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Pierre Normandajc Lecanu, Joël Bréard, Dominique Mouazé. Theoretical calculation of wind (Or water) turbine considering kinetic and potential energy to exceed the Betz limit. 2023. hal-01982516v12

**HAL Id: hal-01982516**

**<https://hal.science/hal-01982516v12>**

Preprint submitted on 15 Jun 2023 (v12), last revised 21 Jun 2023 (v13)

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# Theoretical calculation of wind (Or water) turbine considering kinetic and potential energy to exceed the Betz limit

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June 9, 2023

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## Abstract :

The Betz limit sets a theoretical upper limit for the energy efficiency of turbines. The energy efficiency of turbines, expressed as a maximum power coefficient of 16/27.

Betz's theory is precise and is based on the calculation of kinetic energy. However, if the potential energy is taken into account the theoretical energy efficiency of a turbine can be higher. Fast wind turbines recover the kinetic energy of the wind in an optimal way. A large amount of potential energy is created without being recovered. The notion of potential energy is fundamental, it is not possible to recover energy, if we do not create a constraint.

This article examines this potential energy and the possibility for a wind turbine to transform it into kinetic energy. The Betz theory has been defined from the model of fast moving turbines. This theory has been generalized to slow and fast moving turbines and it has been defined as a law.

The conservation of energy implies that if a variation of kinetic energy increases, the variation of potential energy decreases. In the case of slow moving turbines, the conservation of energy applies, but not for the case of fast moving turbines, however this is the reality.

This paper proposes a new formulation of the turbine power with a notion of temporal, in order to be able to verify the conservation of energy.

**Keywords :** Betz limit, Betz's law, Wind turbine, Tidal turbine, HAWT, VAWT, ALWT .

# Nomenclature

## Acronyms :

*ALWT* Active lift wind turbine

*HAWT* Horizontal-axis wind turbine

*VAWT* Vertical-axis wind turbine

## specific symbol to ALWT :

$\sigma_R$  stiffness coefficient ( $\sigma_R = \frac{N \cdot c}{R}$ )

$b$  coefficient ( $b = \sigma_R \lambda \quad b \leq \frac{4}{3}$ )

$e$  eccentric distance [ $m$ ]

## Symbols :

$\beta$  Rotation angle of the blades [ $rad$ ]

$\lambda$  Tip ratio ( $\lambda = \frac{\omega R}{V_{fluid}}$ )

$\omega$  Angular Velocity ( $\omega = \frac{d\beta}{dt}$ ) [ $\frac{rad}{s}$ ]

$\rho$  Fluid density [ $\frac{kg}{m^3}$ ]

$\sigma$  Rotation angle of the blades [ $\frac{N}{m^2}$ ]

$c$  chord length of blades [ $m$ ]

$C_{kinBetz}$  Maximum kinetic power coefficient ( $C_{kinBetz} = \frac{16}{27}$ )

$C_{kin}$  kinetic power coefficient

$C_{pot}$  Potential power coefficient

$C_{total}$  or  $C_T$  Total power coefficient

$E_{kin}$  Kinetic energy [ $J$ ]

$E_{pot}$  Potential energy [ $J$ ]

$E_{total}$  Total energy ( $E_{total} = E_{kin} + E_{pot}$ ) [ $J$ ]

$F_a$  Axial force (axial to the blade profile) [ $N$ ]

$F_n$  Normal force (normal to the blade profile) [ $N$ ]

$H$  Blade height [ $m$ ]

$P_{fluid}$  Fluid power ( $P_{fluid} = \frac{1}{2}\rho S_{fluid} V_{fluid}^3$ ) [ $W$ ]

$P_{kin}$  Kinetic power [ $W$ ]

$P_{pot}$  Potential power [ $W$ ]

$S$  Swept area  $S = 2 R H$  (*VAWT*)  $S = \pi R^2$  (*HAWT*) [ $m^2$ ]

$S_{fluid}$  upstream stream-tube cross-sectionnal area  $S_{fluid} = \frac{S}{a}$  [ $m^2$ ]

$U$  Rotation speed ( $U = \omega R$ ) [ $\frac{m}{s}$ ]

$W$  Relative speed [ $\frac{m}{s}$ ]

- $a$  Velocity factor  $a = \frac{V}{V_{fluid}}$
- $V_{fluid}$  Velocity wind (upwind) [ $\frac{m}{s}$ ]
- $V$  Fluid velocity at the position of the turbine [ $\frac{m}{s}$ ]

# 1 Introduction

Lanchester, Betz, Joukowsky have defined the maximum power coefficient of wind turbines (article "The Lanchester–Betz–Joukowsky limit" van Kuik (2007)). This limit is commonly called the Betz limit. Considerable research efforts have been deployed to optimize wind turbines in order to reach this limit, for instance by optimizing the angle of incidence, the shape of the blade profile etc. One may for example refer to "Wind Energy Handbook "Burton et al. for fast moving horizontal axis wind turbines (HAWT) Burton et al. (2006), or to "Hydrodynamic modelling of marine renewable energy devices: A state of the art review" Day et al. (2015) Day et al. (2015) or to "Wind Turbine Design: With Emphasis on Darrieus Concept" for vertical axis wind turbines (VAWT) Paraschivoiu (2002) , or to "Les éoliennes : théorie, conception et calcul pratique" Le Gourieres (2008).

Research has been made to reach the Betz limit:

by ducting the turbine Georgiou and Theodoropoulos (2016)

by placing a water turbine in a narrow channel, which allows the water level upstream to be increased by the resistance to the advancing fluid Quaranta (2018)

by grouping turbines in wind farms in order to create an excess of power due to the proximity between machines (ducting effect) ) Vennell (2012) Broberg et al. (2018).

Studies are still being carried out to try to obtain the maximum gain and exceed the Betz limit, for example :

"Performance analysis of various types of ducted wind turbines – A review" Aravindhnan et al. (2023),

"Towards development and optimization of the Savonius wind turbine incorporated with a wind-lens" Hesami and Nikseresht (2023),

"Finding an absolute maximum theoretical power coefficient for ducted wind turbines" Quispe-Abad and Müller (2023),

"The complementary Betz Theory" Pellegrini (2022),

"Theoretical framework to surpass the Betz limit using unsteady fluid mechanics" Dabiri (2020),

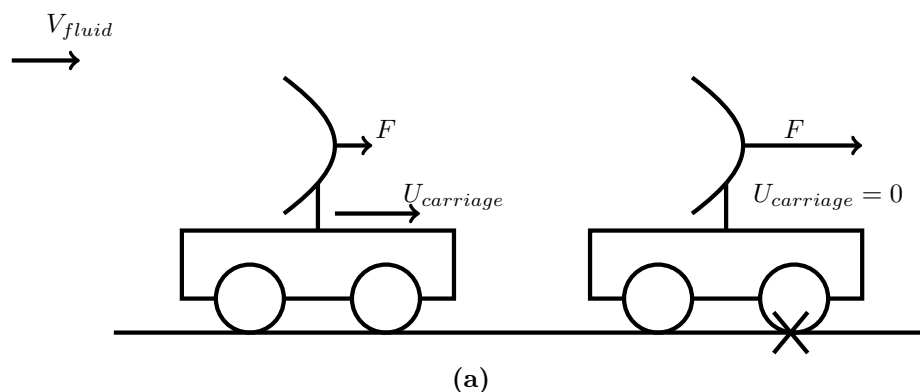
"A comprehensive review of wind resource assessment" Murthy and Rahi (2017),

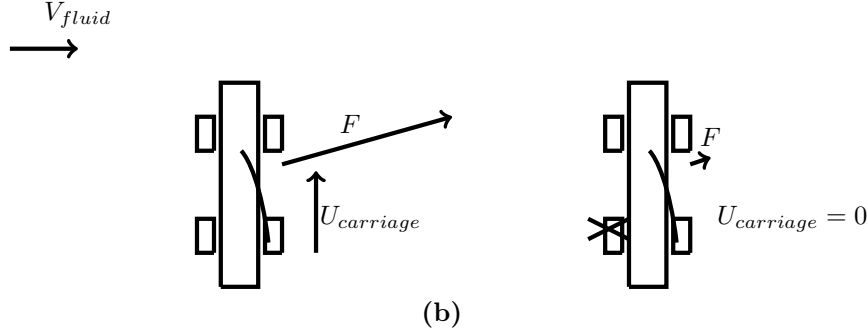
"Fluid structure interaction analysis of the operating Savonius wind turbine" Łukasz Marzec et al. (2021),

"Towards a standard approach for future Vertical Axis Wind Turbine aerodynamics research and development" Barnes et al. (2021)

have shown a desire to surpass Betz's coefficient, but have not taken into account the notion of potential energy, nor the notion of temporal energy transfer.

