



HAL
open science

Protoperads I: Combinatorics and definitions

Johan Leray

► **To cite this version:**

| Johan Leray. Protoperads I: Combinatorics and definitions. 2019. hal-01982314

HAL Id: hal-01982314

<https://hal.science/hal-01982314>

Preprint submitted on 15 Jan 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

PROTOPERADS I: COMBINATORICS AND DEFINITIONS

JOHAN LERAY

ABSTRACT. This paper is the first of two articles which develop the notion of protoperads. In this one, we construct a new monoidal product on the category of reduced \mathfrak{S} -modules. We study the associated monoids, called *protoperads*, which are a generalization of operads. As operads encode algebraic operations with several inputs and one outputs, protoperads encode algebraic operations with the same number of inputs and outputs. We describe the underlying combinatorics of protoperads, and show that there exists a notion of free protoperad. We also show that the monoidal product introduced here is related to Vallette's one on the category of \mathfrak{S} -bimodules, via the induction functor.

INTRODUCTION

This is the first of two papers in which the author develops the notion of *protoperad*, which is a kind of properad (see [Val03, Val07]), and the homotopy theory of these new objects (see [Ler18]). Properad is an algebraic notion which encodes types of bialgebras, i.e. operations with several inputs and several outputs.

The motivation for this work is to determine what is a double Poisson bracket up to homotopy. Double Poisson structure, defined by Van den Bergh in [VdB08a], give a Poisson structure in noncommutative algebraic geometry (see [Gin05, VdB08b]) under the *Kontsevich-Rosenberg principle*, i.e. if A is a double Poisson algebra, then the family of affine representation schemes $\text{Rep}_n(A)$, represented by commutative algebras A_n , has a Poisson structure.

Berest, et al define derived representation schemes (see [BKR13, BCER12]) as the left derived functor $\mathbb{L}(-)_n$. A natural question is the following: what is a double Poisson structure compatible with the derived side of the Berest's construction? What is a double Poisson bracket up to homotopy? Double Poisson structure is *properadic* in nature. It is encoded by the properad \mathcal{DPois} (see [Ler18]); it gives us a good framework to study what is its version up to homotopy (cf. [Val03, MV09a, MV09b]). The structure of double Poisson algebras up to homotopy is controlled by a cofibrant resolution \mathcal{DPois}_∞ of the properad \mathcal{DPois} (cf. [MV09a, MV09b] for the model structure of the category of properads). Properads are algebraic objects which encode some algebraic structures, there are included in a large family of such objects:

Date: January 15, 2019.

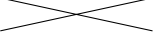



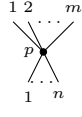
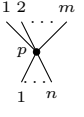
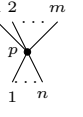

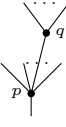
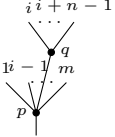
2010 Mathematics Subject Classification. 05E25,18D50,18G35,55U10.

Key words and phrases. Combinatorics, Species, Properad, Protoperad.

This article is the combinatorial part of the PhD thesis of the author, supported by the project "Nouvelle Équipe", convention n°2013-10203/10204 between La Région des Pays de Loire and the University of Angers. The author thanks the Centre Henri Lebesgue ANR-11-LABX-0020-01 for its stimulating mathematical research programs. This paper was finished at the University Paris 13, where the author was financed by a postdoctoral allocation given by DIM Math Innov. The author is indebted to G. Powell who has carefully read and corrected the first version of this paper.

Associative alg. \subset NS-Operads \subset Operads \subset Dioperads \subset Properads \subset Props.

Let V be a k -vector space: algebras encode algebraic structures with one input and one output $V \rightarrow V$; non-symmetric and (symmetric) operads encode algebraic structures with several input and one output $V^{\otimes m} \rightarrow V$; dioperads, properads and props encodes algebraic structures with several input and outputs $V^{\otimes m} \rightarrow V^{\otimes n}$. We can resume that by expliciting in which categories these objects live and there underlying combinatorics.

	Associative Algebras	NS-Operads	Operads	Dioperads	Properads	Props
Live in the category	\mathbf{Vect}_k	$\mathbf{N-mod}_k^{\text{red}}$	$\mathfrak{S-mod}_k^{\text{red}}$	$\mathfrak{S-bimod}_k^{\text{red}}$	$\mathfrak{S-bimod}_k^{\text{red}}$	$\mathfrak{S-bimod}_k$
Monoid for the product	\otimes_k	\circ^{ns}	\circ	$\boxtimes_{c, \emptyset}^{\text{Gan}}$	\boxtimes_c^{Val}	
Generators						
Composition controlled by				connected oriented graphs <i>without genus</i>	connected oriented graphs with genus	oriented graphs with genus
Example of encoded structures	Chain complexes	Associative algebras	Lie algebras	Involutive Lie bialgebras	Lie bialgebras	Loday infinitesimal bialgebras
A reference	[LV12, Chapter 1]	[LV12, Chapter 5]	[LV12, Chapter 5]	[Gan03]	[Val07]	[ML65]

For such an object \mathcal{P} , e.g. \mathcal{P} an operad, a properad, etc... , the notion of \mathcal{P} -algebra up to homotopy is given by cofibrant resolution of \mathcal{P} (see the Homotopy Tranfert Theorem for operads [LV12]). The homotopy theory of such objects is more or less complicated.

These successive algebraic structures are increasingly more complicate since they encode more and more type of algebras. As a consequence, their homotopy theory is also more and more elaborate. If props encode the largest category of algebraic structures, we do not yet have the same homotopical tools, like the Koszul duality theory, that hold for properads, operads, etc. The chain complex structures are encoded by the algebra of dual numbers. The non-symmetric operad framework is the minimal one to encode associative algebras, symmetric operad framework is the minimal one to encode associative commutative algebras, and so on: dioperads to encode bigebras without genus in the underlying combinatorics, as involutive Lie bialgebras.

In general, it is a hard problem to determine a usable (minimal) cofibrant resolution of a properad. However, if a (quadratic) properad \mathcal{P} is Koszul, (i.e. the homology of its bar construction is concentrated in a certain weight), then we have an explicit minimal resolution of \mathcal{P} : Vallette defines the Koszul duality for properads (see [Val03, Val07, MV09a]). Recall that Koszul duality theory does not exists for props. Unfortunately, it is also a hard problem to show that a properad is

Koszul. Almost all known examples of Koszul properads come from Koszulness results for operads. For operads, there are technical tools for showing that an operad is Koszul, including rewriting methods, PBW or Gröbner bases and distributive laws (see [Hof10, LV12, DK10]). The only other known example of a Koszul properad which does not come from an operadic result is the Frobenius properad (and its Koszul dual) (see [CMW16]).

A double Poisson bracket on an associative algebra A is a double Lie bracket with some compatibilities with the product of A . The properad \mathcal{DLie} , which encodes double Lie structure, is defined by generators and relations, i.e. $\mathcal{DLie} = \mathcal{F}(V_{\mathcal{DLie}})/\langle R_{\mathcal{DJ}} \rangle$, with $V_{\mathcal{DLie}}$ concentrated in arity $(2, 2)$:

$$V_{\mathcal{DLie}} = \begin{array}{c} 1 \quad 2 \\ \hline \hline \\ \hline 1 \quad 2 \end{array} \otimes \text{sgn}(\mathfrak{S}_2) \uparrow_{\mathfrak{S}_2^{op}}^{\mathfrak{S}_2 \times \mathfrak{S}_2^{op}}$$

and the relation

$$R_{\mathcal{DJ}} = \begin{array}{c} 1 \quad 2 \quad 3 \\ \hline \hline \\ \hline 1 \quad 2 \quad 3 \end{array} + \begin{array}{c} 2 \quad 3 \quad 1 \\ \hline \hline \\ \hline 2 \quad 3 \quad 1 \end{array} + \begin{array}{c} 3 \quad 1 \quad 2 \\ \hline \hline \\ \hline 3 \quad 1 \quad 2 \end{array} .$$

A double Lie bracket on a chain complex A is given by a morphism of properads $\mathcal{DLie} \rightarrow \text{End}_A$ where End_A is the properad of endomorphisms of A (see [Val07] for the definition). The operad $\mathcal{Pois} \cong \text{Com} \circ \mathcal{Lie}$, which encodes Poisson structures, is Koszul because the operads Com and \mathcal{Lie} , which respectively encode commutative and Lie algebras, are Koszul and satisfy a distributive law (see [LV12, Sect. 8.6.3]). We have an analogous statement for double Poisson structures: as the properad \mathcal{As} , which encodes associative algebras, is Koszul (see [LV12]), the properad $\mathcal{DPois} \cong \mathcal{As} \boxtimes_c^{\text{Val}} \mathcal{DLie}$ is Koszul if, and only if, the properad \mathcal{DLie} is Koszul (see [Ler18, Sect. 4]). As the study of the Koszulness of \mathcal{DLie} is still difficult, the idea is to use the diagonal symmetry of the generator and the relation of this properad to simplify the problem. In fact, the \mathfrak{S} -bimodules (which are families of representations of permutation groups) $V_{\mathcal{DLie}}$ and $R_{\mathcal{DJ}}$ are respectively induced from representations of the groups \mathfrak{S}_2 and \mathfrak{S}_3 , by the diagonal morphism

$$(1) \quad \begin{array}{ccc} G & \longrightarrow & G \times G^{op} \\ g & \longmapsto & g \times g^{-1} \end{array} ,$$

for G in $\{\mathfrak{S}_2, \mathfrak{S}_3\}$ respectively. This fundamental observation allows us to reduce the problem to one in the category of \mathfrak{S} -modules, and to define the minimal framework, called *protoperaad*, which encode double Lie algebras. We will see (in [Ler18]) that the homotopy theory of these new objects is more simpler than the homotopy of properads. We resume the most important results of this article in the following theorem.

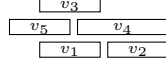
Theorem (see Def. 2.8, Thm. 4.16, Prop. 5.21). *The category $\mathfrak{S}\text{-mod}_k^{\text{red}}$ of reduced \mathfrak{S} -modules, i.e. the full sub-category of functors $P : \text{Fin}^{\text{op}} \rightarrow \text{Ch}_k$ such that $P(\emptyset) = 0$ is monoidal for the connected composition product \boxtimes_c . The monoids in this category are called **protoperaads**. There exists the free protoperaad functor, denoted by $\mathcal{F}(-)$ and the functor*

$$\text{Ind} : (\mathfrak{S}\text{-mod}_k^{\text{red}}, \boxtimes_c) \longrightarrow (\mathfrak{S}\text{-bimod}_k^{\text{red}}, \boxtimes_c^{\text{Val}})$$

which is exact and satisfies $\text{Ind} \circ \mathcal{F} = \mathcal{F}^{\text{Val}} \circ \text{Ind}$, where \mathcal{F}^{Val} is the functor of free protoperaad.

Using the framework of protoperads is successful: we prove in [Ler18] that there is a Koszul duality theory for protoperads which is compatible with that for properads via Ind, and we use it to prove that the properads \mathcal{DLie} and hence \mathcal{DPois} are Koszul.

SECTION 1 – BRICKS AND WALLS. We develop the combinatorics for protoperads. The combinatorics is controlled by *walls*. A wall over a non-empty finite set S is a set of subsets of S , equipped with a particular partial order and such that the union of these subsets is S . We represent a wall diagrammatically as follows:



the width of each brick giving us the number of entries and exits of each of the operations v_i which is represented. This is encoded by the functor $\mathcal{W}^{\text{conn}}$ and certain subfunctors. Let S be a non-empty finite set and n a natural number, a *wall of n bricks* over S is the datum of a set $W = \{W_\alpha\}_{\alpha \in A}$ of $|A| = n$ non-empty subsets of S , which are called *bricks of W* , satisfying:

- the union of all bricks is S ;
- for all element s of S , the set of bricks W_α which contains the element s is totally ordered
- a *compatibility between orders*.

We define also the notion of *connectedness* for a wall and denote by $\mathcal{W}^{\text{conn}}(S)$, the set of connected walls over S .

SECTION 2 – PRODUCTS ON \mathfrak{S} -MODULES. We review two monoidal products on $\mathfrak{S}\text{-mod}_k^{\text{red}} := \text{Func}(\text{Fin}^{\text{op}}, \text{Ch}_k)^{\text{red}}$ which is the full sub-category of functors $P : \text{Fin}^{\text{op}} \rightarrow \text{Ch}_k$ from the category Fin of finite sets with bijections, to the category Ch_k of k -chain complexes such that $P(\emptyset) = 0$. These product are *the composition product* \square , also called the Hadamard product, and *the concatenation product* \otimes^{conc} (see Section 2.2).

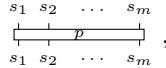
We also define the *connected composition product* on $\mathfrak{S}\text{-mod}_k^{\text{red}}$ (see Definition 2.8), denoted by \boxtimes_c , which encodes algebraic structures which have the same number of inputs and outputs and a diagonal symmetry. It is the bifunctor

$$- \boxtimes_c - : \mathfrak{S}\text{-mod}_k^{\text{red}} \times \mathfrak{S}\text{-mod}_k^{\text{red}} \longrightarrow \mathfrak{S}\text{-mod}_k^{\text{red}}$$

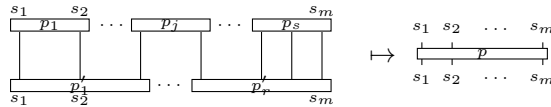
defined, for all reduced \mathfrak{S} -modules P and Q and for all non-empty finite sets S , by:

$$P \boxtimes_c Q(S) := \bigoplus_{(I, J) \in \mathcal{X}^{\text{conn}}(S)} \bigotimes_{\alpha} P(I_\alpha) \otimes \bigotimes_{\beta} Q(J_\beta).$$

Protoperads are the monoids for this monoidal structure. For a protoperad \mathcal{P} , we can view a homogeneous element p of $\mathcal{P}(S)$ as a labelled brick



and the product $\mathcal{P} \boxtimes_c \mathcal{P}(S) \rightarrow \mathcal{P}(S)$ as the composition of two rows of bricks which are connected:



SECTION 3 – CONNECTED PRODUCT ON \mathfrak{S} -BIMODULES. In this section, we also recall three monoidal structures on the category of reduced \mathfrak{S} -bimodules which are analogous to ones on the \mathfrak{S} -module category: the concatenation product, the composition product, and the connected composition product \boxtimes_c^{Val} , defined by Vallette in [Val03, Val07]. We do this review with an other point of view than the original one: we give an equivalent definition of the connected composition product, which is more adapted to *species* and the functorial point of view of \mathfrak{S} -bimodules.

SECTION 4 – INDUCTION FUNCTOR. The new product \boxtimes_c is the avatar of the product \boxtimes_c^{Val} on the category $\mathfrak{S}\text{-bimod}_k^{\text{red}}$. The most important property of the product \boxtimes_c is its compatibility with the product of Vallette via the induction functor $\text{Ind} : \mathfrak{S}\text{-mod}_k^{\text{red}} \rightarrow \mathfrak{S}\text{-bimod}_k^{\text{red}}$, defined using Equation (1). We prove the following.

Theorem (see Theorem 4.16). *The induction functor*

$$\text{Ind} : (\mathfrak{S}\text{-mod}_k^{\text{red}}, \boxtimes_c) \longrightarrow (\mathfrak{S}\text{-bimod}_k^{\text{red}}, \boxtimes_c^{\text{Val}})$$

is monoidal. In particular, it sends protoperads to properads,

SECTION 5 – PROTOPERADS. In this section, we give an equivalent definition of a protoperad, generalizing the definition of operads in terms of partial compositions.

Proposition (see Proposition 5.8). *A protoperad \mathcal{P} has canonically a partial composition system. Conversely, a partial composition system on a \mathfrak{S} -module P canonically extends to a protoperad structure.*

Using the work of Vallette on free monoids in abelian monoidal categories (see [Val09]), we show that there exists a free protoperad functor. We also have a combinatorial description of the free protoperad.

Theorem (see Theorem 5.21). *Let V be a reduced \mathfrak{S} -module. The free protoperad functor is the graded functor $\mathcal{F}^*(-)$ given, for all finite set S , and for all natural number ρ , by the isomorphism of right $\text{Aut}(S)$ -modules*

$$\mathcal{F}^\rho(V)(S) \cong \bigoplus_{\substack{\{W_\alpha\}_{\alpha \in A, \leq} \\ \in \mathcal{W}_\rho^{\text{conn}}(S)}} \bigotimes_{\alpha \in A} V(W_\alpha),$$

with $\mathcal{W}_\rho^{\text{conn}}(S)$, the set of connected walls with ρ bricks.

