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Minimal time for the exact controllability of one-dimensional first-order linear hyperbolic systems by one-sided boundary controls

Long Hu*, Guillaume Olive†

January 15, 2019

Abstract

In this article we study the minimal time for the exact controllability of one-dimensional first-order linear hyperbolic systems when all the controls are acting on the same side of the boundary. We establish an explicit and easy-to-compute formula for this time with respect to all the coupling parameters of the system. This partially solves an open problem raised in the celebrated survey [Rus78] and in the article [Hu15]. The proof relies on the introduction of a canonical UL -decomposition and the compactness-uniqueness method.

Keywords: Hyperbolic systems, Boundary controllability, Minimal control time, UL -decomposition, Compactness-uniqueness method.

1 Introduction and main result

In this article we are interested in the controllability properties of the following class of one-dimensional first-order linear hyperbolic systems, which appears for instance in linearized Saint-Venant equations and many other physical models of balance laws (see e.g. [BC16, Chapter 1]):

$$\begin{cases} \frac{\partial y}{\partial t}(t, x) = \Lambda(x) \frac{\partial y}{\partial x}(t, x) + M(x)y(t, x), \\ y_+(t, 0) = Qy_-(t, 0), \quad y_-(t, 1) = u(t), \quad t \in (0, +\infty), x \in (0, 1). \\ y(0, x) = y^0(x), \end{cases} \quad (1)$$

In (1), $y(t, \cdot)$ is the state at time t , y^0 is the initial data and $u(t)$ is the control at time t . We denote by $n \geq 2$ the total number of equations of the system. The matrix $\Lambda \in C^{0,1}([0, 1])^{n \times n}$ is assumed to be diagonal:

$$\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n), \quad (2)$$

with $p \geq 1$ negative eigenvalues and $m \geq 1$ positive eigenvalues (so that $p + m = n$) such that:

$$\lambda_1(x) \leq \dots \leq \lambda_p(x) < 0 < \lambda_{p+1}(x) \leq \dots \leq \lambda_{p+m}(x), \quad \forall x \in [0, 1], \quad (3)$$

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and we assume that, in case two eigenvalues agree somewhere, they agree everywhere:

$$\forall i, j \in \{1, \dots, n\}, \quad i \neq j, \quad (\exists x \in [0, 1], \quad \lambda_i(x) = \lambda_j(x)) \implies (\lambda_i(x) = \lambda_j(x), \quad \forall x \in [0, 1]). \quad (4)$$

All along this paper, for a vector (or vector-valued function) $y \in \mathbb{R}^n$ we use the notation

$$y = \begin{pmatrix} y_+ \\ y_- \end{pmatrix},$$

where $y_+ \in \mathbb{R}^p$ and $y_- \in \mathbb{R}^m$. Finally, the matrix $M \in L^\infty(0, 1)^{n \times n}$ couples the equations of the system inside the domain and the constant matrix $Q \in \mathbb{R}^{p \times m}$ couples the equations of the system on the boundary $x = 0$.

Taking formally the inner product in \mathbb{R}^n (denoted by \cdot) of (1) with a smooth function φ and integrating in time and space, we are lead to the following definition of solution (see e.g. [BC16, pp. 250-251]):

Definition 1.1. Let $y^0 \in L^2(0, 1)^n$ and $u \in L^2(0, +\infty)^m$. We say that a function y is a (weak) solution to (1) if $y \in C^0([0, +\infty); L^2(0, 1)^n)$ and, for every $T > 0$,

$$\begin{aligned} & \int_0^1 y(T, x) \cdot \varphi(T, x) dx - \int_0^1 y^0(x) \cdot \varphi(0, x) dx \\ &= \int_0^T \int_0^1 y(t, x) \cdot \left(\frac{\partial \varphi}{\partial t}(t, x) - \Lambda(x) \frac{\partial \varphi}{\partial x}(t, x) + \left(-\frac{\partial \Lambda}{\partial x}(x) + M(x)^* \right) \varphi(t, x) \right) dx dt \\ & \quad + \int_0^T u(t) \cdot \Lambda_-(1) \varphi_-(t, 1) dt, \end{aligned} \quad (5)$$

for every $\varphi \in C^1([0, T] \times [0, 1])^n$ such that $\varphi_+(\cdot, 1) = 0$ and $\varphi_-(\cdot, 0) = R^* \varphi_+(\cdot, 0)$, where $R \in \mathbb{R}^{p \times m}$ is defined by

$$R = -\Lambda_+(0) Q \Lambda_-(0)^{-1}, \quad (6)$$

and $\Lambda_+ = \text{diag}(\lambda_1, \dots, \lambda_p)$ and $\Lambda_- = \text{diag}(\lambda_{p+1}, \dots, \lambda_n)$.

We recall that $\Lambda \in C^{0,1}([0, 1])^{n \times n} = W^{1,\infty}(0, 1)^{n \times n}$ so that $\frac{\partial \Lambda}{\partial x}$ exists and belongs to $L^\infty(0, 1)^{n \times n}$. We can establish that system (1) is well-posed, that is, for every $y^0 \in L^2(0, 1)^n$ and $u \in L^2(0, +\infty)^m$, there exists a unique solution $y \in C^0([0, +\infty); L^2(0, 1)^n)$ to (1) and this solution depends continuously on y^0 and u on compact time intervals (see e.g. Section 2 below). The regularity of the solution to (1) allows us to consider control problems in $L^2(0, 1)^n$. We say that the system (1) is:

- exactly controllable in time T if, for every $y^0, y^1 \in L^2(0, 1)^n$, there exists $u \in L^2(0, +\infty)^m$ such that the corresponding solution $y \in C^0([0, +\infty); L^2(0, 1)^n)$ to system (1) satisfies $y(T) = y^1$.
- null controllable in time T if the previous property holds at least for $y^1 = 0$.
- approximately controllable in time T if, for every $\varepsilon > 0$ and every $y^0, y^1 \in L^2(0, 1)^n$, there exists $u \in L^2(0, +\infty)^m$ such that the corresponding solution $y \in C^0([0, +\infty); L^2(0, 1)^n)$ to system (1) satisfies $\|y(T) - y^1\|_{L^2(0, 1)^n} \leq \varepsilon$.

Clearly, exact controllability in time T implies null and approximate controllability in the same time. For any (Λ, M, Q) that satisfies the above standing assumptions, we denote by $T_{\text{inf}}(\Lambda, M, Q) \in [0, +\infty]$ the minimal time for the exact controllability of (1), that is

$$T_{\text{inf}}(\Lambda, M, Q) = \inf \{T > 0, \quad (1) \text{ is exactly controllable in time } T\}. \quad (7)$$

The time $T_{\text{inf}}(\Lambda, M, Q)$ is named “minimal time” according to the current literature, despite it is not always a minimal element of the set. We keep this naming here, but we use the notation with the “inf” to avoid eventual confusions. Since exact controllability in time T_1 clearly implies exact controllability in time T_2 for every $T_2 \geq T_1$, the time $T_{\text{inf}}(\Lambda, M, Q) \in [0, +\infty]$ is also the unique time that satisfies the following two properties:

- If $T > T_{\text{inf}}(\Lambda, M, Q)$, then (1) is exactly controllable in time T .
- If $T < T_{\text{inf}}(\Lambda, M, Q)$, then (1) is not exactly controllable in time T .

The goal of the present article is precisely to explicitly characterize $T_{\text{inf}}(\Lambda, M, Q)$ in terms of Λ , M and Q . To the best of our knowledge, finding the minimal time for the controllability of one-dimensional first-order linear hyperbolic systems is a problem that dates back at least to the celebrated survey [Rus78]. In this article, the author introduced two times, one for which we have exact controllability after this time, and another one for which we do not have exact controllability before this other time (this is also done for the null controllability, with different times). There, the author also observed that in general these two times do not agree, thus leaving a gap in between where the situation was not clear. There are not a lot of other works in the literature devoted to a characterization of the minimal time of control for this class of systems. It seems that the attention was mainly directed towards the controllability of quasilinear versions of such systems afterwards, see for instance the book [Li10] and the references therein. More recently in [Hu15], the author also addressed the question of finding the minimal time of control and he showed that the time of [Rus78] giving the positive result can actually be reduced if we require stronger assumptions on the boundary coupling parameter Q . The author also showed that, for some very particular Q within this class, the time he obtained is in fact the minimal time of control. Extending the condition of [Hu15], a new and smaller time of control was then introduced in [CN18], but it was again shown to be the minimal time only in some particular situations. To summarize, most of the works in the literature require conditions on Q to obtain a better time for the exact (or null) controllability of such systems and, as a result, they are unable to deal with general Q not meeting their conditions. In addition, none of them are actually able to show what happens just before their time without considering particular cases of systems (this is done only for $M = 0$ in [Hu15, CN18]), thus legitimately questioning the optimality of the times they introduced in these articles. In this paper, we will finally completely solve this long-standing problem (regarding the exact controllability property) by developing the original ideas of [Rus78], combined with some results of [DO18] and [NRL86], and by introducing an accurate factorization of Q similar to the one of [DJM06].

Before going further and stating the main result of this paper, we need to introduce some notations and concepts. We start with the characteristics associated with system (1). For every $i \in \{1, \dots, n\}$, every $t \geq 0$ and $x \in [0, 1]$ fixed, we introduce the characteristic $\chi_i(\cdot; t, x) \in C^1([s_i^{\text{in}}(t, x), s_i^{\text{out}}(t, x)])$ passing through (t, x) , that is the solution to the ordinary differential equation:

$$\begin{cases} \frac{d}{ds}\chi_i(s; t, x) = \lambda_i(\chi_i(s; t, x)), & s \in [s_i^{\text{in}}(t, x), s_i^{\text{out}}(t, x)], \\ \chi_i(t; t, x) = x, \end{cases} \quad (8)$$

where $s_i^{\text{in}}(t, x), s_i^{\text{out}}(t, x) \in \mathbb{R}$ (with $s_i^{\text{in}}(t, x) < t < s_i^{\text{out}}(t, x)$) are the enter and exit parameters of the domain $[0, 1]$, that is the unique respective solutions to

$$\begin{cases} \chi_i(s_i^{\text{in}}(t, x); t, x) = 1, & \chi_i(s_i^{\text{out}}(t, x); t, x) = 0, & \text{if } i \in \{1, \dots, p\}, \\ \chi_i(s_i^{\text{in}}(t, x); t, x) = 0, & \chi_i(s_i^{\text{out}}(t, x); t, x) = 1, & \text{if } i \in \{p+1, \dots, n\}. \end{cases} \quad (9)$$

Their existence and uniqueness are guaranteed by the assumption (3). We then introduce

$$T_i(\Lambda) = \begin{cases} s_i^{\text{out}}(0, 1) & \text{if } i \in \{1, \dots, p\}, \\ s_i^{\text{out}}(0, 0) & \text{if } i \in \{p+1, \dots, n\}. \end{cases}$$

Since the speeds do not depend on time, the exact value of $T_i(\Lambda)$ can actually be obtained by integrating over $[0, 1]$ the differential equation satisfied by the inverse function $\xi \mapsto \chi_i^{-1}(\xi; t, x)$ (see e.g. (30) and (31) below):

$$T_i(\Lambda) = \begin{cases} -\int_0^1 \frac{1}{\lambda_i(\xi)} d\xi & \text{if } i \in \{1, \dots, p\}, \\ \int_0^1 \frac{1}{\lambda_i(\xi)} d\xi & \text{if } i \in \{p+1, \dots, n\}. \end{cases} \quad (10)$$

For the rest of this article it is important to keep in mind that the assumption (3) implies the following order relation between the $T_i(\Lambda)$:

$$\begin{cases} T_1(\Lambda) \leq \dots \leq T_p(\Lambda), \\ T_{p+m}(\Lambda) \leq \dots \leq T_{p+1}(\Lambda). \end{cases} \quad (11)$$

It is known that the combination of the two largest times $T_p(\Lambda) + T_{p+1}(\Lambda)$ yields a time for which the exact controllability of (1) holds if $\text{rank } Q = p$. This was proved for instance in [Rus78, Theorem 3.2] but with a slightly different boundary condition at $x = 1$, namely $y_-(t, 1) = D_1 y_+(t, 1) + Du(t)$ with $D_1 \in \mathbb{R}^{m \times p}$ and $D \in \mathbb{R}^{m \times m}$ invertible (so that $m = p$), which makes the corresponding system time-reversible and has clearly equivalent controllability properties. Let us also mention [Li10, Theorem 3.2] for a constructive method, moreover for quasilinear systems. It is then not difficult to see that $T_p(\Lambda) + T_{p+1}(\Lambda)$ is the sharpest time for the exact controllability of (1) which is uniform with respect to all possible choices of M and Q , i.e.

$$\max_M \max_{\substack{Q \\ \text{rank } Q=p}} T_{\text{inf}}(\Lambda, M, Q) = T_p(\Lambda) + T_{p+1}(\Lambda). \quad (12)$$

In [Rus78], the author already tried to improve the time (12) by imposing some conditions on Q . Notably, he showed in [Rus78, Theorem 3.7] (still in the above context) that the exact controllability holds in time T_1 , where

$$T_1 = \max_{i \in \{p+1, \dots, n\}} (\widehat{T}_i, T_p(\Lambda)), \quad \widehat{T}_i = \begin{cases} T_i(\Lambda) + T_{j_i}(\Lambda) & \text{if } j_i \text{ exists,} \\ T_i(\Lambda) & \text{if not,} \end{cases} \quad (13)$$

where $j_i \in \{1, \dots, p\}$ is the largest index such that $q_{j_i, i-p} \neq 0$, if it exists¹. In [Rus78, Theorem 3.6], he also introduced a time $T_0 > 0$ for which the exact controllability fails for $T < T_0$ but he already observed that in general these two times T_1 and T_0 do not agree.

More recently, it has been shown in [Hu15], and then in [CN18], that the time (12) may be sharpen for another particular class of matrices Q . To be more precise, let us introduce the following condition for every $i \in \{1, \dots, p\}$:

$$\text{the } i \times i \text{ matrix formed from the last } i \text{ rows and the last } i \text{ columns of } Q \text{ is invertible.} \quad (14)$$

¹There is a missprint in the definition of j_i in [Rus78, Proposition 3.4]: it should be $(D_0^*)_{j_i}^{i-p}$ in place of $(D_0^*)_{j_i}^i$ (which does not always make sense since $i \in \{p+1, \dots, n\}$).

It was then proved in [Hu15, Theorem 1.1], by developing the constructive approach of [Li10, Theorem 3.2], that for every Q such that (14) holds for $i = p$, the system (1) with $M = 0$ (in fact, a quasilinear version of (1)) is exactly controllable in time T for every $T > T_2$, where

$$T_2 = \max \{T_{m+1}(\Lambda) + T_p(\Lambda), T_{p+1}(\Lambda)\}. \quad (15)$$

We emphasize that the condition (14) for $i = p$ is stronger than just assuming that $\text{rank } Q = p$ when $m > p$. In [Hu15, Remark 1.3], the author then showed an example of Q that satisfies (14) for $i = p$ for which the system (1) with $M = 0$ is not exactly controllable in time T for every $T < T_2$.

On the other hand, using the so-called backstepping method, it was proved in [CN18, Theorem 2] that for every Q such that (14) holds for every $i \in \{1, \dots, p\}$ and every but a possible countable number of M , the system (1) is exactly controllable in time T_3 , where

$$T_3 = \max_{i \in \{1, \dots, p\}} (T_i(\Lambda) + T_{m+i}(\Lambda), T_{p+1}(\Lambda)). \quad (16)$$

It is also shown in [CN18, Theorem 3] that the part “but a possible countable number of” in the previous statement can be removed if we look for exact controllability in times $T > T_3$, but it is done under obviously too restrictive assumptions ($m = 2$, $p = 1$ or $p = 2$, Λ constant, M analytic in a neighborhood of $x = 0$, etc.). Finally, in [CN18, Proposition 1] the authors showed that the system (1) with $M = 0$ is not exactly controllable in time T for every $T < T_3$ but, once again, only for Q satisfying (14) for every $i \in \{1, \dots, p\}$.

In this article we will obtain the minimal time of control for any fixed Q , without assuming anything more than $\text{rank } Q = p$. This will generalize the previous results. As already mentioned before, we use a different approach than in the two last mentioned articles and we go back to the original ideas of the first paper [Rus78].