The Betz limit is based on the calculation of kinetic energy. Designing a wind turbine, the energy of the fluid is taken into account in order to calculate the recoverable energy and to design a structure withstanding the stress.





(b)  
Fig. 1

Comparison of carriages with a sail.

A carriage with a square sail (a) or a modern sail (b) that can move freely, will be set in motion by the wind and thus be subject to very low wind stress (a) and to high stress from the wind (b). If the carriage is blocked, its structure will be subject to high stress (a) or to low stress on its structure (b).

In the case of fast-moving wind turbines, when the wind increases, the structure of the wind turbine is subjected to high stresses in a similar way to the carriage in case (b) of Figure 1.

Large wind turbines are stopped when the wind becomes too strong, not because they produce too much energy but because excessive stress could damage or destroy their structure. The stress induced in a structure is potential energy.

The pressure exerted by a fluid on the sides of an object, causes stresses inside this object. When the object is no longer under the pressure of the fluid, the stresses are released and the object returns to its original shape. Under pressure, the object has stored potential energy.

Betz's theory does not take into account this potential energy. It exclusively takes into consideration kinetic energy.

In the case of a sailboat, the sole sail cannot recover all the energy from the wind. Betz's theory applies to the sail that can not retrieve more than 16/27 of the wind's kinetic energy. Many books explore the Betz limit, for instance John Kimball's "Physics of Sailing" Kimball (2009).

However, the America's Cup is a good example showing that the Betz limit can be exceeded. Indeed, a sailboat equipped with hydrofoils transforms potential energy into kinetic energy. The kinetic energy of the wind sets the boat in motion. The fluid flowing around the hydrofoils lifts the boat out of the water, thus reducing the drag and improving the efficiency. The wind power has not increased but the potential energy that applied constraints on the keel has been converted into kinetic energy by the flow of fluid around the profiles of the foils.

This article introduces the notions of kinetic and potential energy. It suggests that a vertical axis wind turbine (VAWT) with an appropriate design can transform potential energy into kinetic energy.

## 2 Preliminary considerations of Betz's theory

The German mathematician A. Betz has demonstrated that the power of a turbine is :

$$P_{kin} = F V = \left[ \frac{C_{kin}}{a} \frac{1}{2} \rho S V_{fluid}^2 \right] a V_{fluid} = C_{kin} \frac{1}{2} \rho S V_{fluid}^3 \quad (1)$$

According to Betz's work (preliminarily by Lanchester Lanchester (1915)), a kinetic energy approach shows that the maximum power coefficient  $C_k$  can not exceed a maximum of  $\frac{16}{27}$

$$C_{kin \text{ Betz}} = \frac{16}{27} \quad (2)$$

Figure 2 shows the forces applied to a HAWT turbine.

The relative speed due to the rotation speed of the wind turbine and the fluid speed creates an induced force  $F$  on the profile. This induced force  $F$  is composed of an axial force  $F_a$  and of a normal force  $F_n$ . The

axial force  $F_a$  associated with the radius  $R$  creates a driving torque to produce energy. The normal force  $F_n$  associated with the same radius  $R$  creates a bending stress on the blade.

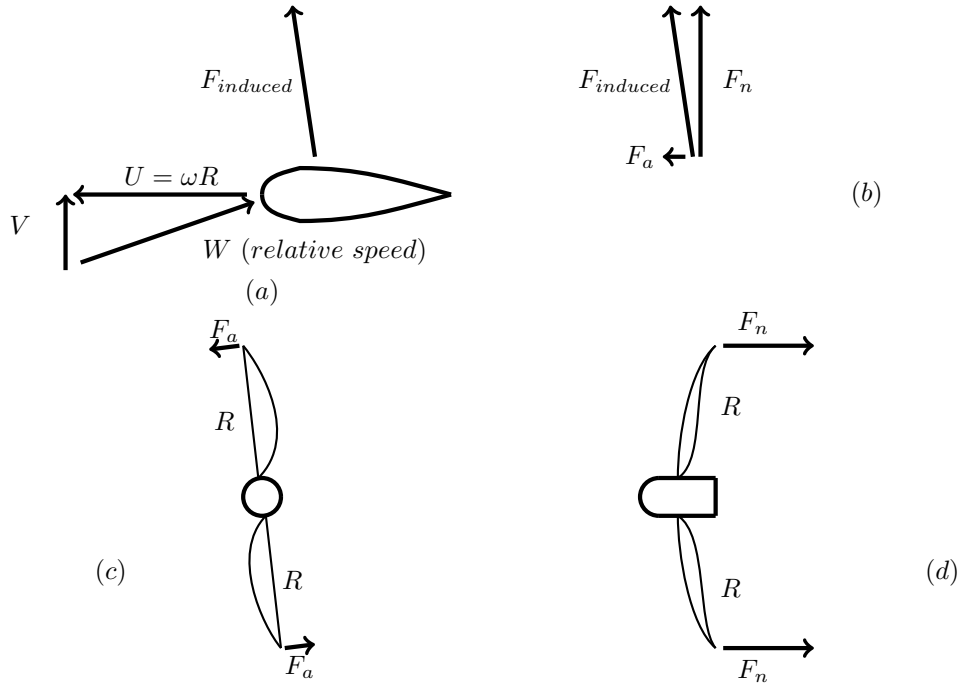


Fig. 2  
 Forces applied to a HAWT turbine  
 (a) *The relative speed creates an induced force on the profile*  
 (b) *Decomposition of the induced force*  
 (c) *Driving torque*  
 (d) *Bending stress in the blades*

The power of the torque due to the axial forces and the angular rotation speed of the wind turbine is limited to  $16/27$  of the wind's kinetic power. The forces  $F_n$  and  $F_a$  are associated with the same radius and have the same origin: the fluid velocity.  $F_n$  creates constraints that are the source of internal energy. These constraints are potential energy. Betz's theory does not take into account this potential energy which is as important as the kinetic energy. Betz limit is correct. However, it only takes into account kinetic energy. In order to increase the efficiency of wind and hydrokinetic turbines, they could be designed to transform the potential energy into kinetic energy.

It is necessary to dissociate slow speed turbines and fast speed turbines.

$$\lambda = = \frac{V}{V_{fluid}} = \frac{\omega R}{V_{fluid}} \quad (3)$$

is an important parameter which makes a difference in the behaviour of the turbines when exposed to wind. As for the sail, it is necessary to dissociate the navigation in thrust (square sail, spinnaker) (unstuck flow) and the navigation in smoothness (wing profile sail) (laminar flow).

### 3 Difference between a low speed turbine and a high speed turbine

#### 3.1 The advantage of using an airplane wing

Fast moving turbines can not be described as slow moving turbines such as Savionus. The comparison is made between the airfoil of a fast moving turbine and the cup of an anemometer in figure 3.

If the movement of an anemometer is not impeded, then its cups are almost free of stress. If the rotation of an anemometer is hampered, then its cups are subject to stress. In the case of a wing profile, this effect is totally different. Indeed, the tangential rotation speed  $U$  of the profile is transverse to the flow of the fluid. The relative speed  $W$  due to the rotation speed  $U$  and the fluid speed  $V_{fluid}$  creates an induced force  $F$  on the wing profile. The axial component  $F_a$  of this force  $F$ , combined with the radius, creates a driving torque.