SECTION 6 – COLOURS ON WALLS. In [Ler18], the author develops the Koszul duality for protoperads, which is related to Koszul duality theory of properads (see [Val03, Val07]) via the functor Ind . In this last section, we define the notion of a coloured wall, and we associate to a wall W over a totally ordered set S , the *colouring complex*, denoted by $C_\bullet^{\text{col}}(W)$. This is motivated by the combinatorial description of the bar construction of the free protoperad (see also [Ler17, Ler18]). The principal result of this section is the following:

Theorem (see Theorem 6.15). *Let S be a finite totally ordered set, and W a wall over S . If the set $\text{Succ}(W)$ (see Section 1.1 for the definition of Succ) is non empty, then the colouring complex $C_\bullet^{\text{col}}(W)$ is acyclic.*

This theorem corresponds to the acyclicity result for bar construction of free protoperad (see [Ler18] for the definition of the bar construction of a protoperad).

CONTENTS

Notation	6
1. Bricks and walls	6
2. Products on \mathfrak{S} -modules	13
3. Connected product on \mathfrak{S} -bimodules	18
4. Induction functor	26
5. Protoperads	31
6. Colours on walls	45
References	53

NOTATION

We use the notation \mathbb{N}^* for the set $\mathbb{N} - \{0\}$. We denote by \mathbf{Fin} , the category with finite sets as objects and bijections as morphisms and \mathbf{Set} , the category of all sets and all applications. For two integers a and b , we denote by $\llbracket a, b \rrbracket$ the set $[a, b] \cap \mathbb{Z}$, and, for $n \in \mathbb{N}^*$, \mathfrak{S}_n is the automorphism group of $\llbracket 1, n \rrbracket$, i.e. $\mathfrak{S}_n = \text{Aut}_{\mathbf{Fin}}(\llbracket 1, n \rrbracket)$. We denote by \mathbf{Ch}_k the category of \mathbb{Z} -graded chain complexes over the field k .

Let (\mathbf{C}, \odot) be a monoidal category: we denote by $\mathcal{A}s(\mathbf{C}, \odot)$ the category of monoids without unit (e.g. semi-groups) in \mathbf{C} and $\mathcal{U}\mathcal{A}s(\mathbf{C}, \odot)$ the category of unital monoids in \mathbf{C} . If (\mathbf{C}, \odot) is symmetric monoidal, we also denote by $\mathcal{C}om(\mathbf{C}, \odot)$ the category of commutative monoids without unit and $\mathcal{U}\mathcal{C}om(\mathbf{C}, \odot)$ the category of commutative unital monoids in \mathbf{C} .

A monoidal category (\mathbf{C}, \odot, I) is an *abelian monoidal category* if \mathbf{C} is also abelian: we do not suppose any compatibility between the monoidal product \odot and the abelian structure.

1. BRICKS AND WALLS

We begin by describing the combinatorial framework of this paper. The first section is about posets and, after that, we introduce the *functor of walls*. Walls encode the combinatorics of "diagonal properads", as rooted trees govern the combinatorics of operads. In this section, we define two important functors:

$$\mathcal{X}^{\text{conn}} : \mathbf{Fin}^{\text{op}} \rightarrow \mathbf{Set}^{\text{op}} \quad \text{and} \quad \mathcal{W}^{\text{conn}} : \mathbf{Fin}^{\text{op}} \rightarrow \mathbf{Set}^{\text{op}} .$$

The first one, $\mathcal{X}^{\text{conn}}$, encodes the combinatorics of the new monoidal structure on the category of \mathfrak{S} -modules, *the connected composition product* (see Section 2.3). The second, $\mathcal{W}^{\text{conn}}$ encodes the combinatorics of the free monoid for this monoidal structure (see Theorem 5.21).

Remark 1.1. In this section, we construct (covariant) functors from the opposite category of finite sets to the opposite category of sets, i.e. $F : \mathbf{Fin}^{\text{op}} \rightarrow \mathbf{Set}^{\text{op}}$ or the category of chain complexes, i.e. $F : \mathbf{Fin}^{\text{op}} \rightarrow \mathbf{Ch}_k$. We choose to consider the opposite category of \mathbf{Fin} to get a right action of the automorphism group $\text{Aut}(S)$ on $F(S)$. This right action mimics the actions of symmetric groups on the leaves of trees in the operadic case.

1.1. Recollections on posets. Let k and l be two elements of a poset (K, \leq) . We say that k and l are *successors* if $k < l$ and if there does not exist an element t in K such that $k < t < l$. We denote by $\text{Succ}(K)$, the set of pairs of successors of K . A *chain* of a poset K is an increasing sequence of elements of K and the *length* of

the chain is the number of elements of the chain: we denote the length of a chain $k_1 < k_2 < \dots < k_r$ by $\text{len}(k_1 < k_2 < \dots < k_r)$. The *height* of an element k of a poset (K, \leq) is the element $\mathfrak{h}(k)$ of $\mathbb{N} \cup \{\infty\}$ defined by

$$\mathfrak{h}(k) := \max\{\text{len}(c) \in \mathbb{N}^* \mid c = (\lambda_1 < \lambda_2 < \dots < \lambda_{r-1} < k)\}.$$

Proposition 1.2. *Let (K, \leq) be a poset and (k, l) in $\text{Succ}(K)$. Then the surjection*

$$\pi_k^l : K \rightarrow K/k \sim l$$

induces a partial order on $K/k \sim l$ defined, for all r and s in K , by

- $[r] \leq [s]$ if $r \leq s$ and $r, s \notin \{k, l\}$
- $[s] \leq [k \sim l]$ (resp. $s \geq [k \sim l]$) if $s \leq k$ or $s \leq l$ (resp. $s \geq k$ or $s \geq l$).

Proof. Left to the reader. □

Lemma 1.3. *Let (R, \leq_R) and (S, \leq_S) be two posets with injections $R \hookrightarrow T \hookleftarrow S$. If, for all a and b in $R \cap S$, $a \leq_R b$ if and only if $a \leq_S b$, then $R \cup S$ has a canonical partial order which extends the partial orders \leq_R et \leq_S .*

Proof. For any x and y in $R \cup S$, we have $x \leq_{R \cup S} y$ if and only if one of the following assumption holds:

- x and y are in R and $x \leq_R y$;
 - x and y are in S and $x \leq_S y$;
 - x is in R , y is in S and there exists t in $R \cap S$ such that $x \leq_R t \leq_S y$.
-

1.2. The functors of walls. In the rest of this section, we define some (covariant) functors from the category Fin^{op} to the category Set^{op} , called *functors of walls*. Let \mathcal{F} be a functor of walls (see below for definitions) and S a finite set with n elements. An element W of $\mathcal{F}(S)$ should represent a morphism $\text{Hom}_{\mathcal{C}}(V^{\otimes n}, V^{\otimes n})$, for V a chain complex, with a diagonal action of \mathfrak{S}_n by permutations inputs and outputs at the same time.

Definition 1.4 (Functor of ordered walls \mathcal{W}^{or}). For n in \mathbb{N}^* , the covariant functor $\mathcal{W}_n^{\text{or}} : \text{Fin}^{\text{op}} \rightarrow \text{Fin}^{\text{op}}$ is given, for all finite set S , by

$$\mathcal{W}_n^{\text{or}}(S) := \left\{ (W = (W_1, \dots, W_n)) \left| \begin{array}{l} \cup_i W_i = S; \forall i \in [1, n], W_i \neq \emptyset; \\ \forall s, t \in S, \Gamma_s^W := \{W_i \mid s \in W_i\} \\ \text{is totally ordered (by } \leq_s \text{);} \\ \forall a, b \in \Gamma_s^W \cap \Gamma_t^W, a \leq_s b \Leftrightarrow a \leq_t b \end{array} \right. \right\}$$

where an element W of $\mathcal{W}_n^{\text{or}}(S)$ is a poset for \leq which is the induced partial order (cf. lemma 1.3) on $\cup_{s \in S} \Gamma_s^W = \{W_1, \dots, W_n\}$. We denote by (W, \leq) , the elements of $\mathcal{W}_n^{\text{or}}(S)$. The action of σ , an element of $\text{Aut}(S)$, on $((W_1, \dots, W_n), \leq)$ in $\mathcal{W}_n^{\text{or}}$ is induced by the canonical action on S ,

$$((W_1, \dots, W_n), \leq) \cdot \sigma = ((W_1 \cdot \sigma, \dots, W_n \cdot \sigma), \leq^\sigma)$$

where \leq^σ is induced by the total orders of the sets $\Gamma_s^{W \cdot \sigma} := \{W_i \cdot \sigma \mid s \in W_i \cdot \sigma\}$. The functor \mathcal{W}^{or} , defined by

$$\begin{array}{ccc} \mathcal{W}^{\text{or}} : \text{Fin}^{\text{op}} & \longrightarrow & \text{Sets}^{\text{op}} \\ S & \longmapsto & \coprod_{n \in \mathbb{N}^*} \mathcal{W}_n^{\text{or}}(S) \end{array} .$$

Remark 1.5. For all non-empty sets S , we have $\mathcal{W}_0^{\text{or}}(S) = \emptyset$. For all integers $n > 0$, the group \mathfrak{S}_n acts freely on $\mathcal{W}_n^{\text{or}}(S)$ by permuting the position of elements, i.e. for τ in \mathfrak{S}_n , we have $\tau \cdot ((W_1, \dots, W_n), \leq) = ((W_{\tau^{-1}(1)}, \dots, W_{\tau^{-1}(n)}), \leq)$. The partial order is the same because it doesn't depend of the W_i 's indexes.

Example 1.6. We consider $W_a = \{1, 2\}$, $W_b = \{3, 4\}$ and $W_c = \{2, 3\}$, three subset of $S = \llbracket 1, 4 \rrbracket$. Then, we have the following four elements of $\mathcal{W}_3^{\text{or}}(S)$:

- $((W_1, W_2, W_3), <^1)$ with $<^1$ given by $W_1 <^1 W_3$ and $W_2 <^1 W_3$;
- $((W_1, W_2, W_3), <^2)$ with $<^2$ given by $W_3 <^2 W_1$ and $W_3 <^2 W_2$;
- $((W_1, W_2, W_3), <^3)$ with $<^3$ given by $W_1 <^3 W_3$ and $W_3 <^3 W_2$;
- $((W_1, W_2, W_3), <^4)$ with $<^4$ given by $W_3 <^4 W_1$ and $W_2 <^4 W_3$.

This four elements of $\mathcal{W}_3^{\text{or}}(S)$ are distinct.

The *vertical composition product* on \mathcal{W}^{or} , is the natural transformation:

$$\mathcal{V} : (\mathcal{W}^{\text{or}} \times \mathcal{W}^{\text{or}})(-) \longrightarrow \mathcal{W}^{\text{or}}(-)$$

given, for all finite set S , by $\mathcal{V}_{n,m,S} : \mathcal{W}_n^{\text{or}}(S) \times \mathcal{W}_m^{\text{or}}(S) \longrightarrow \mathcal{W}_{m+n}^{\text{or}}(S)$ which sends the pair $((W, \leq_W), (L, \leq_L))$ on $(R = (W_1, \dots, W_n, L_1, \dots, L_m), \leq_W^L)$ where, for all s in S , the total order of the poset Γ_s^R is induced by the ones of Γ_s^W and Γ_s^L and by extension, for all W_i in Γ_s^W and all L_j in Γ_s^L , we have $W_i \leq_W^L L_j$. This product is associative, so, for all finite set S , we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{W}^{\text{or}}(S) \times \mathcal{W}^{\text{or}}(S) \times \mathcal{W}^{\text{or}}(S) & \xrightarrow{\mathcal{V} \times \text{id}} & \mathcal{W}^{\text{or}}(S) \times \mathcal{W}^{\text{or}}(S) \\ \text{id} \times \mathcal{V} \downarrow & \circlearrowleft & \downarrow \mathcal{V} \\ \mathcal{W}^{\text{or}}(S) \times \mathcal{W}^{\text{or}}(S) & \xrightarrow{\mathcal{V}} & \mathcal{W}^{\text{or}}(S). \end{array}$$

The *(horizontal) concatenation product* on \mathcal{W}^{or} , is the natural transformation between bifunctors:

$$\mathcal{H} : \mathcal{W}^{\text{or}}(-_1) \times \mathcal{W}^{\text{or}}(-_2) \longrightarrow \mathcal{W}^{\text{or}}(-_1 \amalg -_2)$$

given, for all finite sets S and T , by $\mathcal{H}_{n,m,S,T} : \mathcal{W}_n^{\text{or}}(S) \times \mathcal{W}_m^{\text{or}}(T) \longrightarrow \mathcal{W}_{m+n}^{\text{or}}(S \amalg T)$ which sends $((W, \leq_W), (L, \leq_L))$ to $(R = (W_1, \dots, W_n, L_1, \dots, L_m), \leq_{W,L})$ where, for all s in S and t in T , we have the equalities $\Gamma_s^R = \Gamma_s^W$ and $\Gamma_t^R = \Gamma_t^L$. This product is associative and commutative, so we have the following commutative diagrams:

$$\begin{array}{ccc} \mathcal{W}^{\text{or}}(-_1) \times \mathcal{W}^{\text{or}}(-_2) \times \mathcal{W}^{\text{or}}(-_3) & \xrightarrow{\mathcal{H} \times \text{id}} & \mathcal{W}^{\text{or}}(-_1) \times \mathcal{W}^{\text{or}}(-_2 \amalg -_3) \\ \text{id} \times \mathcal{H} \downarrow & \circlearrowleft & \downarrow \mathcal{H} \\ \mathcal{W}^{\text{or}}(-_1 \amalg -_2) \times \mathcal{W}^{\text{or}}(-_3) & \xrightarrow{\mathcal{H}} & \mathcal{W}^{\text{or}}(-_1 \amalg -_2 \amalg -_3) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{W}^{\text{or}}(-_1) \times \mathcal{W}^{\text{or}}(-_2) & \xrightarrow{\mathcal{H}} & \mathcal{W}^{\text{or}}(-_1 \amalg -_2) \\ \cong \downarrow & \circlearrowleft & \downarrow \cong \\ \mathcal{W}^{\text{or}}(-_2) \times \mathcal{W}^{\text{or}}(-_1) & \xrightarrow{\mathcal{H}} & \mathcal{W}^{\text{or}}(-_2 \amalg -_1). \end{array}$$

We also have the following commutative diagram of natural transformations, called *the interchanging law*:

$$\begin{array}{ccc}
 (\mathcal{W}^{\text{or}})^{\times 2}(-1) \times (\mathcal{W}^{\text{or}})^{\times 2}(-2) & \xrightarrow{\text{id} \times \sigma \times \text{id}} & (\mathcal{W}^{\text{or}}(-1) \times \mathcal{W}^{\text{or}}(-2))^{\times 2} \\
 \downarrow \nu \times \nu & & \downarrow \mathcal{H} \times \mathcal{H} \\
 \mathcal{W}^{\text{or}}(-1) \times \mathcal{W}^{\text{or}}(-2) & \xrightarrow{\mathcal{H}} & \mathcal{W}^{\text{or}}(-1 \amalg -2) \\
 & \searrow \mathcal{H} & \swarrow \nu \\
 & & \mathcal{W}^{\text{or}}(-1 \amalg -2)
 \end{array}$$

Pass to \mathcal{W} , the functor of unordered walls:

Definition 1.7 (Functor of walls \mathcal{W}). We define the functor $\mathcal{W}_n : \text{Fin}^{\text{op}} \rightarrow \text{Fin}^{\text{op}}$ by $\mathcal{W}_n := (\mathcal{W}_n^{\text{or}})_{\mathfrak{S}_n}$ which is given, for all finite sets S , by $\mathcal{W}_n(S) :=$

$$\left\{ \left(W = \{W_\alpha\}_{\alpha \in A}, \leq \right) \mid \begin{array}{l} |A| = n; \forall \alpha \in A, W_\alpha \neq \emptyset; \cup_\alpha W_\alpha = S; \\ \forall s \in S, \Gamma_s^W := \{W_\alpha \mid s \in W_\alpha\} \\ \text{is totally ordered (by } \leq_s) \\ \forall s, t \in S, \forall a, b \in \Gamma_s \cap \Gamma_t, a \leq_s b \Leftrightarrow a \leq_t b \end{array} \right\}$$

where \leq is the canonical partial order of $\cup_{s \in S} \Gamma_s^W = \{W_\alpha\}_{\alpha \in A}$. We have the natural projection $\pi : \mathcal{W}^{\text{or}} \rightarrow \mathcal{W}$. The action of σ , an element of $\text{Aut}(S)$, on $(\{W_\alpha\}_{\alpha \in A}, \leq) \in \mathcal{W}_n$ is induced by the canonical action on S , i.e.

$$(\{W_\alpha\}_{\alpha \in A}, \leq) \cdot \sigma = (W = \{W_\alpha \cdot \sigma\}_{\alpha \in A}, \leq^\sigma)$$

where \leq^σ is induced by total orders of $\Gamma_{s \cdot \sigma}^{W \cdot \sigma} := \{W_\alpha \cdot \sigma \mid s \cdot \sigma \in W_\alpha \cdot \sigma\}$. We also define the functor

$$\begin{array}{ccc}
 \mathcal{W} : \text{Fin}^{\text{op}} & \longrightarrow & \mathbf{Sets}^{\text{op}} \\
 S & \longmapsto & \coprod_{n \in \mathbb{N}^*} \mathcal{W}_n(S)
 \end{array}$$

An element W of $\mathcal{W}(S)$ is called a *wall* over S , and an element of a wall W is called a *brick* of W .

