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Theoretical calculation of the power of wind turbine or tidal turbine

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1 Abstract

The subject of this article is to define the power of a wind turbine or marine current turbine. The German mathematician A. Betz [1] demonstrates that the power P of such turbine will reach the followinf law (cf. annex 1):

$$P = F\ V_{fluid} = [\frac{C_p}{a}\ \frac{1}{2}\ \rho\ S\ V_{fluid}^2]\ V = C_p\ \frac{1}{2}\ \rho\ S\ V_{fluid}^3$$

$$with \quad V\ =\ a\ V_{fluid}$$

$P_{turbine}$ the turbine power,
C_p the power coefficient,
ρ the fluid density,
V_{fluid} the fluid velocity.
V the fluid velocity at the position of the turbine.

According to the work of Betz, a kinetic energy approach shows that the maximum power coefficient can not exceed a maximum of $\frac{16}{27}$

$$Cp_{maxi} = Cp_{Betz} = \frac{16}{27}$$

Many people consider that Betz limit is a stated law.
This article will attempt to demonstrate that this theoretical power coefficient can be reached and exceeded.

The theory of the Betz limit is correct, it is based on the calculation of the kinetic energy. Transforming potential energy into kinetic energy provides additional motor energy.

2 Power calculation

2.1 Stress in turbine blade :

The blades are subjected to stresses due to a thrust force of wind

$$F_x = C_x\ \frac{1}{2}\ \rho\ S\ V_{fluid}^2$$

The power of this thrust force is equal to

$$P_x = V\ F_x \quad with \quad V\ =\ a\ V_{fluid} \quad and \quad \frac{1}{2}\ \leq\ a\ \leq\ 1$$

(cf. annexe 2)

$$P_x\ =\ a\ C_x\ \frac{1}{2}\ \rho\ S\ V_{fluid}^3$$

with V the fluid velocity at the position of the turbine.

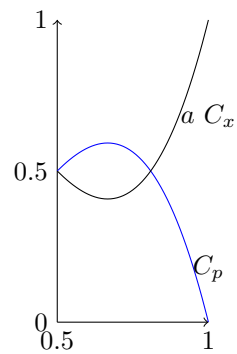
The total power is equal to

$$P_T = P \ + P_x = (C_p \ + a \ C_x) \ \frac{1}{2} \ \rho \ S \ V_{fluid}^3$$

$$with \quad (C_p \ + a \ C_x) \ \leq \ 1$$

(cf. annexe 3)

$$C_p \ = \ 4 \ a^2 \ (1 \ - \ a) \quad C_x \ = \ \frac{1}{a} \ (1 \ - \ C_p)$$



The power P should be superior to the thrust power F_x in order that additional energy to be produced.

$$P \ \geq \ P_x \qquad C_p \ \geq \ a \ C_x$$

Previous equation is resolved by this way : $C_p \ = \ a \ C_x$

$$-\frac{1}{4} \ [4a^2 \ (1-a) - \frac{1}{2}] \ = \ 0$$

$$-\frac{1}{4} \ [4a^2 \ (1-a) - \frac{1}{2}] \ = \ (x - \frac{1}{2})(x - a_2)(x - a_3)$$

$$a_1 = \frac{1}{2} \quad a_2 = \frac{2 + (20)^{0.5}}{8} \quad a_3 = \frac{2 - (20)^{0.5}}{8}$$

$$P \ \geq \ P_x \quad if \quad \frac{1}{2} \leq \ a \leq \ \frac{2 + (20)^{0.5}}{8} \ (= 0.8)$$

The calculations of the C_p maximum (Betz coefficient) and the corresponding C_x are following

$$\frac{dC_p}{da} \ = \ \frac{d(a \ C_x)}{da} \ = \ 0 \quad a \ (2 \ - \ 3 \ a) \ = \ 0 \qquad as \quad a \ \geq \ 0.5 \quad a \ = \ \frac{2}{3}$$

$$a \ = \ \frac{2}{3} \qquad C_p = Cp_{Betz} = \frac{16}{27} \approx 60\%^{-} \qquad C_x = \frac{3}{2}(1 - \frac{16}{27}) \approx 60\%^{+}$$

In the case of horizontal wind turbines (HAWT, fast wind turbine type), the stresses in the blades for a defined wind speed, are constant.

$$\frac{d\sigma}{d\beta} = 0$$

σ Stress in turbine blade
β rotation angle of the blades.
In fact, some variations of the stresses are existing due to gravitational forces and the differencial velocity within the boundary layer depending on the elevation.