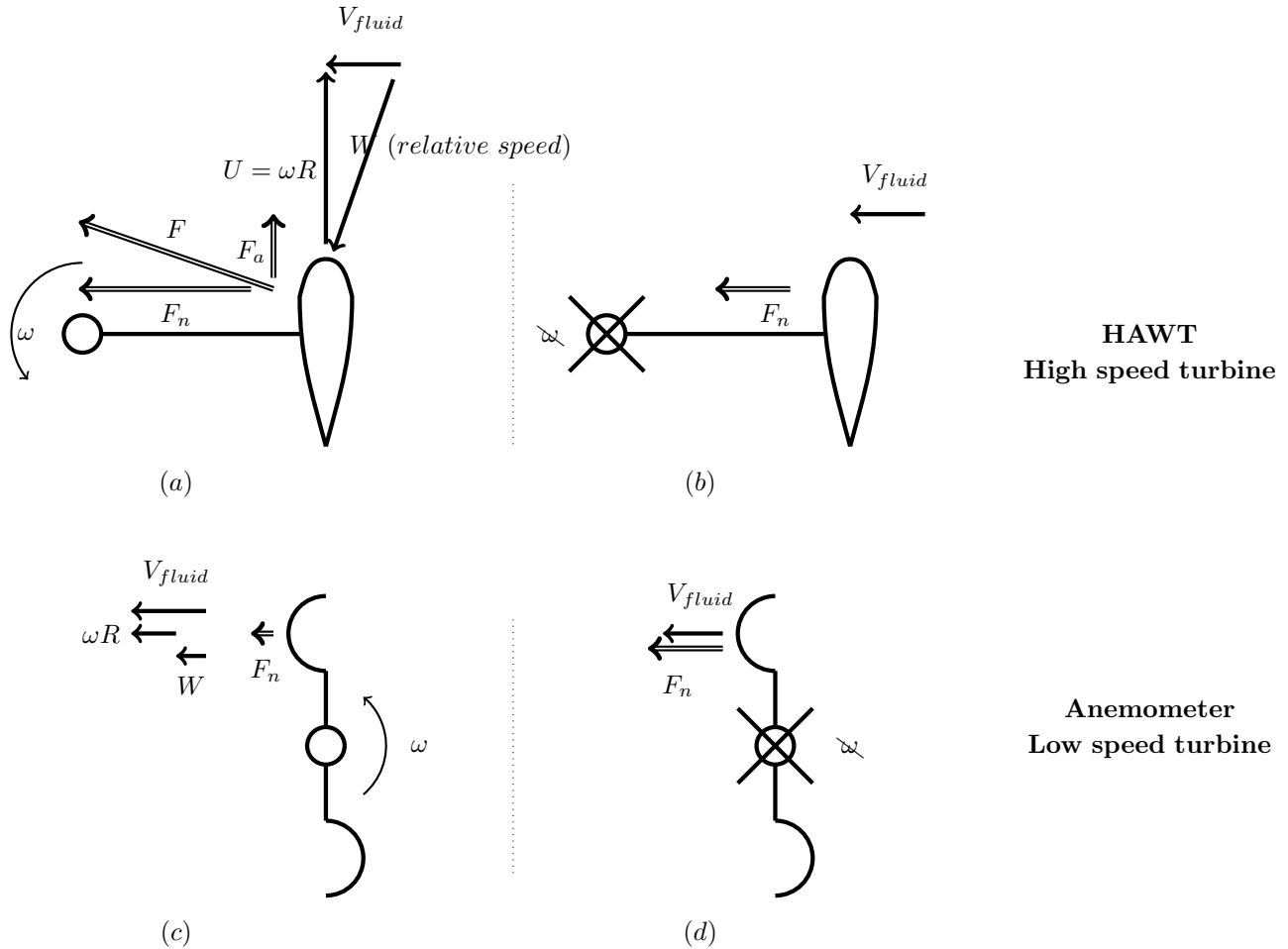


Fig. 3  
Comparison between (a)-(b) and (c)-(d) speed turbine  
(b) and (d) are the cases blocked in rotation.

For *horizontal axis wind turbines* (HAWT) (fig. 3), this torque improves the efficiency which can then approach the Betz limit. Adding a transverse speed  $U$  will optimize the efficiency defined by Betz, but will add significant stress on the wind turbine due to the normal component  $F_n$  of force  $F$ . That's why HAWT wind turbines have to be stopped when the speed of the fluid is too high. They do not produce too much but they are subject to excessive bending stress.

Appendix A detail flow and energy conservation and Table 1 present energy variations.

**Anemometer** ( $\lambda < 1$ )

case	$U = \omega R$	$W$	$F_n$	$Stress$	$torque$	$potential\ energy$	$kinetic\ energy$
<i>blocked</i>	0	$= V_{fluid}$	<i>max</i>	<i>max</i>	<i>null</i>	<i>max</i>	<i>null</i>
<i>free</i>	$\nearrow$	$\searrow$	$\searrow$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

As soon as the turbine is free, the stresses decrease and the torque increases.

### **Horizontal axis wind turbines ( $\lambda > 1$ ) (HAWT)**

case	$U = \omega R$	$W$	$F_n$	$Stress$	$torque$	$potential\ energy$	$kinetic\ energy$
<i>blocked</i>	0	$= V_{fluid}$	<i>min</i>	<i>min</i>	<i>null</i>	<i>min</i>	<i>null</i>
<i>free</i>	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

As soon as the turbine is free, the stresses increase and the torque increases.

Table 1 : Energy variations between the blocked and rotating turbine situations.

In the case of an anemometer, the kinetic energy varies in a direction opposite to the potential energy. This is not the case for HAWT turbines.

In the case of HAWT turbines, we must not only consider the speed of the fluid. The kinetic energy and potential energy are related to the fluid speed and to the tangential rotation speed  $U$ . The direction of  $U$  is perpendicular to the direction of the fluid.

#### **3.1.1 Low speed turbines $\lambda < 1$**

Using the example of an anemometer with cups, when there is no resisting torque, the turbine rotates at maximum speed ( $\omega R = V_{fluid}$   $W = 0$ ), there is no energy production and the kinetic energy is at maximum. When the turbine is blocked, the stress on the turbine is at maximum ( $W = V_{fluid}$ ) and the speed of rotation is null. The potential energy is at maximum.

To obtain energy production, both kinetic and potential energy are required. If the kinetic energy is higher than the potential energy, the energy production is limited by the potential energy.

Similarly, if the potential energy is higher than the kinetic energy, the energy production is limited by this latter.

There can be no production of kinetic energy without potential energy.

$$E_{kin-max-productive} \leq \min( E_{kin} , E_{pot} )$$

If the variation in kinetic energy increases, the variation in potential energy decreases and vice versa

$$\frac{dE_{kin}}{dt} \nearrow \frac{dE_{pot}}{dt} \searrow \quad or \quad \frac{dE_{kin}}{dt} \searrow \frac{dE_{pot}}{dt} \nearrow$$

Energy production is at its maximum when the kinetic energy is equal to the potential energy. The total energy is equal to the sum of the kinetic energy and the potential energy.

$$E_{total} = E_{kin} + E_{pot}$$

Conservation of energy ( ref. 6 of appendix A ) can be applied and verified.

$$\frac{dE_{total}}{dt} = 0 \quad \rightarrow \quad \frac{dE_{kin}}{dt} = - \frac{dE_{pot}}{dt} \quad (4)$$

For a low speed, the kinetic energy of the wind is partially transformed into kinetic and potential energy.

$$E_{kwind} \Rightarrow E_{pot} \& E_{kin}$$



### 3.1.2 High speed turbines $\lambda > 1$

Using the example of a turbine with wings profiles, the variation of kinetic and potential energy increases simultaneously or decreases simultaneously

$$\frac{dE_{kin}}{dt} \nearrow \frac{dE_{pot}}{dt} \nearrow \quad \text{or} \quad \frac{dE_{kin}}{dt} \searrow \frac{dE_{pot}}{dt} \searrow$$

If we apply the conservation of energy  $\frac{dE_{kin}}{dt} = -\frac{dE_{pot}}{dt}$  This relationship is not verified, yet it is reality. It lacks the consideration of a parameter to be able to verify the conservation of energy

### 3.2 Consideration of a supplementary parameter

The analysis is based with temporal steps; Appendix B detail kinetic energy

The change of direction of the fluid is not done naturally, it is necessary to create constraints on the blades, in order to be able to make a change of direction to the fluid and to be able to create a driving torque

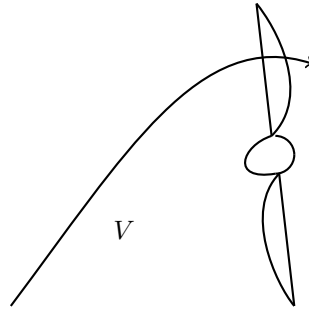


Fig. 4  
Change of fluid direction

The fluid is in motion and has kinetic energy. The fluid in contact with the turbine exerts a pressure field and creates stresses in it. These stresses allow the fluid to change direction. The change of direction of the fluid gives the possibility to create a driving torque.