Proposition 1.8 (Products on \mathcal{W}). *The products \mathcal{V} and \mathcal{H} on \mathcal{W}^{or} pass through the quotient by the actions of the symmetric groups on the indexes of bricks, hence induce natural transformations*

$$\mathcal{V} : (\mathcal{W} \times \mathcal{W})(-) \longrightarrow \mathcal{W}(-) \quad \text{and} \quad \mathcal{H} : \mathcal{W}(-1) \times \mathcal{W}(-2) \longrightarrow \mathcal{W}(-1 \amalg -2),$$

respectively called the *composition product* and *concatenation product* on \mathcal{W} , such that we have the following commutative diagrams

$$\begin{array}{ccc}
 (\mathcal{W}^{\text{or}} \times \mathcal{W}^{\text{or}})(-) & \xrightarrow{\mathcal{V}} & \mathcal{W}^{\text{or}}(-) & \mathcal{W}^{\text{or}}(-1) \times \mathcal{W}^{\text{or}}(-2) & \xrightarrow{\mathcal{H}} & \mathcal{W}^{\text{or}}(-1 \amalg -2) \\
 \pi^{\times 2} \downarrow & & \downarrow \pi & \pi^{\times 2} \downarrow & & \downarrow \pi \\
 (\mathcal{W} \times \mathcal{W})(-) & \xrightarrow{\mathcal{V}} & \mathcal{W}(-) & \mathcal{W}(-1) \times \mathcal{W}(-2) & \xrightarrow{\mathcal{H}} & \mathcal{W}(-1 \amalg -2)
 \end{array}$$

$$\begin{array}{ccc}
 (\mathcal{W} \times \mathcal{W})(-1) \times (\mathcal{W} \times \mathcal{W})(-2) & \xrightarrow{\text{id} \times \sigma \times \text{id}} & \mathcal{W}(-1) \times \mathcal{W}(-2) \times \mathcal{W}(-1) \times \mathcal{W}(-2) \\
 \downarrow \nu \times \nu & & \downarrow \mathcal{H} \times \mathcal{H} \\
 \mathcal{W}(-1) \times \mathcal{W}(-2) & \xrightarrow{\mathcal{H}} & \mathcal{W}(-1 \amalg -2) \\
 & \searrow \mathcal{H} & \swarrow \mathcal{V} \\
 & & \mathcal{W}(-1 \amalg -2)
 \end{array}$$

1.3. Two subfunctors of \mathcal{W} . We introduce here two important subfunctors of \mathcal{W} . For all finite non-empty sets S and all n in \mathbb{N}^* , we define the functor of ordered partitions $\mathcal{Y}_n^{\text{or}} : \text{Fin}^{\text{op}} \rightarrow \text{Fin}^{\text{op}}$, by

$$\mathcal{Y}_n^{\text{or}}(S) := \{ (K_1, \dots, K_n) \mid \coprod_{i=1}^n K_i = S; \forall i \in \llbracket 1, n \rrbracket, K_i \neq \emptyset \}$$

equipped with the natural injection $\mathcal{Y}_n^{\text{or}} \hookrightarrow \mathcal{W}_n^{\text{or}}$. By disjoint union, we also define the functor \mathcal{Y}^{or} by

$$\mathcal{Y}^{\text{or}} : \begin{array}{ccc} \text{Fin}^{\text{op}} & \longrightarrow & \text{Set}^{\text{op}} \\ S & \longmapsto & \coprod_{n \in \mathbb{N}^*} \mathcal{Y}_n^{\text{or}}(S) \end{array} .$$

Via the vertical composition, we have, for all finite sets S and all m and n in \mathbb{N}^* , the isomorphism:

$$\mathcal{Y}_m^{\text{or}}(S) \times \mathcal{Y}_n^{\text{or}}(S) \cong \left\{ (R = (K_1, \dots, K_m, L_1, \dots, L_n)) \left| \begin{array}{l} \coprod_i K_i = S = \coprod_j L_j; \\ \forall i \in \llbracket 1, m \rrbracket, K_i \neq \emptyset; \forall j \in \llbracket 1, n \rrbracket, L_j \neq \emptyset; \\ \forall s \in S, \exists ! i \in \llbracket 1, m \rrbracket, \exists ! j \in \llbracket 1, n \rrbracket \\ \quad \text{s.t. } \Gamma_s^R := \{K_i, L_j\} \text{ with } K_i \leq_s L_j \\ \forall s, t \in S, \forall a, b \in \Gamma_s^R \cap \Gamma_t^R, \\ \quad a \leq_s b \Leftrightarrow a \leq_t b \end{array} \right. \right\},$$

which gives us the natural injection $\mathcal{Y}_m^{\text{or}}(S) \times \mathcal{Y}_n^{\text{or}}(S) \hookrightarrow \mathcal{W}_{m+n}^{\text{or}}(S)$. Hence, we define, for all non-empty finite sets S , the functor \mathcal{X}^{or} of ordered pairs of partitions of finite sets, by

$$\mathcal{X}^{\text{or}}(S) := \coprod_{m, n \in \mathbb{N}^*} \mathcal{Y}_n^{\text{or}}(S) \times \mathcal{Y}_m^{\text{or}}(S).$$

equipped with the natural injection $\mathcal{X}^{\text{or}} \hookrightarrow \mathcal{W}^{\text{or}}$. This functor is important: it encodes the combinatorics of our new monoidal product, up to a property of connectedness (see Section 1.4)

The natural surjection $\pi : \mathcal{W}^{\text{or}} \rightarrow \mathcal{W}$ gives the following commutative diagrams of natural transformations:

$$\begin{array}{ccccc} \mathcal{Y}^{\text{or}} & \hookrightarrow & \mathcal{W}^{\text{or}} & \longleftarrow & \mathcal{X}^{\text{or}} \\ \downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow \\ \mathcal{Y} & \hookrightarrow & \mathcal{W} & \longleftarrow & \mathcal{X} \end{array},$$

where \mathcal{Y} (resp. \mathcal{X}) is the quotient of \mathcal{Y}^{or} (resp. \mathcal{X}^{or}) by the action of the symmetric group on the indexes of bricks. The concatenation product restricts to the subfunctors \mathcal{X} and \mathcal{Y} :

$$\begin{array}{ccccc} \mathcal{Y}(-1) \times \mathcal{Y}(-2) & \hookrightarrow & \mathcal{W}(-1) \times \mathcal{W}(-2) & & \mathcal{X}(-1) \times \mathcal{X}(-2) \hookrightarrow \mathcal{W}(-1) \times \mathcal{W}(-2) \\ \mu^{\text{conc}} \downarrow & \circlearrowleft & \downarrow \mathcal{H} & \text{and} & \mu^{\text{conc}} \downarrow \quad \circlearrowleft \quad \downarrow \mathcal{H} \\ \mathcal{Y}(-1 \amalg -2) & \hookrightarrow & \mathcal{W}(-1 \amalg -2) & & \mathcal{X}(-1 \amalg -2) \hookrightarrow \mathcal{W}(-1 \amalg -2) \end{array} .$$

1.4. Connected walls. Now, we introduce the notion of *connectedness* of a wall. Let $(W = \{W_\alpha\}_{\alpha \in A, \leq})$ be a wall in $\mathcal{W}(S)$. We define on W the *equivalence relation of connectedness* \sim^{conn} : for two elements a and b of A , we say $W_a \sim^{\text{conn}} W_b$ if there

exist an integer $n \geq 2$ and a sequence $W_0, W_1, \dots, W_{n-1}, W_n$ of elements of \mathcal{W} with $W_0 = W_a$ and $W_n = W_b$ such that, for all i in $\llbracket 0, n-1 \rrbracket$,

$$W_i \cap W_{i+1} \neq \emptyset \text{ and } (W_i, W_{i+1}) \in \text{Succ}(\mathcal{W}) \text{ or } (W_{i+1}, W_i) \in \text{Succ}(\mathcal{W}).$$

Definition 1.9 (Projection \mathcal{K}). We define the projection \mathcal{K} as follows: for a finite set S , we have

$$\begin{aligned} \mathcal{K}_S : \mathcal{W}(S) &\longrightarrow \mathcal{Y}(S) \subset \mathcal{W}(-) \\ W &\longmapsto \left\{ \bigcup_{B_\alpha \in \pi^{-1}([B])} B_\alpha \mid [B] \in \pi(W) \right\}, \end{aligned}$$

where π is the projection of W to its quotient by $\overset{\text{conn.}}{\sim}$.

We have the natural commutative diagram

$$\begin{array}{ccc} \mathcal{X}^{\text{C}} & \longrightarrow & \mathcal{W} \\ \mathcal{K} \downarrow & \circlearrowleft & \downarrow \mathcal{K} \\ \mathcal{Y}^{\text{C}} & \longrightarrow & \mathcal{W}. \end{array}$$

Lemma 1.10. *The projection \mathcal{K} is associative, i.e. the following diagram of natural transformation commutes:*

$$\begin{array}{ccc} \mathcal{Y}^{\times 3} & \xrightarrow{\mathcal{Y} \times \mathcal{K}} & \mathcal{Y}^{\times 2} \\ \mathcal{K} \times \mathcal{Y} \downarrow & \circlearrowleft & \downarrow \mathcal{K} \\ \mathcal{Y}^{\times 2} & \xrightarrow{\mathcal{K}} & \mathcal{Y} \end{array}.$$

Definition 1.11 (The functor $\mathcal{W}^{\text{conn}}$). We define the functor

$$\mathcal{W}_n^{\text{conn,or}} : \text{Fin}^{\text{op}} \longrightarrow \text{Fin}^{\text{op}}$$

by, for all non-empty set S , the fiber of $\mathcal{K}_S : \mathcal{W}^{\text{or}}(S) \rightarrow \mathcal{Y}^{\text{or}}(S)$ over the wall with one brick $\{S\}$, i.e. the subfunctor of $\mathcal{W}_n^{\text{or}}$ giving by $\mathcal{W}_n^{\text{conn,or}}(S) :=$

$$\left\{ (W, <_W) \in \mathcal{W}_n^{\text{or}} \mid \begin{array}{l} \forall \alpha, \beta \in \llbracket 1, n \rrbracket, \exists W_\alpha =: W_{i_0}, \dots, W_{i_{m-1}}, W_m := W_\beta \\ \text{s.t. } \forall j \in \llbracket 0, m-1 \rrbracket, W_{i_j} \in W, W_{i_j} \cap W_{i_{j+1}} \neq \emptyset \\ \text{and } (W_{i_j}, W_{i_{j+1}}) \text{ or } (W_{i_{j+1}}, W_{i_j}) \in \text{Succ}(W) \end{array} \right\}.$$

The natural surjection $\mathcal{W}^{\text{or}} \twoheadrightarrow \mathcal{W}$ gives us the subfunctor

$$\begin{array}{ccc} \mathcal{W}_n^{\text{conn,or}} & \longrightarrow & \mathcal{W}^{\text{or}} \\ \downarrow & & \downarrow \\ \mathcal{W}_n^{\text{conn}} & \longrightarrow & \mathcal{W} \end{array}$$

called *the functor of connected walls*: an element of $\mathcal{W}_n^{\text{conn}}(S)$ is called a *connected wall* on S .

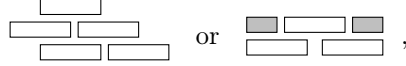
Remark 1.12. By the same arguments as in remark 1.3, we have the natural injection of $\mathcal{X}^{\text{conn}}$ in $\mathcal{W}_n^{\text{conn}}$.

Proposition 1.13. *Let W be a wall in $\mathcal{W}(S)$. Then, there exist n in \mathbb{N} and $S_1 \amalg \dots \amalg S_n$ a unique non-ordered partition of S such that*

$$W \in \text{im} \left(\mathcal{H} : \prod_{i=1}^n \mathcal{W}_n^{\text{conn}}(S_i) \longrightarrow \mathcal{W}(S) \right).$$

Proof. Let S be a finite set and W be in $\mathcal{W}(S)$, a wall over S . The partition $\mathcal{K}(W)$ in $\mathcal{Y}(S)$ gives the result. \square

Remark 1.14. *About the diagrammatic representation of walls.* The terminology introduced in definition 1.11 comes from the diagrammatic representation of the elements of $\mathcal{W}(S)$. Just as the combinatorics of operads is controlled by rooted trees (cf. [LV12, Sect. 5.6]), the combinatorics of protooperads is controlled by a stack of bricks:



the gray colour indicates a brick that is not connected in this representation. We have some examples.

- First, we consider the set $S = \llbracket 1, 4 \rrbracket$ and the wall $W = \{W_a, W_b, W_c\}$ in $\mathcal{W}(\llbracket 1, 4 \rrbracket)$ over S with the three bricks

$$W_a = \{1, 2\}, W_b = \{3, 4\} \text{ et } W_c = \{1, 2\}$$

with the partial order $W_a < W_c$. We represent this wall by



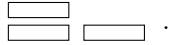
This wall is not connected: W is in the image of the product

$$\mathcal{H} : \mathcal{W}^{\text{conn}}(\llbracket 1, 2 \rrbracket) \times \mathcal{W}^{\text{conn}}(\llbracket 3, 4 \rrbracket) \rightarrow \mathcal{W}(\llbracket 1, 4 \rrbracket).$$

Note that this graphical representation of the wall W depends on the choice of an order on S : so, the diagrams



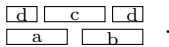
represent the same wall W . In the second diagram, the brick W_b is not connected. The choice of the natural order on $\llbracket 1, 4 \rrbracket$ corresponds to Equation (2). In cases where there is no ambiguity, we can omit the elements of S and the names of bricks to obtain the following diagram



- We consider the connected wall with four bricks $W = \{W_a, W_b, W_c, W_d\}$ in $\mathcal{W}^{\text{conn}}(\llbracket 1, 4 \rrbracket)$ with

$$W_a = \{1, 2\}, W_b = \{3, 4\}, W_c = \{2, 3\} \text{ and } W_d = \{1, 4\}$$

and the partial order $W_a < W_c$, $W_a < W_d$, $W_b < W_c$ and $W_b < W_d$. We represent this wall by



2. PRODUCTS ON \mathfrak{S} -MODULES

2.1. \mathfrak{S} -modules. Recall that the category \mathbf{Fin} is a groupoid which gives us the equivalence of categories $\mathbf{Fin} \cong \mathbf{Fin}^{\text{op}}$ by passage to the inverse. One of the key points of the constructions of this section is that $(\mathbf{Fin}, \mathbb{I})$ is a symmetric monoidal category. We denote by \mathfrak{S} its skeleton i.e. the category where objects are natural numbers, i.e. $\text{Ob } \mathfrak{S} = \mathbb{N}$ and where morphisms are given by $\text{Hom}_{\mathfrak{S}}(n, n) = \mathfrak{S}_n$ for $n \neq 0$ and $\text{Hom}(0, 0) = \{\text{id}\}$, and which is equivalent to \mathbf{Fin} .

Definition 2.1 (\mathfrak{S} -module, \mathfrak{S} -bimodule). A (right) \mathfrak{S} -*module* is an object of $\mathfrak{S}\text{-mod}_k \stackrel{\text{not.}}{:=} \text{Func}(\mathbf{Fin}^{\text{op}}, \text{Ch}_k)$, the category of contravariant functors from \mathbf{Fin} to the category of k -chain complexes. A \mathfrak{S} -*bimodule* is an object of $\mathfrak{S}\text{-bimod}_k \stackrel{\text{not.}}{:=} \text{Func}(\mathbf{Fin} \times \mathbf{Fin}^{\text{op}}, \text{Ch}_k)$.

Let V be a \mathfrak{S} -module (respectively W a \mathfrak{S} -bimodule). We say that V (resp. W) is *locally finite* if, for all finite sets S , $V(S)$ has finite total dimension (resp. for all finite sets S, E , the chain complex $W(S, E)$ has finite dimension).