To deal with general Q and state our main result we need to introduce the concept of canonical form for full row rank matrices (a related notion can be found in [DJM06, Definition 2]):

Definition 1.2. We say that a matrix $Q^0 \in \mathbb{R}^{p \times m}$ is in canonical form if there exist distinct column indices $c_1(Q^0), \dots, c_p(Q^0) \in \{1, \dots, m\}$ such that:

$$\forall i \in \{1, \dots, p\}, \quad \begin{cases} q_{i, c_i(Q^0)}^0 \neq 0, \\ q_{i, j}^0 = 0, \quad \forall j > c_i(Q^0), \quad j \notin \{c_{i+1}(Q^0), \dots, c_p(Q^0)\}, \\ q_{i, j}^0 = 0, \quad \forall j < c_i(Q^0). \end{cases} \quad (17)$$

Example 1.3. Consider the following matrices

$$Q_1^0 = \begin{pmatrix} 0 & \boxed{1} & 4 & -1 \\ 0 & 0 & \boxed{2} & 3 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix}, \quad Q_2^0 = \begin{pmatrix} 0 & 0 & \boxed{4} \\ \boxed{1} & 2 & 0 \\ 0 & \boxed{1} & 0 \end{pmatrix}, \quad Q_3^0 = \begin{pmatrix} 1 & 4 & -1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

The matrices Q_1^0 and Q_2^0 are both in canonical form, with $c_3(Q_1^0) = 4$, $c_2(Q_1^0) = 3$, $c_1(Q_1^0) = 2$ and $c_3(Q_2^0) = 2$, $c_2(Q_2^0) = 1$, $c_1(Q_2^0) = 3$. However, Q_3^0 is not in canonical form because there is no $c_3(Q_3^0)$ that simultaneously satisfies the second and third conditions of (17).

Remark 1.4. If $Q^0 \in \mathbb{R}^{p \times m}$ is in canonical form, then necessarily:

- (i) The indices $c_1(Q^0), \dots, c_p(Q^0)$ are unique.

(ii) $q_{i,j}^0 = 0$ for every $i \in \{1, \dots, p\}$ and $j \notin \{c_1(Q^0), \dots, c_p(Q^0)\}$.

(iii) $\text{rank } Q^0 = p$.

(iv) We have

$$q_{k,c_i(Q^0)}^0 = 0, \quad \forall k > i, \quad \forall i \in \{1, \dots, p\}. \quad (18)$$

The first point is clear since $c_i(Q^0)$ is the column index of the unique non-zero entry of the i -th row of Q^0 that is not in the columns with indices $c_{i+1}(Q^0), \dots, c_p(Q^0)$. The second point immediately follows from the two last conditions in (17). The third point is also clear by considering a linear combination of the only p non-zero columns of Q^0 and looking first at its last row, then at its last but one row, etc. For the last point, first note that for $i = p$, (18) is clear since there is no condition ($k \in \{1, \dots, p\}$). For $i = p - 1$, we have to check that $q_{p,c_{p-1}(Q^0)}^0 = 0$. Since $c_{p-1}(Q^0) \neq c_p(Q^0)$ we have two possibilities, either $c_{p-1}(Q^0) < c_p(Q^0)$ so that the equality follows from the last condition in (17), either $c_{p-1}(Q^0) > c_p(Q^0)$ so that the equality follows from the second condition in (17). Repeating the reasoning for $i = p - 2, p - 3$, etc. eventually leads to (18).

Next, we present a result that comes from the Gaussian elimination and that we will call in this article “canonical UL -decomposition” (U for upper and L for lower, see also Remark 1.12 below for this naming):

Proposition 1.5. *Let $Q \in \mathbb{R}^{p \times m}$ with $\text{rank } Q = p$. Then, there exists a unique $Q^0 \in \mathbb{R}^{p \times m}$ such that the following two properties hold:*

(i) *There exists $L \in \mathbb{R}^{m \times m}$ such that $QL = Q^0$ with L lower triangular ($\ell_{ij} = 0$ if $i < j$) and with only ones on its diagonal ($\ell_{ii} = 1$ for every i).*

(ii) *Q^0 is in canonical form.*

We call Q^0 the canonical form of Q .

We mention that, because of possible zero columns of Q , the matrix L is in general not unique. The proof of Proposition 1.5 is given in Appendix A. With this proposition, we can extend the definition of the c_i indices in Definition 1.2 to any full row rank matrix:

Definition 1.6. Let $Q \in \mathbb{R}^{p \times m}$ with $\text{rank } Q = p$. We define $c_1(Q), \dots, c_p(Q) \in \{1, \dots, m\}$ by

$$c_i(Q) = c_i(Q^0),$$

where Q^0 is the canonical form of Q provided by Proposition 1.5.

Example 1.7. *We illustrate how to find the decomposition of Proposition 1.5 in practice. Consider*

$$Q_1 = \begin{pmatrix} 4 & 6 & 3 & -1 \\ 8 & -1 & 5 & 3 \\ 2 & -1 & 1 & 1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 4 & -4 & 4 \\ 5 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix}.$$

Let us deal with Q_1 first. We look at the last row, we take the last nonzero entry as pivot. We remove the entries to the left on the same row by doing the column substitutions $C_3 \leftarrow C_3 - C_4$, $C_2 \leftarrow C_2 + C_4$

and $C_1 \leftarrow C_1 - 2C_4$ so that

$$Q_1 L_1 = Q_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 5 & 4 & -1 \\ 2 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We now move up one row and take as new pivot the last nonzero entry that is not in C_4 . We remove the entries to the left on the same row by doing the column substitutions $C_2 \leftarrow C_2 - C_3$ and $C_1 \leftarrow C_1 - C_3$ so that

$$Q_1 L_1 L_2 = Q_1 L_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 4 & -1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally, a last substitution shows that Q_1 becomes Q_1^0 of Example 1.3, namely:

$$Q_1 L = Q_1 L_1 L_2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 4 & -1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = Q_1^0.$$

Similarly, it can be checked the canonical form of Q_2 is in fact Q_2^0 of Example 1.3.

Remark 1.8. Where we want to put entries to zero in Example 1.7 in fact depends on the way the times are ordered (11). This will be more clear during the proof of Theorem 3.1 below. We mention this point to highlight the fact that the definition of the canonical form is linked to this ordering and a change in (11) may then require another kind of factorization of Q .

After such a long but necessary preparation we can now clearly state the main result of this paper:

Theorem 1.9. *Let $\Lambda \in C^{0,1}([0, 1])^{n \times n}$ satisfy (2), (3) and (4), $M \in L^\infty(0, 1)^{n \times n}$ and $Q \in \mathbb{R}^{p \times m}$ be fixed. We have:*

- (i) $T_{\inf}(\Lambda, M, Q) < +\infty$ if, and only if, $\text{rank } Q = p$.
- (ii) If $\text{rank } Q = p$, then

$$T_{\inf}(\Lambda, M, Q) = \max_{i \in \{1, \dots, p\}} (T_{p+1}(\Lambda), T_i(\Lambda) + T_{p+c_i(Q)}(\Lambda)), \quad (19)$$

where $c_1(Q), \dots, c_p(Q) \in \{1, \dots, m\}$ are defined in Definition 1.6.

To the best of our knowledge, this is the first result that completely characterizes the minimal time for the exact controllability of (1). Not only this, but this result also shows that the time (19) is explicit in terms of Λ (recall (10)) and in terms of Q as well, since the computation of the indices $c_i(Q)$ rely on the Gaussian elimination, which is a very efficient algorithm that shows that the minimal time (19) is actually easy to compute in practice.

Example 1.10. *A comparison with the other times of the literature (13), (15) and (16) can be made. For $Q_1 \in \mathbb{R}^{3 \times 4}$ of Example 1.7 we have*

$$T_{\inf}(\Lambda, M, Q_1) = T_3 = \max(T_4(\Lambda), T_1(\Lambda) + T_5(\Lambda), T_2(\Lambda) + T_6(\Lambda), T_3(\Lambda) + T_7(\Lambda)),$$

whereas $T_1 = T_4(\Lambda) + T_3(\Lambda)$ and $T_2 = \max(T_4(\Lambda), T_3(\Lambda) + T_5(\Lambda))$. On the contrary, the case of $Q_2 \in \mathbb{R}^{3 \times 3}$ of Example 1.7 is not covered by the results [CN18], and for this parameter we have

$$\begin{aligned} T_{\inf}(\Lambda, M, Q_2) &= \max(T_4(\Lambda), T_1(\Lambda) + T_6(\Lambda), T_2(\Lambda) + T_4(\Lambda), T_3(\Lambda) + T_5(\Lambda)) \\ &= \max(T_2(\Lambda) + T_4(\Lambda), T_3(\Lambda) + T_5(\Lambda)), \end{aligned}$$

whereas $T_1 = T_2 = T_4(\Lambda) + T_3(\Lambda)$.

Remark 1.11. Observe that the expression (19) of $T_{\inf}(\Lambda, M, Q)$ does not depend on M . This means that the internal coupling terms $M(x)y(t, x)$ in (1) have almost no impact on the controllability properties of this system. All our attention should then be on the coupling on the boundary Q . Let us however mention that whether the infimum in the definition (7) of $T_{\inf}(\Lambda, M, Q)$ is or is not a minimum depends on the values of M . In fact, by using the techniques we will develop below, it can be shown that the minimum is reached if, and only if, (1) is approximately controllable in time (19), and it is known that this latter property depends on M , as for instance illustrated in [Rus78, pp. 659-661] (see also item 2. of [CN18, Theorem 1]). A complete characterization of the parameters M and Q for which the infimum is equal to the minimum seems an open problem (some partial results can be found in [CN18]).

Remark 1.12. We have seen that $T_p(\Lambda) + T_{p+1}(\Lambda)$ is the worst possible time of control. On the other hand, it can be checked that

$$\min_{\substack{(c_1, \dots, c_p) \in \{1, \dots, m\} \\ c_j \neq c_k, j \neq k}} \left(\max_{i \in \{1, \dots, p\}} (T_{p+1}(\Lambda), T_i(\Lambda) + T_{p+c_i}(\Lambda)) \right) = T_3,$$

where we recall that T_3 is defined in (16), and the minimum is reached for c_i satisfying

$$c_i = m - p + i, \quad \forall i \in \{1, \dots, p\}. \quad (20)$$

The condition (20) means that the canonical form Q^0 of Q is an upper triangular matrix, see e.g. Q_1^0 of Example 1.3. Thus in this case Q has a “standard” UL -decomposition. Moreover, it can be shown with the Gaussian elimination that a full row-rank matrix Q admits such a decomposition if, and only if, Q satisfies (14) for every $i \in \{1, \dots, p\}$ (see e.g. [Gan59, Theorem II.1]). As a result, we see that we recover the time and the assumption given in [CN18, Theorem 2]. Note as well that our observation justifies the name of “optimal time” given in this article (before it, there were no real justification to such a naming).

The rest of the paper is organized as follows. In the next section we simply recast the system (1) into its abstract form and prove basic properties. In Section 3, we make use of the notion of canonical UL -decomposition to establish necessary and sufficient conditions for the system (1) to be exact controllable in a given time when there are no internal coupling terms, i.e. when $M = 0$. In Section 4 we use compactness-uniqueness arguments to show that the minimal time of control remains the same when we add a bounded perturbation M .

2 Abstract setting

It is well-known that the system (1) can equivalently be rewritten as an abstract evolution system:

$$\begin{cases} \frac{d}{dt}y(t) = A_M y(t) + Bu(t), & t \in (0, +\infty), \\ y(0) = y^0, \end{cases} \quad (21)$$

also to be referred to as (A_M, B) in the sequel, where we can identify the operators A_M and B through their adjoints by formally taking the inner product of (21) with a smooth function φ and then comparing with (5). The state and control spaces are

$$H = L^2(0, 1)^n, \quad U = \mathbb{R}^m.$$

They are equipped with their usual inner products and identified with their dual. The unbounded linear operator $A_M : D(A_M) \subset H \rightarrow H$ is defined, for every $y \in D(A_M)$ by

$$A_M y(x) = \Lambda(x) \frac{\partial y}{\partial x}(x) + M(x)y(x), \quad x \in (0, 1),$$

with domain

$$D(A_M) = \{y \in H^1(0, 1)^n, \quad y_+(0) = Qy_-(0), \quad y_-(1) = 0\}.$$

It is clear that $D(A_M)$ is dense in H since it contains $C_c^\infty(0, 1)^n$. A computation shows that

$$D(A_M^*) = \{z \in H^1(0, 1)^n, \quad z_+(1) = 0, \quad z_-(0) = R^* z_+(0)\},$$

where we recall that $R \in \mathbb{R}^{p \times m}$ is defined in (6), and we have, for every $z \in D(A_M^*)$,

$$A_M^* z(x) = -\Lambda(x) \frac{\partial z}{\partial x}(x) + \left(-\frac{\partial \Lambda}{\partial x}(x) + M(x)^*\right) z(x), \quad x \in (0, 1). \quad (22)$$

Note that in fact $D(A_M^*)$ does not depend on M . On the other hand, the control operator $B \in \mathcal{L}(U, D(A_M^*)')$ is given for every $u \in U$ and $z \in D(A_M^*)$ by

$$\langle Bu, z \rangle_{D(A_M^*)', D(A_M^*)} = u \cdot \Lambda_-(1) z_-(1). \quad (23)$$

Note that B is well-defined since Bu is continuous on $H^1(0, L)^n$ (by the trace theorem $H^1(0, 1)^n \hookrightarrow C^0([0, 1]^n)$) and since $\|\cdot\|_{D(A_M^*)}$ and $\|\cdot\|_{H^1(0, 1)^n}$ are equivalent norms on $D(A_M^*)$. Finally, the adjoint $B^* \in \mathcal{L}(D(A_M^*), U)$ is given for every $z \in D(A_M^*)$ by

$$B^* z = \Lambda_-(1) z_-(1).$$

Using the method of characteristics, it is not difficult to show that the operator A_M generates a C_0 -semigroup when M is diagonal and we even have an explicit formula for it. Since we will mainly perform computations on the adjoint semigroup in the sequel, it is then when $M = \frac{\partial \Lambda}{\partial x}$ that the adjoint semigroup will have the simplest expression (see (22)).

Proposition 2.1. *For every $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, m\}$, let $\phi_i, \phi_{p+j} \in C^{1,1}([0, 1])$ be the non-negative and increasing functions defined for every $x \in [0, 1]$ by*

$$\phi_i(x) = - \int_0^x \frac{1}{\lambda_i(\xi)} d\xi, \quad \phi_{p+j}(x) = \int_0^x \frac{1}{\lambda_{p+j}(\xi)} d\xi, \quad (24)$$

(note that $\phi_i(1) = T_i(\Lambda)$ and $\phi_{p+j}(1) = T_{p+j}(\Lambda)$, see (10)). Then, the operator $A_{\frac{\partial\Lambda}{\partial x}}^*$ generates a C_0 -semigroup on H given, for every $t \geq 0$ and $z^0 \in H$, by

$$\left(S_{A_{\frac{\partial\Lambda}{\partial x}}}(t)^* z^0\right)_i(x) = \begin{cases} z_i^0(\phi_i^{-1}(t + \phi_i(x))), & \text{if } t + \phi_i(x) < \phi_i(1), \\ 0, & \text{if } t + \phi_i(x) > \phi_i(1), \end{cases} \quad (25)$$

for every $i \in \{1, \dots, p\}$ and a.e. $x \in (0, 1)$, and by

$$\begin{aligned} & \left(S_{A_{\frac{\partial\Lambda}{\partial x}}}(t)^* z^0\right)_{p+j}(x) \\ &= \begin{cases} z_{p+j}^0(\phi_{p+j}^{-1}(\phi_{p+j}(x) - t)), & \text{if } t - \phi_{p+j}(x) < 0, \\ \sum_{i=1}^p r_{i,p+j} z_i^0(\phi_i^{-1}(t - \phi_{p+j}(x))), & \text{if } 0 < t - \phi_{p+j}(x) < \phi_1(1), \\ \vdots & \vdots \\ \sum_{i=k+1}^p r_{i,p+j} z_i^0(\phi_i^{-1}(t - \phi_{p+j}(x))), & \text{if } \phi_k(1) < t - \phi_{p+j}(x) < \phi_{k+1}(1), \\ \vdots & \vdots \\ 0 & \text{if } \phi_p(1) < t - \phi_{p+j}(x), \end{cases} \quad (26) \end{aligned}$$

for every $j \in \{1, \dots, m\}$ and a.e. $x \in (0, 1)$.

