

Theoretical calculation of the power of wind turbine or tidal turbine

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January 18, 2020

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1 Abstract

The subject of this article is to define the power of a wind turbine or marine current turbine. As it was admitted that the maximum power coefficient for a turbine type wind turbine or tidal, is that defined by Betz. Many people consider that Betz limit is even a stated law.

The research has long been for efficient turbines to optimize the angle of incidence, the shape of the profiles, the shape of the ends of the blade tips, the wakes caused by the turbine, etc. The number of references to these subjects are numerous see PARASCHIVOIU, LI et al. ((2018)), BURTON et al., JAMIESON, PAILLARD et al. ((2013)).

The article presented below is not an optimal search of the power coefficient in order to approach or to exceed the coefficient defined by Betz, but to reformulate the definition of the power coefficient The power factor can be much higher than that defined by Betz, which is presented in this article

2 preliminary

The German mathematician A. Betz has shown that the power of this turbine is(cf. annex 1): see BETZ ((1920))

$$P_k = F V = \left[\frac{C_k}{a} \frac{1}{2} \rho S V_{fluid}^2 \right] V = C_k \frac{1}{2} \rho S V_{fluid}^3$$

with $V = a V_{fluid}$

P_k the kinetic turbine power,

C_k the kinetic power coefficient,

ρ the fluid density,

V_{fluid} the fluid velocity.

V the fluid velocity at the position of the turbine.

According to the work of Betz, a kinetic energy approach shows that the maximum power coefficient C_T can not exceed a maximum of $\frac{16}{27}$

$$C_{T \text{ maxi}} = C_{k \text{ Betz}} = \frac{16}{27} \quad C_k \leq C_{T \text{ maxi}} \quad P_{T \text{ maxi}} = C_{k \text{ Betz}} \frac{1}{2} \rho S V_{fluid}^3$$

The theory of the Betz limit is correct, it is based on the calculation of the kinetic energy. The objective to increase the efficiency of the turbine is to transform the potential energy into kinetic.

3 theoretical preliminary

3.1 continuity equation :

The mass flow rate is the same everywhere along the streamtube and so

$$\rho S_{fluid} V_{fluid} = \rho S V = \rho S_{wake} V_{wake}$$

ρ the fluid density (constant),

V_{fluid} the fluid velocity.

V the fluid velocity at the position of the turbine.

V_{wake} the streamwise velocity in the far wake.

3.2 kinetic energy :

k index for kinetic

$$E_k = \frac{1}{2} m v^2$$

$$P_k = \frac{dE_k}{dt} = \frac{1}{2} \frac{dm}{dt} v^2 + \frac{1}{2} m \frac{dv^2}{dt} \quad \frac{dv}{dt} = 0$$

$$P_k = \frac{dE_k}{dt} = \frac{1}{2} \dot{m} v^2 = \frac{1}{2} \rho S v^3$$

3.3 potential energy :

p index for potential

$$E_p = m \frac{p}{\rho}$$

$$P_p = \frac{dE_p}{dt} = \frac{dm}{dt} \frac{p}{\rho} + m \frac{1}{\rho} \frac{dp}{dt} \quad \frac{dp}{dt} = 0$$

$$P_p = \frac{dE_p}{dt} = \dot{m} \frac{p}{\rho} = S v p$$

4 power calculation

4.1 power calculation from kinetic energy :

Force applied by the fluid on the rotor.

$$F_k = m \frac{dV}{dt} = \dot{m} \Delta V = \rho S V$$

Power

$$P_k = F_k V = \rho S V^2 (V_{fluid} - V_{wake})$$

$$P_k = \frac{\Delta E_k}{\Delta t} = \frac{1}{2} \rho S V^2 (V_{fluid}^2 - V_{wake}^2)$$