As \mathfrak{S} is the skeletal category of \mathbf{Fin} , we can view an \mathfrak{S} -module M as a collection $(M(n))_{n \in \mathbb{N}^*}$ of chain complexes indexed by natural numbers, where the the group \mathfrak{S}_n acts (on the left) on $M(n)$, for $n \neq 0$. Similarly, an \mathfrak{S} -bimodule P is a collection of chain complexes $(P(m, n))_{m, n \in \mathbb{N}}$ indexed by pairs of integers where $P(m, n)$ has an action of \mathfrak{S}_m on the left and an action of \mathfrak{S}_n on the right, or equivalently, has an action of the group $\mathfrak{S}_m \times \mathfrak{S}_n^{\text{op}}$ on the left.

Definition 2.2 (Reduced \mathfrak{S} -(bi)module). A \mathfrak{S} -module (resp. \mathfrak{S} -bimodule) P which satisfies $P(\emptyset) = 0$ (resp. $P(\emptyset, S) = 0$ and $P(S, \emptyset) = 0$ for all finite set S) is called *reduced*. We respectively note by $\mathfrak{S}\text{-mod}_k^{\text{red}}$ and $\mathfrak{S}\text{-bimod}_k^{\text{red}}$, the full subcategories of $\mathfrak{S}\text{-mod}_k$ and $\mathfrak{S}\text{-bimod}_k$ of reduced \mathfrak{S} -modules and \mathfrak{S} -bimodules.

Remark 2.3. We have the equivalence of categories $\mathfrak{S}\text{-mod}_k \cong \mathfrak{S}^{\text{op}}\text{-mod}_k$, induced by taking the inverse of elements in symmetric groups. We use this equivalence without mention.

2.2. Composition and concatenation products on $\mathfrak{S}\text{-mod}_k$. In this subsection, we recall the classical constructions of composition and concatenation product of \mathfrak{S} -modules. The *composition product* (or *vertical product*) is the bifunctor

$$-\square- : \mathfrak{S}\text{-mod}_k^{\text{red}} \times \mathfrak{S}\text{-mod}_k^{\text{red}} \longrightarrow \mathfrak{S}\text{-mod}_k^{\text{red}}$$

defined, for P and Q two reduced \mathfrak{S} -modules and S a non-empty finite set, by

$$(P \square Q)(S) := P(S) \otimes Q(S).$$

This bi-additive bifunctor gives the category $\mathfrak{S}\text{-mod}^{\text{red}}$ a symmetric monoidal structure, with the identity I_{\square} , defined, for all non-empty sets S , by $I_{\square}(S) := k$ concentrated in degree 0. In the litterature or algebraic operads (cf. [LV12, Sect. 5.1.12]), the composition product of \mathfrak{S} -modules is also called the *Hadamard product*. The *concatenation product* is the bifunctor

$$-\otimes^{\text{conc}}- : \mathfrak{S}\text{-mod}_k^{\text{red}} \times \mathfrak{S}\text{-mod}_k^{\text{red}} \longrightarrow \mathfrak{S}\text{-mod}_k^{\text{red}}$$

defined, for all finite set S and all reduced \mathfrak{S} -modules P and Q , by:

$$(P \otimes^{\text{conc}} Q)(S) := \bigoplus_{\{S', S''\} \in \mathcal{Y}_2^{\text{or}}(S)} P(S') \otimes Q(S'').$$

This product has no identity.

Remark 2.4. This product is called the *concatenation product* because it corresponds to a concatenation of operations. It is functorial: it is a particular case of Day's convolution product. For P and Q two reduced \mathfrak{S} -modules, we have

$$P \otimes^{\text{conc}} Q(-) := \int^{(S', S'') \in \text{Ob}(\text{Fin}^{\text{op}})^{\times 2}} k[\text{Hom}_{\text{Fin}}(S' \amalg S'', -)] \otimes P(S') \otimes Q(S'').$$

Proposition 2.5. *The concatenation product is symmetric, i.e. for all reduced \mathfrak{S} -modules P and Q , we have the following isomorphism of \mathfrak{S} -modules*

$$P \otimes^{\text{conc}} Q \cong Q \otimes^{\text{conc}} P.$$

Proof. If (S', S'') is an element of $\mathcal{Y}_2^{\text{or}}(S)$, then (S'', S') too. As the monoidal product \otimes_k of the category Ch_k is symmetric, we have the natural equivalence τ defined as follows: for all finite sets S ,

$$\tau_S : (P \otimes^{\text{conc}} Q)(S) \xrightarrow{\cong} (Q \otimes^{\text{conc}} P)(S)$$

given by $\tau_S(p \otimes q) = (-1)^{|p||q|} q \otimes p$ for $p \otimes q$ in $P(S') \otimes^{\text{conc}} Q(S'')$. \square

We can extend the concatenation product:

$$(3) \quad - \otimes^{\text{conc}} - : \mathfrak{S}\text{-mod}_k \times \mathfrak{S}\text{-mod}_k^{\text{red}} \longrightarrow \mathfrak{S}\text{-mod}_k^{\text{red}}.$$

This extension is induced by the equivalence of categories

$$\mathfrak{S}\text{-mod}_k \cong \text{Ch}_k \times \mathfrak{S}\text{-mod}_k^{\text{red}},$$

by the injection $(-)^{\mathfrak{S}} : \text{Ch}_k \hookrightarrow \mathfrak{S}\text{-mod}_k$ defined, for all chain complexes C and all finite sets S , by

$$(C)^{\mathfrak{S}}(S) := \begin{cases} C & \text{if } S = \emptyset, \\ 0 & \text{sinon;} \end{cases}$$

and by the action of the category Ch_k on $\mathfrak{S}\text{-mod}_k^{\text{red}}$ defined, for all chain complexes C and all finite sets S , by

$$(C \otimes^{\text{conc}} V)(S) := C \otimes V(S).$$

This extension allows us to define the suspension of a \mathfrak{S} -module.

Definition 2.6 (Suspension and desuspension of a \mathfrak{S} -module). Let Σ (respectively Σ^{-1}) be the chain complex k concentrated in degree 1 (resp. in degree -1). For V a reduced \mathfrak{S} -module, the *suspension of V* (resp. *desuspension of V*) is the reduced \mathfrak{S} -module $\Sigma V \stackrel{\text{not.}}{:=} \Sigma \otimes^{\text{conc}} V$ (resp. $\Sigma^{-1} V \stackrel{\text{not.}}{:=} \Sigma^{-1} \otimes^{\text{conc}} V$).

2.2.1. *Free monoids associated to \otimes^{conc} .* Recall that the bifunctor $- \otimes^{\text{conc}} -$ is linear in each of its inputs. We have the functor:

$$\mathbb{T}_{\otimes}(-) : \mathfrak{S}\text{-mod}_k^{\text{red}} \longrightarrow \mathcal{A}s(\mathfrak{S}\text{-mod}_k^{\text{red}}, \otimes^{\text{conc}})$$

defined, for all finite sets S and all reduced \mathfrak{S} -modules P , by

$$(\mathbb{T}_{\otimes} P)(S) := \bigoplus_{r \in \mathbb{N}^*} (\mathbb{T}_{\otimes}^r P)(S)$$

with

$$(\mathbb{T}_{\otimes}^r P)(S) := P^{\otimes^{\text{conc}} r}(S) = \bigoplus_{I \in \mathcal{Y}_r^{\text{or}}(S)} P(I_1) \otimes \dots \otimes P(I_r).$$

As \otimes^{conc} is symmetric, we have also the *commutative free monoid functor*

$$\mathbb{S}(-) : \mathfrak{S}\text{-mod}_k^{\text{red}} \longrightarrow \text{Com}(\mathfrak{S}\text{-mod}_k^{\text{red}}, \otimes^{\text{conc}}).$$

defined, for all reduced \mathfrak{S} -modules P , by:

$$\mathbb{S}(P) := \bigoplus_{b \in \mathbb{N}^*} \mathbb{S}^b(P) \text{ with } \mathbb{S}^b(P) := (\overline{\mathbb{T}}_{\otimes}^b(P))_{\mathfrak{S}_b}$$

where the action of \mathfrak{S}_b is given by the symmetry τ of the product \otimes^{conc} : explicitly, for $p_1 \otimes \cdots \otimes p_b$ an element of $P(I_1) \otimes^{\text{conc}} \cdots \otimes^{\text{conc}} P(I_b)$ which is included in $(\overline{\mathbb{T}}_{\otimes}^b P)(S)$, and σ , a permutation in \mathfrak{S}_b , we have

$$\sigma \cdot (p_1 \otimes \cdots \otimes p_b) := (-1)^{|\sigma(m)|} p_{\sigma^{-1}(1)} \otimes \cdots \otimes p_{\sigma^{-1}(b)}$$

which lives in $P(I_{\sigma^{-1}(1)}) \otimes^{\text{conc}} \cdots \otimes^{\text{conc}} P(I_{\sigma^{-1}(b)})$ with $(-1)^{|\sigma(m)|}$ the Koszul sign induced by τ .

Notation 2.7. Let S be a finite set and P be a reduced \mathfrak{S} -module, as in [LV12, Sect. 5.1.14], we use the following notation

$$\bigoplus_{I \in \mathcal{Y}_r(S)} \bigotimes_{\alpha \in A} P(I_\alpha) \stackrel{\text{not.}}{:=} \left(\bigoplus_{I \in \mathcal{Y}_r^{\text{pr}}(S)} P(I_1) \otimes \cdots \otimes P(I_r) \right)_{\mathfrak{S}_r}.$$

Let S be a finite set and P and Q be two reduced \mathfrak{S} -modules. Since

$$(\mathbb{S}^r P)(S) := \left((\overline{\mathbb{T}}^r P)(S) \right)_{\mathfrak{S}_r} \cong \left(\bigoplus_{I \in \mathcal{Y}_r^{\text{pr}}(S)} P(I_1) \otimes \cdots \otimes P(I_r) \right)_{\mathfrak{S}_r}$$

then, we have

$$(\mathbb{S}P)(S) \cong \bigoplus_{\{I_\alpha\}_{\alpha \in A} \in \mathcal{Y}(S)} \bigotimes_{\alpha \in A} P(I_\alpha).$$

We also have the following isomorphism:

$$(\mathbb{S}P \square \mathbb{S}Q)(S) \cong \bigoplus_{(\{I_\alpha\}_{\alpha \in A}, \{J_\beta\}_{\beta \in B}) \in \mathcal{X}(S)} \bigotimes_{\alpha \in A} P(I_\alpha) \otimes \bigotimes_{\beta \in B} Q(J_\beta).$$

Moreover, as the bifunctor $-\otimes^{\text{conc}}-$ is biadditive, the functor \mathbb{S} has the exponential property:

$$(4) \quad \mathbb{S}(P \oplus Q) \cong \mathbb{S}(P) \oplus \mathbb{S}(Q) \oplus \mathbb{S}(P) \otimes^{\text{conc}} \mathbb{S}(Q).$$

2.3. Connected composition product of \mathfrak{S} -modules. In this section, we define the new monoidal structure on the category of reduced \mathfrak{S} -modules, which is called the connected composition product. This monoidal structure is the analogue of the product defined by Vallette in [Val03, Val07].

Definition 2.8 (Connected composition product of \mathfrak{S} -modules). The *connected composition product* of reduced \mathfrak{S} -modules is the bifunctor

$$-\boxtimes_c - : \mathfrak{S}\text{-mod}_k^{\text{red}} \times \mathfrak{S}\text{-mod}_k^{\text{red}} \longrightarrow \mathfrak{S}\text{-mod}_k^{\text{red}}$$

defined, for all reduced \mathfrak{S} -modules P and Q and for all non-empty finite set S , by:

$$P \boxtimes_c Q(S) := \bigoplus_{(I, J) \in \mathcal{X}^{\text{conn}}(S)} \bigotimes_{\alpha} P(I_\alpha) \otimes \bigotimes_{\beta} Q(J_\beta).$$

Denote by I_{\boxtimes} , the \mathfrak{S} -module given by

$$I_{\boxtimes}(S) \stackrel{\text{def}}{=} \begin{cases} k & \text{if } |S| = 1 \\ 0 & \text{otherwise} \end{cases},$$

which is the unit of the product \boxtimes_c .

Below, we require a description of elements of $P \boxtimes_c Q(S)$: we represent by

$$(p_1 \otimes \dots \otimes p_r) \otimes (q_1 \otimes \dots \otimes q_s)$$

a class of $\bigotimes_{\alpha} P(I_{\alpha}) \otimes \bigotimes_{\beta} Q(J_{\beta})$ with (I, J) in $\mathcal{X}^{\text{conn}}(S)$. Recall that we identify $(p_1 \otimes \dots \otimes p_r) \otimes (q_1 \otimes \dots \otimes q_s)$ with

$$(-1)^{|\sigma(p)|+|\tau(q)|} (p_{\sigma^{-1}(1)} \otimes \dots \otimes p_{\sigma^{-1}(r)}) \otimes (q_{\tau^{-1}(1)} \otimes \dots \otimes q_{\tau^{-1}(s)})$$

for all permutations σ in \mathfrak{S}_r , τ in \mathfrak{S}_s and $(-1)^{|\sigma(p)|}, (-1)^{|\tau(q)|}$, the Koszul signs induced by these permutations.

Lemma 2.9. *The product \boxtimes_c is associative and, for all reduced \mathfrak{S} -modules A and B , the endofunctor*

$$\begin{array}{ccc} \Phi_{A,B} : \mathfrak{S}\text{-mod}_k^{\text{red}} & \longrightarrow & \mathfrak{S}\text{-mod}_k^{\text{red}} \\ X & \longmapsto & A \boxtimes_c X \boxtimes_c B \end{array}$$

is split analytic (in the sense of [Val09]).

Proof. The associativity of the product \boxtimes_c follow from the associativity of \mathcal{K}_S : for P, Q and R three reduced \mathfrak{S} -modules, and S a non-empty set, we have the following isomorphism of chain complexes:

$$\begin{aligned} ((P \boxtimes_c Q) \boxtimes_c R)(S) &= \bigoplus_{(I,J) \in \mathcal{X}^{\text{conn}}(S)} \bigotimes_{\alpha} (P \boxtimes_c Q)(I_{\alpha}) \otimes \bigotimes_{\beta} R(J_{\beta}) \\ &= \bigoplus_{(I,J) \in \mathcal{X}^{\text{conn}}(S)} \bigotimes_{\alpha} \bigoplus_{\substack{(K^{\alpha}, L^{\alpha}) \\ \in \mathcal{X}^{\text{conn}}(I_{\alpha})}} \bigotimes_{\gamma} P(K_{\gamma}^{\alpha}) \otimes \bigotimes_{\delta} Q(L_{\delta}^{\alpha}) \otimes \bigotimes_{\beta} R(J_{\beta}) \\ &\cong \bigoplus_{(I,J) \in \mathcal{X}^{\text{conn}}(S)} \bigoplus_{\substack{\{(K^{\alpha}, L^{\alpha})\} \\ \in \coprod_{\alpha \in A} \mathcal{X}^{\text{conn}}(I_{\alpha})}} \bigotimes_{\gamma, \alpha} P(K_{\gamma}^{\alpha}) \otimes \bigotimes_{\delta, \alpha} Q(L_{\delta}^{\alpha}) \otimes \bigotimes_{\beta} R(J_{\beta}) \\ &\cong \bigoplus_{(I,J) \in \mathcal{X}^{\text{conn}}(S)} \bigoplus_{\substack{(K,L) \in \mathcal{Y}(S) \times \mathcal{Y}(S) \\ \mathcal{K}_S(K,L)=I}} \bigotimes_{\gamma, \alpha} P(K_{\gamma}^{\alpha}) \otimes \bigotimes_{\delta, \alpha} Q(L_{\delta}^{\alpha}) \otimes \bigotimes_{\beta} R(J_{\beta}) \\ &\cong \bigoplus_{\substack{(K,L,J) \in \mathcal{Y}(S)^{\times 3} \\ \mathcal{K}_S(\mathcal{K}_S(K,L), J)=(S)}} \bigotimes_{\gamma} P(K_{\gamma}) \otimes \bigotimes_{\delta} Q(L_{\delta}) \otimes \bigotimes_{\beta} R(J_{\beta}) \\ &\stackrel{(1.10)}{\cong} \bigoplus_{\substack{(K,L,J) \in \mathcal{Y}(S)^{\times 3} \\ \mathcal{K}_S(K, \mathcal{K}_S(L, J))=(S)}} \bigotimes_{\gamma} P(K_{\gamma}) \otimes \bigotimes_{\delta} Q(L_{\delta}) \otimes \bigotimes_{\beta} R(J_{\beta}) \\ &\cong \bigoplus_{\substack{(K,I) \\ \in \mathcal{X}^{\text{conn}}(S)}} \bigoplus_{\substack{(J,L) \in \mathcal{Y}(S) \times \mathcal{Y}(S) \\ \mathcal{K}_S(J,L)=I}} \bigotimes_{\gamma} P(K_{\gamma}) \otimes \bigotimes_{\delta, \alpha} Q(L_{\delta}^{\alpha}) \otimes \bigotimes_{\beta, \alpha} R(J_{\beta}^{\alpha}) \\ &\cong \bigoplus_{(K,I) \in \mathcal{X}^{\text{conn}}(S)} \bigotimes_{\alpha} P(K_{\gamma}) \otimes \bigotimes_{\alpha} (Q \boxtimes_c R)(I_{\alpha}) \\ &= (P \boxtimes_c (Q \boxtimes_c R))(S). \end{aligned}$$