Proof. We only show how to find the formula (25) and (26). It can be checked afterwards that these formula define a C_0 -semigroup and that $A_{\frac{\partial\Lambda}{\partial x}}^*$ is indeed the corresponding generator (by using the very definition of what is a C_0 -semigroup). We recall that $\tilde{z}(t) = S_{A_{\frac{\partial\Lambda}{\partial x}}}(t)^* z^0$ is the unique solution to the following abstract O.D.E. when $z^0 \in D(A_{\frac{\partial\Lambda}{\partial x}}^*)$ (see e.g. [EN00, Lemma II.1.3]):

$$\begin{cases} \frac{d}{dt} \tilde{z}(t) = A_{\frac{\partial\Lambda}{\partial x}}^* \tilde{z}(t), & t \in [0, +\infty), \\ \tilde{z}(0) = z^0. \end{cases}$$

Therefore, we expect \tilde{z} to solve

$$\begin{cases} \frac{\partial \tilde{z}}{\partial t}(t, x) = -\Lambda(x) \frac{\partial \tilde{z}}{\partial x}(t, x), \\ \tilde{z}_+(t, 1) = 0, \quad \tilde{z}_-(t, 0) = R^* \tilde{z}_+(t, 0), & t \in [0, +\infty), x \in (0, 1), \\ \tilde{z}(0, x) = z^0(x), \end{cases} \quad (27)$$

Let us first find \tilde{z}_i for $i \in \{1, \dots, p\}$. Since \tilde{z}_i solves

$$\begin{cases} \frac{\partial \tilde{z}_i}{\partial t}(t, x) + \lambda_i(x) \frac{\partial \tilde{z}_i}{\partial x}(t, x) = 0, \\ \tilde{z}_i(t, 1) = 0, & t \in [0, +\infty), x \in (0, 1), \\ \tilde{z}_i(0, x) = z_i^0(x), \end{cases}$$

along the characteristic χ_i we have

$$\frac{d}{ds} \tilde{z}_i(s, \chi_i(s; t, x)) = 0, \quad \forall s \in [s_i^{\text{in}}(t, x), s_i^{\text{out}}(t, x)], \quad s \in [0, +\infty).$$

It follows that

$$\tilde{z}_i(t, x) = \begin{cases} z_i^0(\chi_i(0; t, x)), & \text{if } s_i^{\text{in}}(t, x) < 0, \\ 0, & \text{if } s_i^{\text{in}}(t, x) > 0. \end{cases} \quad (28)$$

On the other hand, for $j \in \{1, \dots, m\}$, similar computations lead to

$$\tilde{z}_{p+j}(t, x) = \begin{cases} z_{p+j}^0(\chi_{p+j}(0; t, x)), & s_{p+j}^{\text{in}}(t, x) < 0, \\ \sum_{i=1}^p r_{i,p+j} \tilde{z}_i(s_{p+j}^{\text{in}}(t, x), 0), & s_{p+j}^{\text{in}}(t, x) > 0. \end{cases} \quad (29)$$

Now, since λ_i does not depend on time, we have a more explicit formula for $\chi_i(0; t, x)$ and $s_i^{\text{in}}(t, x)$. Indeed, the inverse function $\xi \mapsto \chi_i^{-1}(\xi; t, x)$ solves

$$\begin{cases} \frac{\partial \chi_i^{-1}}{\partial \xi}(\xi; t, x) = \frac{1}{\frac{\partial \chi_i}{\partial s}(\chi_i^{-1}(\xi; t, x); t, x)} = \frac{1}{\lambda_i(\xi)}, & \xi \in [0, 1], \\ \chi_i^{-1}(x; t, x) = t. \end{cases} \quad (30)$$

Therefore, $\chi_i^{-1}(y; t, x) = t + \int_x^y \frac{1}{\lambda_i(\xi)} d\xi$. Using the functions (24), we have

$$\chi_i^{-1}(y; t, x) = \begin{cases} t + \phi_i(x) - \phi_i(y), & \text{if } i \in \{1, \dots, p\}, \\ t - \phi_i(x) + \phi_i(y), & \text{if } i \in \{p+1, \dots, n\}. \end{cases} \quad (31)$$

Recalling the definition (9) of $s_i^{\text{in}}(t, x)$, we then have

$$s_i^{\text{in}}(t, x) = \begin{cases} t + \phi_i(x) - \phi_i(1), & \text{if } i \in \{1, \dots, p\}, \\ t - \phi_i(x), & \text{if } i \in \{p+1, \dots, n\}, \end{cases} \quad (32)$$

and

$$\chi_i(0; t, x) = \begin{cases} \phi_i^{-1}(t + \phi_i(x)), & \text{if } i \in \{1, \dots, p\} \text{ and } s_i^{\text{in}}(t, x) < 0, \\ \phi_i^{-1}(\phi_i(x) - t), & \text{if } i \in \{p+1, \dots, n\} \text{ and } s_i^{\text{in}}(t, x) < 0. \end{cases} \quad (33)$$

Plugging these formula in (28) and (29), and taking into account that $\phi_i(1) \leq \phi_{i+1}(1)$ for every $i \in \{1, \dots, p-1\}$ by (11), we obtain (25) and (26). \square

Remark 2.2. Observe that the right-hand sides in (25) and (26), considered as functions of t and x , make sense for $z^0 \in L^2(0, 1)^n$ only (i.e. the compositions are well-defined), either for every $t \geq 0$ and a.e. $x \in [0, 1]$, or for every $x \in [0, 1]$ and a.e. $t \geq 0$. For instance for (25) this follows from the fact that the maps $x \in (0, \phi_i^{-1}(\phi_i(1) - t)) \mapsto \phi_i^{-1}(t + \phi_i(x))$ and $t \in (0, \phi_i(1) - \phi_i(x)) \mapsto \phi_i^{-1}(t + \phi_i(x))$ are C^1 -diffeomorphisms (for every $t \in [0, \phi_i(1)]$ and $x \in [0, 1)$, respectively). For the rest of this article, we then abuse the notation $S_{A \frac{\partial \Delta}{\partial x}}(t)^* z^0(x)$ to denote either of these functions when $z^0 \in L^2(0, 1)^n$.

Let us now turn out to the properties of the control operator B . First of all, it can be checked directly from the formula (26) that, when $z^0 \in D(A_{\frac{\partial \Delta}{\partial x}}^*)$, the function $x \mapsto \left(S_{A_{\frac{\partial \Delta}{\partial x}}}(t)^* z^0 \right)_-(x)$ belongs to $H^1(0, 1)^m$ and has a trace at $x = 1$ equal to $\left(S_{A_{\frac{\partial \Delta}{\partial x}}}(t)^* z^0 \right)_-(1)$ since the right-hand side of (26) is a continuous function of x on $[0, 1]$ for such z^0 . A simple change of variable then easily shows that, for any $0 < T < \phi_n(1)$, there exists $C > 0$ such that

$$\int_0^T \left\| B^* S_{A_{\frac{\partial \Delta}{\partial x}}}(t)^* z^0 \right\|_U^2 dt \leq C \|z^0\|_H^2, \quad \forall z^0 \in D(A_{\frac{\partial \Delta}{\partial x}}^*). \quad (34)$$

This property shows that B is a so-called admissible control operator for $A_{\frac{\partial \Delta}{\partial x}}$ (see e.g. [TW09, Theorem 4.4.3]).

Since the operator A_M is nothing but a bounded perturbation of $A_{\frac{\partial \Delta}{\partial x}}$, it follows that A_M also generates a C_0 -semigroup on H (see e.g. [EN00, Theorem III.1.3]) and that B is also admissible for A_M (see e.g. [DO18, p. 401]). It also follows that the abstract system (21) is well-posed in the sense that: for every $y^0 \in H$ and every $u \in L^2(0, +\infty; U)$, there exists a unique solution $y \in C^0([0, +\infty); H)$ to (21) given by the Duhamel formula (see e.g. [TW09, Proposition 4.2.5]):

$$y(T) = S_{A_M}(T)y^0 + \Phi_M(T)u, \quad \forall T \geq 0, \quad (35)$$

where $\Phi_M(T)$ is the so-called input map of (A_M, B) , that is the linear operator defined for every $u \in L^2(0, +\infty; U)$ by

$$\Phi_M(T)u = \int_0^T S_{A_M}(T-s)Bu(s) ds.$$

We recall that a priori $\text{Im } \Phi_M(T) \subset D(A_M^*)'$ but the admissibility of B in fact means that $\text{Im } \Phi_M(T) \subset H$ for some (and hence all) $T > 0$ (see e.g. [TW09, Definition 4.2.1]). From this assumption it follows that the function $T \in [0, +\infty) \mapsto \Phi_M(T)u \in H$ is continuous for every $u \in L^2(0, +\infty; U)$ (see e.g. [TW09, Proposition 4.2.4]), so that the function y defined by (35) indeed belongs to $C^0([0, +\infty); H)$. From the admissibility of B it also follows that $\Phi_M(T) \in \mathcal{L}(L^2(0, +\infty; U), H)$ (see e.g. [TW09, Proposition 4.2.2]). The adjoint $\Phi_M(T)^* \in \mathcal{L}(H, L^2(0, +\infty; U))$ is nothing but the unique continuous linear extension to H of the map that takes $z^1 \in D(A_M^*)$ and associates to it the following function of $L^2(0, +\infty; U)$ (see e.g. [TW09, Proposition 4.4.1]):

$$t \in (0, +\infty) \mapsto \begin{cases} B^* S_{A_M}(T-t)^* z^1, & \text{if } t \in (0, T), \\ 0, & \text{if } t > T. \end{cases}$$

Finally, it can be checked that the function y defined by (35) satisfies (5) and is thus the (weak) solution to (1) in the sense of Definition 1.1 (see e.g. [Cor07, pp. 63-65]).

Let us now recall that all the notions of controllability can be reformulated in terms of $\text{Im } \Phi_M(T)$. Indeed, it is not difficult to see that (A_M, B) is exactly (resp. approximately) controllable in time T if, and only if, $\text{Im } \Phi_M(T) = H$ (resp. $\overline{\text{Im } \Phi_M(T)} = H$). It is also well-known that the controllability has a dual concept named observability. More precisely (see e.g. [TW09, Theorem 11.2.1]), (A_M, B) is exactly controllable in time T if, and only if, there exists $C > 0$ such that

$$\|z^1\|_H^2 \leq C \int_0^T \|\Phi_M(T)^* z^1(t)\|_U^2 dt, \quad \forall z^1 \in H, \quad (36)$$

and (A_M, B) is approximately controllable in time T if, and only if,

$$(\Phi_M(T)^* z^1(t) = 0, \quad \text{a.e. } t \in (0, T)) \implies z^1 = 0, \quad \forall z^1 \in H. \quad (37)$$

Finally, for $M = \frac{\partial \Lambda}{\partial x}$, the adjoint of the input map $\Phi_{\frac{\partial \Lambda}{\partial x}}(T)^*$ is explicit. Indeed, we see from the formula (26) that the operator $z^1 \in L^2(0, 1)^n \mapsto \Lambda_-(1) \left(S_{A_{\frac{\partial \Lambda}{\partial x}}}(T - \cdot)^* z^1 \right)_-(1)$ (extended by zero outside $(0, T)$) belongs to $\mathcal{L}(H, L^2(0, +\infty; U))$. Since we have already seen that it agrees with $B^* S_{A_{\frac{\partial \Lambda}{\partial x}}}(T - \cdot)^* z^1$ for $z^1 \in D(A_{\frac{\partial \Lambda}{\partial x}}^*)$, by uniqueness of the continuous extension, this shows that the adjoint of the input map is given, for every $z^1 \in H$, by

$$\Phi_{\frac{\partial \Lambda}{\partial x}}(T)^* z^1(t) = \Lambda_-(1) \left(S_{A_{\frac{\partial \Lambda}{\partial x}}}(T - t)^* z^1 \right)_-(1), \quad \text{a.e. } t \in (0, T).$$

3 Controllability of the unperturbed system

The goal of this section is to characterize the minimal time for the exact controllability of the unperturbed system (A_0, B) , i.e. of the system

$$\begin{cases} \frac{\partial y}{\partial t}(t, x) = \Lambda(x) \frac{\partial y}{\partial x}(t, x), \\ y_+(t, 0) = Q y_-(t, 0), \quad y_-(t, 1) = u(t), \quad t \in (0, +\infty), x \in (0, 1). \\ y(0, x) = y^0(x), \end{cases} \quad (38)$$

It is indeed natural to first investigate what happens when $M = 0$ and constitutes a first step towards our main result Theorem 1.9. We will then use a perturbation argument in the next section to deal with internal couplings $M \neq 0$. For the system (38) we will actually establish an even more precise result, namely:

Theorem 3.1. *Let $\Lambda \in C^{0,1}([0, 1])^{n \times n}$ satisfy (2) and (3), and $Q \in \mathbb{R}^{p \times m}$ be fixed. For every $T > 0$, (38) is exactly controllable in time T if, and only if, the following two properties hold:*

- (i) $\text{rank } Q = p$.
- (ii) $T \geq \max_{i \in \{1, \dots, p\}} (T_{p+1}(\Lambda), T_i(\Lambda) + T_{p+c_i(Q)}(\Lambda))$.

Remark 3.2. Note that the assumption (4) is not needed in Theorem 3.1. We also point out that the assumption (3) could be weakened all along this section into the following:

$$\lambda_i(x) < 0 < \lambda_{p+j}(x), \quad \forall x \in [0, 1], \quad \forall i \in \{1, \dots, p\}, \quad \forall j \in \{1, \dots, m\},$$

as long as we assume that the eigenvalues are ordered in such a way that (11) holds, which can always be done without loss of generality.

As simple as system (38) looks like (the equations are not coupled inside the domain), a necessary and sufficient condition for the exact controllability for general boundary couplings Q was not known before, to the best of our knowledge. As already mentioned in the introduction, the sufficiency is established in [Hu15, Theorem 1.1] for couplings Q satisfying the condition (14) for $i = p$ (which is stronger than (i)) and for times $T > T_2$, where we recall that T_2 is defined by (15) (possibly larger than in (ii), see e.g. Example 1.10). Moreover, the necessity is only shown for a particular choice of Q in [Hu15, Remark 1.3]. On the other hand, it is proved in [CN18, Proposition 1], under the standing assumption that Q satisfies (14) for every $i \in \{1, \dots, p\}$, that (ii) is a sufficient condition for the null controllability in time T , and that it also becomes necessary if we add (i). Theorem 3.1 is thus the first result to provide a complete answer to this problem. It also partially solves the open problem in [Rus78, Remark p. 656] (we say ‘‘partially’’ because this paper was focused on the null controllability): ‘‘This raises the question, unresolved at the moment, concerning the identification

of a “critical time” T_c such that observability holds if $T \geq T_c$ and does not hold if $T < T_c$. Such a critical time T_c can readily be shown to exist but no satisfactory characterization of it is available at this writing.” The key point to solve this problem is to carefully investigate the boundary condition at $x = 0$, which somehow allows to transfer the actions of the controls to the indirectly controlled components (i.e. to the components associated with a positive speed in our framework). This is where the introduction of the canonical UL -decomposition of Q is crucial.

Before giving the proof of Theorem 3.1 we mention that we can add any diagonal matrix to the system (38) without changing its controllability properties. We use it to simplify the diagonal terms in the adjoint system, and thus the computations below (in other words, we can use the formula (25) and (26)).

Lemma 3.3. *Let $\Lambda \in C^{0,1}([0, 1])^{n \times n}$ satisfy (2) and (3) and $Q \in \mathbb{R}^{p \times m}$. For every $T > 0$, $(A_{\frac{\partial \Lambda}{\partial x}}, B)$ is exactly controllable in time T if, and only if, (A_0, B) is exactly controllable in time T .*

The proof of Lemma 3.3 is a simple change of variable. It is contained in Appendix C.

3.1 Sufficient conditions

In this part we establish the positive result, that is we assume that $\text{rank } Q = p$ and that $T \geq \max_{i \in \{1, \dots, p\}} (T_{p+1}(\Lambda), T_i(\Lambda) + T_{p+c_i(Q)}(\Lambda))$ and we are going to prove that in this case (A_0, B) is exactly controllable in time T . Thanks to Lemma 3.3, it is equivalent to prove the exact controllability of $(A_{\frac{\partial \Lambda}{\partial x}}, B)$. Now, to prove that $(A_{\frac{\partial \Lambda}{\partial x}}, B)$ is exactly controllable in time T , we will use the duality and show that there exists $C > 0$ such that, for every $z^1 \in L^2(0, 1)^n$, we have

$$\|z^1\|_{L^2(0,1)^n}^2 \leq C \int_0^T \|z_-(t, 1)\|_{\mathbb{R}^m}^2 dt, \quad (39)$$

where $z \in C^0([0, T]; L^2(0, 1)^n)$ is the solution to the adjoint system, i.e. $z(t) = S_{A_{\frac{\partial \Lambda}{\partial x}}}(T-t)^* z^1$.

In what follows, $C > 0$ is a positive constant that may change from line to line but that does not depend on z^1 .

- 1) For $j \in \{1, \dots, m\}$, since in particular $T \geq T_{p+1}(\Lambda) \geq T_{p+j}(\Lambda)$, using the method of characteristics (see e.g. (26) with $z^0 = z^1$ and $T-t$ in place of t), we have

$$\|z_{p+j}^1\|_{L^2(0,1)}^2 \leq C \int_{T-T_{p+j}(\Lambda)}^T |z_{p+j}(t, 1)|^2 dt. \quad (40)$$

These terms are good because it concerns $z_-(t, 1)$ (see (39)). Similarly, for $i \in \{1, \dots, p\}$, since $T \geq T_i(\Lambda)$, we have (see e.g. (25))

$$\|z_i^1\|_{L^2(0,1)}^2 \leq C \int_{T-T_i(\Lambda)}^T |z_i(t, 0)|^2 dt. \quad (41)$$

These terms are not good because it concerns $z_+(t, 0)$. We would like to get rid of it. The only information that we know about $z_+(t, 0)$ is through the boundary condition

$$z_-(t, 0) = R^* z_+(t, 0). \quad (42)$$

Since $\text{rank } Q = p$ we also have $\text{rank } R = p$. Therefore, $R^* \in \mathbb{R}^{m \times p}$ has at least one left-inverse and we can express $z_+(t, 0)$ in function of $z_-(t, 0)$. However, we do not really want to completely inverse this relation without looking more closely at it as it will eventually lead to the observability inequality (39) only for times T larger or equal than the time $T_p(\Lambda) + T_{p+1}(\Lambda)$, which is not the minimal one in general.