From these equalities

$$V = \frac{V_{fluid} + V_{wake}}{2}$$

defining $a = \frac{V_{wake}}{V_{fluid}}$ $0 \leq a \leq 1$

$$V_{wake} = V_{fluid}(2a - 1) \quad V_{wake} \geq 0 \quad a \geq \frac{1}{2}$$

$$F = \rho S V (V_{fluid} - V_{wake}) = \frac{1}{2} \rho S (V_{fluid}^2 - V_{wake}^2)$$

$$P = FV = \rho S V^2 (V_{fluid} - V_{wake})$$

defining $a = \frac{V_{wake}}{V_{fluid}}$

$$V_{wake} = V_{fluid} (2a - 1) \quad \text{as } V_{wake} \geq 0 \quad a \geq \frac{1}{2}$$

$$P_k = 4 a^2 (1 - a) \frac{1}{2} \rho S V_{fluid}^3$$

defining power coefficient $C_k = \frac{P_k}{\frac{1}{2} \rho S V_{fluid}^3} = 4 a^2 (1 - a)$

Search of maximum power coefficient

$$\frac{dC_k}{da} = 0 \quad a(2 - 3a) = 0 \quad a = 0 \quad \text{or} \quad a = \frac{2}{3}$$

$$a = \frac{2}{3} \quad C_k = \frac{16}{27} = 0.593$$

The maximum power coefficient C_{kmaxi} is defined by Betz

$$C_{kmaxi} = C_{kBetz} = \frac{16}{27} \approx 60\%$$

4.2 power calculation from potential energy :

The pressure difference is equal to

$$p_{fluid} - p_{wake} = \frac{1}{2} \rho (V_{fluid}^2 - V_{wake}^2)$$

this pressure difference creates on the surface of the turbine a force.

$$F_p = (p_{fluid} - p_{wake})S$$

The power of this force is equal to

$$P_p = (p_{fluid} - p_{wake})SV = a(p_{fluid} - p_{wake})SV_{fluid} = 4 a^2 (1 - a) \frac{1}{2} \rho S V_{fluid}^3$$

defining potential power coefficient $C_p = 4 a^2 (1 - a)$

4.3 Objective :

this pressure difference creates on the surface of the turbine a force but stresses in the rotor. The stresses are the source of a potential energy because of the position of each internal particles that interact with each other see JAIN and C.

Using piezo-electric materials see BRISSAUD, it is possible to transform these stresses into electric energy.

The objective is rather to transform this potential energy into kinetic mechanical energy

4.4 Conversion :

In the case of horizontal wind turbines (HAWT, fast wind turbine type), the stresses in the blades for a defined wind speed, are constant.

$$\frac{d\sigma}{d\beta} = 0$$

σ Stress in turbine blade

β rotation angle of the blades.

In fact, some variations of the stresses are existing due to gravitational forces and the differential velocity within the boundary layer depending on the elevation.

In the case of vertical axis turbines (VAWT, Darrieus type) the blade and arm stresses are depending on the rotation angle of the blades (for a given wind speed).

$$\frac{d\sigma}{d\beta} \neq 0 \quad \frac{1}{2\pi} \int_0^{2\pi} \sigma d\beta = \epsilon \quad (\epsilon \text{ small})$$

During a half-turn, the arms are submitted to compression stresses whereas extending stresses are dominant during the next half-turn.

In the case of a HAWT, it is not possible to convert with a dynamic mechanical system, these stresses into additional energy as the stresses are constant for a given wind speed. Alternative stresses, encountered in a vertical axis wind turbine VAWT can allow the extraction of additional energy.

4.5 Coefficient of power of a turbine with a conversion system :

Moving fluid creates stresses on rotor. Energy conservation can be applied. Energy conservation : the total energy remains constant see JOULE

Total energy is equal to the sum of kinetic energy and potential energy.

$$E_T = E_k + E_p$$

The variation of the total energy during the time is zero.

$$\frac{dE_{total}}{dt} = 0 \quad \frac{dE_{kinetic}}{dt} = - \frac{dE_{potential}}{dt}$$

The variations of energy kinetic and potential vary simultaneously and in opposite sense.

$$\frac{dE_{kinetic}}{dt} = P_k = \frac{1}{2} \rho s v^3 \quad \frac{dE_{potential}}{dt} = P_p = s v p$$

$$\frac{dE_{kinetic}}{dt} = - \frac{dE_{potential}}{dt} \quad P_k = -P_p \quad p = -\frac{1}{2} \rho v^2$$

Pressure variations vary with variations in speed

the pressure difference creates stresses on the turbine.