Also, for all reduced \mathfrak{S} -modules A and B , the endofunctor $\Phi_{A,B}$ is well defined by

$$\begin{aligned}
 \Phi_{A,B}(X) &\cong \bigoplus_{\substack{(K,L,J) \in \mathcal{Y}(S)^{\times 3} \\ \mathcal{K}_S(\mathcal{K}_S(K,L),J)=(S)}} \bigotimes_{\gamma} A(K_{\gamma}) \otimes \bigotimes_{\delta} X(L_{\delta}) \otimes \bigotimes_{\beta} B(J_{\beta}) \\
 &\cong \bigoplus_{n \in \mathbb{N}} \bigoplus_{L \in \mathcal{Y}_n(S)} \bigoplus_{\substack{(K,J) \in \mathcal{Y}(S)^{\times 2} \\ \mathcal{K}_S(\mathcal{K}_S(K,L),J)=(S)}} \bigotimes_{\gamma} A(K_{\gamma}) \otimes \bigotimes_{\delta} X(L_{\delta}) \otimes \bigotimes_{\beta} B(J_{\beta}) \\
 &=: \bigoplus_{n \in \mathbb{N}} (\Phi_{A,B})_n(X).
 \end{aligned}$$

where $(\Phi_{A,B})_n$ is an homogeneous polynomial functor of degree n ; so, for all reduced \mathfrak{S} -modules A and B , the functor $\Phi_{A,B}$ is a split analytic functor (in the sense of [Val09, Sect. 4]). \square

Proposition 2.10. *The category $(\mathfrak{S}\text{-mod}^{\text{red}}, \boxtimes_c, \tau, I_{\boxtimes})$ is an abelian symmetric monoidal category that preserves reflexive coequalizers and sequential colimits.*

Proof. Let P and Q be two reduced \mathfrak{S} -modules, we have, for all non-empty finite sets S , the isomorphism of $\mathfrak{S}_{|S|}$ -modules

$$\begin{aligned}
 P \boxtimes_c Q(S) &:= \bigoplus_{(I,J) \in \mathcal{X}^{\text{conn}}(S)} \bigotimes_{\alpha} P(I_{\alpha}) \otimes \bigotimes_{\beta} Q(J_{\beta}) \\
 &\cong \bigoplus_{(I,J) \in \mathcal{X}^{\text{conn}}(S)} \bigotimes_{\beta} Q(J_{\beta}) \otimes \bigotimes_{\alpha} P(I_{\alpha})
 \end{aligned}$$

by symmetry of \otimes of the category Ch_k ; since (I, J) is in $\mathcal{X}(S)$ if, and only if, (J, I) is in $\mathcal{X}(S)$, we have the isomorphism

$$P \boxtimes_c Q(S) \cong \bigoplus_{(J,I) \in \mathcal{X}^{\text{conn}}(S)} \bigotimes_{\beta} Q(J_{\beta}) \otimes \bigotimes_{\alpha} P(I_{\alpha}) \cong Q \boxtimes_c P(S).$$

We denote this isomorphism $\tau_{P,Q}(S) : P \boxtimes_c Q(S) \longrightarrow Q \boxtimes_c P(S)$, which gives us the symmetry of the product. The rest of the proof is similar to [Val09, Prop 13]. \square

We have the following compatibility between these products:

Proposition 2.11. *Let P and Q be two reduced \mathfrak{S} -modules. We have the following natural isomorphism of \mathfrak{S} -modules:*

$$\mathbb{S}(P \boxtimes_c Q) \cong \mathbb{S}P \square \mathbb{S}Q.$$

Proof. Let S be a finite set, we have

$$\begin{aligned}
\mathbb{S}(P \boxtimes_c Q)(S) &= \bigoplus_{\Lambda \in \mathcal{Y}(S)} \bigotimes_{\gamma} (P \boxtimes_c Q)(\Lambda_{\gamma}) \\
&\stackrel{2.8}{=} \bigoplus_{\Lambda \in \mathcal{Y}(S)} \bigotimes_{\gamma} \bigoplus_{(I_{\gamma}, J_{\gamma}) \in \mathcal{X}^{\text{conn}}(\Lambda_{\gamma})} \bigotimes_{\alpha} P(I_{\alpha}^{\gamma}) \otimes \bigotimes_{\beta} Q(J_{\beta}^{\gamma}) \\
&\cong \bigoplus_{\Lambda \in \mathcal{Y}(S)} \bigoplus_{\{(I_{\gamma}, J_{\gamma})\} \in \coprod_{\gamma} \mathcal{X}^{\text{conn}}(\Lambda_{\gamma})} \left(\bigotimes_{\alpha, \gamma} P(I_{\alpha}^{\gamma}) \otimes \bigotimes_{\beta, \gamma} Q(J_{\beta}^{\gamma}) \right) \\
&\cong \bigoplus_{\Lambda \in \mathcal{Y}(S)} \bigoplus_{(\tilde{I}, \tilde{J}) \in \mathcal{K}_{\mathbb{S}}^{-1}(\Lambda)} \left(\bigotimes_a P(\tilde{I}_a) \otimes \bigotimes_b Q(\tilde{J}_b) \right) \\
&\cong \bigoplus_{(\tilde{I}, \tilde{J}) \in X(S)} \left(\bigotimes_{\alpha} P(\tilde{I}_{\alpha}) \otimes \bigotimes_{\beta} Q(\tilde{J}_{\beta}) \right) = (\mathbb{S}P \square \mathbb{S}Q)(S)
\end{aligned}$$

which conclude the proof. \square

3. CONNECTED PRODUCT ON \mathfrak{S} -BIMODULES

Just as in the \mathfrak{S} -modules, we give a description of the three monoidal structures on the category of (reduced) \mathfrak{S} -bimodules which are the equivalent of the three previous ones defined in $\mathfrak{S}\text{-mod}_k^{\text{red}}$. We start with a section on the functors that encode the combinatorics of our monoidal products. We express the combinatorics of the different monoidal structures on the $\mathfrak{S}\text{-bimod}_k$ category, using the same formalism as in the previous section.

3.1. Combinatorics of the connected composition of \mathfrak{S} -bimodules.

Definition 3.1 (Functors \mathbb{Y}^{or} and \mathbb{Y}). Let S and E be two finite sets. We define

- (1) the bifunctor $\mathbb{Y}^{\text{or}}(-, -) : \text{Fin} \times \text{Fin}^{\text{op}} \rightarrow \text{Set}$ given on objects by $\mathbb{Y}^{\text{or}}(S, E) = \prod_{r \in \mathbb{N}^*} \mathbb{Y}_r^{\text{or}}(S, E)$, with $\mathbb{Y}_r^{\text{or}}(S, E) := \mathcal{Y}_r^{\text{or}}(S) \times \mathcal{Y}_r^{\text{or}}(E) \cong$

$$\left\{ (I, K) = ((I_j, K_j))_{j \in [1, r]} \mid \begin{array}{l} \prod_{j=1}^r I_j = S; \prod_{j=1}^r K_j = E; \\ \forall j \in [1, r], I_j \neq \emptyset, K_j \neq \emptyset \end{array} \right\},$$

where the elements of $\mathbb{Y}_r^{\text{or}}(S, E)$ are ordered sets of pairs of sets.

- (2) $\mathbb{Y}(-, -) : \text{Fin} \times \text{Fin}^{\text{op}} \rightarrow \text{Set}$ the bifunctor given by $\mathbb{Y}(S, E) = \prod_{r \in \mathbb{N}^*} \mathbb{Y}_r(S, E)$ with $\mathbb{Y}_r(S, E) =$

$$\left\{ \{I, K\} \stackrel{\text{not}}{:=} \{(I_{\alpha}, K_{\alpha})\}_{\alpha \in A} \mid \begin{array}{l} A \in \text{Ob Fin}, |A| = r \\ \prod_{\alpha \in A} I_{\alpha} = S, \prod_{\alpha \in A} K_{\alpha} = E \\ \forall \alpha \in A, I_{\alpha} \neq \emptyset, K_{\alpha} \neq \emptyset \end{array} \right\},$$

where the elements of $\mathbb{Y}_r(S, E)$ are non ordered sets of pairs of sets.

Note that $\mathbb{Y}_r(S, E) \cong \mathcal{Y}(S) \times \mathcal{Y}(E)$. As in the case of $\mathcal{Y}^{\text{or}}(-)$, we have a free action of \mathfrak{S}_r on $\mathbb{Y}_r^{\text{or}}(S, E)$ given, for all (I, K) in $\mathbb{Y}_r^{\text{or}}(S, E)$ and all permutation σ in \mathfrak{S}_r , by

$$\sigma \cdot (I, K) = ((I_{\sigma^{-1}(j)}, K_{\sigma^{-1}(j)}))_{j \in [1, r]}$$

which induces the surjection $\mathbb{Y}_r^{\text{or}}(S, E) \twoheadrightarrow \mathbb{Y}_r(S, E)$.

As in the case of \mathfrak{S} -modules, we define a new bifunctor, denoted by \mathbb{X}^{conn} which encodes connectedness. For r and s in \mathbb{N}^* and for S and E two finite sets, the bifunctor $\mathbb{X}_{r,s}^{n,\text{conn}}(S, E)$ is equal to

$$\left\{ \begin{array}{l} (\{I, K'\}, \{K'', J\}) \\ \in \mathbb{Y}_r(S, \llbracket 1, n \rrbracket) \times \mathbb{Y}_s(\llbracket 1, n \rrbracket, E) \end{array} \middle| (K', K'') \in \mathcal{X}^{\text{conn}}(\llbracket 1, n \rrbracket) \right\} /_{\mathfrak{S}_n},$$

where, for all n in \mathbb{N}^* , by the functoriality of $\mathcal{X}^{\text{conn}}$, the quotient by \mathfrak{S}_n which identifies

$$(\{I, K'\} \cdot \sigma, \{K'', J\}) \sim (\{I, K'\}, \sigma^{-1} \cdot \{K'', J\})$$

is well defined. We also have the functors

$$\mathbb{X}^{n,\text{conn}}(S, E) := \coprod_{r,s \in \mathbb{N}^*} \mathbb{X}_{r,s}^{n,\text{conn}}(S, E) \text{ and } \mathbb{X}^{\text{conn}}(S, E) := \coprod_{n \in \mathbb{N}^*} \mathbb{X}^{n,\text{conn}}(S, E).$$

3.2. Monoidal products of the category $\mathfrak{S}\text{-bimod}_k$.

3.2.1. Composition product of \mathfrak{S} -bimodules \square .

Definition 3.2 (The composition product of \mathfrak{S} -bimodules). The *composition product of \mathfrak{S} -bimodules* is the bifunctor of

$$- \square - : \mathfrak{S}\text{-bimod}_k^{\text{red}} \times \mathfrak{S}\text{-bimod}_k^{\text{red}} \longrightarrow \mathfrak{S}\text{-bimod}_k^{\text{red}}$$

defined, for all reduced \mathfrak{S} -bimodules P and Q , and for all finite sets S and E , by

$$\begin{aligned} (P \square Q)(S, E) &:= \bigoplus_{n \in \mathbb{N}^*} \left(\bigoplus_{(\{I, K'\}, \{K'', J\}) \in \mathbb{X}_{1,1}^n(S, E)} P(I, K') \otimes Q(K'', J) \right) /_{\mathfrak{S}_n} \\ &\cong \bigoplus_{n \in \mathbb{N}^*} P(S, \llbracket 1, n \rrbracket) \otimes_{\mathfrak{S}_n} Q(\llbracket 1, n \rrbracket, E); \end{aligned}$$

where the action of \mathfrak{S}_n is induced by the action on $\llbracket 1, n \rrbracket$.

Remark 3.3. The composition product \square is defined by a coend. In fact, the tensorial product of the category Ch_k gives us the external product

$$\begin{array}{ccc} \text{Func}(\text{Fin} \times \text{Fin}^{\text{op}}, \text{Ch}_k)^{\times 2} & \longrightarrow & \text{Func}(\text{Fin} \times \text{Fin}^{\text{op}} \times \text{Fin} \times \text{Fin}^{\text{op}}, \text{Ch}_k) \\ (P, Q) & \longmapsto & \{(S_1, E_1, S_2, E_2) \mapsto P(S_1, E_1) \otimes Q(S_2, E_2)\} \end{array} \quad ,$$

and, by taking the coend of the functors $P(S_1, -) \otimes Q(-, E_2) : \text{Fin}^{\text{op}} \times \text{Fin} \rightarrow \text{Fin}$ for S_1 and E_2 two finite sets, we have

$$- \square - : \text{Func}(\text{Fin} \times \text{Fin}^{\text{op}}, \text{Ch}_k)^{\times 2} \longrightarrow \text{Func}(\text{Fin} \times \text{Fin}^{\text{op}}, \text{Ch}_k) \\ (P, Q) \longmapsto \int^{S \in \text{Fin}} P(-, S) \otimes Q(S, -) \quad .$$

As the category \mathfrak{S} is a skeleton of the category Fin , we finally have

$$- \square - : \text{Func}(\text{Fin} \times \text{Fin}^{\text{op}}, \text{Ch}_k)^{\times 2} \longrightarrow \text{Func}(\text{Fin} \times \text{Fin}^{\text{op}}, \text{Ch}_k) \\ (P, Q) \longmapsto \bigoplus_{n \in \mathbb{N}} P(-, \llbracket 1, n \rrbracket) \otimes_{\mathfrak{S}_n} Q(\llbracket 1, n \rrbracket, -) \quad .$$

Remark 3.4. Let P and Q be two reduced \mathfrak{S} -modules. For all m and n in \mathbb{N}^* , we have the following isomorphism of $\mathfrak{S}_m \times \mathfrak{S}_n^{\text{op}}$ -bimodules:

$$P \square Q(m, n) \cong \bigoplus_{N \in \mathbb{N}^*} P(m, N) \otimes_{k[\mathfrak{S}_N]} Q(N, n).$$

Proposition 3.5. *The category $(\mathfrak{S}\text{-bimod}_k, \square, I_{\square})$ with $I_{\square}(S, E) = k[\text{Aut}(S)]$ for $S \cong E$ and 0 otherwise, is a monoidal category.*

Proof. Let P , Q and R be three reduced \mathfrak{S} -bimodules and S and E be two non-empty finite sets. We note by e the cardinal of the set E . By definition of I_{\square} , we have the following isomorphisms:

$$\begin{aligned} P \square I_{\square}(S, E) &\cong \bigoplus_{n \in \mathbb{N}} P(S, [1, n]) \otimes_{\mathfrak{S}_n} I_{\square}([1, n], E) \\ &\cong P(S, [1, e]) \otimes_{\mathfrak{S}_e} I_{\square}([1, e], E) \\ &\cong P(S, [1, e]) \otimes_{\mathfrak{S}_e} k[\text{Aut}([1, e])] \\ &\cong P(S, E). \end{aligned}$$

By Remark 3.3, we also have the isomorphisms:

$$\begin{aligned} (P \square Q) \square R(S, E) &= \int^{U \in \text{Fin}} (P \square Q)(S, U) \otimes R(U, E) \\ &\cong \int^{U \in \text{Fin}} \int^{V \in \text{Fin}} P(S, V) \otimes Q(V, U) \otimes R(U, E) \\ &\stackrel{\text{Fubini}}{\cong} \int^{(U, V) \in \text{Fin}^{\times 2}} P(S, V) \otimes Q(V, U) \otimes R(U, E) \\ &\cong P \square (Q \square R)(S, E) \end{aligned}$$

□

3.2.2. *The concatenation product \otimes^{conc} .* As in the case of \mathfrak{S} -modules, we define the concatenation product of two reduced \mathfrak{S} -bimodules.