In the case of vertical axis turbines (VAWT, Darrieus type) the blade and arm stresses are depending on the rotation angle of the blades (for a given wind speed).

$$\frac{d\sigma}{d\beta} \neq 0 \qquad \frac{1}{2 \ \pi} \int_0^{2 \ \pi} \sigma d\beta \ \ = \ \ \epsilon \ \ (\epsilon \ small)$$

During a half-turn, the arms are submitted to compression stresses whereas extending stresses are dominant during the next half-turn.

In the case of a HAWT, it is not possible to convert these stresses into additional energy as the stresses will be kept constant for a given wind speed. Alternative stresses, encountered in a vertical axis wind turbine will allow the extraction of additionnal energy.

2.2 HAWT-VAWT comparison

Following the work of E.Hau, the power coefficient of different turbines is compared

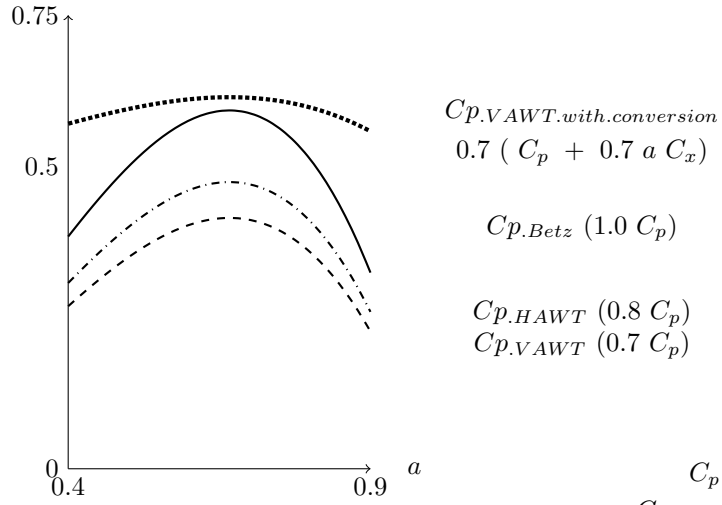
(A performance of 0.7 is applied for the supplementary energy recovery system.)

$$With\ a = \frac{2}{3} \qquad C_p \approx 60\% \qquad a\ C_x \approx 40\%$$

case	Coef.	HAWT	VAWT(Darrieus)	VAWT(with conversion)
$perfect$	C_p	60%	60%	60%
$perfect$	$a\ C_x$	0%	0%	40%
$perfect$	$C_p + a\ C_x$	60%	60%	100%
in practice	C_p	$0.8 \times 60\% \approx 48\%$	$0.7 \times 60\% \approx 42\%$	$0.7 \times 60\% \approx 42\%$
in practice	$a\ C_x$	0%	0%	$0.7 \times 0.7 \times 40\% \approx 20\%$
in practice	$C_p + a\ C_x$	48%	42%	62%
gain / HAWT		+ 0%	−12 %	+ 29%

$$With\ a \approx 0.8 \qquad C_p \approx 50\% \qquad a\ C_x \approx 50\%$$

case	Coef.	HAWT	VAWT(Darrieus)	VAWT(with conversion)
in practice	C_p	$0.8 \times 50\% \approx 40\%$	$0.7 \times 50\% \approx 35\%$	$0.7 \times 50\% \approx 35\%$
in practice	$a\ C_x$	0%	0%	$0.7 \times 0.7 \times 50\% \approx 12\%$
in practice	$C_p + a\ C_x$	40%	35%	52%
gain / HAWT		+ 0%	−12 %	+ 30%



$$C_p\ =\ 4\ a^2\ (1\ -\ a)$$
$$C_x\ =\ \frac{1}{a}\ [1\ -\ 4\ a^2\ (1\ -\ a)]$$

3 Conclusion :

For a Horizontal Axis Turbine HAWT the power is equal to

$$P_{HAWT} \;=\; C_p \; \frac{1}{2} \; \rho \; S \; V_{fluid}^3 \quad with \quad 0.5 \leq \; a \leq \; 1$$

For a vertical axis turbine VAWT the power is equal to

$$P_{VAWT} \;=\; (C_p \; + a \; C_x) \; \frac{1}{2} \; \rho \; S \; V_{fluid}^3 \quad with \quad 0.5 \leq \; a \leq \; \frac{2 + (20)^{0.5}}{8} (\approx 0.8)$$

$$a = \frac{V}{V_{fluid}} \qquad a \; C_x \; \leq \; C_p \qquad C_p \; \leq \; \frac{16}{27} \quad C_x \; \leq \; \frac{1}{2 \; a}$$

with

$$C_p \; = \; 4 \; a^2 \; (1 \; - \; a) \qquad C_x \; = \; \frac{1}{a} \; [1 \; - \; 4 \; a^2 \; (1 \; - \; a)]$$

Compared to a HAWT turbine, the gain of a VAWT Turbine with an energy recovery system is in practice from 20% to 30%.