The stresses are maximum when the forces due to the kinetic energy are maximum.

The coefficient C_p can not be greater than or equal to C_k .

4.6 total power of the turbine :

At the power level, taking into account the differential of kinetic energy and the differential of the potential energy, the power coefficient is

$$C_T = C_k + C_p$$

$$C_T = C_k + C_p$$

$$P_T = C_T \frac{1}{2} \rho S V_{fluid}^3 = (C_k + C_p) \frac{1}{2} \rho S V_{fluid}^3$$

4.7 Balance sheet of the powers

In the case of horizontal wind turbines (HAWT, fast wind turbine type), the power coefficient $C_{T \text{ HAWT}}$ is

$$C_{T \text{ HAWT}} = C_k = 4 a^2 (1 - a) \quad C_p = 0$$

In the case of vertical axis turbines (VAWT, Darrieus type) the power coefficient $C_{T \text{ VAWT Darrieus}}$ is

$$C_{T \text{ VAWT Darrieus}} = C_k = 4 a^2 (1 - a) \quad C_p = 0$$

In the case of vertical axis turbines (VAWT with conversion) with convert, these stresses into additional energy, the power coefficient $C_{T \text{ VAWT with conversion}}$ is

$$C_{T \text{ VAWT with conversion}} = C_k + C_p = 8 a^2 (1 - a)$$

At the level of Powers

$$P = C_T \frac{1}{2} \rho S V_{fluid}^3$$

$$P_{T \text{ HAWT}} = 4 a^2 (1 - a) \frac{1}{2} \rho S V_{fluid}^3$$

$$P_{T \text{ VAWT Darrieus}} = 4 a^2 (1 - a) \frac{1}{2} \rho S V_{fluid}^3$$

$$P_{T \text{ VAWT with conversion}} = 8 a^2 (1 - a) \frac{1}{2} \rho S V_{fluid}^3$$

4.8 HAWT-VAWT comparison

Following the work of E.Hau , the power coefficient of different turbines is compared see HAU ((2000))

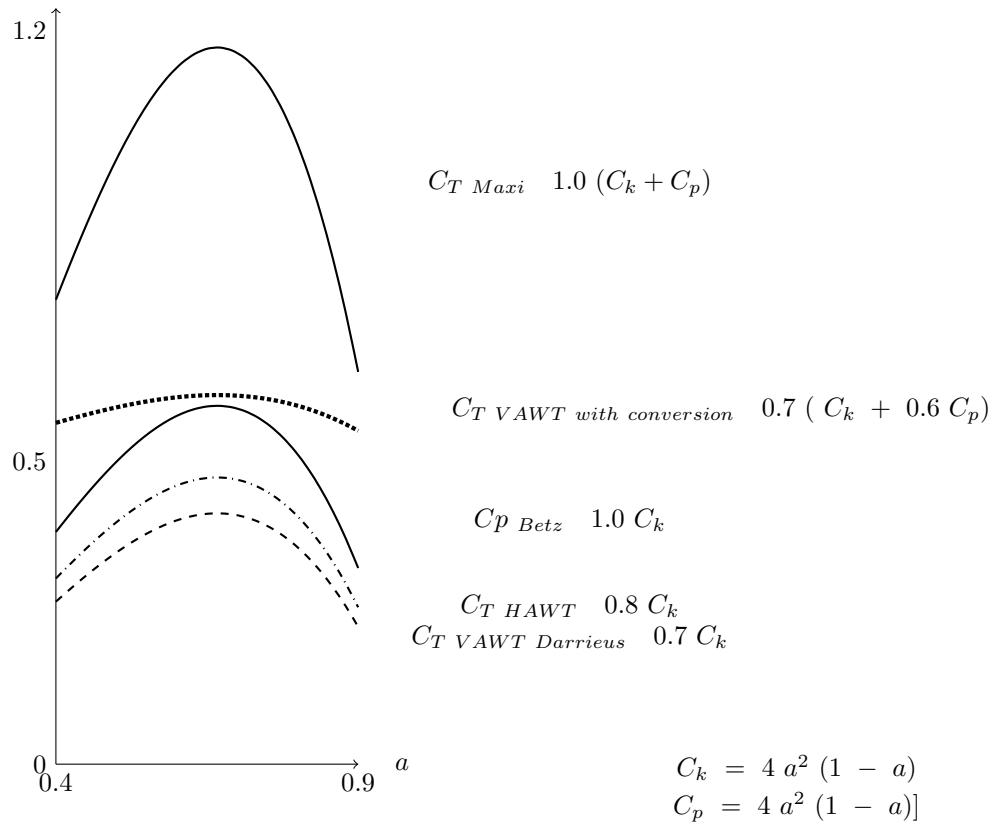
(A performance of 0.6 is applied for the supplementary energy recovery system.)