Definition 3.6 (Concatenation product of \mathfrak{S} -bimodules). The *concatenation product* is the bifunctor

$$- \otimes^{\text{conc}} - : \mathfrak{S}\text{-bimod}_k^{\text{red}} \times \mathfrak{S}\text{-bimod}_k^{\text{red}} \longrightarrow \mathfrak{S}\text{-bimod}_k^{\text{red}},$$

defined, for P and Q two reduced \mathfrak{S} -bimodules and for all finite sets S and E , by

$$(P \otimes^{\text{conc}} Q)(S, E) := \bigoplus_{(I, K) \in \mathbb{Y}_2^{\text{ord}}(S, E)} P(I_1, K_1) \otimes Q(I_2, K_2).$$

Remark 3.7. This product is a particular case of a Day convolution product: let P and Q be two reduced \mathfrak{S} -bimodules, the bifunctor $P \otimes^{\text{conc}} Q(-_1, -_2)$ is given by

$$\int^{(I_1, I_2, J_1, J_2)} k [\text{Hom}_{\text{Fin}^{\text{op}} \times \text{Fin}}((I_1 \amalg I_2, J_1 \amalg J_2), (-_1, -_2))] \otimes P(I_1, J_1) \otimes Q(I_2, J_2).$$

The two product \square and \otimes^{conc} satisfy the interchanging law.

Proposition 3.8 (Interchanging law). *Let A, B, C, D, E and F be reduced \mathfrak{S} -bimodules. We have the natural injection of \mathfrak{S} -bimodules*

$$\Phi_{A, B, C, D} : (A \square B) \otimes^{\text{conc}} (C \square D) \hookrightarrow (A \otimes^{\text{conc}} C) \square (B \otimes^{\text{conc}} D),$$

which is associative, i.e. we have

$$\Phi_{A \otimes C, B \otimes D, E, F} ((\Phi_{A, B, C, D}) \otimes (E \square F)) = \Phi_{A, B, C \otimes E, D \otimes F} ((A \square B) \otimes \Phi_{C, D, E, F}).$$

Proof. Let A, B, C and D be reduced \mathfrak{S} -bimodules. We have

$$\begin{aligned}
 & (A \square B) \otimes^{\text{conc}} (C \square D)(-, -) \\
 &= \int^{(I_1, I_2, J_1, J_2) \in \text{Fin}^{\times 4}} k[\text{Hom}((I_1 \amalg I_2, J_1 \amalg J_2), (-, -))] \\
 & \quad \otimes (A \square B)(I_1, J_1) \otimes (C \square D)(I_2, J_2) \\
 &= \int^{(I_1, I_2, J_1, J_2) \in \text{Fin}^{\times 4}} k[\text{Hom}((I_1 \amalg I_2, J_1 \amalg J_2), (-, -))] \\
 & \quad \otimes \int^{S \in \text{Fin}} A(I_1, S) \otimes B(S, J_1) \otimes \int^{T \in \text{Fin}} C(I_2, T) \otimes D(T, J_2) \\
 & \stackrel{\cong}{\underset{\text{Fubini}}{=}} \int^{(I_1, I_2, J_1, J_2, S, T) \in \text{Fin}^{\times 6}} k[\text{Hom}((I_1 \amalg I_2, J_1 \amalg J_2), (-, -))] \\
 & \quad \otimes A(I_1, S) \otimes C(I_2, T) \otimes B(S, J_1) \otimes D(T, J_2).
 \end{aligned}$$

The natural injections

$$\begin{aligned}
 & k[\text{Hom}((I_1 \amalg I_2, J_1 \amalg J_2), (-, -))] \\
 & \hookrightarrow \prod_S k[\text{Hom}((I_1 \amalg I_2, J_1 \amalg J_2), (-, S))] \otimes k[\text{Hom}((I_1 \amalg I_2, J_1 \amalg J_2), (S, -))] \\
 & \hookrightarrow \prod_{\substack{S \\ U \xrightarrow{S} S \\ S \rightarrow V}} k[\text{Hom}((I_1 \amalg I_2, U), (-, S))] \otimes k[\text{Hom}((V, J_1 \amalg J_2), (S, -))]
 \end{aligned}$$

and

$$A(I_1, S) \otimes C(I_2, T) \otimes B(S, J_1) \otimes D(T, J_2) \hookrightarrow \prod_{J_1, J_2, K_1, K_2} A(I_1, J_1) \otimes C(I_2, J_2) \otimes B(K_1, L_1) \otimes D(K_2, L_2)$$

give us the natural injection

$$\begin{aligned}
 & (A \square B) \otimes^{\text{conc}} (C \square D)(-, -) \\
 & \hookrightarrow \int^{(S, I_1, I_2, J_1, J_2, K_1, K_2, L_1, L_2)} k[\text{Hom}((I_1 \amalg I_2, J_1 \amalg J_2), (-, S))] \\
 & \quad \otimes k[\text{Hom}((K_1 \amalg K_2, L_1 \amalg L_2), (S, -))] \\
 & \quad \otimes A(I_1, J_1) \otimes C(I_2, J_2) \otimes B(K_1, L_1) \otimes D(K_2, L_2) \\
 & \stackrel{\cong}{\underset{\text{Fubini}}{=}} \int^S \int^{I_i, J_i} k[\text{Hom}((I_1 \amalg I_2, J_1 \amalg J_2), (-, S))] \otimes \int^{K_i, L_i} k[\text{Hom}((K_1 \amalg K_2, L_1 \amalg L_2), (S, -))] \\
 & \quad \otimes A(I_1, J_1) \otimes C(I_2, J_2) \otimes B(K_1, L_1) \otimes D(K_2, L_2) \\
 &= \int^S (A \otimes^{\text{conc}} C)(-, S) \otimes (B \otimes^{\text{conc}} D)(S, -) \\
 &= (A \otimes^{\text{conc}} C) \square (B \otimes^{\text{conc}} D)(-, -).
 \end{aligned}$$

Then we have the injective natural transformation between bifunctors:

$$\Phi_{A, B, C, D} : (A \square B) \otimes^{\text{conc}} (C \square D) \hookrightarrow (A \otimes^{\text{conc}} C) \square (B \otimes^{\text{conc}} D).$$

Moreover, Φ is associative, because the following diagram is commutative:

$$\begin{array}{ccc}
\left((A \square B) \otimes^{\text{conc}} (C \square D) \right) \otimes^{\text{conc}} (E \square F) & \xrightarrow{\cong} & (A \square B) \otimes^{\text{conc}} \left((C \square D) \otimes^{\text{conc}} (E \square F) \right) \\
\Phi_{A,B,C,D} \otimes 1 \downarrow & & \downarrow 1 \otimes \Phi_{C,D,E,F} \\
\left((A \otimes^{\text{conc}} C) \square (B \otimes^{\text{conc}} D) \right) \otimes^{\text{conc}} (E \square F) & \circlearrowleft & (A \square B) \otimes^{\text{conc}} \left((C \otimes^{\text{conc}} E) \square (D \otimes^{\text{conc}} F) \right) \\
\Phi_{A \otimes C, B \otimes D, E, F} \searrow & & \swarrow \Phi_{A, B, C \otimes E, D \otimes F} \\
(A \otimes^{\text{conc}} C \otimes^{\text{conc}} E) \square (B \otimes^{\text{conc}} D \otimes^{\text{conc}} F) & &
\end{array}$$

□

Corollary 3.9. *The categories of monoids $(\mathcal{A}s(\mathfrak{S}\text{-bimod}_k, \otimes^{\text{conc}}), \square, I_\square)$ and commutative monoids $(\text{Com}(\mathfrak{S}\text{-bimod}_k, \otimes^{\text{conc}}), \square, I_\square)$ are monoidal. In other words, if (P, c_P) and (Q, c_Q) are two monoids in the symmetric monoidal category (without unit) $(\mathfrak{S}\text{-bimod}_k, \otimes^{\text{conc}})$, then*

- (1) $(P \square Q, c_{P \square Q})$ is a monoid in $(\mathfrak{S}\text{-bimod}_k, \otimes^{\text{conc}})$;
- (2) if P and Q are commutative monoids, then $P \square Q$ is too.

Proof. (1) The injection Φ of the Proposition 3.8 gives us the product $c_{P \square Q}$:

$$\begin{array}{ccc}
(P \square Q) \otimes^{\text{conc}} (P \square Q) & \xrightarrow{\Phi} & (P \otimes^{\text{conc}} P) \square (Q \otimes^{\text{conc}} Q) \\
& \searrow c_{P \square Q} & \downarrow c_{P \square Q} \\
& & P \square Q
\end{array}$$

and the associativity of the product \square gives us the associativity of the product $c_{P \square Q}$.

- (2) The commutativity of $c_{P \square Q}$ follows from the commutativity of c_P and c_Q . □

Definition 3.10 (Free monoids). Let P be a reduced \mathfrak{S} -bimodule and S and E be two finite sets.

- (1) The free associative monoid without unit on P is defined by

$$\mathbb{T}^r P(S, E) := \bigoplus_{(I, K) \in \mathbb{Y}_r^{\text{ord}}(S, E)} P(I_1, K_1) \otimes \dots \otimes P(I_r, K_r).$$

- (2) The free commutative monoid without unit on P is defined by the quotient of the free associative monoid by the action of the symmetric groups; namely

$$\mathbb{S}^r P(S, E) := \left(\mathbb{T}^r P(S, E) \right)_{\mathfrak{S}_r} \cong \bigoplus_{\{I, K\} \in \mathbb{Y}_r(S, E)} \bigotimes_{\alpha} P(I_\alpha, K_\alpha).$$

3.3. Connected composition product of \mathfrak{S} -bimodules. We define the connected composition product of \mathfrak{S} -bimodules. The product was defined the first time by Vallette in his PhD thesis [Val03], for studying the homotopic comportment of algebraic structures with several inputs and outputs. Our definition is not the original one, but we show (cf. proposition 3.16) that they are equivalent.

Definition 3.11 (Product of connected composition \boxtimes_c^b). The *product of connected composition* of reduced \mathfrak{S} -bimodules is the bifunctor

$$- \boxtimes_c^b - : \mathfrak{S}\text{-bimod}_k^{\text{red}} \times \mathfrak{S}\text{-bimod}_k^{\text{red}} \longrightarrow \mathfrak{S}\text{-bimod}_k^{\text{red}}$$

defined, for all reduced \mathfrak{S} -bimodules P and Q and all pairs (S, E) of finite sets, by $(P \boxtimes_c^b Q)(S, E) :=$

$$\bigoplus_{n \in \mathbb{N}^*} \left(\bigoplus_{(\{I, K'\}, \{K'', J\}) \in \mathbb{X}^{n, \text{conn}}(S, E)} \bigotimes_{\alpha} P(I_{\alpha}, K'_{\alpha}) \otimes \bigotimes_{\beta} Q(K''_{\beta}, J_{\beta}) \right) /_{\mathfrak{S}_n},$$

where the quotient by \mathfrak{S}_n identifies, for all σ in \mathfrak{S}_n , the terms

$$\bigotimes_{\alpha} P(I_{\alpha}, K'_{\alpha}) \otimes \bigotimes_{\beta} Q(K''_{\beta}, J_{\beta}) \sim \left(\bigotimes_{\alpha} P(I_{\alpha}, K'_{\alpha}) \right) \cdot \sigma^{-1} \otimes \sigma \cdot \left(\bigotimes_{\beta} Q(K''_{\beta}, J_{\beta}) \right).$$

Remark 3.12. This construction is functorial, because $(P \boxtimes_c^b Q)(-, -)$ is a sub-bifunctor of $(\mathbb{S}P \square \mathbb{S}Q)(-, -)$ (see Lemma 3.18).

Notation 3.13. We denote by $/_{\mathfrak{S}}$, the quotient by symmetric groups for

$$(P \boxtimes_c^b Q)(S, E) := \left(\bigoplus_{\substack{(\{I, K'\}, \{K'', J\}) \\ \in \mathbb{X}^{\text{conn}}(S, E)}} \bigotimes_{\alpha} P(I_{\alpha}, K'_{\alpha}) \otimes \bigotimes_{\beta} Q(K''_{\beta}, J_{\beta}) \right) /_{\mathfrak{S}}$$

The following proposition says that our definition of the connected composition product of \mathfrak{S} -bimodules is equivalent to that of Vallette in [Val03, Val07]. First, recall the notion of connected permutation.

Definition 3.14 (Connected permutation – [Val07, Sect 1.3]). Let a, b and N be three integers with a and b in \mathbb{N}^* , let $\bar{\alpha} = (\alpha_1, \dots, \alpha_a)$ in $(\mathbb{N}^*)^a$ be an a -tuple and $\bar{\beta} = (\beta_1, \dots, \beta_b)$ in $(\mathbb{N}^*)^b$ be a b -tuple such that $|\bar{\alpha}| = N = |\bar{\beta}|$. A $(\bar{\alpha}, \bar{\beta})$ -connected permutation σ of \mathfrak{S}_N is a permutation of \mathfrak{S}_N such that the graph of a geometric representation of σ is connected if one gathers the inputs labelled by $\alpha_1 + \dots + \alpha_i + 1, \dots, \alpha_1 + \dots + \alpha_i + \alpha_{i+1}$ for $0 \leq i \leq a-1$, and the outputs labelled by $\beta_1 + \dots + \beta_i + 1, \dots, \beta_1 + \dots + \beta_i + \beta_{i+1}$ for $0 \leq i \leq b-1$. The set of $(\bar{\alpha}, \bar{\beta})$ -connected permutations is denoted by $S_c^{\bar{\beta}, \bar{\alpha}}$.

Lemma 3.15. Let r, s and N be in \mathbb{N}^* , and let \bar{k} be an r -tuple in $(\mathbb{N}^*)^r$ and \bar{j} be a s -tuple in $(\mathbb{N}^*)^s$ such that $\sum_{\alpha=1}^r k_{\alpha} = N = \sum_{\beta=1}^s j_{\beta}$. The map

$$\begin{aligned} \varphi : \left\{ (K, J) \in \mathcal{X}_{r,s}^{\text{conn,ord}}(N) \mid \begin{array}{l} (|K_1|, \dots, |K_r|) = \bar{k}, \\ (|J_1|, \dots, |J_s|) = \bar{j} \end{array} \right\} &\longrightarrow S_c^{\bar{k}, \bar{j}} \\ (K, J) &\longmapsto \sigma_K^{-1} \sigma_J \end{aligned}$$

is surjective.

Proof. By the definition of the connectedness of a pair (K, J) in $\mathcal{X}_{r,s}^{\text{conn,ord}}(N)$, if we consider the graph of a geometric representation of the permutation $\sigma_K^{-1} \sigma_J$, where we gather the inputs and the outputs as in the definition of connected permutation, then there exists a path between every input labelled by i and every output labelled by j . Then, we have $\sigma_K^{-1} \sigma_J$ in $S_c^{\bar{k}, \bar{j}}$.