- 2) This is where we use the decomposition of Proposition 1.5. According to it, there exist a canonical form $Q^0 \in \mathbb{R}^{p \times m}$ and a lower triangular matrix $L \in \mathbb{R}^{m \times m}$ such that

$$QL = Q^0.$$

As a result, (42) implies that (we recall that $R = -\Lambda_+(0)Q\Lambda_-(0)^{-1}$)

$$(Q^0)^* \Lambda_+(0) z_+(t, 0) = -L^* \Lambda_-(0) z_-(t, 0). \quad (43)$$

We now look carefully at this relation row by row for the row indices $c_i(Q)$. Let $i \in \{1, \dots, p\}$ be fixed. The $c_i(Q)$ -th row of (43) is

$$\sum_{k=1}^p q_{k, c_i(Q)}^0 \lambda_k(0) z_k(t, 0) = - \sum_{j=1}^m \ell_{j, c_i(Q)} \lambda_{p+j}(0) z_{p+j}(t, 0).$$

Using some of the structural properties of Q^0 and L , namely, $q_{k, c_i(Q)}^0 = 0$ for $k > i$ (see (18) in Remark 1.4) and $\ell_{i, j} = 0$ for $i < j$, this is equivalent to

$$\sum_{k < i} q_{k, c_i(Q)}^0 \lambda_k(0) z_k(t, 0) + q_{i, c_i(Q)}^0 \lambda_i(0) z_i(t, 0) = - \sum_{j \geq c_i(Q)} \ell_{j, c_i(Q)} \lambda_{p+j}(0) z_{p+j}(t, 0).$$

Using now the fact that $q_{i, c_i(Q)}^0 \neq 0$, we obtain

$$z_i(t, 0) = \frac{1}{q_{i, c_i(Q)}^0 \lambda_i(0)} \left(- \sum_{k < i} q_{k, c_i(Q)}^0 \lambda_k(0) z_k(t, 0) - \sum_{j \geq c_i(Q)} \ell_{j, c_i(Q)} \lambda_{p+j}(0) z_{p+j}(t, 0) \right). \quad (44)$$

We recall that the goal is to estimate $z_i(t, 0)$ on the time interval $(T - T_i(\Lambda), T)$ (see (41)). Therefore, we estimate each term in the brackets in (44) on this interval.

- 3) To estimate the first term, we first observe, using the method of characteristics and the boundary condition $z_+(\cdot, 1) = 0$, that we have $z_k(\cdot, 0) = 0$ in $(0, T - T_k(\Lambda))$ for every $k \in \{1, \dots, p\}$ (see (25)), so that

$$\int_{T - T_i(\Lambda)}^T |z_k(t, 0)|^2 dt = \int_{T - T_k(\Lambda)}^T |z_k(t, 0)|^2 dt, \quad \forall k \leq i.$$

Therefore, for the first term in the right-hand side of (44), we have

$$\begin{aligned} \int_{T - T_i(\Lambda)}^T \left| \sum_{k < i} q_{k, c_i(Q)}^0 \lambda_k(0) z_k(t, 0) \right|^2 dt &\leq C \int_{T - T_i(\Lambda)}^T \sum_{k < i} |z_k(t, 0)|^2 dt \\ &= C \sum_{k < i} \int_{T - T_i(\Lambda)}^T |z_k(t, 0)|^2 dt \\ &= C \sum_{k < i} \int_{T - T_k(\Lambda)}^T |z_k(t, 0)|^2 dt. \end{aligned}$$

The important point is that it is estimated by a similar expression than the one we want to estimate but that contains only terms for $k < i$.

The second term to be estimated is the one which fixes the time $T_i(\Lambda) + T_{p+c_i(Q)}(\Lambda)$. Indeed, we first observe that, using the method of characteristics (see (26)) and the fact that $T \geq T_i(\Lambda) + T_{p+c_i(Q)}(\Lambda) \geq T_i(\Lambda) + T_{p+j}(\Lambda)$ for $j \geq c_i(Q)$, we have

$$\int_{T - T_i(\Lambda)}^T |z_{p+j}(t, 0)|^2 dt \leq C \int_{T - T_i(\Lambda) - T_{p+j}(\Lambda)}^{T - T_{p+j}(\Lambda)} |z_{p+j}(t, 1)|^2 dt, \quad \forall j \geq c_i(Q).$$

Therefore, for the second term in the right-hand side of (44), we have

$$\begin{aligned}
\int_{T-T_i(\Lambda)}^T \left| \sum_{j \geq c_i(Q)} \ell_{j, c_i(Q)} \lambda_{p+j}(0) z_{p+j}(t, 0) \right|^2 dt &\leq C \int_{T-T_i(\Lambda)}^T \sum_{j \geq c_i(Q)} |z_{p+j}(t, 0)|^2 dt \\
&= C \sum_{j \geq c_i(Q)} \int_{T-T_i(\Lambda)}^T |z_{p+j}(t, 0)|^2 dt \\
&\leq C \sum_{j \geq c_i(Q)} \int_{T-T_i(\Lambda)-T_{p+j}(\Lambda)}^{T-T_{p+j}(\Lambda)} |z_{p+j}(t, 1)|^2 dt \\
&\leq C \int_0^T \|z_-(t, 1)\|_{\mathbb{R}^m}^2 dt.
\end{aligned}$$

To summarize, we have obtained the following estimate, valid for every $i \in \{1, \dots, p\}$:

$$\int_{T-T_i(\Lambda)}^T |z_i(t, 0)|^2 dt \leq C \sum_{k < i} \int_{T-T_k(\Lambda)}^T |z_k(t, 0)|^2 dt + C \int_0^T \|z_-(t, 1)\|_{\mathbb{R}^m}^2 dt.$$

By induction (starting with $i = 1$) we easily deduce that, for every $i \in \{1, \dots, p\}$,

$$\int_{T-T_i(\Lambda)}^T |z_i(t, 0)|^2 dt \leq C \int_0^T \|z_-(t, 1)\|_{\mathbb{R}^m}^2 dt.$$

Combined with (41) and (40) this establishes (39) and conclude the proof of the positive result.

3.2 Necessary conditions

We now turn out to the proof of the negative result, that is we assume that (A_0, B) is exactly controllable in a time $T > 0$ and we show that both conditions (i) and (ii) of Theorem 3.1 necessary hold. Once again, thanks to Lemma 3.3, it is equivalent assume the exact controllability of $(A_{\frac{\partial \Lambda}{\partial x}}, B)$. We then argue by contraposition and show that, if one of the conditions (i) or (ii) fails, then $(A_{\frac{\partial \Lambda}{\partial x}}, B)$ is not approximately controllable in time T for any $T > 0$. Now, to prove that $(A_{\frac{\partial \Lambda}{\partial x}}, B)$ is not approximately controllable in time T , we will use the duality and show that there exists $z^1 \in L^2(0, 1)^n$ such that

$$z_-(t, 1) = 0, \quad \text{a.e. } t \in (0, T), \quad z^1 \neq 0,$$

where as usual $z \in C^0([0, T]; L^2(0, 1)^n)$ is the solution to the adjoint system, i.e. $z(t) = S_{A_{\frac{\partial \Lambda}{\partial x}}}(T-t)^* z^1$.

- 1) First we show that, if $\text{rank } Q < p$, then $(A_{\frac{\partial \Lambda}{\partial x}}, B)$ is not approximately controllable in time T for any $T > 0$. Let then $T > 0$ be fixed. Since $\text{rank } R = \text{rank } Q$, by assumption, there exists $\eta \in \mathbb{R}^p$ such that

$$R^* \eta = 0, \quad \eta \neq 0.$$

Let us then define $z^1 \in L^2(0, 1)^n$ for every $x \in (0, 1)$ by

$$z_i^1(x) = \begin{cases} \eta_i & \text{if } x \in (0, \phi_i^{-1}(T_1(\Lambda))), \\ 0 & \text{otherwise,} \end{cases} \quad \forall i \in \{1, \dots, p\}, \quad z_-^1(x) = 0.$$

Note that it is well-defined since $T_1(\Lambda) = \phi_1(1) \leq \phi_i(1)$ for every $i \in \{1, \dots, p\}$. Let $z \in C^0([0, T]; L^2(0, 1)^n)$ be the solution to the adjoint system corresponding to this data. The

method of characteristics (see (25)) gives

$$\forall i \in \{1, \dots, p\}, \quad z_i(t, 0) = \begin{cases} \eta_i & \text{if } t \in (T - T_1(\Lambda), T) \text{ and } t > 0, \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$R^* z_+(t, 0) = 0, \quad \text{a.e. } t \in (0, T). \quad (45)$$

It follows that $z_- = 0$ (see (26)). In particular, $z_-(t, 1) = 0$ a.e. $t \in (0, T)$. Since it is clear that $z^1 \neq 0$, this shows that $(A_{\frac{\partial \Lambda}{\partial x}}, B)$ is not approximately controllable in time T for any $T > 0$ if $\text{rank } Q < p$.

2) Let us now assume that $\text{rank } Q = p$ but

$$T < \max_{i \in \{1, \dots, p\}} (T_{p+1}(\Lambda), T_i(\Lambda) + T_{p+c_i(Q)}(\Lambda)).$$

Let us show once again that, in this case, the system $(A_{\frac{\partial \Lambda}{\partial x}}, B)$ is not approximately controllable in time T . First of all, we can always assume that

$$T \geq \max(T_p(\Lambda), T_{p+1}(\Lambda)).$$

Indeed, let $T < \max(T_p(\Lambda), T_{p+1}(\Lambda))$ be fixed. Assume for instance that $T_p(\Lambda) \geq T_{p+1}(\Lambda)$. Let us then define $z^1 \in L^2(0, 1)^n$ for every $x \in (0, 1)$ by

$$z_i^1(x) = 0, \quad \forall i \in \{1, \dots, p-1\}, \quad z_p^1(x) = \begin{cases} 0 & \text{if } x \in (0, \phi_p^{-1}(T)), \\ 1 & \text{if } x \in (\phi_p^{-1}(T), 1), \end{cases} \quad z_-^1(x) = 0.$$

Note that $z^1 \neq 0$ since $z_p^1 = 1$ in the open set $(\phi_p^{-1}(T), 1)$, which is not empty by assumption. Let $z \in C^0([0, T]; L^2(0, 1)^n)$ be the solution to the adjoint system corresponding to this data. The method of characteristics (see (25)) gives

$$z_+(t, 0) = 0, \quad \text{a.e. } t \in (0, T).$$

In particular, we have again (45) and we conclude as in the previous step. The case $T_p(\Lambda) \leq T_{p+1}(\Lambda)$ can be treated similarly.

3) From now on, let $i_0 \in \{1, \dots, p\}$ be fixed such that

$$T_{i_0}(\Lambda) + T_{p+c_{i_0}(Q)}(\Lambda) = \max_{i \in \{1, \dots, p\}} (T_i(\Lambda) + T_{p+c_i(Q)}(\Lambda)).$$

Let us now construct the final data $z^1 \in L^2(0, 1)^n$ for which the controllability will fail. Let $\alpha_+ \in \mathbb{R}^p$ be defined by

$$\alpha_i = \begin{cases} 0 & \text{if } i \in \{1, \dots, i_0 - 1\}, \\ 1 & \text{if } i = i_0, \\ \frac{-1}{q_{i, c_i(Q)}^0 \lambda_i(0)} \sum_{k=1}^{i-1} q_{k, c_i(Q)}^0 \lambda_k(0) \alpha_k & \text{if } i \in \{i_0 + 1, \dots, p\}. \end{cases}$$

Let then $\beta \in \mathbb{R}^m$ be defined by

$$\beta = (Q^0)^* \Lambda_+(0) \alpha_+,$$

that is, $\beta_j = \sum_{k=1}^p q_{k, j}^0 \lambda_k(0) \alpha_k$ for every $j \in \{1, \dots, m\}$. Using the second property of Remark 1.4, we see first that

$$\beta_j = 0, \quad \forall j \notin \{c_1(Q), \dots, c_p(Q)\}.$$

On the other hand, using the property (18) and the definition of α_+ , we have

$$\beta_{c_i(Q)} = \sum_{k < i} q_{k, c_i(Q)}^0 \lambda_k(0) \alpha_k + q_{i, c_i(Q)}^0 \lambda_i(0) \alpha_i = \begin{cases} 0 & \text{if } i \in \{1, \dots, i_0 - 1\}, \\ q_{i_0, c_{i_0}(Q)}^0 \lambda_{i_0}(0) & \text{if } i = i_0, \\ 0 & \text{if } i \in \{i_0 + 1, \dots, p\}. \end{cases}$$

Let us now introduce $\alpha_- = (\alpha_{p+1}, \dots, \alpha_n) \in \mathbb{R}^m$ defined by

$$\alpha_- = -\Lambda_-(0)^{-1} (L^*)^{-1} \beta,$$

so that

$$\alpha_- = -\Lambda_-(0)^{-1} (L^*)^{-1} (Q^0)^* \Lambda_+(0) \alpha_+. \quad (46)$$

Note that, since L^* is an upper triangular matrix and $\beta_j = 0$ for $j > c_{i_0}(Q)$, we have

$$\alpha_{p+j} = 0, \quad \forall j > c_{i_0}(Q). \quad (47)$$

We now define $z^1 \in L^2(0, 1)^n$ for every $x \in (0, 1)$ by

$$z_i^1(x) = \begin{cases} \alpha_i & \text{if } i \in \{i_0, \dots, p\} \text{ and } \phi_i^{-1} \left(T - T_{p+c_{i_0}(Q)}(\Lambda) \right) < x < \phi_i^{-1} (T_{i_0}(\Lambda)), \\ 0 & \text{otherwise.} \end{cases}$$

Note that z^1 is well-defined since $0 \leq T - T_{p+c_{i_0}(Q)}(\Lambda) < T_{i_0}(\Lambda)$ by assumption and since $T_{i_0}(\Lambda) = \phi_{i_0}(1) \leq \phi_i(1)$ for $i \in \{i_0, \dots, p\}$. It is also clear that $z^1 \neq 0$ since $z_{i_0}^1(x) = \alpha_{i_0} = 1 \neq 0$ for every x in the non-empty open set $(\phi_{i_0}^{-1}(T - T_{p+c_{i_0}(Q)}(\Lambda)), \phi_{i_0}^{-1}(T_{i_0}(\Lambda)))$.

- 4) Let $z \in C^0([0, T]; L^2(0, 1)^n)$ be the solution to the adjoint system corresponding to this data. Let us now show that

$$z_-(t, 1) = 0, \quad \text{a.e. } t \in (0, T). \quad (48)$$

Since $z_-^1 = 0$ and $T \geq T_{p+1}(\Lambda) \geq T_{p+j}(\Lambda)$, the method of characteristics (see (26)) gives

$$z_{p+j}(t, 1) = 0, \quad \text{a.e. } t \in (T - T_{p+j}(\Lambda), T), \quad \forall j \in \{1, \dots, m\}. \quad (49)$$

On the other hand, for every $i \in \{1, \dots, i_0 - 1\}$, since $z_i^1 = 0$ and $z_i(t, 1) = 0$, the method of characteristics (see (25)) gives

$$z_i(t, 0) = 0 = \alpha_i, \quad \text{a.e. } t \in (0, T), \quad \forall i \in \{1, \dots, i_0 - 1\}. \quad (50)$$

Finally, since $T \geq T_p(\Lambda) \geq T_i(\Lambda) \geq T_{i_0}(\Lambda)$ for $i \in \{i_0, \dots, p\}$, the method of characteristics (see (25)) gives

$$\forall i \in \{i_0, \dots, p\}, \quad z_i(t, 0) = \begin{cases} \alpha_i & \text{if } T - T_{i_0}(\Lambda) < t < T_{p+c_{i_0}(Q)}(\Lambda), \\ 0 & \text{otherwise.} \end{cases} \quad (51)$$

Let us now recall the boundary condition at $x = 0$ that z satisfies:

$$z_-(t, 0) = -\Lambda_-(0)^{-1} (L^*)^{-1} (Q^0)^* \Lambda_+(0) z_+(t, 0).$$

Combined with (50), (51) and the relation (46) between α_- and α_+ , we see that

$$\forall j \in \{1, \dots, m\}, \quad z_{p+j}(t, 0) = \begin{cases} \alpha_{p+j} & \text{if } T - T_{i_0}(\Lambda) < t < T_{p+c_{i_0}(Q)}(\Lambda), \\ 0 & \text{otherwise.} \end{cases} \quad (52)$$

In particular, thanks to (47), we have

$$z_{p+j}(t, 0) = 0, \quad \text{a.e. } t \in (0, T), \quad \forall j \in \{c_{i_0}(Q) + 1, \dots, m\}.$$

On the other hand, if $t > T_{p+j}(\Lambda)$ and $j \leq c_{i_0}(Q)$ we then have $t > T_{p+c_{i_0}(Q)}(\Lambda)$ so that, by (52),

$$z_{p+j}(t, 0) = 0, \quad \text{a.e. } t \in (T_{p+j}(\Lambda), T), \quad \forall j \in \{1, \dots, c_{i_0}(Q)\}.$$

As a result,

$$z_{p+j}(t, 0) = 0, \quad \text{a.e. } t \in (T_{p+j}(\Lambda), T), \quad \forall j \in \{1, \dots, m\}.$$

The method of characteristics (see (26)) then gives

$$z_{p+j}(t, 1) = 0, \quad \text{a.e. } t \in (0, T - T_{p+j}(\Lambda)), \quad \forall j \in \{1, \dots, m\}.$$

Combined with (49), we see that (48) holds. Since we have seen that $z^1 \neq 0$, the system $(A_{\frac{\partial \Lambda}{\partial x}}, B)$ is not approximately controllable in time T .

