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# Theoretical calculation of the power of wind turbine or tidal turbine

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# 1 Abstract

The subject of this article is to define the power of a wind turbine or marine current turbine. The German mathematician A. Betz [1] demonstrates that the power  $P$  of such turbine will reach the following law (cf. annex 1):

$$P = F V_{fluid} = \left[ \frac{C_p}{a} \frac{1}{2} \rho S V_{fluid}^2 \right] V = C_p \frac{1}{2} \rho S V_{fluid}^3$$

*with*  $V = a V_{fluid}$

$P_{turbine}$  the turbine power,  
 $C_p$  the power coefficient,  
 $\rho$  the fluid density,  
 $V_{fluid}$  the fluid velocity.  
 $V$  the fluid velocity at the position of the turbine.

According to the work of Betz, a kinetic energy approach shows that the maximum power coefficient can not exceed a maximum of  $\frac{16}{27}$

$$C_{p_{maxi}} = C_{p_{Betz}} = \frac{16}{27}$$

Many people consider that Betz limit is a stated law. This article will attempt to demonstrate that this theoretical power coefficient can be reached and exceeded.

The theory of the Betz limit is correct, it is based on the calculation of the kinetic energy. Transforming potential energy into kinetic energy provides additional motor energy.

## 2 Power calculation

### 2.1 Stress in turbine blade :

The blades are subjected to stresses due to a thrust force of wind

$$F_x = C_x \frac{1}{2} \rho S V_{fluid}^2$$

The power of this thrust force is equal to

$$P_x = V F_x \quad \text{with} \quad V = a V_{fluid} \quad \text{and} \quad \frac{1}{2} \leq a \leq 1$$

( cf. annexe 2)

$$P_x = a C_x \frac{1}{2} \rho S V_{fluid}^3$$

with  $V$  the fluid velocity at the position of the turbine.

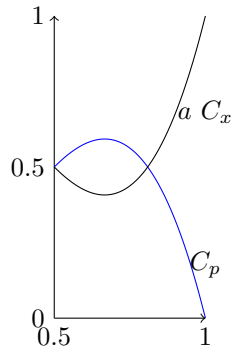
The total power is equal to

$$P_T = P + P_x = (C_p + a C_x) \frac{1}{2} \rho S V_{fluid}^3$$

$$\text{with } (C_p + a C_x) \leq 1$$

( cf. annexe 3)

$$C_p = 4 a^2 (1 - a) \quad C_x = \frac{1}{a} (1 - C_p)$$



The power  $P$  should be superior to the thrust power  $F_x$  in order that additional energy to be produced.

$$P \geq P_x \quad C_p \geq a C_x$$

Previous equation is resolved by this way :  $C_p = a C_x$

$$-\frac{1}{4} [4a^2 (1 - a) - \frac{1}{2}] = 0$$

$$-\frac{1}{4} [4a^2 (1 - a) - \frac{1}{2}] = (x - \frac{1}{2})(x - a_2)(x - a_3)$$

$$a_1 = \frac{1}{2} \quad a_2 = \frac{2 + (20)^{0.5}}{8} \quad a_3 = \frac{2 - (20)^{0.5}}{8}$$

$$P \geq P_x \quad \text{if} \quad \frac{1}{2} \leq a \leq \frac{2 + (20)^{0.5}}{8} \quad (= 0.8)$$

The calculations of the  $C_p$  maximum (Betz coefficient) and the corresponding  $C_x$  are following

$$\frac{dC_p}{da} = \frac{d(a C_x)}{da} = 0 \quad a(2 - 3a) = 0 \quad \text{as} \quad a \geq 0.5 \quad a = \frac{2}{3}$$

$$a = \frac{2}{3} \quad C_p = C_{p_{Betz}} = \frac{16}{27} \approx 60\%^- \quad C_x = \frac{3}{2} \left(1 - \frac{16}{27}\right) \approx 60\%^+$$

In the case of horizontal wind turbines (HAWT, fast wind turbine type), the stresses in the blades for a defined wind speed, are constant.

$$\frac{d\sigma}{d\beta} = 0$$

$\sigma$  Stress in turbine blade

$\beta$  rotation angle of the blades.

In fact, some variations of the stresses are existing due to gravitational forces and the differential velocity within the boundary layer depending on the elevation.