$$\text{With } a = \frac{2}{3} \quad C_k \approx 60\% \quad C_p \approx 60\%$$

case	Coef.	HAWT	VAWT(Darrieus)	VAWT(with conversion)
<i>perfect</i>	C_k	60%	60%	60%
<i>perfect</i>	C_p with conversion	0%	0%	60%
<i>perfect</i>	C_T	60%	60%	120%
<i>in practice</i>	C_k	$0.8 \times 60\% \approx 48\%$	$0.7 \times 60\% \approx 42\%$	$0.7 \times 60\% \approx 42\%$
<i>in practice</i>	C_p with conversion	0%	0%	$0.6 \times 0.7 \times 60\% \approx 25\%$
<i>in practice</i>	C_T	48%	42%	67%
.....				
gain / HAWT		+ 0%	-12 %	+ 39%
gain / C_k (Betz)		- 20%	-30 %	+ 11%

$$\text{With } a \approx 0.8 \quad C_k \approx 50\% \quad C_p \approx 50\%$$

case	Coef.	HAWT	VAWT(Darrieus)	VAWT(with conversion)
<i>in practice</i>	C_k	$0.8 \times 50\% \approx 40\%$	$0.7 \times 50\% \approx 35\%$	$0.7 \times 50\% \approx 35\%$
<i>in practice</i>	C_p with conversion	0%	0%	$0.6 \times 0.7 \times 50\% \approx 21\%$
<i>in practice</i>	C_T	40%	35%	56%
.....				
gain / HAWT		+ 0%	-12 %	+ 27%
gain / C_k		- 20%	-30 %	+ 12%



5 Project - Active Lift Turbine :

The project "Active lift turbine" is an transformation' example of potential energy into kinetic energy.

$$C_k = \frac{9\pi}{27} b^3 - \frac{2^3}{3} b^2 + \frac{\pi}{2} b$$

$$C_p = \frac{e}{R} \lambda \left(\frac{9\pi}{27} b^3 - \frac{2^2}{3} b^2 + \frac{\pi}{2} b \right)$$

with $b = \sigma\lambda$

σ stiffness coefficient

λ Tip speed ratio

e eccentric distance

R Turbine radius

See preprint : Simplified theory of an active lift turbine with controlled displacement see LECANU et al. ((2016)) LECANU et al. ((2018)).

6 Synthesis :

With a mechanical conversion sytem, the powers for the different turbines are

$$P_{T \text{ HAWT}} = 4 a^2 (1 - a) \frac{1}{2} \rho S V_{fluid}^3$$

$$P_{T \text{ VAWT Darrieus}} = 4 a^2 (1 - a) \frac{1}{2} \rho S V_{fluid}^3$$

$$P_{T \text{ VAWT with conversion}} = 8 a^2 (1 - a) \frac{1}{2} \rho S V_{fluid}^3$$

Compared to a HAWT turbine, the gain of a VAWT Turbine with an energy recovery system is in practice from 20% to 50%.

For a vertical axis turbine with a conversion system, the power factor is higher than that defined by betz.

In the comparative table, a yield of 0.6 was chosen for the mechanical conversion system of the potential energy into mechanical energy. By choosing a powerful technology, this yield can be greatly increased, which will increase the performance of the turbine.