4 Stability of the minimal time of control

In this section we show that the internal coupling term M in (1) has almost no impact on the exact controllability properties of (1) and that it can be completely removed without affecting the minimal time of control. More precisely, the goal of this section is to establish the following perturbation result:

Theorem 4.1. *For every $\Lambda \in C^{0,1}([0, 1])^{n \times n}$ that satisfies (2), (3) and (4), $Q \in \mathbb{R}^{p \times m}$ and $M \in L^\infty(0, 1)^{n \times n}$, we have*

$$T_{\inf}(\Lambda, M, Q) = T_{\inf}(\Lambda, 0, Q). \quad (53)$$

Note that this will achieve the proof of our main result Theorem 1.9, when combined with Theorem 3.1 of the previous section.

4.1 Idea of the proof and preliminary results

The key point in the proof of Theorem 4.1 is to show that the difference between the input maps of two systems (not exactly (A_0, B) and (A_M, B) , but some perturbations of them) is a compact operator. Indeed, the conclusion will then follow from the following general abstract result:

Theorem 4.2. *Let H and U be two complex Hilbert spaces. Let $A_1 : D(A_1) \subset H \rightarrow H$ be the generator of a C_0 -semigroup on H and let $B \in \mathcal{L}(U, D(A_1)')$ be admissible for A_1 . Let $P \in \mathcal{L}(H)$ be a bounded operator and let us form the unbounded operator $A_2 = A_1 + P$ with $D(A_2) = D(A_1)$. For $i = 1, 2$, let $\Phi_i(T) \in \mathcal{L}(L^2(0, +\infty; U), H)$ be the input map of (A_i, B) at time $T \geq 0$, and let*

$$T_{\inf}(A_i, B) = \inf \{T > 0, \quad (A_i, B) \text{ is exactly controllable in time } T\} \in [0, +\infty].$$

We assume that:

(i) For $i = 1, 2$, (A_i, B) satisfies the Fattorini-Hautus test, i.e.

$$\ker(\lambda - A_i^*) \cap \ker B^* = \{0\}, \quad \forall \lambda \in \mathbb{C}. \quad (54)$$

(ii) $\Phi_1(T)^* - \Phi_2(T)^*$ is compact for every $T > 0$.

Then, we have $T_{\inf}(A_2, B) = T_{\inf}(A_1, B)$.

This general result was already noticed in [DO18, Remarks 2.4 and 1.5] and similar ideas have also been used earlier in [Rus78, p. 657, p. 659] (with a stronger assumption than (i) though). The proof of Theorem 4.2 is a simple application of the compactness-uniqueness result [DO18, Theorem 4.1], it is detailed at the beginning of Appendix B for the sake of completeness.

Let us now point out that concerning our system (1) it is actually claimed (without proof) in [Rus78, p. 657] that “A somewhat involved, but not conceptually difficult, argument allows one to see that the operator differences $S^* - S_d^*$, $C^* - C_d^*$ are both compact.” (see also [Rus78, p. 659]), where $C^* - C_d^*$ corresponds to $\Phi_M(T)^* - \Phi_{M_d}(T)^*$ in our notation, where M_d denotes the diagonal part of M (strictly speaking it is only almost true, since we recall that a different boundary condition at $x = 1$ is considered in [Rus78]). However, it appears to us that the proof of this claim is not straightforward at all, in particular because the solution to the adjoint system of (A_M, B) is not explicit if M has no particular structure. We also think that it deserves more than these three lines since it is in fact the key point to transfer the controllability properties of one system onto another, thanks to Theorem 4.2. The main goal of Section 4 is thus to provide a complete proof of this fact. As already mentioned, once this is done, Theorem 4.1 will be an immediate consequence of Theorem 4.2, because the assumption (i) will be easily checked in our case. Now, in order to check that the difference between the input maps of two systems is compact, we developed the following practical sufficient condition involving only the unperturbed system:

Lemma 4.3. *Under the framework of Theorem 4.2 (we do not assume (i) and (ii) here though), we assume that:*

(ii)' *There exist $\varepsilon > 0$, a Hilbert space \widehat{H} , a function $G \in L^2(0, \varepsilon; \mathcal{L}(H, \widehat{H}))$ with $G(t)$ compact for a.e. $t \in (0, \varepsilon)$ and $C > 0$ such that, for a.e. $t \in (0, \varepsilon)$,*

$$\|B^*V\tilde{z}(t)\|_U + \|V\tilde{z}(t)\|_H \leq C \|G(t)z^0\|_{\widehat{H}}, \quad \forall z^0 \in D(A_1^*),$$

where $V\tilde{z}(t) = \int_0^t K(t, s)\tilde{z}(s) ds$ is the Volterra operator with kernel $K(t, s) = S_{A_1}(t-s)^*P^*$ and $\tilde{z}(t) = S_{A_1}(t)^*z^0$.

Then, the assumption (ii) of Theorem 4.2 holds.

The proof of Lemma 4.3 is postponed to Appendix B for the sake of the presentation. It relies on some ideas of [NRL86] and an estimate that can be found in [DO18].

Remark 4.4. It is crucial to observe that the assumption (ii)' in Lemma 4.3 only concerns the semigroup of the unperturbed system (A_1, B) . This is what makes this result usable in practice. Note as well that this assumption has to be checked only for small times, which makes the computation easier in our case. Finally, let us also mention that another more general condition than (ii)' can be found in Proposition B.2 below.

Roughly speaking, the proof of Theorem 4.1 will then be reduced to check the assumption (ii)' of Lemma 4.3. We will see in the next section that the computation of $V\tilde{z}(t)$ will reveal some integral operators of a particular form, for which we will need the following technical result to conclude (see also [NRL86, Lemma 4]):

Lemma 4.5. *Let $\Omega \subset \mathbb{R}^2$ be the bounded open subset defined by*

$$\Omega = \{(s, x) \in \mathbb{R}^2, \quad x \in (0, 1), \quad a(x) < b(x), \quad s \in (a(x), b(x))\},$$

for some functions $a, b \in C^{0,1}([0, 1])$. We assume that $\Omega \neq \emptyset$. Let $\beta \in C^1(\overline{\Omega})$ with $\beta(\Omega) \subset (0, 1)$ and

$$\left(\frac{\partial \beta}{\partial s}(s, x) > 0, \quad \forall (s, x) \in \overline{\Omega} \right) \quad \text{or} \quad \left(\frac{\partial \beta}{\partial s}(s, x) < 0, \quad \forall (s, x) \in \overline{\Omega} \right).$$

Denoting the inverse of the map $s \mapsto \beta(s, x)$ by $\beta^{-1}(\cdot, x)$, we also assume that $x \mapsto \frac{\partial \beta^{-1}}{\partial \xi}(\xi, x)$ does not depend on x . For every $x \in [0, 1]$, let $J(x) \subset \mathbb{R}$ be the bounded open subset defined by

$$J(x) = \begin{cases} \{s \in (a(x), b(x)), & f_1(\beta(s, x)) < f_2(x)\} & \text{if } a(x) < b(x), \\ \emptyset & \text{otherwise,} \end{cases}$$

for some $f_1, f_2 \in C^{1,1}([0, 1])$ with $\frac{\partial f_1}{\partial \xi} > 0$ in $[0, 1]$ or $\frac{\partial f_1}{\partial \xi} < 0$ in $[0, 1]$. Let then $\Omega' \subset \Omega$ be the bounded open subset defined by

$$\Omega' = \{(s, x) \in \mathbb{R}^2, \quad x \in (0, 1), \quad s \in J(x)\}.$$

Let $\alpha \in C^1(\overline{\Omega'})$ with $\alpha(\Omega') \subset (0, 1)$ and

$$\frac{\partial \alpha}{\partial s}(s, x) \neq 0, \quad \forall (s, x) \in \overline{\Omega'}. \quad (55)$$

Finally, let $k \in L^\infty(0, 1)$.

Then, for every $f \in L^2(0, 1)$ and $x \in [0, 1]$, the function $s \mapsto k(\beta(s, x))f(\alpha(s, x))$ belongs to $L^1(J(x))$ with the estimate

$$\left| \int_{J(x)} k(\beta(s, x))f(\alpha(s, x)) ds \right| \leq \frac{\|k\|_{L^\infty(0,1)}}{\inf_{s \in J(x)} \left| \frac{\partial \alpha}{\partial s}(s, x) \right|} \|f\|_{L^2(0,1)}. \quad (56)$$

Moreover, the linear operator defined for every $f \in L^2(0, 1)$ and $x \in [0, 1]$ by

$$Kf(x) = \int_{J(x)} k(\beta(s, x))f(\alpha(s, x)) ds, \quad (57)$$

has the following properties:

- (i) $K(L^2(0, 1)) \subset L^2(0, 1)$ and the operator $f \in L^2(0, 1) \mapsto Kf \in L^2(0, 1)$ is compact.
- (ii) $K(H^1(0, 1)) \subset H^1(0, 1)$ and, for every $f \in H^1(0, 1)$ and $x \in [0, 1]$, the trace of Kf at x is equal to $Kf(x)$.
- (iii) For every $x \in [0, 1]$, the operator $f \in L^2(0, 1) \mapsto Kf(x) \in \mathbb{R}$ is compact.

Proof.

- 1) By assumption (55), the function $s \in J(x) \mapsto \alpha(s, x)$ is a C^1 -diffeomorphism for every $x \in [0, 1]$ such that $a(x) < b(x)$. Its inverse will be denoted by $\alpha^{-1}(\cdot, x)$. Using the change of variable $s \mapsto \alpha(s, x)$ we see that the function $s \mapsto k(\beta(s, x))f(\alpha(s, x))$ belongs to $L^1(J(x))$ and

$$Kf(x) = \int_0^1 h(\xi, x) f(\xi) d\xi, \quad h(\xi, x) = \begin{cases} \frac{k(\beta(\alpha^{-1}(\xi, x), x))}{\left| \frac{\partial \alpha}{\partial s}(\alpha^{-1}(\xi, x), x) \right|} \mathbf{1}_{\alpha(J(x), x)}(\xi) & \text{if } a(x) < b(x), \\ 0 & \text{otherwise.} \end{cases} \quad (58)$$

The Cauchy-Schwarz inequality immediately gives the estimate (56). Since the kernel $h \in L^\infty((0, 1) \times (0, 1))$, it is well-known that the operators of the form (58) are compact, so that (i) holds.

- 2) For the proof of item (ii) we assume for instance that we are in the case $\frac{\partial \beta}{\partial s} > 0$ in $\bar{\Omega}$ and $\frac{\partial f_1}{\partial \xi} > 0$ in $[0, 1]$. Using then the change of variable $s \mapsto \beta(s, x)$ when $a(x) < b(x)$ shows that

$$Kf(x) = \int_{\beta(a(x), x)}^{c(x)} k(\xi) f(\alpha(\beta^{-1}(\xi, x), x)) \frac{\partial \beta^{-1}}{\partial \xi}(\xi, x) d\xi,$$

where

$$c(x) = \begin{cases} \beta(b(x), x) & \text{if } a(x) < b(x) \text{ and } f_1(\beta(b(x), x)) < f_2(x), \\ f_1^{-1}(f_2(x)) & \text{if } a(x) < b(x) \text{ and } f_1(\beta(a(x), x)) \leq f_2(x) \leq f_1(\beta(b(x), x)), \\ \beta(a(x), x) & \text{otherwise.} \end{cases}$$

Thanks to our regularity assumptions, we see that, when $f \in H^1(0, 1)$, Kf is continuous on $[0, 1]$ and piecewise $H^1(0, 1)$, which yields $Kf \in H^1(0, 1)$ with trace at $x \in [0, 1]$ equal to $Kf(x)$.

- 3) Finally, the compactness of $f \in L^2(0, 1) \mapsto Kf(x)$ is immediate since this operator is bounded by the estimate (56) and its range is a finite-dimensional space. □

We conclude this section with the statement of a last lemma. We will see during the proof of Theorem 4.1 below that it is crucial to have only integral terms on subsets of the form $J(x)$ satisfying the assumptions of the previous lemma. Since these subsets do not in general agree with $(0, 1)$, we may have other undesirable integral terms. The goal of the next lemma is to show that we can remove these possible other “bad” integral terms if we assume (4), which is the main purpose of this assumption.

Lemma 4.6. *For every $i \in \{1, \dots, n\}$, let*

$$E_i = \{j \in \{1, \dots, n\} \mid \exists x \in [0, 1], \lambda_j(x) = \lambda_i(x)\}.$$

Assume that (4) holds, i.e.

$$E_i = \{j \in \{1, \dots, n\} \mid \lambda_j(x) = \lambda_i(x), \quad \forall x \in [0, 1]\}.$$

Then, for every $M \in L^\infty(0, 1)^{n \times n}$, there exists $\widetilde{M} \in L^\infty(0, 1)^{n \times n}$ such that the following two properties hold:

- (i) *For every $T > 0$, $(A_{\widetilde{M}}, B)$ is exactly controllable in time T if, and only if, (A_M, B) is exactly controllable in time \widetilde{T} .*
- (ii) *For every $i \in \{1, \dots, n\}$ and every $j \in E_i$, we have*

$$\widetilde{m}_{i,j}(x) = \delta_{i,j} \frac{\partial \lambda_i}{\partial x}(x), \quad \text{a.e. } x \in (0, 1),$$

where $\delta_{i,j}$ denotes the Kronecker delta, i.e. $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise.

In fact, we can prescribe any L^∞ function on the diagonal of \widetilde{M} , we chose $\frac{\partial \lambda_i}{\partial x}$ only for later computational purposes. The proof of Lemma 4.6 is technical and it is postponed to Appendix C for the sake of clarity. It is essentially an appropriate change of variable.

Remark 4.7. Let us mention that it is assumed in [NRL86, p. 322] that $m_{i,j} = 0$ for every $i \in \{1, \dots, n\}$ and $j \in E_i$ with $j \neq i$. Roughly speaking, Lemma 4.6 shows that the assumptions in [Rus78] are stronger than the assumptions in [NRL86].

4.2 Proof of Theorem 4.1

The main steps of the proof of Theorem 4.1 have been explained in the previous section. Let us now go into the details.

- 1) Let $M \in L^\infty(0, 1)^{n \times n}$ be fixed and let $\widetilde{M} \in L^\infty(0, 1)^{n \times n}$ be the corresponding matrix provided by Lemma 4.6. The idea is to apply Theorem 4.2 with

$$A_1 = A_{\frac{\partial \Lambda}{\partial x}}, \quad A_2 = A_{\widetilde{M}}, \quad P = \widetilde{M} - \frac{\partial \Lambda}{\partial x}.$$

Once the assumptions of this theorem will be checked, we will obtain

$$T_{\text{inf}}(\Lambda, \widetilde{M}, Q) = T_{\text{inf}}\left(\Lambda, \frac{\partial \Lambda}{\partial x}, Q\right).$$

The desired identity (53) will then follow from item (i) of Lemma 4.6 and Lemma 3.3.

- 2) First of all, we have to check that $(A_{\frac{\partial \Lambda}{\partial x}}, B)$ and $(A_{\widetilde{M}}, B)$ satisfy the Fattorini-Hautus test. This is an easy step. In fact, let us show that (A_M, B) satisfies the Fattorini-Hautus test for every $M \in L^\infty(0, 1)^{n \times n}$. Let $\lambda \in \mathbb{C}$ and $z \in D(A_M^*)$ be such that $A_M^* z = \lambda z$ and $B^* z = 0$. Thus, $z \in H^1(0, 1)^n$ solves the system of O.D.E.

$$\begin{cases} \frac{\partial z}{\partial x}(x) = -\Lambda(x)^{-1} \left(\lambda \text{Id}_{\mathbb{R}^{n \times n}} + \frac{\partial \Lambda}{\partial x}(x) - M(x)^* \right) z(x), & x \in (0, 1), \\ z(1) = 0, \end{cases}$$

so that $z = 0$ by uniqueness.