These three successive phenomena can be interpreted as follows

The fluid ① creates stress on the surface swept ② by the turbine, then there is a change of direction ③ of the fluid .

- ① Kinetic energy of the fluid
- ② Stresses are created on the turbine
- ③ The change of direction of the fluid creates a driving torque to generate energy

Figure 5 is a visual schematic of these 3 steps.

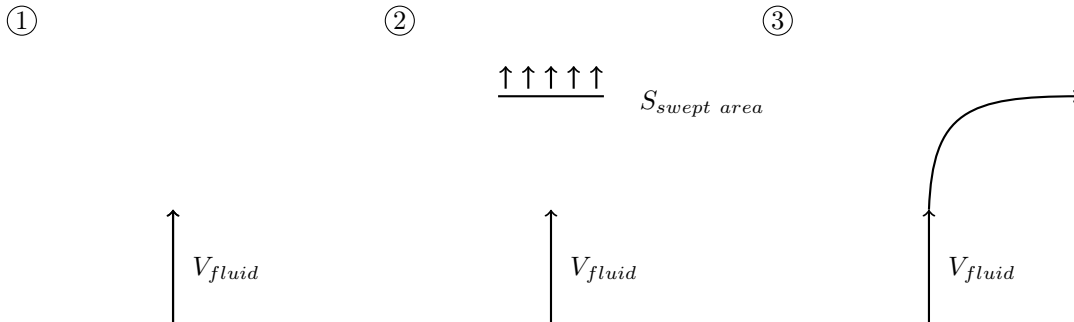
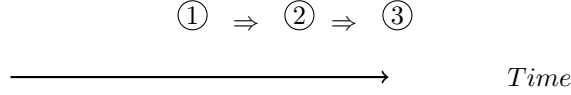


Fig. 5  
Symbolization of the 3 steps

Successive phenomena involve a notion of time.



In terms of energy, these transfers are

There is a transformation of energy from the kinetic energy of the fluid into potential energy, then into kinetic energy.

$$E_{kwind} \Rightarrow E_{pot} \Rightarrow E_{kin}$$

Figure 6 shows the energy transfers, taking instantaneous times into account.

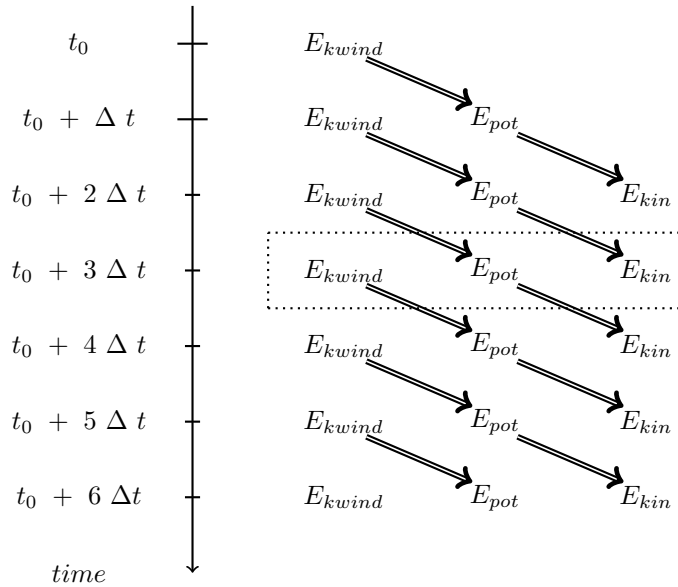


Fig. 6  
Energy transfert with temporal steps

As there is a transfer of temporal energy  $E_{kwind}(t - 2\Delta t) \Rightarrow E_{pot}(t - \Delta t) \Rightarrow E_{kin}(t)$ , in the case of an increase in the speed of the fluid, we have successively these energy variations

$$\frac{dE_{kwind}(t - 2\Delta t)}{dt} > 0 \Rightarrow \frac{dE_{pot}(t - \Delta t)}{dt} > 0 \Rightarrow \frac{dE_{kin}(t)}{dt} > 0 \quad (5)$$

In the paragraph 3.1, it was shown that there was a lack of respect for the conservation of energy. By introducing the temporal notion, the formulation 5 allows to be consistent with the conservation of energy.

For a sand yacht, if you set the sail directly to the right position, it will not move forward or it will take a long time to reach its optimum speed. If you tuck in the sail little by little to put it in the right position, the sand yacht has the right speed.

When stopped, the sail with the sail directly in the right position undergoes pressure forces due to the wind and the wind is forced to go around the obstacle which is the sail.

By gradually putting the sail in the right position, the tank picks up speed, the wind changes direction.

When stopped with the sail in the right position, the wind creates stresses on the sail and creates a force that tries to tilt.

By gradually putting the sail in the right position, the wind combined with the forward speed, creates an induced force which allows the sailboat to move forward and which also creates a rocking force.

In fact, if there is no stage step ② (constraints on the sail to create an induced force which allows the sand yacht to move forward), stage step ③ which allows the fluid to change direction does not perform. The notion of time is important.

In steady state, the induced force allows the sailboat to move forward (step ② ) and then allows the fluid to change direction (step ③). As step ③ is done, the step ② is maintained. We have a steady state of operation.

## 4 Conversion of potential energy into additional kinetic energy

There can be energy production only if we have kinetic energy and potential energy.

### 4.1 Conversion of potential energy

Potential energy is the source of stress in the turbine. It is possible to convert potential energy into kinetic energy

In the case of horizontal wind turbines (HAWT, fast wind turbine type), the stress in the blades for a defined wind speed, is constant.

$$\frac{d\sigma}{d\beta} = 0$$

With  $\sigma$  Stress in turbine blade ( $\frac{N}{m^2}$ ) and  $\beta$  rotation angle of the blades (*rad*) In fact, some variations of the stress exist due to gravitational forces and the differential velocity within the boundary layer depending on the elevation.

In the case of vertical axis turbines (VAWT, Darrieus type) the stress on the blades and arms depends on the rotation angle of the blades (for a given wind speed).

$$\frac{d\sigma}{d\beta} \neq 0 \quad \frac{1}{2\pi} \int_0^{2\pi} \sigma d\beta = \epsilon \quad (\epsilon \text{ small})$$

During a half-turn, the arms are submitted to compression stress whereas extending stress is dominant during the next half-turn.

In the case of a HAWT, the conversion is not possible with a dynamic mechanical system. The additional stress is constant during a rotation for a given wind speed.

Alternative stress, encountered in a vertical axis wind turbine VAWT can allow the extraction of additional energy.

### 4.2 Total recoverable power of the turbine

To determine the power, one selects the power defined from the kinetic energy and, in the case of conversion of stress into a mechanical movement, the power defined from the potential energy. It is possible to convert potential energy into kinetic energy.