In the case of vertical axis turbines (VAWT, Darrieus type) the blade and arm stresses are depending on the rotation angle of the blades (for a given wind speed).

$$\frac{d\sigma}{d\beta} \neq 0 \quad \frac{1}{2\pi} \int_0^{2\pi} \sigma d\beta = \epsilon \quad (\epsilon \text{ small})$$

During a half-turn, the arms are submitted to compression stresses whereas extending stresses are dominant during the next half-turn.

In the case of a HAWT, it is not possible to convert these stresses into additional energy as the stresses will be kept constant for a given wind speed. Alternative stresses, encountered in a vertical axis wind turbine will allow the extraction of additional energy.

## 2.2 HAWT-VAWT comparison

Following the work of E.Hau, the power coefficient of different turbines is compared

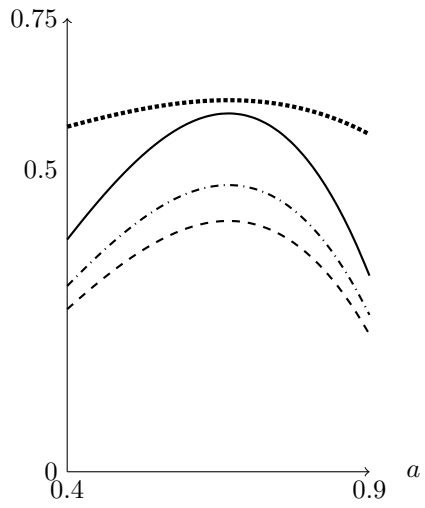
(A performance of 0.7 is applied for the supplementary energy recovery system.)

$$\text{With } a = \frac{2}{3} \quad C_p \approx 60\% \quad a C_x \approx 40\%$$

| case               | Coef.         | HAWT                           | VAWT(Darrieus)                 | VAWT(with conversion)                     |
|--------------------|---------------|--------------------------------|--------------------------------|---|
| <i>perfect</i>     | $C_p$         | 60%                            | 60%                            | 60%                                       |
| <i>perfect</i>     | $a C_x$       | 0%                             | 0%                             | 40%                                       |
| <i>perfect</i>     | $C_p + a C_x$ | 60%                            | 60%                            | 100%                                      |
| <i>in practice</i> | $C_p$         | $0.8 \times 60\% \approx 48\%$ | $0.7 \times 60\% \approx 42\%$ | $0.7 \times 60\% \approx 42\%$            |
| <i>in practice</i> | $a C_x$       | 0%                             | 0%                             | $0.7 \times 0.7 \times 40\% \approx 20\%$ |
| <i>in practice</i> | $C_p + a C_x$ | 48%                            | 42%                            | 62%                                       |
| gain / HAWT        |               | + 0%                           | -12 %                          | + 29%                                     |

$$\text{With } a \approx 0.8 \quad C_p \approx 50\% \quad a C_x \approx 50\%$$

| case               | Coef.         | HAWT                           | VAWT(Darrieus)                 | VAWT(with conversion)                     |
|--------------------|---------------|--------------------------------|--------------------------------|---|
| <i>in practice</i> | $C_p$         | $0.8 \times 50\% \approx 40\%$ | $0.7 \times 50\% \approx 35\%$ | $0.7 \times 50\% \approx 35\%$            |
| <i>in practice</i> | $a C_x$       | 0%                             | 0%                             | $0.7 \times 0.7 \times 50\% \approx 12\%$ |
| <i>in practice</i> | $C_p + a C_x$ | 40%                            | 35%                            | 52%                                       |
| gain / HAWT        |               | + 0%                           | -12 %                          | + 30%                                     |



$$C_{p.VAWT,with.conversion}$$

$$0.7 ( C_p + 0.7 a C_x )$$

$$C_{p.Betz} (1.0 C_p)$$

$$C_{p.HAWT} (0.8 C_p)$$

$$C_{p.VAWT} (0.7 C_p)$$

$$C_p = 4 a^2 (1 - a)$$

$$C_x = \frac{1}{a} [1 - 4 a^2 (1 - a)]$$

### 3 Conclusion :

For a Horizontal Axis Turbine HAWT the power is equal to

$$P_{HAWT} = C_p \frac{1}{2} \rho S V_{fluid}^3 \quad \text{with} \quad 0.5 \leq a \leq 1$$