- 3) We now turn out to the proof of the second condition (ii) in Theorem 4.2. We recall that it is enough to check the assumption (ii)' of Lemma 4.3. In our case, we will do it for

$$\varepsilon = \phi_1(1),$$

so that the expression (26) of the unperturbed semigroup has only two possibilities when $t \in (0, \varepsilon)$, which will make the computations below easier. In order to check this condition (ii)', we will show that $(V\tilde{z}(t))_i(x)$ is in fact a sum of integral terms of the form (57), with the corresponding assumptions of Lemma 4.5 being satisfied. The conclusion will then follow from this lemma (see below).

First of all, we recall that, for every $t \geq 0$ and $f \in L^2(0, t; L^2(0, 1)^n)$, we have the identity

$$\left(\int_0^t f(s) ds \right)_i(x) = \int_0^t f_i(s, x) ds, \quad \text{a.e. } x \in (0, 1), \quad \forall i \in \{1, \dots, n\}.$$

This can be seen using for instance the property $L_{i, \varphi} \left(\int_0^t f(s) ds \right) = \int_0^t L_{i, \varphi}(f(s)) ds$ with the continuous linear forms $L_{i, \varphi} g = \langle g_i, \varphi \rangle_{L^2(0, 1)}$, where $\varphi \in L^2(0, 1)$, and Fubini's theorem. Therefore, for every $i \in \{1, \dots, n\}$, we can write

$$(V\tilde{z}(t))_i(x) = \int_0^t \left(S_{A_{\frac{\partial \Lambda}{\partial x}}}(t-s)^* P^* \tilde{z}(s) \right)_i(x) ds.$$

- 4) We first perform the computations for $i \in \{1, \dots, p\}$. From the expression (28) of the semigroup, we have

$$(V\tilde{z}(t))_i(x) = \int_{J_i^-(t, x)} (P^* \tilde{z}(s))_i(\chi_i(0; t-s, x)) ds,$$

where $J_i^-(t, x)$ is open set defined for every $t \geq 0$ and $x \in [0, 1]$ by

$$J_i^-(t, x) = \{s \in (0, t), \quad s_i^{\text{in}}(t - s, x) < 0\}.$$

On the other hand, denoting the entries of $\widetilde{M}^* - \frac{\partial \Delta}{\partial x}$ by $(p_{i,j}^*)_{1 \leq i, j \leq n}$, we have,

$$(P^* \tilde{z}(s))_i (\chi_i(0; t - s, x)) = \sum_{k=1}^n p_{i,k}^* (\chi_i(0; t - s, x)) \tilde{z}_k (s, \chi_i(0; t - s, x)).$$

As a result, combining both expressions yields

$$(V \tilde{z}(t))_i (x) = \int_{J_i^-(t, x)} \sum_{k=1}^n p_{i,k}^* (\chi_i(0; t - s, x)) \tilde{z}_k (s, \chi_i(0; t - s, x)) ds.$$

Let us now recall that \widetilde{M} has been constructed in such a way that (see item (ii) of Lemma 4.6)

$$p_{i,k}^*(\xi) = 0, \quad \text{a.e. } \xi \in (0, 1), \quad \forall k \in E_i,$$

so that

$$(V \tilde{z}(t))_i (x) = \sum_{\substack{k=1 \\ k \notin E_i}}^n \int_{J_i^-(t, x)} p_{i,k}^* (\chi_i(0; t - s, x)) \tilde{z}_k (s, \chi_i(0; t - s, x)) ds.$$

We split the sum into two sums, according to whether $k \in \{1, \dots, p\}$ or $k \in \{p+1, \dots, n\}$: $(V \tilde{z}(t))_i (x) = (V \tilde{z}(t))_{i, \leq p}(x) + (V \tilde{z}(t))_{i, > p}(x)$ with

$$(V \tilde{z}(t))_{i, \leq p}(x) = \sum_{\substack{k=1 \\ k \notin E_i}}^p \int_{J_i^-(t, x)} p_{i,k}^* (\chi_i(0; t - s, x)) \tilde{z}_k (s, \chi_i(0; t - s, x)) ds,$$

and

$$(V \tilde{z}(t))_{i, > p}(x) = \sum_{k=p+1}^n \int_{J_i^-(t, x)} p_{i,k}^* (\chi_i(0; t - s, x)) \tilde{z}_k (s, \chi_i(0; t - s, x)) ds.$$

Let us deal with the first sum $(V \tilde{z}(t))_{i, \leq p}(x)$. Thanks to the semigroup formula (28), we have

$$(V \tilde{z}(t))_{i, \leq p}(x) = \sum_{\substack{k=1 \\ k \notin E_i}}^p \int_{J_{i,k}^{--}(t, x)} p_{i,k}^* (\chi_i(0; t - s, x)) z_k^0 (\chi_k(0; s, \chi_i(0; t - s, x))) ds,$$

where $J_{i,k}^{--}(t, x) \subset J_i^-(t, x)$ is open set defined by

$$J_{i,k}^{--}(t, x) = \{s \in J_i^-(t, x), \quad s_k^{\text{in}}(s, \chi_i(0; t - s, x)) < 0\}, \quad \forall k \in \{1, \dots, p\}.$$

Let us now deal with the second sum $(V \tilde{z}(t))_{i, > p}(x)$. Thanks to the semigroup formula (29) and (28) (here we use the fact that $t < \phi_1(1)$), we have

$$\begin{aligned} (V \tilde{z}(t))_{i, > p}(x) &= \sum_{k=p+1}^n \int_{J_{i,k}^{--}(t, x)} p_{i,k}^* (\chi_i(0; t - s, x)) z_k^0 (\chi_k(0; s, \chi_i(0; t - s, x))) ds, \\ &+ \sum_{k=p+1}^n \int_{J_{i,k}^{+-}(t, x)} p_{i,k}^* (\chi_i(0; t - s, x)) \sum_{\ell=1}^p r_{\ell,k} z_\ell^0 (\chi_\ell(0; s_k^{\text{in}}(s, \chi_i(0; t - s, x)), 0)) ds, \end{aligned}$$

where $J_{i,k}^{--}(t,x), J_{i,k}^{-+}(t,x) \subset J_i^-(t,x)$ are the open sets defined by

$$\begin{aligned} J_{i,k}^{--}(t,x) &= \{s \in J_i^-(t,x), \quad s_k^{\text{in}}(s, \chi_i(0; t-s, x)) < 0\}, \quad \forall k \in \{p+1, \dots, n\}, \\ J_{i,k}^{-+}(t,x) &= \{s \in J_i^-(t,x), \quad s_k^{\text{in}}(s, \chi_i(0; t-s, x)) > 0\}, \quad \forall k \in \{p+1, \dots, n\}. \end{aligned}$$

In summary, for every $i \in \{1, \dots, p\}$, we have

$$\begin{aligned} (V\tilde{z}(t))_i(x) &= (V\tilde{z}(t))_{i, \leq p}(x) + (V\tilde{z}(t))_{i, > p}(x) \\ &= \sum_{\substack{k=1 \\ k \notin E_i}}^n \int_{J_{i,k}^{--}(t,x)} p_{i,k}^*(\chi_i(0; t-s, x)) z_k^0(\chi_k(0; s, \chi_i(0; t-s, x))) ds \\ &+ \sum_{k=p+1}^n \int_{J_{i,k}^{-+}(t,x)} p_{i,k}^*(\chi_i(0; t-s, x)) \sum_{\ell=1}^p r_{\ell,k} z_\ell^0(\chi_\ell(0; s_k^{\text{in}}(s, \chi_i(0; t-s, x)), 0)) ds. \end{aligned} \quad (59)$$

5) Similar computations for $j \in \{1, \dots, m\}$ show that

$$\begin{aligned} (V\tilde{z}(t))_{p+j}(x) &= \sum_{\substack{k=1 \\ k \notin E_{p+j}}}^n \int_{J_{p+j,k}^{--}(t,x)} p_{p+j,k}^*(\chi_{p+j}(0; t-s, x)) z_k^0(\chi_k(0; s, \chi_{p+j}(0; t-s, x))) ds \\ &+ \sum_{\substack{k=p+1 \\ k \notin E_{p+j}}}^n \int_{J_{p+j,k}^{-+}(t,x)} p_{p+j,k}^*(\chi_{p+j}(0; t-s, x)) \sum_{i=1}^p r_{i,k} z_i^0(\chi_i(0; s_k^{\text{in}}(s, \chi_{p+j}(0; t-s, x)), 0)) ds \\ &+ \sum_{i=1}^p r_{i,p+j} \sum_{\substack{k=1 \\ k \notin E_i}}^n \int_{J_{p+j,k,i}^{+-}(t,x)} p_{i,k}^*(\chi_i(0; s_{p+j}^{\text{in}}(t-s, x), 0)) \\ &\quad \times z_k^0(\chi_k(0; s, \chi_i(0; s_{p+j}^{\text{in}}(t-s, x), 0))) ds \\ &+ \sum_{i=1}^p r_{i,p+j} \sum_{k=p+1}^n \int_{J_{p+j,k,i}^{++}(t,x)} p_{i,k}^*(\chi_i(0; s_{p+j}^{\text{in}}(t-s, x), 0)) \\ &\quad \times \sum_{\ell=1}^p r_{\ell,k} z_\ell^0(\chi_\ell(0; s_k^{\text{in}}(s, \chi_i(0; s_{p+j}^{\text{in}}(t-s, x), 0)), 0)) ds, \end{aligned} \quad (60)$$

where $J_{p+j,k}^{--}(t,x), J_{p+j,k}^{-+}(t,x)$ and $J_{p+j,k,i}^{+-}(t,x), J_{p+j,k,i}^{++}(t,x)$ are the open sets defined for every $t \geq 0$ and $x \in [0, 1]$ by

$$\begin{aligned} J_{p+j,k}^{-\mp}(t,x) &= \{s \in J_{p+j}^-(t,x), \quad \pm s_k^{\text{in}}(s, \chi_{p+j}(0; t-s, x)) < 0\}, \\ J_{p+j,k,i}^{+\mp}(t,x) &= \{s \in J_{p+j}^+(t,x), \quad \pm s_k^{\text{in}}(s, \chi_i(0; s_{p+j}^{\text{in}}(t-s, x), 0)) < 0\}, \end{aligned}$$

and where $J_{p+j}^-(t,x), J_{p+j}^+(t,x)$ are the open sets defined by

$$J_{p+j}^{\mp}(t,x) = \{s \in (0, t), \quad \pm s_{p+j}^{\text{in}}(t-s, x) < 0\}.$$

6) We have just seen that, for every $t \in (0, \varepsilon)$, $z^0 \in L^2(0, 1)^n$, $i \in \{1, \dots, n\}$ and a.e. $x \in (0, 1)$, $(V\tilde{z}(t))_i(x)$ is a sum of terms of the form (57). If we manage to prove that each of these

terms satisfies the assumptions of Lemma 4.5, then this will show that the expressions in the right-hand sides of (59) and (60) make sense for every $x \in [0, 1]$ (not only a.e.) and belong to $H^1(0, 1)$ when $z^0 \in H^1(0, 1)^n$, with a trace at $x = 1$ equal to the same expression but with x changed into 1. A natural candidate for the function G of Lemma 4.3 will then be the function defined for every $t \in (0, \varepsilon)$ and $z^0 \in L^2(0, 1)^n$ by

$$G(t)z^0 = \left((V\tilde{z}(t))_{p+1}(1), \dots, (V\tilde{z}(t))_n(1), (V\tilde{z}(t))_1, \dots, (V\tilde{z}(t))_n \right), \quad (61)$$

where $G(t)$ is considered as an operator from the space $H = L^2(0, 1)^n$ onto the product space $\hat{H} = \mathbb{R}^m \times L^2(0, 1)^n$ and where, by abuse of notation, $(V\tilde{z}(t))_i$ in (61) denotes in fact the function defined for every $x \in [0, 1]$ by the expression in the right-hand side of (59) (if $i \in \{1, \dots, p\}$) or (60) (if $i = p + j \in \{p + 1, \dots, n\}$). We use a similar abuse of notation for $(V\tilde{z}(t))_{p+j}(1)$.

- 7) Let us now check that each of the integral terms in (59) and (60) satisfies the assumptions of Lemma 4.5. We focus on the terms in $(V\tilde{z}(t))_{p+j}(x)$ since they are the most important ones (because $(V\tilde{z}(t))_{p+j}(1)$ appears in (61)) and since the terms in (59) can be treated similarly to the first two terms in (60). Let then $j \in \{1, \dots, m\}$ be fixed. For obvious reasons of presentation we will also only treat one type of integrals in $(V\tilde{z}(t))_{p+j}(x)$. Let us point out that the a priori extra assumptions in Lemma 4.5 are used to treat all the other cases. We choose to deal with the first type of integrals in (60), namely,

$$\int_{J_{p+j,k}^-(t,x)} P_{p+j,k}^*(\chi_{p+j}(0; t-s, x)) z_k^0(\chi_k(0; s, \chi_{p+j}(0; t-s, x))) ds = K(t)z_k^0(x).$$

Let then $k \in \{1, \dots, n\}$ with $k \notin E_{p+j}$ be fixed. We are in the configuration of Lemma 4.5 with

$$J(x) = J_{p+j,k}^-(t, x), \quad \beta(s, x) = \chi_{p+j}(0; t-s, x), \quad \alpha(s, x) = \chi_k(0; s, \chi_{p+j}(0; t-s, x)),$$

$$\beta^{-1}(\xi, x) = \phi_{p+j}(\xi) - \phi_{p+j}(x) + t,$$

$$\Omega = \{(s, x) \in \mathbb{R}^2, \quad x \in (0, 1), \quad s \in J_{p+j}^-(t, x)\}, \quad a(x) = \max(0, t - \phi_{p+j}(x)), \quad b(x) = t,$$

$$f_1(\xi) = \begin{cases} \phi_{p+j}(\xi) + \phi_k(\xi) - \phi_k(1), & \text{if } k \leq p, \\ \phi_{p+j}(\xi) - \phi_k(\xi), & \text{if } k > p, \quad k \notin E_{p+j}, \end{cases} \quad f_2(x) = \phi_{p+j}(x) - t.$$

The regularities of these functions are clear. Note that, for this case, we have $a(x) < b(x)$ for every $x \in (0, 1]$ since $t > 0$. Recalling the definition (24) of the ϕ_k , and thanks to (3), we can check that, if $k \leq p$, then $\frac{\partial f_1}{\partial \xi} > 0$ in $[0, 1]$ and, if $k > p$ with $k \notin E_{p+j}$, then either $\frac{\partial f_1}{\partial \xi} > 0$ in $[0, 1]$ (if $k > p + j$) or $\frac{\partial f_1}{\partial \xi} < 0$ in $[0, 1]$ (if $k < p + j$). Let us now compute the derivatives of β and α . First of all, it can be checked (using for instance the explicit formula (32) and (33)) that, for every $i \in \{1, \dots, n\}$, every $t > 0$ and $x \in (0, 1)$ such that $s_i^{\text{in}}(t, x) < 0$, we have

$$\begin{cases} \frac{\partial \chi_i}{\partial t}(0; t, x) = -\lambda_i(\chi_i(0; t, x)), \\ \frac{\partial \chi_i}{\partial x}(0; t, x) = \lambda_i(\chi_i(0; t, x)) \frac{1}{\lambda_i(x)}. \end{cases}$$

It follows that

$$\frac{\partial \beta}{\partial s}(s, x) = -\frac{\partial \chi_{p+j}}{\partial t}(0; t-s, x) = \lambda_{p+j}(\chi_{p+j}(0; t-s, x)), \quad (62)$$

and

$$\begin{aligned}\frac{\partial \alpha}{\partial s}(s, x) &= \frac{\partial \chi_k}{\partial t}(0; s, \chi_{p+j}(0; t-s, x)) - \frac{\partial \chi_k}{\partial x}(0; s, \chi_{p+j}(0; t-s, x)) \frac{\partial \chi_{p+j}}{\partial t}(0; t-s, x), \\ &= -\lambda_k(\chi_k(0; s, \chi_{p+j}(0; t-s, x))) \left(1 - \frac{\lambda_{p+j}(\chi_{p+j}(0; t-s, x))}{\lambda_k(\chi_{p+j}(0; t-s, x))}\right).\end{aligned}\tag{63}$$

From these computations, we see that none of these terms are equal to zero. For the first term (62), this follows from the basic assumption (3). For the second term (63) this is where we use in a crucial way that $k \notin E_{p+j}$. As a result, all the assumptions of Lemma 4.5 are satisfied.

8) Finally, thanks again to the fact that $k \notin E_{p+j}$, we have

$$\min_{\xi \in [0,1]} |\lambda_k(\xi)| > 0, \quad \min_{\xi \in [0,1]} \left|1 - \frac{\lambda_{p+j}(\xi)}{\lambda_k(\xi)}\right| > 0.$$

Thus, we see from (63) that $\left|\frac{\partial \alpha}{\partial s}\right|$ can be estimated from below by a positive constant that does not depend on t, s or x . As a consequence, from the estimate (56) of Lemma 4.5 we obtain that there exists $C > 0$ such that

$$|K(t)z_k^0(1)| + \|K(t)z_k^0\|_{L^2(0,1)} \leq C \|z_k^0\|_{L^2(0,1)}, \quad \forall t \in (0, \varepsilon).$$

Since similar estimates hold for the other integrals and the other components, this shows that for the function G defined by (61) we also have $G \in L^\infty(0, \varepsilon; \mathcal{L}(H, \widehat{H})) \subset L^2(0, \varepsilon; \mathcal{L}(H, \widehat{H}))$. All the assumptions of Lemma 4.3 are now satisfied. This completes the proof of Theorem 4.1.