The total recoverable power coefficient is

$$C_{total} = C_{kin} + C_{pot}$$

With  $C_p = 0$  when constraints are not converted

$$P_{total} = C_{total} \frac{1}{2} \rho S V_{fluid}^3 = (C_{kin} + C_{pot}) \frac{1}{2} \rho S V_{fluid}^3$$

In the case of horizontal wind turbines (HAWT, fast wind turbine type), the power coefficient  $C_{T \text{ HAWT}}$  is (reference equation 7 )

$$C_{Total\ HAWT} = C_{kin} = 4 a^2 (1 - a) \quad C_{pot} = 0$$

In the case of vertical axis turbines (VAWT, Darrieus type)  
the power coefficient  $C_{T\ VAWT\ Darrieus}$  is

$$C_{Total\ VAWT\ Darrieus} = C_{kin} = 4 a^2 (1 - a) \quad C_{pot} = 0$$

In the case of vertical axis turbines (VAWT with conversion), this stress is converted into additional energy.

Referring to the equations 6 of appendix A and to the equations 10 and 14 of appendix B ( $|\frac{dE_{kin}}{dt}| = |\frac{dE_{pot}}{dt}|$ ), whether for a slow or fast moving turbine, the optimum is when the kinetic power is equal to the potential power ( $E_{kin} = E_{pot}$ )

The power coefficient  $C_{T\ VAWT\ with\ conversion}$  is therefore equal to

$$C_{Total\ VAWT\ with\ conversion} = C_{kin} + C_{pot} = 8 a^2 (1 - a)$$

## 5 HAWT-VAWT comparison and discussion

Following the work of Hau (2000), the power coefficient of different turbines is compared (a performance of 0.6 is applied for the supplementary energy recovery system).

The perormance comparison (table 2) is made in 2 cases. The first when the optimum is obtained for high-speed turbines ( $\lambda > 1$ ), the second case for slow-speed turbines ( $\lambda < 1$ ).

First case : optimum for high-speed turbines

$$With\ a = \frac{2}{3} \quad C_{kin} \approx 60\% \quad C_{pot} \approx 60\%$$

notation  $C_{pot\ w.c} = C_{pot\ with\ conversion}$

case	Coef.	HAWT	VAWT Darrieus	VAWT with conversion
<i>perfect</i>	$C_{kin}$	60%	60%	60%
<i>perfect</i>	$C_{pot\ w.c}$	0%	0%	60%
<i>perfect</i>	$C_{total}$	= 60%	= 60%	= 120%
<i>in practice</i>	$C_{kin}$	$0.8 \times 60\% \approx 48\%$	$0.7 \times 60\% \approx 42\%$	$0.7 \times 60\% \approx 42\%$
<i>in practice</i>	$C_{pot\ w.c}$	0%	0%	$0.6 \times 0.7 \times 60\% \approx 25\%$
<i>in practice</i>	$C_{total}$	= 48%	= 42%	= 67%
.....				
gain/HAWT		+ 0%	-12 %	+ 39%
gain/ $C_{kin}$ (Betz)		- 20%	-30 %	+ 11%

Second case : optimum for slow-speed turbines

$$With\ a \approx 0.8 \quad C_{kin} \approx 50\% \quad C_{pot} \approx 50\%$$

case	Coef.	HAWT	VAWT Darrieus	VAWT with conversion
<i>in practice</i>	$C_{kin}$	$0.8 \times 50\% \approx 40\%$	$0.7 \times 50\% \approx 35\%$	$0.7 \times 50\% \approx 35\%$
<i>in practice</i>	$C_{pot\ w.c}$	0%	0%	$0.6 \times 0.7 \times 50\% \approx 21\%$
<i>in practice</i>	$C_{total}$	= 40%	= 35%	= 56%
.....				
gain / HAWT		+ 0%	-12 %	+ 27%
gain / $C_k$		- 20%	-30 %	+ 12%

Table 2 : Performance comparison

Figure 7 is a graphical representation of the power coefficients as a function of a (velocity factor).

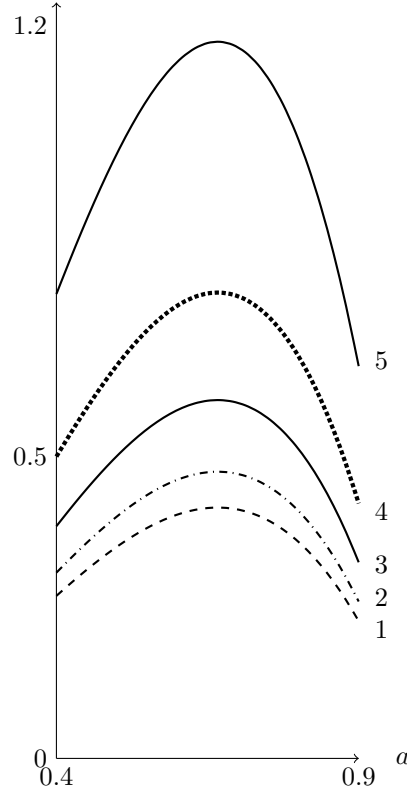


Fig. 7

Tip speed ratio performance curve;

Curve number	Coefficient name	Mathematic function
5	$C_T Maxi$	$1.0 (C_{kin} + C_{pot})$
4	$C_T VAWT with conversion$	$0.7 (C_{kin} + 0.6 C_{pot})$
3	$C_{kin Betz}$	$1.0 C_{kin}$
2	$C_T HAWT$	$0.8 C_{kin}$
1	$C_T VAWT Darrieus$	$0.7 C_{kin}$
	$C_{kin}$	$4 a^2 (1 - a)$
	$C_{pot}$	$4 a^2 (1 - a)$

Table 3 : legend for figure 7

## 5.1 The Active lift turbine project ALWT

The "Active lift turbine" project" is an example of transformation of potential energy into kinetic energy (see preprint : Simplified theory of an active lift turbine with controlled displacement (Lecanu et al. (2016)). The power coefficient is equal to

$$C_{active.lift.turbine} = (1 + \frac{e}{R}\lambda) (\frac{9\pi}{27}b^3 - \frac{2^2}{3}b^2 + \frac{\pi}{2}b)$$

$$with \quad b = \sigma_R \lambda \quad \sigma_R = \frac{Nc}{R} \quad \lambda = \frac{R\dot{\beta}}{V_{fluid}} \quad b \leq \frac{4}{3}$$

NB: When e =0, the power coefficient corresponds to that of a Darrieus turbine.

Figure 8 shows a graphical representation of the "rod and crank" conversion system.

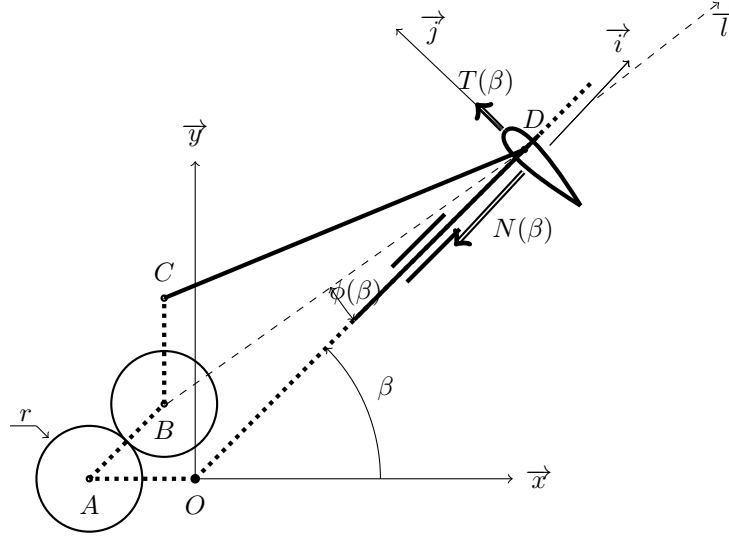


Fig. 8  
Schematization of the crank-rod system;

$$|\bar{OD}| = R(\beta) \quad |\bar{OA}| = e \quad |\bar{AB}| = 2r \quad |\bar{BC}| = r_b \quad |\bar{CD}| = L \quad |\bar{BD}| = K$$

The Active Lift Turbine (ALWT) is a horizontal turbine (HAWT) with a crank and connecting rod system that converts the normal forces on the airfoils into additional energy-recovery torque.