Let σ be a permutation in $S_c^{\bar{k}, \bar{j}}$. We consider

$$[[1, N]]_{\bar{j}} \stackrel{\text{not}}{:=} \left\{ [[1, j_1]], [[j_1 + 1, j_1 + j_2]], \dots, [[j_1 + \dots + j_{s-1} + 1, j_1 + \dots + j_s]] \right\},$$

and we denote by $([1, N] \cdot \sigma)_{\bar{k}}$, the following set

$$\left\{ \{\sigma^{-1}(1), \sigma^{-1}(2), \dots, \sigma^{-1}(k_1)\}, \dots, \{\sigma^{-1}(k_1 + \dots + k_{r-1} + 1), \dots, \sigma^{-1}(k_1 + \dots + k_r)\} \right\};$$

the pair $([1, N] \cdot \sigma)_{\bar{k}}, [1, N]_{\bar{j}}$ is an element of $\mathcal{X}_{r,s}^{\text{conn,ord}}(N)$ by the connectedness of the permutation σ . So the application

$$\begin{aligned} \psi : S_c^{\bar{k}, \bar{j}} &\longrightarrow \left\{ (K, J) \in \mathcal{X}_{r,s}^{\text{conn,ord}}(N) \mid \begin{array}{l} (|K_1|, \dots, |K_r|) = \bar{k}, \\ (|J_1|, \dots, |J_s|) = \bar{j} \end{array} \right\} \\ \sigma &\longmapsto \left(([1, N] \cdot \sigma)_{\bar{k}}, [1, N]_{\bar{j}} \right) \end{aligned}$$

is well-defined and gives us a section of φ , so that φ is surjective. \square

Proposition 3.16. *For all integers m and n in \mathbb{N}^* , we have the isomorphism of $\mathfrak{S}_m \times \mathfrak{S}_n^{\text{op}}$ -modules:*

$$(P \boxtimes_c^b Q)([1, m], [1, n]) \cong (P \boxtimes_c^{\text{val}} Q)(m, n).$$

Proof. Let m and n be two integers in \mathbb{N}^* . We have

$$\begin{aligned} &(P \boxtimes_c^b Q)([1, m], [1, n]) := \\ &\left(\bigoplus_{N, r, s \in \mathbb{N}} \bigoplus_{\substack{\{I, K'\}, \{K'', J\} \\ \in \mathbb{X}^{N, \text{conn}}([1, m], [1, n])}} \bigotimes_{\alpha} P(I_{\alpha}, K'_{\alpha}) \otimes \bigotimes_{\beta} Q(K''_{\beta}, J_{\beta}) \right) /_{\mathfrak{S}} \\ &\cong \left(\bigoplus_{N \in \mathbb{N}} \left(\bigoplus_{\substack{\{I, K'\} \in \mathbb{Y}_r^{\text{or}}([1, m], [1, N]) \\ \{K'', J\} \in \mathbb{Y}_s^{\text{or}}([1, N], [1, n]) \\ (K', K'') \in \mathcal{X}_{r,s}^{\text{conn}}([1, N])}} \left(\bigotimes_{\alpha=1}^r P(I_{\alpha}, K'_{\alpha}) \right)_{\mathfrak{S}_r} \otimes \left(\bigotimes_{\beta=1}^s Q(K''_{\beta}, J_{\beta}) \right)_{\mathfrak{S}_s} \right) \right) /_{\mathfrak{S}} \\ &\cong \left(\bigoplus_{N \in \mathbb{N}} \left(\bigoplus_{\substack{\{I, K'\} \in \mathbb{Y}_r^{\text{or}}([1, m], [1, N]) \\ \{K'', J\} \in \mathbb{Y}_s^{\text{or}}([1, N], [1, n]) \\ (K', K'') \in \mathcal{X}_{r,s}^{\text{conn}}([1, N])}} \bigotimes_{\alpha=1}^r P(I_{\alpha}, K'_{\alpha}) \otimes \bigotimes_{\beta=1}^s Q(K''_{\beta}, J_{\beta}) \right)_{\mathfrak{S}_r \times \mathfrak{S}_s} \right) /_{\mathfrak{S}} \\ &\cong \left(\bigoplus_{N \in \mathbb{N}} \left(\bigoplus_{\substack{\bar{l} \in (\mathbb{N}^*)^r, |\bar{l}|=m \\ \bar{i} \in (\mathbb{N}^*)^s, |\bar{i}|=n \\ (K', K'') \in \\ \mathcal{X}_{r,s}^{\text{conn,ord}}([1, N])}} k[\mathfrak{S}_m] \otimes_{\prod \mathfrak{S}_{l_{\alpha}}} \bigotimes_{\alpha=1}^r P(l_{\alpha}, K'_{\alpha}) \otimes \bigotimes_{\beta=1}^s Q(K''_{\beta}, i_{\beta}) \otimes_{\prod \mathfrak{S}_{i_{\beta}}} k[\mathfrak{S}_n] \right)_{\mathfrak{S}_r \times \mathfrak{S}_s} \right) /_{\mathfrak{S}} \\ &\xrightarrow{\varphi} \bigoplus_{N \in \mathbb{N}} \left(\bigoplus_{\substack{\bar{l}, \bar{k} \in (\mathbb{N}^*)^r, |\bar{l}|=m \\ \bar{i}, \bar{j} \in (\mathbb{N}^*)^s, |\bar{i}|=n \\ |\bar{k}|=N=|\bar{j}|}} k[\mathfrak{S}_m] \otimes_{\prod \mathfrak{S}_{l_{\alpha}}} \left(\bigotimes_{\alpha=1}^r P(l_{\alpha}, k_{\alpha}) \right)_{\mathfrak{S}_{\bar{k}}} \otimes_{\mathfrak{S}_{\bar{j}}} k[S_c^{\bar{k}, \bar{j}}] \otimes_{\prod \mathfrak{S}_{i_{\beta}}} \left(\bigotimes_{\beta=1}^s Q(j_{\beta}, i_{\beta}) \right) \otimes_{\prod \mathfrak{S}_{i_{\beta}}} k[\mathfrak{S}_n] \right)_{\mathfrak{S}_r \times \mathfrak{S}_s} \end{aligned}$$

where φ sends, for $(K', K'') \in \mathcal{X}_{r,s}^{\text{conn}}([1, N])$ fixed, the component

$$\left(k[\mathfrak{S}_m] \otimes_{\prod \mathfrak{S}_{l_{\alpha}}} \bigotimes_{\alpha=1}^r P(l_{\alpha}, K'_{\alpha}) \otimes \bigotimes_{\beta=1}^s Q(K''_{\beta}, i_{\beta}) \right) \otimes_{\prod \mathfrak{S}_{i_{\beta}}} k[\mathfrak{S}_n] /_{\mathfrak{S}_N}$$

to the following $\mathfrak{S}_m \times \mathfrak{S}_n^{\text{op}}$ -module

$$k[\mathfrak{S}_m] \otimes_{\prod \mathfrak{S}_{l_\alpha}} \bigotimes_{\alpha=1}^r P(l_\alpha, [\sum_{j=1}^{\alpha-1} |K'_j| + 1, \sum_{j=1}^{\alpha} |K'_j|]) \otimes_{\mathfrak{S}_{\bar{k}}} \sigma_{K'}^{-1} \otimes_{\mathfrak{S}_N} \sigma_{K''} \\ \otimes_{\mathfrak{S}_{\bar{j}}} \bigotimes_{\beta=1}^s Q([\sum_{j=1}^{\beta-1} |K''_j| + 1, \sum_{j=1}^{\beta} |K''_j|], i_\beta) \otimes_{\prod \mathfrak{S}_{i_\beta}} k[\mathfrak{S}_n],$$

which is isomorphic to

$$\cong k[\mathfrak{S}_m] \otimes_{\prod \mathfrak{S}_{l_\alpha}} \bigotimes_{\alpha=1}^r P(l_\alpha, [\sum_{j=1}^{\alpha-1} |K'_j| + 1, \sum_{j=1}^{\alpha} |K'_j|]) \otimes_{\mathfrak{S}_{\bar{k}}} \sigma_{K'}^{-1} \sigma_{K''} \otimes_{\mathfrak{S}_{\bar{j}}} \\ \bigotimes_{\beta=1}^s Q([\sum_{j=1}^{\beta-1} |K''_j| + 1, \sum_{j=1}^{\beta} |K''_j|], i_\beta) \otimes_{\prod \mathfrak{S}_{i_\beta}} k[\mathfrak{S}_n].$$

However, by Lemma 3.15, the morphism φ is surjective, and the quotient by the group $\mathfrak{S}_{\bar{k}} \times \mathfrak{S}_{\bar{j}}$ gives us the injectivity. \square

Proposition 3.17 ([Val03, Lem. 49]). *The category $(\mathfrak{S}\text{-bimod}_k, \boxtimes_c^{\text{Val}}, I_{\boxtimes}^{\text{Val}})$ where the unit I is defined, for all pairs (S, E) of finite sets, by:*

$$I_{\boxtimes}^{\text{Val}}(S, E) := \begin{cases} k & \text{if } |S| = 1 = |E| \\ 0 & \text{otherwise} \end{cases},$$

is abelian monoidal and preserves coequalizers and sequential colimits.

The \mathfrak{S} -bimodule $P \boxtimes_c^{\text{Val}} Q$ appears as the indecomposables for the product \otimes^{conc} of $\mathbb{S}P \square \mathbb{S}Q$.

Lemma 3.18. *Let P and Q be two reduced \mathfrak{S} -bimodules. We have, for all finite sets S and E , the following isomorphism*

$$(\mathbb{S}P \square \mathbb{S}Q)(S, E) \cong \bigoplus_{n \in \mathbb{N}^*} \left(\bigoplus_{\substack{\{I, K'\}, \{K'', J\} \\ \in \mathbb{X}_n^{\text{Val}}(S, E)}} \bigotimes_{\alpha} P(I_\alpha, K'_\alpha) \otimes \bigotimes_{\beta} Q(K''_\beta, J_\beta) \right) /_{\mathfrak{S}_n}.$$

Proof. Let S and E be two finite sets, then

$$\begin{aligned} (\mathbb{S}P \square \mathbb{S}Q)(S, E) &\cong \bigoplus_{n \in \mathbb{N}^*} \left(\bigoplus_{\{A, B\}, \{C, D\} \in \mathbb{X}_{1,1}^{\text{Val}}(m, n)} \mathbb{S}P(A, B) \otimes \mathbb{S}Q(C, D) \right) /_{\mathfrak{S}_n} \\ &\cong \bigoplus_{n \in \mathbb{N}^*} \left(\bigoplus_{\substack{\{A, B\}, \{C, D\} \\ \in \mathbb{X}_{1,1}^{\text{Val}}(S, E)}} \bigoplus_{\substack{\{I, K'\} \in \mathbb{Y}(A, B) \\ \{K'', J\} \in \mathbb{Y}(C, D)}} \bigotimes_{\alpha} P(I_\alpha, K'_\alpha) \otimes \bigotimes_{\beta} Q(K''_\beta, J_\beta) \right) /_{\mathfrak{S}_n} \\ &\cong \bigoplus_{n \in \mathbb{N}^*} \left(\bigoplus_{\{I, K'\}, \{K'', J\} \in \mathbb{X}_n^{\text{Val}}(S, E)} \bigotimes_{\alpha} P(I_\alpha, K'_\alpha) \otimes \bigotimes_{\beta} Q(K''_\beta, J_\beta) \right) /_{\mathfrak{S}_n}. \end{aligned}$$

\square

Proposition 3.19. *Let P and Q be two reduced \mathfrak{S} -bimodules. We have the natural isomorphism*

$$\mathbb{S}(P \boxtimes_c^{\text{Val}} Q) \cong \mathbb{S}P \square \mathbb{S}Q.$$

Proof. By Lemma 3.18, for all finite sets S and E , we have the isomorphism

$$(\mathbb{S}P \square \mathbb{S}Q)(S, E) \cong \bigoplus_{n \in \mathbb{N}^*} \left(\bigoplus_{\substack{\{I, K'\}, \{K'', J\} \\ \in \mathbb{X}^n(S, E)}} \bigotimes_{\alpha} P(I_{\alpha}, K'_{\alpha}) \otimes \bigotimes_{\beta} Q(K''_{\beta}, J_{\beta}) \right) \Big/_{\mathfrak{S}_n}.$$

□

4. INDUCTION FUNCTOR

We describe the functor Ind , and its right adjoint, the restriction functor Res , which establishes the link between the two previous sections, since $\text{Ind} : \mathfrak{S}\text{-mod}_k \rightarrow \mathfrak{S}\text{-bimod}_k$ is strong monoidal for the different products introduced in the sections 2 and 3.

4.1. Adjunction Ind-Res. For \mathcal{C} a groupoid, we note $\Delta_{\mathcal{C}}$ the functor

$$\Delta_{\mathcal{C}} := (\text{inv}_{\mathcal{C}}, \text{id}_{\mathcal{C}}) : \mathcal{C} \longrightarrow \mathcal{C}^{\text{op}} \times \mathcal{C}$$

where $\text{inv}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$ is the equivalence of categories given by the passage to the inverse. The *restriction functor*, denoted by Res , is given by the following composition:

$$\begin{array}{ccc} \text{Res} : \mathfrak{S}\text{-bimod}_k & \longrightarrow & \mathfrak{S}\text{-mod}_k \\ P & \longmapsto & P \circ \Delta_{\text{Fin}^{\text{op}}} \end{array}.$$

This functor is exact and has a left adjoint functor, the *induction functor*.

Definition 4.1 (Functor Ind). The *induction functor* is given by:

$$\begin{array}{ccc} \text{Ind} : \mathfrak{S}\text{-mod}_k & \longrightarrow & \mathfrak{S}\text{-bimod}_k \\ V & \longmapsto & (\text{Ind}V)(S, E) := \bigoplus_{\text{Hom}_{\text{Fin}}(E, S)} V(E) \end{array}$$

where an element f in $\text{Aut}(S)$ acts on the left by

$$f \cdot \left(\bigoplus_{\varphi \in \text{Hom}_{\text{Fin}}(E, S)} V(E) \right) = \bigoplus_{f\varphi \in \text{Hom}_{\text{Fin}}(E, S)} V(E)$$

and an element g in $\text{Aut}(E)$ acts on the right by

$$\left(\bigoplus_{\varphi \in \text{Hom}_{\text{Fin}}(E, S)} V(E) \right) \cdot g = \bigoplus_{\varphi g \in \text{Hom}_{\text{Fin}}(E, S)} V(f^{-1}(E)).$$

Remark 4.2. Let V be a reduced \mathfrak{S} -module and S and E be two finite sets. If S and E are not isomorphic (i.e. $|S| \neq |E|$) then $(\text{Ind}V)(S, E) = 0$. Finally, we have:

$$(\text{Ind}V)(S, E) \cong \begin{cases} 0 & \text{if } S \not\cong E \\ k[\text{Aut}(S)] \otimes V(S) & \text{otherwise.} \end{cases}$$

Proposition 4.3. *We have the adjunction*

$$\text{Ind} : \mathfrak{S}\text{-mod}_k \rightleftarrows \mathfrak{S}\text{-bimod}_k : \text{Res}.$$

Proof. By the classical result [Ser71, Th.13]. □

One of the fundamental properties of Ind is the following.

Proposition 4.4. *The functor Ind is exact, preserves quasi-isomorphisms and commutes with colimits.*

Proof. The result is induced by the remark 4.2; more generally, for G a finite group and H a subgroup of G , $k[G]$ is a free $k[H]$ -module. As the functor Ind has a right adjoint, it commutes with colimits. \square

4.2. Compatibilities with products. In this subsection, we show compatibilities of the functor Ind with products defined in previous sections. First, we recall the following proposition, about compatibility between monoidal structures and adjunction.

Proposition 4.5. *Let $(\mathbf{C}, \otimes, I_{\mathbf{C}})$ and $(\mathbf{D}, \odot, I_{\mathbf{D}})$ be two monoidal categories with the following adjunction:*

$$L : \mathbf{C} \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} \mathbf{D} : R$$

such that the left adjoint is a strong monoidal functor by the following natural equivalence $\mu_L : L(-1) \odot L(-2) \xrightarrow{\cong} L(-1 \otimes -2)$ and the natural isomorphism $e_L : I_{\mathbf{D}} \xrightarrow{\cong} L(I_{\mathbf{C}})$. Then the right adjoint is a Lax monoidal functor with the natural transformation μ_R and the morphism ϵ_R given by

$$\begin{array}{ccc} R(-1) \otimes R(-2) & \xrightarrow{\eta^{R(-1) \otimes R(-2)}} & RL(R(-1) \otimes R(-2)) \\ \mu_R := \downarrow \text{dotted} & & \downarrow R(\mu_L^{-1}(R(-1), R(-2))) \\ R(-1 \otimes -2) & \xleftarrow{R(\epsilon_{(-1) \odot (-2)})} & R(LR(-1) \odot LR(-2)) \end{array}$$

$$\text{and } e_R : I_{\mathbf{C}} \xrightarrow{\eta(I_{\mathbf{C}})} RL(I_{\mathbf{C}}) \xrightarrow{R(e_L^{-1})} R(I_{\mathbf{D}}).$$

Now, we can study the case of the adjunction given by the functors Ind and Res .

Notation 4.6. We note $\mathfrak{S}\text{-bimod}_k^{\text{Ind}}$, the essential image of the functor Ind .

Proposition 4.7. *Then $(\mathfrak{S}\text{-bimod}_k^{\text{Ind}}, \square)$ is a symmetric monoidal category and the induction functor*

$$\text{Ind} : (\mathfrak{S}\text{-mod}_k, \square) \longrightarrow (\mathfrak{S}\text{-bimod}_k^{\text{Ind}}, \square)$$

is symmetric monoidal.