A Canonical UL -decomposition

In this appendix we give a proof of Proposition 1.5, which is a crucial result to define the key elements in our main result Theorem 1.9. Let $Q \in \mathbb{R}^{p \times m}$ with $\text{rank } Q = p$ be given. We recall that want to prove that there exists a unique $Q^0 \in \mathbb{R}^{p \times m}$ such that the following two properties hold:

- (i) There exists $L \in \mathbb{R}^{m \times m}$ such that $QL = Q^0$ with L lower triangular ($\ell_{ij} = 0$ if $i < j$) and with only ones on its diagonal ($\ell_{ii} = 1$ for every i).
- (ii) Q^0 is in canonical form (Definition 1.2).

Proof of Proposition 1.5.

- 1) The existence follows from the Gaussian elimination, as shown for instance in Example 1.7. We briefly recall the general procedure. Since $\text{rank } Q = p$, the last row of $Q \in \mathbb{R}^{p \times m}$ cannot be zero. Let then $c_p \in \{1, \dots, m\}$ be the last non-zero entry of the last row of Q . We then remove the entries of Q at the left of q_{p,c_p} . In matricial form this means that we multiply Q to the right by a lower triangular matrix with only ones on its diagonal and zero everywhere else, except for its c_p -row whose first $c_p - 1$ entries are equal to $\frac{-q_{p,1}}{q_{p,c_p}}, \dots, \frac{-q_{p,c_p-1}}{q_{p,c_p}}$. We then obtain an equivalent matrix to Q which has only one non zero entry on its last row. We then forget about the last row to obtain a $(p-1) \times m$ matrix with full-row rank and we repeat the procedure (c_{p-1} being the last non-zero entry of such a matrix which is not in the c_p column, etc.). It is not difficult to see that the matrix resulting from these operations is in canonical form.

- 2) To show the uniqueness, we assume that there exist two canonical form $Q^0, \tilde{Q}^0 \in \mathbb{R}^{p \times m}$ and two lower triangular matrices with only ones on their diagonal $L, \tilde{L} \in \mathbb{R}^{m \times m}$ such that $QL = Q^0$ and $Q\tilde{L} = \tilde{Q}^0$ and we prove that $Q^0 = \tilde{Q}^0$. Denoting $L' = \tilde{L}^{-1}L$, we have

$$Q^0 = \tilde{Q}^0 L',$$

and L' is a lower triangular matrix with only ones on its diagonal. Looking at this equality column by column, we have

$$Q_j^0 = \tilde{Q}_j^0 + \sum_{i=j+1}^m \ell'_{i,j} \tilde{Q}_i^0, \quad \forall j \in \{1, \dots, m\}.$$

We want to prove that $Q_j^0 = \tilde{Q}_j^0$ for every j . For $j = m$ it is clear. For $j = m - 1$, we have

$$Q_{m-1}^0 = \tilde{Q}_{m-1}^0 + \ell'_{m,m-1} \tilde{Q}_m^0. \quad (64)$$

If $\tilde{Q}_m^0 = 0$ then we are done. Assume then that $\tilde{Q}_m^0 \neq 0$. This necessarily means that $m \in \{c_1(\tilde{Q}^0), \dots, c_p(\tilde{Q}^0)\}$ by the two last conditions in (17). Let us write $m = c_{i_m}(\tilde{Q}^0)$. Then, $\tilde{q}_{i_m, m-1}^0 = 0$ by the last condition in (17). On the other hand, since $Q_m^0 = \tilde{Q}_m^0$ by the previous step, the same considerations apply to Q_m^0 , i.e. $m = c_{k_m}(Q^0)$ for some k_m . Let us show that we necessarily have $k_m = i_m$. If $k_m > i_m$, then $\tilde{q}_{k_m, m}^0 = 0$ by (18) in Remark 1.4. Since $q_{k_m, m}^0 \neq 0$ by the first condition in (17), the identity $q_{k_m, m}^0 = \tilde{q}_{k_m, m}^0$ would fail. By the same arguments, $i_m > k_m$ is not possible either. As a result, $m = c_{i_m}(Q^0)$ and thus $q_{i_m, m-1}^0 = 0$ as well. Therefore, looking at the i_m -th row of the equality (64), we obtain

$$0 = \ell'_{m, m-1} \tilde{q}_{i_m, m}^0.$$

Since $\tilde{q}_{i_m, m}^0 \neq 0$ by the first condition in (17), we obtain that $\ell'_{m, m-1} = 0$. Coming back to (64) we have established that $Q_{m-1}^0 = \tilde{Q}_{m-1}^0$. Reasoning by induction we easily obtain that $Q_j^0 = \tilde{Q}_j^0$ for every j . This completes the proof of the uniqueness part. \square

B Sufficient conditions for the stability of the minimal time of control

In this appendix we prove Theorem 4.2 and Lemma 4.3, which provide practical sufficient conditions to ensure that the minimal time for exact controllability is invariant under bounded perturbations of the generator. The proof is based on the compactness-uniqueness method and the Volterra integral equation satisfied by semigroups of boundedly perturbed generators.

Let us first briefly recall that the compactness-uniqueness method has been extensively used to prove the exact controllability of various systems governed by partial differential equations, see in particular [Lio88] and the pioneering work [RT74] concerning stability, and it has recently been improved and put in a complete abstract framework in [DO18]. We refer to the latter article and the numerous references therein for more details on this method. We only wish to add the references [DR77, Corollary 4.2] and [Rus78, p. 657, p. 659] to those already present in [DO18]. The proof of Theorem 4.2 is in fact a simple consequence of the following general abstract result, established in [DO18, Theorem 4.1]:

Theorem B.1. *Let H and U be two complex Hilbert spaces. Let $A : D(A) \subset H \rightarrow H$ be the generator of a C_0 -semigroup on H and let $B \in \mathcal{L}(U, D(A^*)')$ be admissible for A . Let $\Phi(T) \in \mathcal{L}(L^2(0, +\infty; U), H)$ be the input map of (A, B) at time $T \geq 0$. Assume that there exist $T_0 > 0$, a complex Hilbert space \hat{H} , a compact operator $G \in \mathcal{L}(H, \hat{H})$ and $C > 0$ such that, for every $z^1 \in H$,*

$$\|z^1\|_H^2 \leq C \left(\int_0^{T_0} \|\Phi(T_0)^* z^1(t)\|_U^2 dt + \|Gz^1\|_{\hat{H}}^2 \right). \quad (65)$$

Assume moreover that (A, B) satisfies the Fattorini-Hautus test. Then, (A, B) is exactly controllable in time T for every $T > T_0$.

Let us now give the proof of Theorem 4.2. In what follows, we use the notation introduced in the statement of Theorem 4.2.

Proof of Theorem 4.2. We first prove that $T_{\inf}(A_2, B) \leq T_{\inf}(A_1, B)$. Let then $T_1 > 0$ be such that (A_1, B) is exactly controllable in time T_1 and let us show that necessarily $T_{\inf}(A_2, B) \leq T_1$. By assumption and duality there exists $C > 0$ such that, for every $z^1 \in H$,

$$\|z^1\|_H^2 \leq C \int_0^{T_1} \|\Phi_1(T_1)^* z^1(t)\|_U^2 dt,$$

so that,

$$\|z^1\|_H^2 \leq 2C \left(\int_0^{T_1} \|\Phi_2(T_1)^* z^1(t)\|_U^2 dt + \int_0^{T_1} \|(\Phi_1(T_1)^* - \Phi_2(T_1)^*) z^1(t)\|_U^2 dt \right).$$

By assumption we know that the remainder $G = \Phi_1(T_1)^* - \Phi_2(T_1)^*$ is compact and that (A_2, B) satisfies the Fattorini-Hautus test. Therefore, we can apply Theorem B.1 and obtain that (A_2, B) is exactly controllable in time $T_1 + \varepsilon$ for every $\varepsilon > 0$. This shows that $T_{\inf}(A_2, B) \leq T_1 + \varepsilon$ for every $\varepsilon > 0$. Letting $\varepsilon \rightarrow 0$ we obtain the claim. The proof of the reversed inequality $T_{\inf}(A_2, B) \geq T_{\inf}(A_1, B)$ is exactly the same by simply changing the roles of (A_2, B) and (A_1, B) . \square

Let us now turn out to the proof of Lemma 4.3. First of all, we shall establish the following result:

Proposition B.2. *Under the framework of Theorem 4.2 (we do not assume (i) and (ii) here though), we assume that:*

(ii)' *For every $T > 0$, there exist a Hilbert space \tilde{H} , a compact operator $F \in \mathcal{L}(H, \tilde{H})$ and $C > 0$ such that*

$$\int_0^T \|B^* V \tilde{z}(t)\|_U^2 dt + \int_0^T \|V \tilde{z}(t)\|_H^2 dt \leq C \|Fz^0\|_{\tilde{H}}^2, \quad \forall z^0 \in D(A_1^*),$$

where $V \tilde{z}(t) = \int_0^t K(t, s) \tilde{z}(s) ds$ is the Volterra operator with kernel $K(t, s) = S_{A_1}(t-s)^ P^*$ and $\tilde{z}(t) = S_{A_1}(t)^* z^0$.*

Then, the assumption (ii) of Theorem 4.2 holds, i.e. $\Phi_1(T)^ - \Phi_2(T)^*$ is compact for every $T > 0$.*

Remark B.3. As in Lemma 4.3, the assumption (ii)' in Proposition B.2 only concerns the semigroup of the unperturbed system (A_1, B) . Thus, this result is also usable in practice. It was for instance proved in [DO18, p. 402] that (ii)' is satisfied if P is compact. However, we emphasize that the perturbation is only assumed to be bounded in Proposition B.2 (it is important because in (1) the perturbation is not compact). The condition (ii)' is an integrated version of (ii)'. It is more general but it has to be checked for any time T .

The proof of Proposition B.2 relies on some ideas of [NRL86] and an estimate that can be found for instance in [DO18]. More precisely, it is based on the two following results:

Lemma B.4. *For every $f \in C^1([0, +\infty); H)$ and $t \geq 0$,*

$$\int_0^t S_{A_1}(t-s)^* f(s) ds \in D(A_1^*). \quad (66)$$

Moreover, for every $T > 0$, there exists $C > 0$ such that, for every $f \in C^1([0, T]; H)$,

$$\int_0^T \left\| B^* \int_0^t S_{A_1}(t-s)^* f(s) ds \right\|_U^2 dt \leq C \|f\|_{L^2(0, T; H)}^2. \quad (67)$$

The estimate (67) is a consequence of the admissibility of B for A_1 . For a proof we refer for instance to [DO18, Appendix A]. The second result we shall need is the following:

Lemma B.5. *For every $T > 0$, there exists $C > 0$ such that, for every $z^0 \in H$,*

$$\int_0^T \|S_{A_1}(t)^* z^0 - S_{A_2}(t)^* z^0\|_H^2 dt \leq C \int_0^T \left\| \int_0^t S_{A_1}(t-s)^* P^* S_{A_1}(s)^* z^0 ds \right\|_H^2 dt. \quad (68)$$

The proof of this second lemma is included at the end of the proof of [NRL86, Lemma 3] but let us briefly recall it for the sake of completeness:

Proof of Lemma B.5. Let $V \in \mathcal{L}(L^2(0, T; H))$ be the bounded linear operator defined for every $y \in L^2(0, T; H)$ by

$$Vy(t) = \int_0^t K(t, s)y(s) ds, \quad t \in (0, T),$$

where the kernel is $K(t, s) = S_{A_1}(t-s)^* P^*$. Since $K \in L^\infty((0, T) \times (0, T); \mathcal{L}(H))$, the operator V is well-defined and $\text{Id} - V$ is invertible (see e.g. [Hoc73, Theorem 2.5]). Therefore, its inverse is bounded by the closed graph theorem, meaning that there exists $C > 0$ such that, for every $y \in L^2(0, T; H)$,

$$\|y\|_{L^2(0, T; H)} \leq C \|(\text{Id} - V)y\|_{L^2(0, T; H)}. \quad (69)$$

Let us now recall the integral equation satisfied by semigroups of boundedly perturbed operators (see e.g. [EN00, Corollary III.1.7]), valid for every $z^0 \in H$ and $t \geq 0$:

$$S_{A_2}(t)^* z^0 = S_{A_1}(t)^* z^0 + \int_0^t S_{A_1}(t-s)^* P^* S_{A_2}(s)^* z^0 ds.$$

Thus, we see that $y(t) = S_{A_1}(t)^* z^0 - S_{A_2}(t)^* z^0$ is the solution to the following Volterra integral equation in $L^2(0, T; H)$:

$$(\text{Id} - V)y(t) = - \int_0^t S_{A_1}(t-s)^* P^* S_{A_1}(s)^* z^0 ds, \quad t \in (0, T), \quad (70)$$

and the desired estimate (68) then follows from (69). \square

We are now ready to prove Proposition B.2.

Proof of Proposition B.2. Let $T > 0$ be fixed. We will show that there exists $C > 0$ such that, for every $z^0 \in H$,

$$\|(\Phi_1(T)^* - \Phi_2(T)^*)z^0\|_{L^2(0, +\infty; U)} \leq C \|Fz^0\|_{\tilde{H}}. \quad (71)$$

Since F is assumed to be compact, this will clearly implies that $\Phi_1(T)^* - \Phi_2(T)^*$ is compact as well. First of all, note that we only have to prove (71) for $z^0 \in D(A_1^*)$ since this set is dense in H and all the operators involved in (71) are actually continuous operators on H . Besides, when $z^0 \in D(A_1^*) = D(A_2^*)$, we have the more explicit expression $(\Phi_1(T)^* - \Phi_2(T)^*)z^0(t) = B^*S_{A_1}(T-t)^*z^0 - B^*S_{A_2}(T-t)^*z^0$ for a.e. $t \in (0, T)$. The starting point to estimate this difference is again the Volterra integral equation (70). Using (66) we see that each term in (70) actually belongs to $D(A_1^*)$ if $z^0 \in D(A_1^*) = D(A_2^*)$. Therefore, we can apply B^* to obtain the following identity:

$$\begin{aligned} B^*S_{A_1}(t)^*z^0 - B^*S_{A_2}(t)^*z^0 &= -B^* \int_0^t S_{A_1}(t-s)^*P^*S_{A_1}(s)^*z^0 ds \\ &\quad + B^* \int_0^t S_{A_1}(t-s)^*P^*(S_{A_1}(s)^*z^0 - S_{A_2}(s)^*z^0) ds. \end{aligned}$$

Using now the estimate (67) and then (68) on the second term of the right-hand side, we obtain

$$\begin{aligned} \int_0^T \|B^*S_{A_1}(t)^*z^0 - B^*S_{A_2}(t)^*z^0\|_U^2 dt &\leq C \left(\int_0^T \left\| B^* \int_0^t S_{A_1}(t-s)^*P^*S_{A_1}(s)^*z^0 ds \right\|_U^2 dt \right. \\ &\quad \left. + \int_0^T \left\| \int_0^t S_{A_1}(t-s)^*P^*S_{A_1}(s)^*z^0 ds \right\|_H^2 dt \right). \end{aligned}$$

Using the assumption (ii)'' this establishes (71) for every $z^0 \in D(A_1^*)$. \square

Let us now conclude this part of the appendix with the proof of Lemma 4.3, which in fact provides sufficient conditions in small time to guarantee that the assumption (ii)'' of Proposition B.2 is satisfied. The proof is essentially a use of the basic functional equation of semigroups.

Proof of Lemma 4.3.

1) By assumption, there exist $C > 0$ and $\delta \in (0, \varepsilon)$ such that, for every $z^0 \in D(A_1^*)$,

$$\left\{ \begin{array}{l} \|V\tilde{z}(\delta)\|_H \leq C \|G(\delta)z^0\|_{\widehat{H}}, \\ \int_0^\delta \|B^*V\tilde{z}(t)\|_U^2 dt + \int_0^\delta \|V\tilde{z}(t)\|_H^2 dt \leq C \int_0^\delta \|G(t)z^0\|_{\widehat{H}}^2 dt, \end{array} \right. \quad (72)$$

where, by abuse of notation, $G \in \mathcal{L}^2(0, \varepsilon; \mathcal{L}(H, \widehat{H}))$ in (72) denotes in fact a representative of the equivalence class $G \in L^2(0, \varepsilon; \mathcal{L}(H, \widehat{H}))$ (so that $\|G(t)\|_{\mathcal{L}(H, \widehat{H})} < +\infty$ for every $t \in (0, \varepsilon)$, in particular for $t = \delta$) with $G(\delta)$ and $G(t)$ compact for a.e. $t \in (0, \delta)$. Note in particular that the right-hand side in the second estimate define a compact operator from H into $L^2(0, \delta; \widehat{H})$ by Lebesgue's dominated convergence theorem. We will show that (72) is enough to imply (ii)'' of Proposition B.2. In what follows, $C > 0$ denotes a positive constant that may change from line to line but that remains independent of z^0 .