A new version is in preparation with a simplification of the mechanical system.

Numerical application

$$\text{if } e = 0.2 \text{ and } \sigma = 0.25$$

$$\frac{dC_{active.lift.turbine}}{db} = \frac{d[(1 + \frac{e}{R} \frac{b}{\sigma_R}) (\frac{9\pi}{27} b^3 - \frac{2^2}{3} b^2 + \frac{\pi}{2} b)]}{db} = 0 \quad b_1 \approx 0.84 \quad C_{active.lift.turbine} \approx .85$$

Figure 9 shows a graphical representation of the turbine power coefficient.

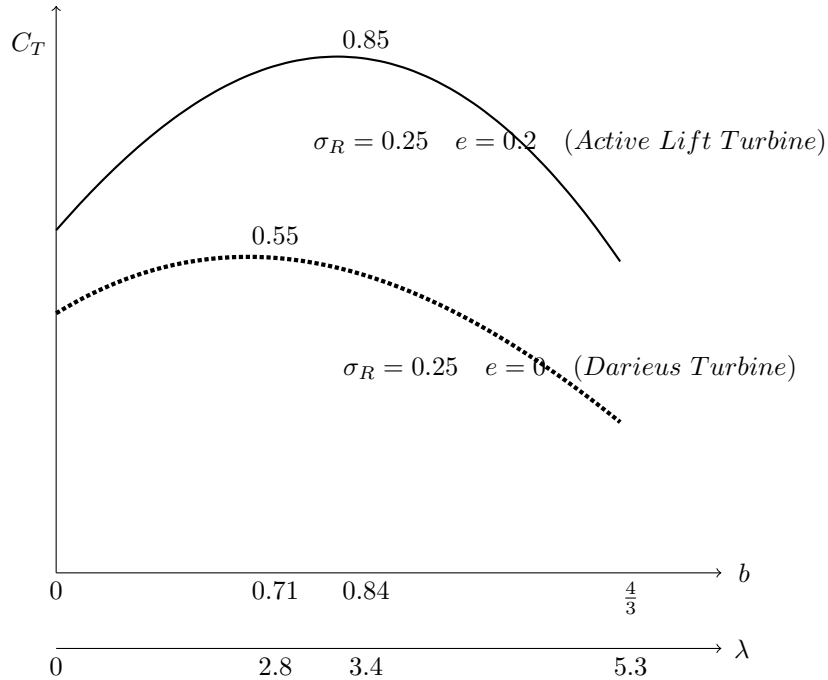


Fig. 9  
Power coefficient

With a large eccentricity value of  $e = 0.2$ , a gain of over 50% is obtained compared to a Darrieus-type turbine.

## 6 analysis & discussion

A graphical diagram illustrates the difference between low- and high-speed turbine energy processing.

### 6.1 Low speed turbine $\lambda < 1$ :

case :  $\lambda < 1$  — turbine type Anemometer with cups  $E_{kin\ wind} \Rightarrow E_{pot} \ \& \ E_{kin}$

For a constant wind speed, if the load of the electric generator is decreased, then we have **simultaneously** these energy variations :

$$\frac{dE_{pot}}{dt} < 0 \quad \& \quad \frac{dE_{kin}}{dt} > 0$$

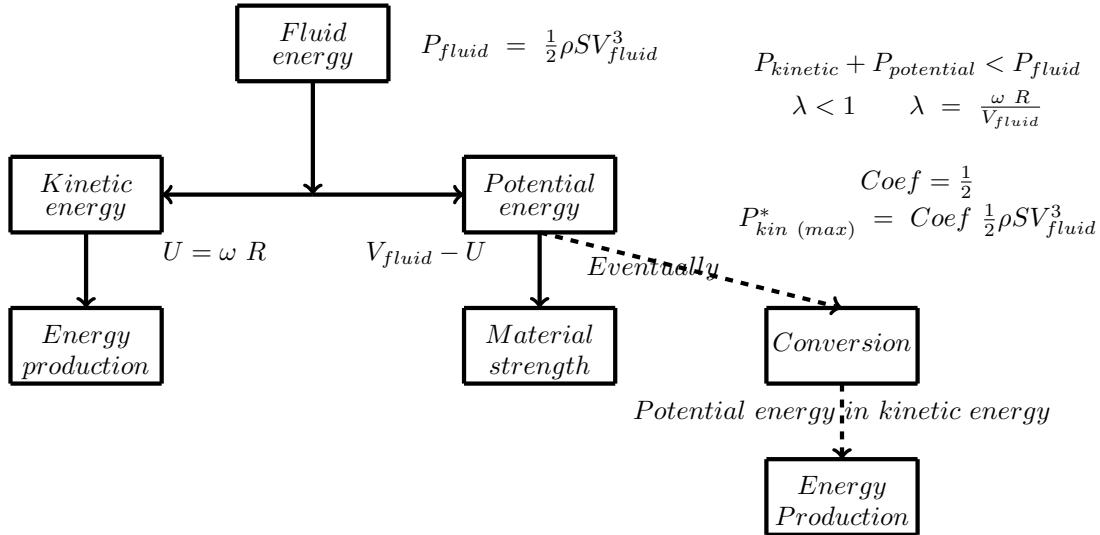


Fig. 10  
Energy schematization for slow-speed turbines

\* to obtain an energy production

### 6.2 Fast speed turbines HAWT or VAWT $\lambda > 1$ :

case :  $\lambda > 1$  — large wind turbines HAWT or VAWT  
Temporal transfer of energy  $E_{kwind} \Rightarrow E_{pot} \Rightarrow E_{kin}$

For a constant wind speed, if the load of the electric generator is decreased, then we have **successively** these energy variations :

$$\frac{dE_{pot}}{dt} > 0 \quad \Rightarrow \quad \frac{dE_{kin}}{dt} > 0$$

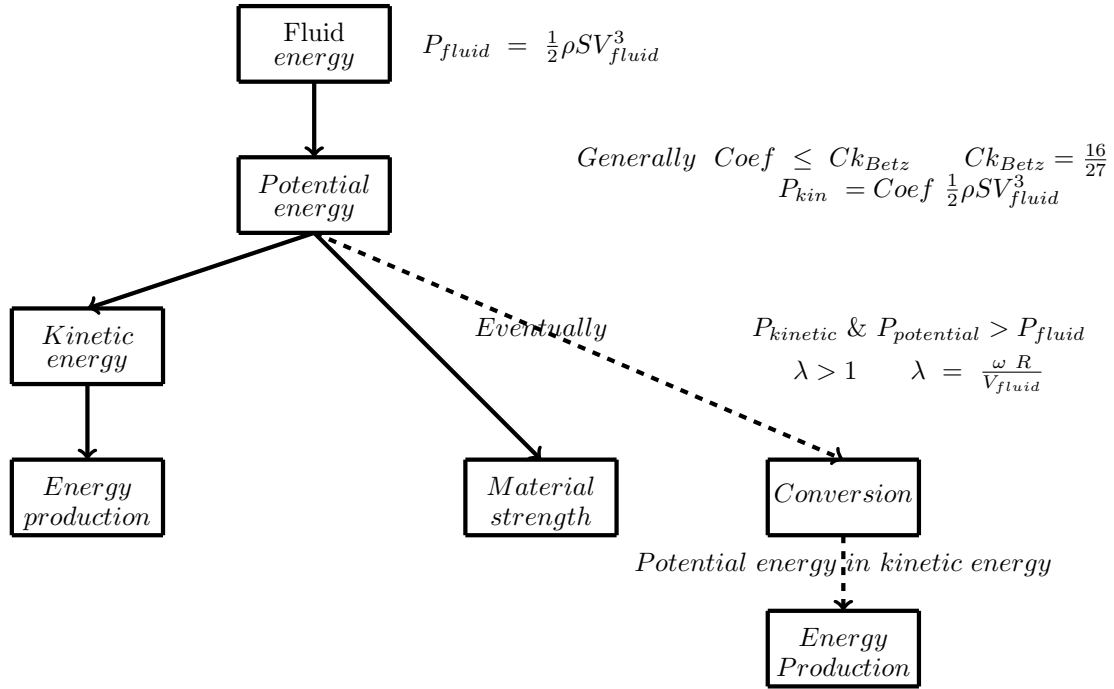


Fig. 11  
Energy schematization for high-speed turbines

In the case of a turbine with a lambda lower than 1, a part of the recoverable energy, is transformed into kinetic energy and the other part into potential energy (stress in the turbine). Eventually a part of this potential energy could be transformed into kinetic energy.