For a vertical axis turbine VAWT the power is equal to

$$P_{VAWT} = (C_p + a C_x) \frac{1}{2} \rho S V_{fluid}^3 \quad \text{with} \quad 0.5 \leq a \leq \frac{2 + (20)^{0.5}}{8} (\approx 0.8)$$

$$a = \frac{V}{V_{fluid}} \quad a C_x \leq C_p \quad C_p \leq \frac{16}{27} \quad C_x \leq \frac{1}{2a}$$

with

$$C_p = 4 a^2 (1 - a) \quad C_x = \frac{1}{a} [1 - 4 a^2 (1 - a)]$$

Compared to a HAWT turbine, the gain of a VAWT Turbine with an energy recovery system is in practice from 20% to 30%.

The stress conversion into an additional recovered energy consists of a potential energy transformed into a kinetic energy

$a C_x = 0$  if the turbine is not equipped with an additional energy recovery system (Darrieus turbine)

$a C_x = \lambda \frac{R_b}{R}$  for a Active lift turbine [2] [3]

$\lambda$  Tip speed ratio

$R$  Turbine radius

$R_b$  Connecting rod radius



## References

- [1] Betz A. Das maximum der theoretisch moglichen ausnutzung des windes-durch windmotoren. *Zeitschrift fur das gesamte Turbinenwesen*, (26):307–309, 1920.
- [2] Pierre Lecanu, Joel Breard, and Dominique Mouazé. Simplified theory of an active lift turbine with controlled displacement. working paper or preprint, April 2016.
- [3] Pierre Lecanu, Joel Breard, and Dominique Mouazé. Operating principle of an active lift turbine with controlled displacement. working paper or preprint, July 2018.

**Maximum wind power (from kinetic energy)**

$$E_c = \frac{1}{2} m V^2$$

$$\frac{dE_c}{dt} = \frac{1}{2} \frac{dm}{dt} V^2 + \frac{1}{2} m \frac{dV^2}{dt} \quad \frac{dV}{dt} = 0$$

$$\frac{dE_c}{dt} = \frac{1}{2} \dot{m} V^2 = \frac{1}{2} \rho S V^3$$

**V Wind speed at the turbine level**

**Force applied by the wind on the rotor**

$$F = m \frac{dV}{dt} = \dot{m} \Delta V = \rho S V (V_{fluid} - V_{wake})$$

**$V_{wake}$  streamwise velocity in the far wake**

$$P = FV = \rho S V^2 (V_{fluid} - V_{wake})$$

$$P = \frac{\Delta E}{\Delta t} = \frac{\frac{1}{2} m V_{fluid}^2 - \frac{1}{2} m V_{wake}^2}{\Delta t}$$

$$P = \frac{\Delta E}{\Delta t} = \frac{1}{2} \dot{m} (V_{fluid}^2 - V_{wake}^2) = \frac{1}{2} \rho S V (V_{fluid}^2 - V_{wake}^2)$$

**From these equalities**

$$V = \bar{V} = \frac{V_{fluid} + V_{wake}}{2}$$

$$F = \rho S V (V_{fluid} - V_{wake}) = \frac{1}{2} \rho S (V_{fluid}^2 - V_{wake}^2)$$

$$P = FV = \rho S V^2 (V_{fluid} - V_{wake})$$

**defining  $a = \frac{V}{V_{fluid}}$**

$$V_{wake} = V_{fluid} (2a - 1) \quad \text{as } V_{wake} \geq 0 \quad a \geq \frac{1}{2}$$

$$P = 4 a^2 (1 - a) \frac{1}{2} \rho S V_{fluid}^3$$

**defining power coefficient  $C_p = \frac{P}{\frac{1}{2} \rho S V_{fluid}^3} = 4 a^2 (1 - a)$**

**Search of maximum power coefficient**

$$\frac{dC_p}{da} = 0 \quad a(2 - 3a) = 0 \quad a = 0 \quad \text{or} \quad a = \frac{2}{3}$$

$$a = \frac{2}{3} \quad C_p = \frac{16}{27} = 0.593$$

The maximum power coefficient  $C_{p_{maxi}}$  is defined by Betz

$$C_{p_{maxi}} = C_{p_{Betz}} = \frac{16}{27} \approx 60\%$$

The maximum power of the fluid is

$$P_{fluid} = \frac{1}{2} \rho S_{fluid} V_{fluid}^3$$

$$S_{fluid} = \frac{S V}{V_{fluid}} = a S$$

The power of the turbine is

$$P = \frac{C_p}{a} P_{fluid} = C_p \frac{1}{2} \rho \frac{S_{fluid}}{a} V_{fluid}^3 = C_p \frac{1}{2} \rho S V_{fluid}^3$$