The stress conversion into an additionnal recovered energy consists of a potential energy transformed into a kinetic energy

$a \; C_x \; = \; 0$ if the turbine is not equipped with an additional energy recovery system (Darrieus turbine)

$a \; C_x \; = \; \lambda \; \frac{R_b}{R}$ for a Active lift turbine [2] [3]
λ Tip speed ratio
R Turbine radius
R_b Connecting rod radius

References

[1] Betz A. Das maximum der theoretisch moglichen ausnutzung des windes-durch windmotoren. *Zeitschrift fur das gesamte Turbinenwesen*, (26):307–309, 1920.

[2] Pierre Lecanu, Joel Breard, and Dominique Mouazé. Simplified theory of an active lift turbine with controlled displacement. working paper or preprint, April 2016.

[3] Pierre Lecanu, Joel Breard, and Dominique Mouazé. Operating principle of an active lift turbine with controlled displacement. working paper or preprint, July 2018.

Maximum wind power (from kinetic energy)

$$E_c = \frac{1}{2}\, m\, V^2$$

$$\frac{dE_c}{dt} = \frac{1}{2}\,\frac{dm}{dt}\,V^2 + \frac{1}{2}\,m\,\frac{dV^2}{dt} \qquad \frac{dV}{dt} = 0$$

$$\frac{dE_c}{dt} = \frac{1}{2}\,\dot{m}\,V^2 = \frac{1}{2}\rho\, S\, V^3$$

V Wind speed at the turbine level

Force applied by the wind on the rotor

$$F = m\frac{dV}{dt} = \dot{m}\Delta V \;=\; \rho S V\,(V_{fluid} \;-\; V_{wake})$$

V_{wake} **streamwise velocity in the far wake**

$$P = FV \;=\; \rho S V^2\,(V_{fluid} \;-\; V_{wake})$$

$$P = \frac{\Delta E}{\Delta t} = \frac{\frac{1}{2}mV_{fluid}^2 - \frac{1}{2}mV_{wake}^2}{\Delta t}$$

$$P = \frac{\Delta E}{\Delta t} \;=\; \frac{1}{2}\dot{m}(V_{fluid}^2 \;-\; V_{wake}^2) = \frac{1}{2}\rho S V(V_{fluid}^2 \;-\; V_{wake}^2)$$

From theses equalities

$$V = \bar{V} = \frac{V_{fluid} + V_{wake}}{2}$$

$$F = \;\rho S V\,(V_{fluid} \;-\; V_{wake}) \;=\; \frac{1}{2}\rho S(V_{fluid}^2 \;-\; V_{wake}^2)$$

$$P = FV \;=\; \rho S V^2\,(V_{fluid} \;-\; V_{wake})$$

defining $a = \frac{V}{V_{fluid}}$

$$V_{wake} \;=\; V_{fluid}\;(2\;a \;-\; 1) \quad as \quad V_{wake} \geq 0 \quad a \geq \frac{1}{2}$$

$$P = \;\; 4\;a^2\;(1-a)\;\frac{1}{2}\;\rho S V_{fluid}^3$$

defining power coefficient $C_p = \frac{P}{\frac{1}{2}\rho S V_{fluid}^3} \;=\; 4\;a^2\;(1-a)$

Search of maximum power coefficient

$$\frac{dC_p}{da} = 0 \quad a\;(2 \;-\; 3\;a) \;=\; 0 \qquad a = 0 \qquad or \qquad a = \frac{2}{3}$$

$$a = \frac{2}{3} \qquad C_p = \frac{16}{27} = 0.593$$

The maximum power coefficient $C_{p_{maxi}}$ is defined by Betz

$$C_{p_{maxi}} \;=\; C_{p_{Betz}} = \frac{16}{27} \approx 60\%$$

The maximum power of the fluid is

$$P_{fluid} = \frac{1}{2}\rho S_{fluid} V_{fluid}^3$$

$S_{fluid} \;=\; \frac{S\ V}{V_{fluid}} \;=\; a\ S$

The power of the turbine is

$$P = \frac{C_p}{a}\ P_{fluid} \;=\; C_p\ \frac{1}{2}\rho \frac{S_{fluid}}{a} V_{fluid}^3 \;=\; C_p\ \frac{1}{2}\rho S V_{fluid}^3$$