2) Let now $T > 0$ be fixed. Let $k \in \mathbb{N}$ be such that $k\delta \leq T \leq (k+1)\delta$. We have

$$\int_0^T \|B^*V\tilde{z}(t)\|_U^2 dt \leq \sum_{j=0}^k \int_{j\delta}^{(j+1)\delta} \left\| B^* \int_0^t S_{A_1}(s)^*P^*S_{A_1}(t-s)^*z^0 ds \right\|_U^2 dt.$$

The change of variable $\tau = t - j\delta$ gives

$$\int_0^T \|B^*V\tilde{z}(t)\|_U^2 dt \leq \sum_{j=0}^k \int_0^\delta \left\| B^* \int_0^{\tau+j\delta} S_{A_1}(s)^*P^*S_{A_1}(\tau+j\delta-s)^*z^0 ds \right\|_U^2 d\tau.$$

Thus, breaking the integral into two parts, we have

$$\int_0^T \|B^*V\tilde{z}(t)\|_U^2 dt \leq \sum_{j=0}^k \left(2 \int_0^\delta \left\| B^* \int_0^\tau S_{A_1}(s)^* P^* S_{A_1}(\tau-s)^* S_{A_1}(j\delta)^* z^0 ds \right\|_U^2 d\tau \right. \\ \left. + 2 \int_0^\delta \left\| B^* \int_\tau^{\tau+j\delta} S_{A_1}(s)^* P^* S_{A_1}(\tau+j\delta-s)^* z^0 ds \right\|_U^2 d\tau \right).$$

The first integral is estimated thanks to the second inequality in (72):

$$\int_0^\delta \left\| B^* \int_0^\tau S_{A_1}(s)^* P^* S_{A_1}(\tau-s)^* S_{A_1}(j\delta)^* z^0 ds \right\|_U^2 d\tau \leq C \int_0^\delta \|G(t)S_{A_1}(j\delta)^* z^0\|_{\hat{H}}^2 dt.$$

For the second integral, we perform the change of variable $\sigma = s - \tau$ and then use the admissibility of B to obtain

$$\int_0^\delta \left\| B^* \int_\tau^{\tau+j\delta} S_{A_1}(s)^* P^* S_{A_1}(\tau+j\delta-s)^* z^0 ds \right\|_U^2 d\tau \\ = \int_0^\delta \left\| B^* S_{A_1}(\tau)^* \int_0^{j\delta} S_{A_1}(\sigma)^* P^* S_{A_1}(j\delta-\sigma)^* z^0 d\sigma \right\|_U^2 d\tau \leq C \|V\tilde{z}(j\delta)\|_H^2.$$

Combining both estimates, we have thus obtained

$$\int_0^T \|B^*V\tilde{z}(t)\|_U^2 dt \leq C \sum_{j=0}^k \left(\int_0^\delta \|G(t)S_{A_1}(j\delta)^* z^0\|_{\hat{H}}^2 dt + \|V\tilde{z}(j\delta)\|_H^2 \right).$$

Note that all the previous computations are also valid for $B = \text{Id}$ since we only used the second inequality in (72) and the admissibility of B . Therefore, we have

$$\int_0^T \|B^*V\tilde{z}(t)\|_U^2 dt + \int_0^T \|V\tilde{z}(t)\|_H^2 dt \leq C \sum_{j=0}^k \left(\int_0^\delta \|G(t)S_{A_1}(j\delta)^* z^0\|_{\hat{H}}^2 dt + \|V\tilde{z}(j\delta)\|_H^2 \right).$$

3) Let us now estimate $V\tilde{z}(j\delta)$. We have

$$V\tilde{z}(j\delta) = \sum_{i=0}^{j-1} \int_{i\delta}^{(i+1)\delta} S_{A_1}(j\delta-s)^* P^* S_{A_1}(s)^* z^0 ds.$$

Doing the change of variables $\sigma = s - i\delta$ we obtain

$$V\tilde{z}(j\delta) = \sum_{i=0}^{j-1} S_{A_1}(j\delta - (i+1)\delta)^* \int_0^\delta S_{A_1}(\delta - \sigma)^* P^* S_{A_1}(\sigma)^* S_{A_1}(i\delta)^* z^0 d\sigma.$$

Using now the first estimate in (72), it follows that

$$\|V\tilde{z}(j\delta)\|_H \leq C \sum_{i=0}^{j-1} \|G(\delta)S_{A_1}(i\delta)^* z^0\|_{\hat{H}}.$$

□

C Removal of the coupling terms where the speeds agree

The goal of this appendix is to give a proof of Lemma 4.6. It is essentially an appropriate change of variable. First of all, it is convenient to introduce the following notion (see also [Bru70]):

Definition C.1. Let $M, \widetilde{M} \in L^\infty(0, 1)^{n \times n}$. We say that the systems $(A_{\widetilde{M}}, B)$ and (A_M, B) are equivalent, and we write

$$(A_{\widetilde{M}}, B) \sim (A_M, B),$$

if there exist two invertible linear transformations $L \in \mathcal{L}(L^2(0, 1)^n)$ and $\Gamma \in \mathbb{R}^{m \times m}$ such that, for every $y^0 \in L^2(0, 1)^n$ and $u \in L^2(0, +\infty)^m$, if $y \in C^0([0, +\infty); L^2(0, 1)^n)$ denotes the solution to (A_M, B) with initial data y^0 and control u , then $\tilde{y} = Ly \in C^0([0, +\infty); L^2(0, 1)^n)$ is the solution to $(A_{\widetilde{M}}, B)$ with initial data $\tilde{y}^0 = Ly^0$ and control $\tilde{u} = \Gamma u$.

It is not difficult to check that \sim is an equivalence relation and that, if $(A_{\widetilde{M}}, B) \sim (A_M, B)$, then, for every $T > 0$, the system $(A_{\widetilde{M}}, B)$ is exactly controllable in time T if, and only if, the system (A_M, B) is exactly controllable in time T .

Proof of Lemma 4.6.

- 1) The goal is to construct \widetilde{M} such that (ii) holds and $(A_{\widetilde{M}}, B) \sim (A_M, B)$, so that (i) will hold as well. Thanks to (3) and (4), we see that there exist $d \in \{1, \dots, n\}$, $n_1, \dots, n_d \in \{1, \dots, n\}$ with $\sum_{k=1}^d n_k = n$ and $\lambda^1, \dots, \lambda^d \in C^{0,1}([0, 1])$ with

$$\lambda^1(x) < \dots < \lambda^d(x), \quad \forall x \in [0, 1],$$

such that, for every $x \in [0, 1]$,

$$\Lambda(x) = \text{diag}(\Lambda^1(x), \dots, \Lambda^d(x)),$$

where

$$\Lambda^k(x) = \lambda^k(x) \text{Id}_{\mathbb{R}^{n_k \times n_k}}. \quad (73)$$

To establish the equivalence between two systems $(A_{\widetilde{M}}, B)$ and (A_M, B) , we will use a transformation of the form

$$\tilde{y}(t, x) = \Psi(x)y(t, x), \quad (74)$$

where $\Psi \in W^{1,\infty}(0, 1)^{n \times n}$ is assumed to be block diagonal:

$$\Psi(x) = \text{diag}(\Psi^1(x), \dots, \Psi^d(x)),$$

where, for every $k \in \{1, \dots, d\}$, $\Psi^k \in W^{1,\infty}(0, 1)^{n_k \times n_k}$ will be determined below. First of all, it is clear that the formula (74) is reversible if we impose that all the matrices $\Psi^1(x), \dots, \Psi^d(x)$ are invertible for every $x \in [0, 1]$, which also implies that $x \mapsto \Psi(x)^{-1} \in C^0([0, 1])^{n \times n} \subset L^\infty(0, 1)^{n \times n}$. Let us now work formally to find what $\Psi^1(x), \dots, \Psi^d(x)$ shall satisfy and what \widetilde{M} is allowed to be. Let us first investigate the boundary conditions. Let us denote by d^+ the index such that

$$\sum_{i=1}^{d^+} n_i = p.$$

At $x = 1$, we see that we should have

$$\tilde{u}(t) = \tilde{y}_-(t, 1) = \begin{pmatrix} \Psi^{d^++1}(1)y^{d^++1}(t, 1) \\ \vdots \\ \Psi^d(1)y^d(t, 1) \end{pmatrix} = \Gamma u(t),$$

with $\Gamma = \text{diag}(\Psi^{d^++1}(1), \dots, \Psi^d(1))$, and where (y^{d^++1}, \dots, y^d) is a block notation to simply denote y_- . On the other hand, at $x = 0$, we see that if we impose the condition $\Psi(0) = \text{Id}_{\mathbb{R}^{n \times n}}$, then

$$\tilde{y}_+(t, 0) - Q\tilde{y}_-(t, 0) = y_+(t, 0) - Qy_-(t, 0).$$

Let us finally look at the equations that Ψ should satisfy. Since $\Psi^k(x)$ and $\Lambda^k(x)$ commute for every $x \in [0, 1]$ and $k \in \{1, \dots, d\}$ (see (73)), so do $\Psi(x)$ and $\Lambda(x)$:

$$\Psi(x)\Lambda(x) = \Lambda(x)\Psi(x), \quad \forall x \in [0, 1].$$

As a result, we have

$$\begin{aligned} \frac{\partial \tilde{y}}{\partial t}(t, x) - \Lambda(x) \frac{\partial \tilde{y}}{\partial x}(t, x) - \widetilde{M}(x) \tilde{y}(t, x) \\ = \Psi(x) \left(\frac{\partial y}{\partial t}(t, x) - \Lambda(x) \frac{\partial y}{\partial x}(t, x) - \Psi(x)^{-1} \left(\Lambda(x) \frac{\partial \Psi}{\partial x}(x) + \widetilde{M}(x) \Psi(x) \right) y(t, x) \right). \end{aligned}$$

Thus, \tilde{y} is a solution to $(A_{\widetilde{M}}, B)$ if y is a solution to (A_M, B) and \widetilde{M} is defined by

$$\widetilde{M}(x) = \left(\Psi(x)M(x) - \Lambda(x) \frac{\partial \Psi}{\partial x}(x) \right) \Psi(x)^{-1}, \quad \text{a.e. } x \in (0, 1). \quad (75)$$

Note that $\widetilde{M} \in L^\infty(0, 1)^{n \times n}$. To summarize, we have $(A_{\widetilde{M}}, B) \sim (A_M, B)$ with \widetilde{M} given by (75) if there exist matrices $\Psi^k \in W^{1, \infty}(0, 1)^{n_k \times n_k}$ such that the following two properties hold for every $k \in \{1, \dots, d\}$:

$$\left\{ \begin{array}{l} \Psi^k(x) \text{ is invertible for every } x \in [0, 1], \\ \Psi^k(0) = \text{Id}_{\mathbb{R}^{n_k \times n_k}}. \end{array} \right.$$

- 2) Our previous discussion was only formal but everything can be established rigorously by coming back to the very definition of weak solution (see Definition 1.1) and using some density arguments. More precisely, let $\tilde{\varphi} \in C^1([0, T] \times [0, 1])^n$ be fixed such that $\tilde{\varphi}_+(\cdot, 1) = 0$ and $\tilde{\varphi}_-(\cdot, 0) = R^* \tilde{\varphi}_+(\cdot, 0)$. Let $H(x) = \Psi(x)^* - (1-x)\Psi(0)^* - x\Psi(1)^*$. Since $H \in H_0^1(0, 1)^{n \times n}$, there exists a sequence $\theta^j \in C_c^\infty([0, 1])^{n \times n}$ such that $\theta^j \rightarrow \Psi^*$ in $H^1(0, 1)^{n \times n}$ as $j \rightarrow +\infty$. Let then φ^j be defined by $\varphi^j(t, x) = (\theta^j(x) + (1-x)\Psi(0)^* + x\Psi(1)^*) \tilde{\varphi}(t, x)$. Clearly, $\varphi^j \in C^1([0, T] \times [0, 1])^n$ with $\varphi_+^j(\cdot, 1) = 0$ and $\varphi_-^j(\cdot, 0) = R^* \varphi_+^j(\cdot, 0)$ (since $\Psi(0) = \text{Id}_{\mathbb{R}^{n \times n}}$). Moreover,

$$\left\{ \begin{array}{l} \varphi^j(T, \cdot) \xrightarrow{j \rightarrow +\infty} \Psi^* \tilde{\varphi}(T, \cdot) \quad \text{and} \quad \varphi^j(0, \cdot) \xrightarrow{j \rightarrow +\infty} \Psi^* \tilde{\varphi}(0, \cdot) \quad \text{in } L^2(0, 1)^n, \\ \varphi^j \xrightarrow{j \rightarrow +\infty} \Psi^* \tilde{\varphi} \quad \text{in } H^1((0, T) \times (0, 1))^n, \\ \varphi_-^j(\cdot, 1) \xrightarrow{j \rightarrow +\infty} \Gamma^* \tilde{\varphi}_-(\cdot, 1) \quad \text{in } L^2(0, T)^m. \end{array} \right.$$

Plugging the test function φ^j in (5) and passing to the limit $j \rightarrow +\infty$, we obtain

$$\begin{aligned} & \int_0^1 y(T, x) \cdot \Psi(x)^* \tilde{\varphi}(T, x) dx - \int_0^1 y^0(x) \cdot \Psi(x)^* \tilde{\varphi}(0, x) dx \\ &= \int_0^T \int_0^1 y(t, x) \cdot \left(\Psi(x)^* \frac{\partial \tilde{\varphi}}{\partial t}(t, x) - \Lambda(x) \Psi(x)^* \frac{\partial \tilde{\varphi}}{\partial x}(t, x) \right. \\ & \quad \left. + \left(-\Lambda(x) \frac{\partial \Psi}{\partial x}(x)^* + \left(-\frac{\partial \Lambda}{\partial x}(x) + M(x)^* \right) \Psi(x)^* \right) \tilde{\varphi}(t, x) \right) dx dt \\ & \quad + \int_0^T u(t) \cdot \Lambda_-(1) \Gamma^* \tilde{\varphi}_-(t, 1) dt. \end{aligned}$$

Since $\Psi(x)^*$ commute with $\Lambda(x)$ and $\frac{\partial \Lambda}{\partial x}(x)$ for every $x \in [0, 1]$, and thanks to the definition (75) of \tilde{M} , we obtain

$$\begin{aligned} & \int_0^1 \tilde{y}(T, x) \cdot \tilde{\varphi}(T, x) dx - \int_0^1 \tilde{y}^0(x) \cdot \tilde{\varphi}(0, x) dx \\ &= \int_0^T \int_0^1 \tilde{y}(t, x) \cdot \left(\frac{\partial \tilde{\varphi}}{\partial t}(t, x) - \Lambda(x) \frac{\partial \tilde{\varphi}}{\partial x}(t, x) + \left(-\frac{\partial \Lambda}{\partial x}(x) + \tilde{M}(x)^* \right) \tilde{\varphi}(t, x) \right) dx dt \\ & \quad + \int_0^T \tilde{u}(t) \cdot \Lambda_-(1) \tilde{\varphi}_-(t, 1) dt. \end{aligned}$$

This show that \tilde{y} defined by (74) is indeed the weak solution of the $(A_{\tilde{M}}, B)$ system.

- 3) The final goal is now to design the matrices Ψ^1, \dots, Ψ^d such that the matrix \tilde{M} given by (75) satisfies the condition (ii) of Lemma 4.6, namely:

$$\tilde{M}^k(x) = \frac{\partial \Lambda^k}{\partial x}(x), \quad \text{a.e. } x \in (0, 1), \quad \forall k \in \{1, \dots, d\}, \quad (76)$$

where $\tilde{M}^k \in L^\infty(0, 1)^{n_k \times n_k}$ denotes the submatrix $(\tilde{m}_{i,j})_{n_{k-1}+1 \leq i, j \leq n_k}$ (with the notation $n_0 = 0$). To this end, for every $k \in \{1, \dots, d\}$, we take $\Psi^k \in W^{1, \infty}(0, 1)^{n_k \times n_k}$ to be the solution to the O.D.E.

$$\begin{cases} \frac{\partial \Psi^k}{\partial x}(x) = \Psi^k(x) \Lambda^k(x)^{-1} \left(M^k(x) - \frac{\partial \Lambda^k}{\partial x}(x) \right), & x \in (0, 1), \\ \Psi^k(0) = \text{Id}_{\mathbb{R}^{n_k \times n_k}}. \end{cases}$$

Again, since $\Psi(x)^*$ commute with $\Lambda(x)$ and $\frac{\partial \Lambda}{\partial x}(x)$, we see that this implies that \tilde{M} given by (75) satisfies (76). Moreover, it is clear that $\Psi^k(x)$ is invertible for every $x \in [0, 1]$. This completes the proof of Lemma 4.6. □

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