The maximum power of the turbine is

$$P_{max} = \frac{C_{p-Betz}}{\frac{2}{3}} P_{fluid} = \frac{8}{9} P_{fluid} = C_{p-Betz} \frac{1}{2} \rho S V_{fluid}^3 = \frac{16}{27} \left( \frac{1}{2} \rho S V_{fluid}^3 \right)$$

$$P_{max} = C_{p-Betz} \frac{1}{2} \rho S V_{fluid}^3 \quad (1)$$

**Additional recovery power (from potential energy)**

The fluid creates stresses in the blade. They are due to thrust force. The energy of this force is

$$E_p = m \frac{f_s}{\rho}$$

$f_s$  thrust force

For HAWT horizontal wind turbines (fast wind turbine type), the thrust force  $F_s$  are constant.

$$\frac{dE_p}{dt} = 0$$

For a VAWT, the thrust force depends on the time or the rotation angle  $F_s(t)$  or  $F_s(\beta)$   $\beta = \omega t$

$\omega = \frac{d\beta}{dt}$  angular frequency

$$\frac{dE_p}{dt} \neq 0 \quad \text{and} \quad \frac{1}{2\pi} \int_0^{2\pi} E_p(\beta) d\beta = \epsilon \quad (\epsilon \text{ small})$$

As

$$F_s = f_s S = C_x \frac{1}{2} \rho S V_{fluid}^2$$

$$E_p = m C_x \frac{1}{2} V_{fluid}^2$$

$V$  fluid speed at the level turbine

The power is

$$P_p = \frac{dE_p}{dt} \quad dm = \rho S V dt$$

$$P_p = \frac{dE_p}{dt} = \frac{dm}{dt} C_x \frac{1}{2} S V_{fluid}^2 + m C_x \frac{1}{2} S \frac{dV_{fluid}^2}{dt} \quad \frac{dV_{fluid}}{dt} = 0$$

$$P_p = a C_x \frac{1}{2} \rho S V_{fluid}^3 \quad \text{with } a = \frac{V}{V_{fluid}} \quad (2)$$

$$\frac{1}{2\pi} \int_0^{2\pi} E_p(\beta) d\beta = \epsilon \quad (\epsilon \text{ small}) \quad E_{p-max} \approx -E_{p-min}$$

the power depends on a potential energy difference

$$P_p = \frac{\Delta E_p}{\Delta t} \quad T = \frac{2\pi}{\omega} \quad P_p \leq \frac{E_{p-max} - E_{p-min}}{T} \quad P_p \leq \frac{E_{p-max}}{\pi} \omega$$

for a half-turn

$$E_p(\beta) R d\beta = dE_p \pi R$$

in particular

$$E_{p-max} = \frac{dE_p}{d\beta} \pi = \frac{dE_p}{dt} \frac{\pi}{\omega}$$

As

$$\frac{dE_p}{dt} = a C_x \frac{1}{2} \rho S V_{fluid}^3 \quad \text{and} \quad P_p \leq \frac{E_{p-max}}{\pi} \omega$$

$$P_p \leq a C_x \frac{1}{2} \rho S V_{fluid}^3$$

**Variation of energy in opposite sense**

**Along a streamline, the Bernoulli's equation is**

$$\frac{p}{\rho} + \frac{v^2}{2} = \text{constant} \quad \text{with } z = 0$$

**By multiplying by m**

$$m \frac{p}{\rho} + m \frac{v^2}{2} = \text{constant}$$

**The differential of this equation is**

$$d\left(\frac{1}{2} m v^2\right) = d\left(m \frac{p}{\rho}\right) \quad (3)$$

**Kinetic energy varies simultaneously and in the opposite sense from potential energy**