{X}; z} b(U_{t \leftarrow 0}^{\mathbf{X}; 0})| \\ &\leq C_J^{\mathbf{X}} \sup_{x \in B(C_U^{\mathbf{X}})} |b(x)| + C_J^{\mathbf{X}} |b(U_{t \leftarrow 0}^{\mathbf{X}; z}) - b(U_{t \leftarrow 0}^{\mathbf{X}; 0})| \\ &\leq C_J^{\mathbf{X}} \left(\|b\|_{\infty, B(C_U^{\mathbf{X}})} + \|\nabla b\|_{\infty} C_J^{\mathbf{X}} |z| \right), \end{aligned}$$

i.e. W has linear growth. This ensures the stability of the solution Z_t to the ODE $\dot{Z}_t = W(t, Z_t)$, and its global existence on any time interval (see e.g. [17, Theorem 3.7]). Thus the process $\{U_{t \leftarrow 0}^{\mathbf{X}; Z_t}\}_{t \in [0, T]}$ is well-defined.

Now to prove that $\{Y_t\}_{t \in [0, \tau]}$ and $\{U_{t \leftarrow 0}^{\mathbf{X}; Z_t}\}_{t \in [0, \tau]}$ coincide, one can follow the scheme of proof of [14, Proposition 3]: let $(\mathbf{X}^k)_{k \in \mathbb{N}}$ a sequence of geometric rough paths such that $(X^k)_{k \in \mathbb{N}}$ is a sequence of Lipschitz paths with uniform β -Hölder bound, which converges pointwise to \mathbf{X} . Denote by Y^k the solution to (2.8) where \mathbf{X}^k replaces \mathbf{X} . Then (Y^k, Z^k) is easily seen to solve (3.3) with \mathbf{X} replaced by \mathbf{X}^k . As stated in [14], it suffices to prove the uniform convergence of Z^k to Z to get the result. Define

$$M^k = \sup_{s \in [0, \tau], z \in \mathbb{R}^e} |J_{0 \leftarrow s}^{\mathbf{X}; z} - J_{0 \leftarrow s}^{\mathbf{X}^k; z}| \vee |U_{s \leftarrow 0}^{\mathbf{X}; z} - U_{s \leftarrow 0}^{\mathbf{X}^k; z}|$$

and denote $J_{\text{Lip}} = \sup_{s \in [0, T]} \|J_{s \leftarrow 0}^{\mathbf{X}; \cdot}\|_{\text{Lip}}$ which is finite (see the discussion of Section 3.1). Now the main difference with [14] lies again in the unboundedness of b : denote by $\bar{Z} = \sup_{t \in [0, T]} |Z_t| < \infty$ and $\bar{C}_U^{\mathbf{X}} = \sup_{t \in [0, T], \sup_{z \in B(0, \bar{Z})}} |U_{t \leftarrow 0}^{\mathbf{X}; z}| < \infty$. Then for $t \leq \tau$,

$$\begin{aligned} |Z_t^k - Z_t| &\leq \int_0^t \left\{ |J_{0 \leftarrow s}^{\mathbf{X}; Z_s} - J_{0 \leftarrow s}^{\mathbf{X}; Z_s^k}| |b(U_{s \leftarrow 0}^{\mathbf{X}; Z_s})| + |J_{0 \leftarrow s}^{\mathbf{X}; Z_s^k} - J_{0 \leftarrow s}^{\mathbf{X}^k; Z_s^k}| |b(U_{s \leftarrow 0}^{\mathbf{X}^k; Z_s^k})| \right. \\ &\quad \left. + |J_{0 \leftarrow s}^{\mathbf{X}; Z_s^k}| |b(U_{s \leftarrow 0}^{\mathbf{X}^k; Z_s^k}) - b(U_{s \leftarrow 0}^{\mathbf{X}; Z_s})| \right\} ds \\ &\leq \int_0^t \left\{ J_{\text{Lip}} |Z_s^k - Z_s| \|b\|_{\infty, B(C_U^{\mathbf{X}})} + M^k \left(\|b\|_{\infty, B(\bar{C}_U^{\mathbf{X}})} + \|\nabla b\|_{\infty} |U_{s \leftarrow 0}^{\mathbf{X}^k; Z_s^k} - U_{s \leftarrow 0}^{\mathbf{X}; Z_s}| \right) \right. \\ &\quad \left. + C_J^{\mathbf{X}} \|\nabla b\|_{\infty} |U_{s \leftarrow 0}^{\mathbf{X}^k; Z_s^k} - U_{s \leftarrow 0}^{\mathbf{X}; Z_s}| \right\} ds \\ &\leq \int_0^t \left\{ J_{\text{Lip}} |Z_s^k - Z_s| \bar{b} + M^k \bar{b} + \|\nabla b\|_{\infty} (C_J^{\mathbf{X}} + M^k) |U_{s \leftarrow 0}^{\mathbf{X}^k; Z_s^k} - U_{s \leftarrow 0}^{\mathbf{X}; Z_s}| \right\} ds \\ &\leq \int_0^t \left\{ J_{\text{Lip}} |Z_s^k - Z_s| \bar{b} + M^k \bar{b} + \|\nabla b\|_{\infty} (C_J^{\mathbf{X}} + M^k) (M^k + C_J^{\mathbf{X}} |Z_s^k - Z_s|) \right\} ds, \end{aligned}$$

where $\bar{b} := \max \left(\|b\|_{\infty, B(C_U^{\mathbf{X}})}, \|b\|_{\infty, B(\bar{C}_U^{\mathbf{X}})} \right)$. Then, denoting $C^{Z,1} := \bar{b} + \|\nabla b\|_{\infty} C_J^{\mathbf{X}}$ and $C^{Z,2} := \bar{b} J_{\text{Lip}} + \|\nabla b\|_{\infty} (C_J^{\mathbf{X}})^2$, one gets that

$$\begin{aligned} \sup_{s \in [0, t]} |Z_s^k - Z_s| &\leq t C^{Z,1} M^k + t \|\nabla b\|_{\infty} (M^k)^2 + C^{Z,2} (1 + M^k) \int_0^t \sup_{u \in [0, s]} |Z_u^k - Z_u| ds \\ &\leq (C^{Z,1} + \|\nabla b\|_{\infty} M^k) T M^k \exp(C^{Z,2} (1 + M^k) T) \end{aligned} \quad (\text{A.1})$$

applying Gronwall's lemma in the last inequality. By the continuity of the mapping $(y, \mathbf{X}) \mapsto U_{\cdot \leftarrow 0}^{\mathbf{X}; y}$ (see [15, Theorem 8.5]), there is $M^k \rightarrow 0$ as $k \rightarrow \infty$, hence the inequality (A.1) implies that Z^k converges uniformly to Z . Since Y^k has the representation (3.3), then so has Y .

Hence $\{Y_t\}_{t \in [0, \tau]}$ and $\{U_{t \leftarrow 0}^{\mathbf{X}; Z_t}\}_{t \in [0, \tau]}$ do coincide and since the latter does not explode in finite time, this implies that there cannot exist $\tau' > \tau$ such that $\lim_{t \rightarrow \tau'} |Y_t| = \infty$. Thus Y is defined on $[0, T]$. \square

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