The definition of the maximum power coefficient is that established by Betz remains valid for the horizontal axis turbine HAWT and no longer makes sense for vertical axis turbines VAWT.

The results given are examined a conversion of the potential energy into kinetic energy through a mechanical system which is not applicable for horizontal axis turbines HAWT.

The calculation of the powers are the sum of the powers which one wants to consider.

The maximum power coefficient for a wind turbine or tidal turbine is

$$C_{T \text{ maxi}} = \frac{32}{27} (\approx 118.5\%)$$

$C_{T \max}$ is a limit value with $C_p \leq C_k$.

The coefficient is greater than 1 because Betz defined its coefficient from only the kinetic power of the fluid.

Using piezo-electric materials as an example, it is possible to transform potential energy into electrical energy. Then the power coefficient is $C_T = C_k + C_p$ whatever the type of turbine.

7 Conclusion :

Betz had the maximum power coefficient $C_{k \text{ Betz}} (= \frac{16}{27})$ of a wind turbine or tidal turbine from the calculation of kinetic energy.

Taking into account the kinetic energy and the potential energy, the coefficient of maximum power becomes $C_{T \text{ maxi}} (= \frac{32}{27})$:

Transforming potential energy into kinetic energy greatly increases turbine performance.

$$P_{T \text{ maxi}} = C_{T \text{ maxi}} \frac{1}{2} \rho S V_{fluid}^3 \quad \text{with} \quad C_{T \text{ maxi}} = \frac{32}{27}$$

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Maximum wind power (from kinetic energy)

$$E_k = \frac{1}{2} m V^2$$

$$\frac{dE_k}{dt} = \frac{1}{2} \frac{dm}{dt} V^2 + \frac{1}{2} m \frac{dV^2}{dt} \quad \frac{dV}{dt} = 0$$

$$\frac{dE_k}{dt} = \frac{1}{2} \dot{m} V^2 = \frac{1}{2} \rho S V^3$$

V Wind speed at the turbine level

Force applied by the wind on the rotor

$$F = m \frac{dV}{dt} = \dot{m} \Delta V = \rho S V (V_{fluid} - V_{wake})$$

V_{wake} streamwise velocity in the far wake

$$P = FV = \rho S V^2 (V_{fluid} - V_{wake})$$

$$P = \frac{\Delta E}{\Delta t} = \frac{\frac{1}{2} m V_{fluid}^2 - \frac{1}{2} m V_{wake}^2}{\Delta t}$$

$$P = \frac{\Delta E}{\Delta t} = \frac{1}{2} \dot{m} (V_{fluid}^2 - V_{wake}^2) = \frac{1}{2} \rho S V (V_{fluid}^2 - V_{wake}^2)$$

From these equalities

$$V = \bar{V} = \frac{V_{fluid} + V_{wake}}{2}$$

$$F = \rho S V (V_{fluid} - V_{wake}) = \frac{1}{2} \rho S (V_{fluid}^2 - V_{wake}^2)$$

$$P = FV = \rho S V^2 (V_{fluid} - V_{wake})$$

defining $a = \frac{V}{V_{fluid}}$

$$V_{wake} = V_{fluid} (2a - 1) \quad \text{as} \quad V_{wake} \geq 0 \quad a \geq \frac{1}{2}$$

$$P = 4a^2(1-a) \frac{1}{2} \rho S V_{fluid}^3$$

defining power coefficient $C_k = \frac{P}{\frac{1}{2} \rho S V_{fluid}^3} = 4a^2(1-a)$

Search of maximum power coefficient

$$\frac{dC_k}{da} = 0 \quad a(2 - 3a) = 0 \quad a = 0 \quad \text{or} \quad a = \frac{2}{3}$$

$$a = \frac{2}{3} \quad C_k = \frac{16}{27} = 0.593$$

The maximum power coefficient C_{kmaxi} is defined by Betz

$$C_{kmaxi} = C_{pBetz} = \frac{16}{27} \approx 60\%$$

The maximum power of the fluid is

$$P_{fluid} = \frac{1}{2} \rho S_{fluid} V_{fluid}^3$$

$$S_{fluid} = \frac{S V}{V_{fluid}} = a S$$

The power of the turbine is

$$P = \frac{C_k}{a} P_{fluid} = C_k \frac{1}{2} \rho \frac{S_{fluid}}{a} V_{fluid}^3 = C_k \frac{1}{2} \rho S V_{fluid}^3$$