The maximum power of the turbine is

$$P_{max} = \frac{C_{p-Betz}}{\frac{2}{3}}\ P_{fluid} \;=\; \frac{8}{9}\ P_{fluid} \;=\; C_{p-Betz}\ \frac{1}{2}\rho S V_{fluid}^3 \;=\; \frac{16}{27}\ (\frac{1}{2}\rho S V_{fluid}^3)$$

$$P_{max} \;=\; C_{p-Betz}\ \frac{1}{2}\rho S V_{fluid}^3 \tag{1}$$

Additional recovery power (from potential energy)

The fluid creates stresses in the blade. They are due to thrust force. The energy of this force is

$$E_p \;=\; m\;\frac{f_s}{\rho}$$

f_s **thrust force**
For HAWT horizontal wind turbines (fast wind turbine type), the thrust force F_s are constant.

$$\frac{dE_p}{dt} \;=\; 0$$

For a VAWT, the thrust force depends on the time or the rotation angle $F_s(t)$ **or** $F_s(\beta)$ $\quad \beta = \omega\; t$
$\omega \;=\; \frac{d\dot{\beta}}{dt}$ **angular frequency**

$$\frac{dE_p}{dt} \;\neq\; 0 \quad and \quad \frac{1}{2\;\pi}\int_0^{2\;\pi} E_p(\beta)d\beta \;=\; \epsilon \;\; (\epsilon\; small)$$

As

$$F_s \;=\; f_s S \;=\; C_x\frac{1}{2}\;\rho\;S\;V_{fluid}^2$$

$$E_p \;=\; m\;C_x\frac{1}{2}\;V_{fluid}^2$$

V **fluid speed at the level turbine**
The power is

$$P_p \;=\; \frac{dE_p}{dt} \quad dm = \rho S V dt$$

$$P_p \;=\; \frac{dE_p}{dt} \;=\; \frac{dm}{dt}\;C_x\;\frac{1}{2}\;S\;V_{fluid}^2 \;+\; m\;C_x\frac{1}{2}\;S\;\frac{dV_{fluid}^2}{dt} \qquad \frac{dV_{fluid}}{dt} \;=\; 0$$

$$P_p \;=\; a\;C_x\;\frac{1}{2}\;\rho\;S\;V_{fluid}^3 \quad with\;a = \frac{V}{V_{fluid}} \tag{2}$$

$$\frac{1}{2\;\pi}\int_0^{2\;\pi} E_p(\beta)d\beta \;=\; \epsilon \;\; (\epsilon\; small) \quad E_{p-max} \;\approx\; -E_{p-min}$$

the power depends on a potential energy difference

$$P_p \;=\; \frac{\Delta E_p}{\Delta t} \quad T \;=\; \frac{2\;\pi}{\omega} \quad P_p \;\leq\; \frac{E_{p-max} - E_{p-min}}{T} \quad P_p \;\leq\; \frac{E_{p-max}}{\pi}\;\omega$$

for a half-turn

$$E_p(\beta)\ R\ d\beta\ =\ dE_p\ \pi\ R$$

in particular

$$E_{p-max}\ =\ \frac{dE_p}{d\beta}\pi\ =\ \frac{dE_p}{dt}\ \frac{\pi}{\omega}$$

As

$$\frac{dE_p}{dt}\ =\ a\ C_x\ \frac{1}{2}\ \rho\ S\ V_{fluid}^3 \quad and \quad P_p\ \leq\ \frac{E_{p-max}}{\pi}\ \omega$$

$$P_p\ \leq\ a\ C_x\ \frac{1}{2}\ \rho\ S\ V_{fluid}^3$$

Variation of energy in opposite sense

Along a streamline, the Bernoulli's equation is

$$\frac{p}{\rho} + \frac{v^2}{2} = constant \quad with \quad z = 0$$

By multiplying by m

$$m\,\frac{p}{\rho} + m\,\frac{v^2}{2} = constant$$

The differential of this equation is

$$d(\frac{1}{2}\,m\,v^2) = d(m\,\frac{p}{\rho}) \tag{3}$$

Kinetic energy varies simultaneously and in the opposite sense from potential energy