In the case of a turbine with a lambda higher than 1, the recoverable energy defined by Betz, is transformed into potential energy. This energy creates kinetic but also potential energy (stress in the turbine). This is possible because a rotation speed which is higher than the speed of the fluid is created. A part of this potential energy (due to the stress) can be transformed into kinetic energy.

### 6.3 Synthesis power with Boolean algebra:

The calculation of the total power recoverable to produce energy can be presented using Boolean algebra. [Boole (1854)]

Whatever type of turbine

$$(P_{k-max-productive} \text{ OR } P_{p-max-productive}) \leq \min(P_k, P_p)$$

For a low speed turbine  $\lambda < 1$

$$P_{Total-productive} \leq (P_{k-max-productive} \text{ OR } P_{p-max-productive})$$

For a fast speed turbines  $\lambda > 1$

$$P_{Total-productive} \leq (P_{k-max-productive} \text{ AND } P_{p-max-productive})$$

For a low speed turbine, the fluid stays in the same direction as the wind.

For a fast speed turbine, the fluid changes direction at 90 degrees to the wind direction.

In practice, the fluid never stays perfectly in the direction and does not change direction at exactly 90 degrees. The total power is a combination of the two types of turbine with more or less predominance depending on the turbine technology.



## 6.4 Synthesis on the extracted power:

In order to obtain productive kinetic energy, kinetic energy and potential energy are needed. The maximum productive kinetic power is equal to

$$P_{k-max-productive} \leq \min( P_k , P_p )$$

The limit of this depends on whether it is a low or high speed turbine

$$P_{kin(max)} = Coef \frac{1}{2} \rho S V_{fluid}^3$$

Where  $Coef = \frac{1}{2}$  for low speed turbines ( $\lambda < 1$ ) and  $Coef = \frac{16}{21}$  for high speed turbines ( $\lambda > 1$ )  
The maximum power coefficient for a wind or tidal turbine is

$$C_{Total\ maxi\ \lambda < 1} = \frac{100}{100} ( 100\% )$$

$$C_{Total\ maxi\ \lambda > 1} = \frac{32}{27} (\approx 118.5\%)$$

Using piezo-electric materials would make it possible to transform potential energy into electrical energy. However, in order to achieve higher efficiency, the potential energy should be transformed into kinetic energy. Thus, the power coefficient will be  $C_{total} = C_{kin} + C_{pot}$  whatever the type of turbine.

Compared to a HAWT turbine, the gain of a VAWT Turbine with an energy recovery system is in practice from 20% to 50%.

Concerning a vertical axis turbine with a conversion system, the power factor is higher than the one defined by Betz.

In the comparative table, a yield of 0.6 was chosen for the mechanical conversion system of the potential energy into mechanical energy. By choosing an efficient technology, this yield can be greatly increased, which will improve the performance of the turbine.

The definition of the maximum power coefficient is the one established by Betz which remains valid for horizontal axis turbines HAWT but not for vertical axis turbines VAWT.

The given results examined a conversion of the potential energy into kinetic energy through a mechanical system which is not applicable for horizontal axis turbines HAWT.

The calculation of the powers are the sum of the powers taken into account.

## 7 Conclusion

The article presents a new formulation for power calculation of wind or water turbines.

Taking into account the potential energy allows to establish that the conservation of energy is well verified. This was not the case in the formulation presented by Betz. Until now, potential energy has never been taken into account when defining turbine power coefficients.

A new power coefficient has been defined for the turbines depending on their type, low or high speed.

As described in the paragraph 5, a gain of 10, 20, 30% is possible. Obtaining much higher gains will require research and technical ingenuity. Currently, a study is underway with the Green River project for a wind turbine with a higher efficiency.

## 8 Acknowledgements

The authors gratefully acknowledge support of this work by CPIER – Région Normandie and ADEME under the Green River project. They would also like to thank Bertil Smorgrav and Guillaume Carpentier for proofreading this article.

# A : Appendix :Steady state, flow conservation and energy conservation

## A.1 Flow conservation

In steady state, the fluid velocity is constant in time:  $\frac{dV}{dt} = 0$

The continuity equation is used to investigate flow rates in incompressible fluid mechanics . It is given by the equation

$$S_1 V_1 = S_i V_i = S_2 V_2$$

the indices 1,2 and i indicate the different points to where the fluid is flowing, S is the area, and V is the corresponding velocity at the points 1 or 2 or i

## A.2 Energy, power and kinetic force

The kinetic force of the fluid can be determined from the kinetic energy The index *kin* for "kinetic" is used

$$E_{kin} = \frac{1}{2} m v^2 \quad P_{kin} = \frac{dE_{kin}}{dt} = \frac{1}{2} \frac{dm}{dt} v^2 + \frac{1}{2} m \frac{dv^2}{dt}$$
$$\frac{dv}{dt} = 0 \quad m = \rho s v dt \quad P_{kin} = \frac{1}{2} \rho s v^3 \quad F_{kin} = \frac{P_{kin}}{v} = \frac{1}{2} \rho s v^2$$

## A.3 Energy, power and potential force

Similarly, the pressure forces of the fluid can be determined from the potential energy.

The index *pot* for "potential" is used

$$E_{pot} = - m \frac{p}{\rho} \quad P_{pot} = \frac{dE_{pot}}{dt} = - \frac{dm}{dt} \frac{p}{\rho} + (-) m \frac{1}{\rho} \frac{dp}{dt}$$
$$\frac{dp}{dt} = 0 \quad m = \rho s v dt \quad P_{pot} = - s v p \quad F_{pot} = \frac{P_{pot}}{v} = - p s$$

## A.4 Energy conservation

James Prescott Joule Prescott Joule (1884) contributed to the definition of the conservation of energy through his work on the equivalence of work and heat. This was established more precisely by James Prescott Joule through his famous experiments of 1843-1847, in which a falling weight heats up the water in a calorimeter thanks to the friction of a weight-driven impeller in the water.

For a fluid in circulation without exchange of energy with the outside, the energy of the fluid can be converted into kinetic energy and potential energy.

$$E_{kin} + E_{pot} = E_{kin \ fluid}$$

Kinetic energy can be a driving energy to drive, for example, an electric generator. In case of variation of the motor resistive torque, the kinetic energy of the fluid is not modified and causes a variation of the kinetic energy of the fluid which results in a variation in the opposite direction of the potential energy (conservation of the energy)

The equation becomes

$$\Rightarrow \frac{dE_{kin}}{dt} = - \frac{dE_{pot}}{dt} \quad (6)$$

## B : Appendix : Calculation of energy

### B.1 Expression of the kinetic power of a turbine

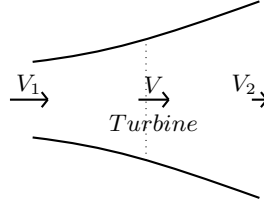


Fig. 12  
Current tube for a turbine

The force applied to the turbine is  $F_{kin} = \dot{m}\Delta V = \rho SV(V_1 - V_2)$

The power of this force is  $P_{kin} = F_{kin}V = \rho SV^2(V_1 - V_2)$

Power could be determined from kinetic energy  $P_{kin} = \frac{\Delta E_{kin}}{\Delta t} = \frac{1}{2}\rho SV^2(V_1^2 - V_2^2)$