The maximum power of the turbine is

$$P_{max} = \frac{C_p \text{ Betz}}{\frac{2}{3}} P_{fluid} = \frac{8}{9} P_{fluid} = C_p \text{ Betz} \frac{1}{2} \rho S V_{fluid}^3 = \frac{16}{27} \left(\frac{1}{2} \rho S V_{fluid}^3 \right)$$

$$P_{max} = C_p \text{ Betz} \frac{1}{2} \rho S V_{fluid}^3 \quad (1)$$

Additional recovery power (from potential energy)

The fluid creates stresses in the blade. They are due to thrust force. The energy of this force is

$$E_p = m \frac{f_s}{\rho}$$

f_s thrust force

For HAWT horizontal wind turbines (fast wind turbine type), the thrust force F_s are constant.

$$\frac{dE_p}{dt} = 0$$

For a VAWT, the thrust force depends on the time or the rotation angle $F_s(t)$ or $F_s(\beta)$ $\beta = \omega t$

$\omega = \frac{d\beta}{dt}$ angular frequency

$$\frac{dE_p}{dt} \neq 0 \quad \text{and} \quad \frac{1}{2\pi} \int_0^{2\pi} E_p(\beta) d\beta = \epsilon \quad (\epsilon \text{ small})$$

As

$$F_s = f_s S = C_x \frac{1}{2} \rho S V_{fluid}^2$$

$$E_p = m C_x \frac{1}{2} V_{fluid}^2$$

V fluid speed at the level turbine

The power is

$$P_p = \frac{dE_p}{dt} \quad dm = \rho S V dt$$

$$P_p = \frac{dE_p}{dt} = \frac{dm}{dt} C_x \frac{1}{2} S V_{fluid}^2 + m C_x \frac{1}{2} S \frac{dV_{fluid}^2}{dt} \quad \frac{dV_{fluid}}{dt} = 0$$

$$P_p = a C_x \frac{1}{2} \rho S V_{fluid}^3 \quad \text{with } a = \frac{V}{V_{fluid}} \quad (2)$$

$$\frac{1}{2\pi} \int_0^{2\pi} E_p(\beta) d\beta = \epsilon \quad (\epsilon \text{ small}) \quad E_{p-max} \approx -E_{p-min}$$

the power depends on a potential energy difference

$$P_p = \frac{\Delta E_p}{\Delta t} \quad T = \frac{2\pi}{\omega} \quad P_p \leq \frac{E_{p-max} - E_{p-min}}{T} \quad P_p \leq \frac{E_{p-max}}{\pi} \omega$$

for a half-turn

$$E_p(\beta) R d\beta = dE_p \pi R$$

in particular

$$E_{p-max} = \frac{dE_p}{d\beta} \pi = \frac{dE_p}{dt} \frac{\pi}{\omega}$$

As

$$\frac{dE_p}{dt} = a C_x \frac{1}{2} \rho S V_{fluid}^3 \quad \text{and} \quad P_p \leq \frac{E_{p-max}}{\pi} \omega$$

$$P_p \leq a C_x \frac{1}{2} \rho S V_{fluid}^3$$

Variation of energy in opposite sense

Along a streamline, the Bernoulli's equation is see BERNOULLI ((1738))

$$\frac{p}{\rho} + \frac{v^2}{2} = \text{constant} \quad \text{with } z = 0$$

By multiplying by m

$$m \frac{p}{\rho} + m \frac{v^2}{2} = \text{constant}$$

The differential of this equation is

$$d\left(\frac{1}{2} m v^2\right) = -d\left(m \frac{p}{\rho}\right) \quad (3)$$

the variations of energy vary simultaneously and in opposite sense.

$$\text{As } \frac{dV}{dt} = 0 \quad \text{and} \quad \frac{dp}{dt} = 0 \quad p = -\frac{1}{2} \rho v^2$$