It is deduced  $V = \frac{V_1 + V_2}{2}$

By defining  $a$  as  $a = \frac{V}{V_{fluid}}$   $V_{fluid} = V_1$   $0 \leq a \leq 1$   $V_2 = V_1(2a - 1)$   $V_2 \geq 0$   $a \geq \frac{1}{2}$

$$P_k = 4a^2(1-a)\frac{1}{2}\rho SV_{fluid}^3 \quad \text{with} \quad C_{kin} = 4a^2(1-a) \quad P_{kin} = C_{kin}\frac{1}{2}\rho SV_{fluid}^3 = \frac{C_{kin}}{a} P_{fluid} \quad (7)$$

### B.2 Calculation of energy $\lambda < 1$

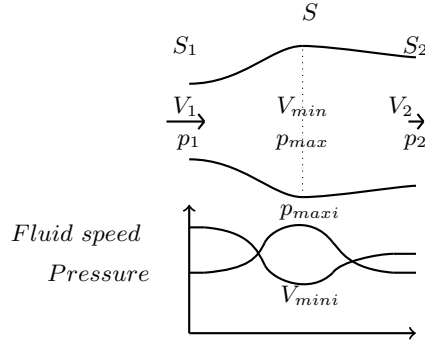


Fig. 13  
Current tube and evolution of speed and pressure

The figure is an example of a curve for pressure and velocity. The curves may be different for different types of technology, but pressure and velocity are continuous functions.

#### B.2.1 Recoverable kinetic power is limited

There can be no production of kinetic energy without potential energy.

$$E_{kin-max-productive} \leq \min( E_{kin} , E_{pot} ) \quad (8)$$

If the variation in kinetic energy increases, the variation in potential energy decreases and vice versa

$$\frac{dE_{kin}}{dt} \nearrow \frac{dE_{pot}}{dt} \searrow \quad \text{or} \quad \frac{dE_{kin}}{dt} \searrow \frac{dE_{pot}}{dt} \nearrow \quad (9)$$

Energy conservation requires

$$\frac{dE_{kin}}{dt} = - \frac{dE_{pot}}{dt}$$

In accordance with conditions 8 and 9 , the optimum is in the case of  $E_{kin} = E_{pot}$

The kinetic power of the turbine is defined by this formula 7

$$P_{kin} = 4 a^2 (1-a) \frac{1}{2} \rho SV_{fluid}^3$$

The possible values of  $a$  are for the solution of the equation  $4 a^2 (1 - a) = \frac{1}{2}$

$$a = 0.8090167 \quad a = \frac{1}{2} \quad a = -0.3090167$$

$a$  must be positive and in the case  $a = \frac{1}{2}$ ,  $V_2$  would be zero and  $S_2$  infinite. The only possible solution is  $a = 0.8090167$

$$P_{kin} = C_{kin} \frac{1}{2} \rho S V_{fluid}^3 \quad C_{kin} \leq \frac{1}{2}$$

In the case of  $a = 0.8090167$

$$P_{kin} = P_{pot} \tag{10}$$

The kinetic energy and the potential energy are equal to

$$E_{kin} = \frac{1}{2} m V^2 \quad E_{pot} = m \frac{p}{\rho} \quad \text{with} \quad m = \rho V S dt$$

$$P_{kin} = \frac{1}{2} \rho S V^3 \quad P_{pot} = p S V$$

For  $a = 0.8090167$

$$\frac{dE_{kin}}{dt} = - \frac{dE_{pot}}{dt} \tag{11}$$

$$\frac{E_{kin}}{dt} = P_{kin} = \frac{1}{2} \rho S V^3 \quad \frac{E_{pot}}{dt} = P_{pot} = - S V p$$

$$P_{kin} = P_{pot} = C \frac{1}{2} \rho S V_{fluid}^3 = \frac{C}{a} P_{fluid} \approx 0.618 P_{fluid}$$

$$C = C_{kin} = C_{pot} = 4 * a^2 (1 - a) = \frac{1}{2}$$

### B.3 Calculation of energy $\lambda > 1$

The pressure difference across the turbine can be schematized as follows

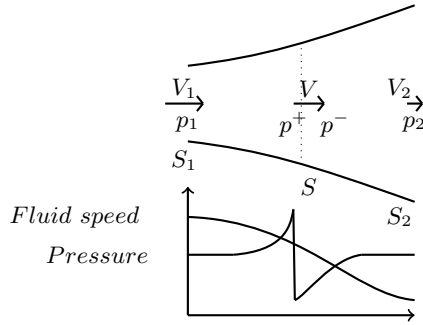


Fig. 14

Current tube and evolution of speed and pressure

#### B.3.1 Energy in stationary state applied in the turbine area

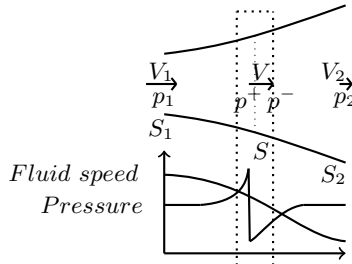


fig. 15

Turbine area

$$\begin{aligned}
E_{kin+} + E_{pot+} &= E_{kin-} + E_{pot-} = E_{pot} \\
\rho m \left[ \frac{V_+^2}{2} - \frac{V_-^2}{2} + \frac{p_+}{\rho} - \frac{p_-}{\rho} \right] &= E_{pot} \quad \text{with} \quad V_+ = V_- = V \\
\rho m \left[ \frac{p_+}{\rho} - \frac{p_-}{\rho} \right] &= E_{pot}
\end{aligned} \tag{12}$$

### B.3.2 Energy in stationary state applied to the whole turbine

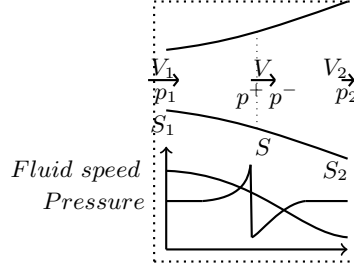


Fig. 16  
Turbine area

$$\begin{aligned}
E_{kin_1} + E_{pot_1} &= E_{kin_2} + E_{pot_2} + E_{kin} \\
\rho m \left[ \frac{V_1^2}{2} - \frac{V_2^2}{2} + \frac{p_1}{\rho} - \frac{p_2}{\rho} \right] &= E_{kin} \quad \text{with} \quad p_1 = p_2 \\
\rho m \left[ \frac{V_1^2}{2} - \frac{V_2^2}{2} \right] &= E_{kin}
\end{aligned} \tag{13}$$

### B.3.3 Bernoulli equations

$$\begin{aligned}
p_1 + \frac{1}{2}\rho s_1 V_1 &= p^+ + \frac{1}{2}\rho s V & p^- + \frac{1}{2}\rho s V &= p_2 + \frac{1}{2}\rho s_2 V_2 \\
V_1 = V_{fluid} \quad p_1 = p_2 &\rightarrow \frac{V_1^2}{2} - \frac{V_2^2}{2} &= \frac{p_+}{\rho} - \frac{p_-}{\rho}
\end{aligned}$$

by using the equations (12 and 13 ),

$$E_{pot} = E_{kin} \quad \frac{dE_{pot}}{dt} = \frac{dE_{kin}}{dt} \tag{14}$$

The energy variations are of the same sign and not in the opposite direction  
Kinetic power can be determined from kinetic energy and its expression 7 is

$$P_{kin} = 4 a^2 (1 - a) \frac{1}{2} \rho S V_{fluid}^3$$

Searching for the maximum kinetic power (Betz limit) :

$$\frac{dP_{kin}}{dt} = 0 \quad a(2 - 3a) = 0 \quad \text{as} \quad a \geq \frac{1}{2} \quad a = \frac{2}{3} \quad C_{kin-maxi} = \frac{16}{27} = C_{kin-Betz}$$

For  $a = \frac{2}{3}$

$$P_{kin} = P_{kin} = C \frac{1}{2} \rho S V_{fluid}^3 = \frac{C}{a} P_{fluid} = \frac{8}{9} P_{fluid}$$

$$C = C_{kin} = C_{pot} = 4 * a^2 (1 - a) = \frac{16}{27}$$

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