Proof. Let P and Q be two \mathfrak{S} -modules and, S and E be two finite sets. We have

$$\begin{aligned} (\text{Ind}P \square \text{Ind}Q)(S, E) &\cong \int^{T \in \text{Fin}} \text{Ind}P(S, T) \otimes \text{Ind}Q(T, E) \\ &\cong \int^{T \in \text{Fin}} \bigoplus_{\text{Hom}_{\text{Fin}}(T, S)} P(T) \otimes \bigoplus_{\text{Hom}_{\text{Fin}}(E, T)} Q(E) \\ &\cong \int^{T \in \text{Fin}} \bigoplus_{\text{Hom}_{\text{Fin}}(T, S) \times \text{Hom}_{\text{Fin}}(E, T)} P(T) \otimes Q(E) . \end{aligned}$$

Also, we have the following commutative diagram:

$$\begin{array}{ccc}
 \bigoplus_{\text{Hom}(E,S)} P(E) \otimes Q(E) & \xrightarrow{\quad} & \bigoplus_{\text{Hom}(T,S) \times \text{Hom}(E,T)} P(T) \otimes Q(E) \\
 \downarrow T=E & & \downarrow \varphi \\
 \prod_T \bigoplus_{\text{Hom}(T,S) \times \text{Hom}(E,T)} P(T) \otimes Q(E) & \xrightarrow{\int^T} & \bigoplus_{\text{Hom}(T,S) \times \text{Hom}(E,T)} P(T) \otimes Q(E) \\
 \downarrow \varphi & & \downarrow \psi \\
 \bigoplus_{\text{Hom}(E,S)} P(E) \otimes Q(E) & \xrightarrow{\quad} & \bigoplus_{\text{Hom}(E,S)} P(E) \otimes Q(E)
 \end{array}$$

, ,

where φ is given by the composition of functions and ψ exists by the universal property of the coend and the commutativity of the following diagram:

$$\begin{array}{ccc}
 \prod_{T_1 \rightarrow T_2} \bigoplus_{\text{Hom}(T_2,S) \times \text{Hom}(E,T_1)} P(T_2) \otimes Q(E) & \xrightleftharpoons[\Phi_2]{\Phi_1} & \prod_T \bigoplus_{\text{Hom}(T,S) \times \text{Hom}(E,T)} P(T) \otimes Q(E) \xrightarrow{\int^T} \bigoplus_{\text{Hom}_{\text{Fin}}(T,S) \times \text{Hom}_{\text{Fin}}(E,T)} P(T) \otimes Q(E) \\
 \searrow p\Phi_1 = p\Phi_2 & & \downarrow p \\
 & & \bigoplus_{\text{Hom}(E,S)} P(E) \otimes Q(E) \xleftarrow{\psi} \bigoplus_{\text{Hom}_{\text{Fin}}(T,S) \times \text{Hom}_{\text{Fin}}(E,T)} P(T) \otimes Q(E)
 \end{array}$$

which gives us the natural isomorphism

$$\int^{T \in \text{Fin}} \bigoplus_{\text{Hom}_{\text{Fin}}(T,S) \times \text{Hom}_{\text{Fin}}(E,T)} P(T) \otimes Q(E) \cong \bigoplus_{\text{Hom}(E,S)} P(E) \otimes Q(E) \cong \text{Ind}(P \square Q)(E) .$$

□

Corollary 4.8. *Let M be a reduced \mathfrak{S} -module. We have the following natural isomorphism of \mathfrak{S} -bimodules $\mathbb{T}_{\square}(\text{Ind}(M)) \cong \text{Ind}(\mathbb{T}_{\square}(M))$ where \mathbb{T}_{\square} is the free unitary associative monoid for the product \square .*

Corollary 4.9. *The functor $\text{Res} : (\mathfrak{S}\text{-bimod}_k^{\text{red}}, \square) \rightarrow (\mathfrak{S}\text{-mod}_k^{\text{red}}, \square)$ is Lax-monoidal.*

Proof. By adjunction of functors Ind and Res , and Proposition 4.5. □

Remark 4.10. The functor Res is not strongly monoidal with respect to \square . For example, if we consider P and Q , the \mathfrak{S} -bimodules defined by

$$P(S, E) := \begin{cases} k & \text{if } |S| = 1 \text{ and } |E| = 1 \text{ or } 2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$Q(S, E) := \begin{cases} k & \text{if } |E| = 1 \text{ and } |S| = 1 \text{ or } 2 \\ 0 & \text{otherwise} \end{cases}$$

then $\text{Res}(P \square Q)(\{*\}, \{*\}) = k^2$ and $(\text{Res}P \square \text{Res}Q)(\{*\}, \{*\}) = k$.

Remark 4.11. We have the monoidal adjunction

$$\text{Ind} : (\mathfrak{S}\text{-mod}_k^{\text{red}}, \square) \rightleftarrows (\mathfrak{S}\text{-bimod}_k^{\text{red}}, \square) : \text{Res}.$$

The functor Ind is also compatible with the concatenation product.

Proposition 4.12. *The functor of induction*

$$\text{Ind} : (\mathfrak{S}\text{-mod}_k^{\text{red}}, \otimes^{\text{conc}}) \longrightarrow (\mathfrak{S}\text{-bimod}_k^{\text{red}}, \otimes^{\text{conc}})$$

is symmetric monoidal.

Proof. Let P and Q be two reduced \mathfrak{S} -modules and S and E be two finite sets. We have the isomorphisms of chain complexes:

$$\begin{aligned} (\text{Ind}P \otimes^{\text{conc}} \text{Ind}Q)(S, E) &\cong \bigoplus_{(I, J) \in \mathcal{Y}_2^{\text{or}}(S, E)} \bigoplus_{\text{Hom}_{\text{Fin}}(J_1, I_1) \times \text{Hom}_{\text{Fin}}(J_2, I_2)} P(J_1) \otimes Q(J_2) \\ &\cong \bigoplus_{J \in \mathcal{Y}_2^{\text{or}}(E)} \bigoplus_{\text{Hom}_{\text{Fin}}(J_1 \amalg J_2, S)} P(J_1) \otimes Q(J_2) \\ &\cong \text{Ind}(P \otimes^{\text{conc}} Q)(S, E). \end{aligned}$$

□

Proposition 4.13. *The functor $\text{Res} : (\mathfrak{S}\text{-bimod}_k^{\text{red}}, \otimes^{\text{conc}}) \rightarrow (\mathfrak{S}\text{-mod}_k^{\text{red}}, \otimes^{\text{conc}})$ is Lax-monoidal.*

Proof. Let P and Q be two \mathfrak{S} -bimodules. We have the natural injections

$$\begin{aligned} &\prod_{I \in \mathcal{Y}_2^{\text{or}}(-)} \text{Hom}_{\text{Fin}^{\text{op}}}(I_1 \amalg I_2, -) \otimes \text{Res}P(I_1) \otimes \text{Res}Q(I_2) \\ &= \prod_{I \in \mathcal{Y}_2^{\text{or}}(-)} \text{Hom}_{\text{Fin}^{\text{op}}}(I_1 \amalg I_2, -) \otimes P(I_1, I_1) \otimes Q(I_2, I_2) \\ &\hookrightarrow \prod_{I \in \mathcal{Y}_2^{\text{or}}(-)} \text{Hom}_{\text{Fin} \times \text{Fin}^{\text{op}}}((I_1 \amalg I_2, I_1 \amalg I_2), \Delta(-)) \otimes P(I_1, I_1) \otimes Q(I_2, I_2) \\ &\hookrightarrow \prod_{(I, J) \in \mathcal{Y}_2^{\text{or}}(-)} \text{Hom}_{\text{Fin} \times \text{Fin}^{\text{op}}}((I_1 \amalg I_2, J_1 \amalg J_2), \Delta(-)) \otimes P(I_1, J_1) \otimes Q(I_2, J_2) \end{aligned}$$

which imply the following natural injection

$$\prod_{I \in \mathcal{Y}_2^{\text{or}}(-)} \text{Hom}_{\text{Fin}^{\text{op}}}(I_1 \amalg I_2, -) \otimes \text{Res}P(I_1) \otimes \text{Res}Q(I_2) \hookrightarrow \text{Res}(P \otimes^{\text{conc}} Q).$$

Also, we have the following commutative diagram

$$\begin{array}{ccc} \prod_{I_1 \rightarrow J_1, I_2 \rightarrow J_2} \text{Hom}_{\text{Fin}^{\text{op}}}(J_1 \amalg J_2, -) \otimes \text{Res}P(I_1) \otimes \text{Res}Q(I_2) & & \\ \Downarrow & & \\ \prod_{I \in \mathcal{Y}_2^{\text{or}}(-)} \text{Hom}_{\text{Fin}^{\text{op}}}(I_1 \amalg I_2, -) \otimes \text{Res}P(I_1) \otimes \text{Res}Q(I_2) & \hookrightarrow & \text{Res}(P \otimes^{\text{conc}} Q) \\ \downarrow & & \nearrow \exists \\ (\text{Res}P) \otimes (\text{Res}Q) & & \end{array}$$

so, by the universal property of the coend, we have the natural morphism

$$\text{Res}P \otimes^{\text{conc}} \text{Res}Q \longrightarrow \text{Res}(P \otimes^{\text{conc}} Q).$$

□

Remark 4.14. The functor Res is not strongly monoidal with respect to \otimes^{conc} . Indeed, if we consider P and Q , the \mathfrak{S} -bimodules defined by

$$P(S, E) := \begin{cases} k & \text{if } |E| = 2 \text{ and } |S| = 1; \\ 0 & \text{otherwise} \end{cases}$$

and

$$Q(S, E) := \begin{cases} k & \text{if } |S| = 2 \text{ and } |E| = 1; \\ 0 & \text{otherwise} \end{cases},$$

then $\text{Res}P \otimes^{\text{conc}} \text{Res}Q = 0$, while $\text{Res}(P \otimes^{\text{conc}} Q)(S) = k$ if the cardinal of S is 3.

We have the following compatibility between functors $\mathbb{S}(-)$ and $\text{Ind}(-)$.

Proposition 4.15. *Let P be a reduced \mathfrak{S} -module. Then, we have the natural isomorphism of reduced \mathfrak{S} -bimodules:*

$$\text{Ind}(\mathbb{S}(P)) \cong \mathbb{S}(\text{Ind}(P)).$$

Proof. The functor Ind commutes with the concatenation product \otimes^{conc} and is compatible with the symmetry, by Proposition 4.12. We conclude by the exactness of the functor Ind . □

One of the most important properties of the functor Ind is that it is compatible with connected composition products.

Theorem 4.16. *The functor*

$$\text{Ind} : (\mathfrak{S}\text{-mod}_k^{\text{red}}, \boxtimes_c) \rightarrow (\mathfrak{S}\text{-bimod}_k^{\text{red}}, \boxtimes_c^{\text{Val}})$$

is monoidal.

Proof. Let S and E be two finite sets, then $(\text{Ind } I_{\boxtimes})(S, E) = k$ if $|S| = 1 = |E|$ and 0 otherwise. Then, the functor Ind respect the unit. Let V and W be two reduced \mathfrak{S} -modules and S and E be two finite sets.

$$\begin{aligned} & \left(\text{Ind}V \boxtimes_c^{\text{Val}} \text{Ind}W \right)(S, E) \\ &= \bigoplus_{n \in \mathbb{N}^*} \bigoplus_{\substack{\{I, K'\}, \{K'', J\} \\ \in \mathbb{X}^{n, \text{conn}}(S, E)}} \bigotimes_{\alpha} \text{Ind}V(I_{\alpha}, K'_{\alpha}) \otimes_{\mathfrak{S}_n} \bigotimes_{\beta} \text{Ind}W(K''_{\beta}, J_{\beta}) \\ &\cong \bigoplus_{n \in \mathbb{N}^*} \bigoplus_{\substack{\{I, K'\}, \{K'', J\} \\ \in \mathbb{X}^{n, \text{conn}}(S, E)}} \bigotimes_{\alpha} \bigoplus_{\text{Hom}_{\text{Fin}}(K'_{\alpha}, I_{\alpha})} V(K'_{\alpha}) \otimes_{\mathfrak{S}_n} \bigotimes_{\beta} \bigoplus_{\text{Hom}_{\text{Fin}}(J_{\beta}, K''_{\beta})} W(J_{\beta}). \end{aligned}$$

Note that the right side is non zero if and only if $I_\alpha \cong K'_\alpha$ for all α and $K''_\beta \cong J_\beta$ for all β . This implies that $(\text{Ind}V \boxtimes_c \text{Ind}W)(S, E) = 0$ if $|S| \neq |E|$.

$$\begin{aligned}
 & \left(\text{Ind}V \boxtimes_c^{\text{Val}} \text{Ind}W \right)(S, E) \\
 & \cong \bigoplus_{n \in \mathbb{N}^*} \bigoplus_{\substack{(\{I, K'\}, \{K'', J\}) \\ \in \mathbb{X}^{n, \text{conn}}(S, E)}} \prod_{\alpha} \text{Hom}_{\text{Fin}}(K'_\alpha, I_\alpha) \times \prod_{\beta} \text{Hom}_{\text{Fin}}(J_\beta, K''_\beta) \bigotimes_{\mathfrak{S}_n} V(K'_\alpha) \bigotimes_{\mathfrak{S}_n} W(J_\beta) \\
 & \cong \bigoplus_{r, s, n \in \mathbb{N}^*} \bigoplus_{\substack{I \in Y_r(S), J \in Y_s(E) \\ (K', K'') \in \mathcal{X}_{r, s}^{\text{conn}}([1, n]) \\ I_\alpha \cong K'_\alpha, K''_\beta \cong J_\beta}} \bigoplus_{\text{Hom}_{\text{Fin}}([1, n], S) \times \text{Hom}_{\text{Fin}}(E, [1, n])} \bigotimes_{\mathfrak{S}_n} V(K'_\alpha) \bigotimes_{\mathfrak{S}_n} W(J_\beta) \\
 & \cong \bigoplus_{r, s, n \in \mathbb{N}^*} \bigoplus_{\substack{I \in Y_r(S), J \in Y_s(E) \\ (K', K'') \in \mathcal{X}_{r, s}^{\text{conn}}([1, n]) \\ I_\alpha \cong K'_\alpha, K''_\beta \cong J_\beta}} \bigoplus_{\text{Hom}_{\text{Fin}}(E, S)} \bigotimes_{\alpha} V(K'_\alpha) \bigotimes_{\mathfrak{S}_n} W(J_\beta) \\
 & \cong \bigoplus_{\text{Hom}_{\text{Fin}}(E, S)} \bigoplus_{r, s, n \in \mathbb{N}^*} \bigoplus_{\substack{I \in Y_r(S), J \in Y_s(E) \\ (K', K'') \in \mathcal{X}_{r, s}^{\text{conn}}([1, n]) \\ I_\alpha \cong K'_\alpha, K''_\beta \cong J_\beta}} \bigotimes_{\alpha} V(K'_\alpha) \bigotimes_{\mathfrak{S}_n} W(J_\beta) \\
 & \cong \bigoplus_{\text{Hom}_{\text{Fin}}(E, S)} \bigoplus_{r, s \in \mathbb{N}^*} \bigoplus_{(I, J) \in \mathcal{X}_{r, s}^{\text{conn}}(E)} \bigotimes_{\alpha} V(I_\alpha) \bigotimes_{\mathfrak{S}_n} W(J_\beta) \\
 & \cong \text{Ind} (V \boxtimes_c W)(S, E).
 \end{aligned}$$

□

Corollary 4.17. *The functor $\text{Res} : (\mathfrak{S}\text{-bimod}_k^{\text{red}}, \boxtimes_c^{\text{Val}}) \longrightarrow (\mathfrak{S}\text{-mod}_k^{\text{red}}, \boxtimes_c)$ is Lax-monoidal.*

Proof. By the Proposition 4.5. □

5. PROTOPERADS

In this section, we study monoids in the monoidal category $(\mathfrak{S}\text{-mod}_k^{\text{red}}, \boxtimes_c, I_{\boxtimes})$.

Definition 5.1 (Protoperad). A *protoperad* is an unital monoid (P, μ, η) in the monoidal category $(\mathfrak{S}\text{-mod}_k^{\text{red}}, \boxtimes_c)$: we note by

$$\text{protoperads}_k := \mathcal{UAs}(\mathfrak{S}\text{-mod}_k^{\text{red}}, \boxtimes_c, I_{\boxtimes}),$$

the category of protoperads.

Proposition 5.2. *The functor Ind induces the functor*

$$\text{Ind} : \text{protoperads}_k \longrightarrow \text{properads}_k.$$

Proof. By the Theorem 4.16. □

Remark 5.3. There exists the notion of *prop* which is more general than the notion of properad: a prop is an object of the category

$$\text{props}_k := \mathcal{UAs}(\text{Com}(\mathfrak{S}\text{-bimod}_k, \otimes^{\text{conc}}), \square, I_{\square}),$$

i.e. a \mathfrak{S} -bimodule with two products, a horizontal and a vertical ones, which satisfy the *interchanging law* (see [Mar08]). A natural question is the following: what

structure puts on a \mathfrak{S} -module P such that $\text{Ind}(P)$ is a prop? As the functor Ind is monoidal for \square and \otimes^{conc} , it induces the functor

$$\text{Ind} : \text{protops}_k := \mathcal{UAs}(\text{Com}(\mathfrak{S}\text{-mod}, \otimes^{\text{conc}}), \square, I_\square) \longrightarrow \text{propts}_k.$$

We also have the dual notion.

Definition 5.4 (Coprotoperad). A *coprotoperad* is a co-unital comonoid (Q, Δ, ϵ) in the monoidal category $(\mathfrak{S}\text{-mod}_k^{\text{red}}, \boxtimes_c, I_\boxtimes)$: we note coprotoperads_k , the category $\text{coUAs}(\mathfrak{S}\text{-mod}_k^{\text{red}}, \boxtimes_c, I_\boxtimes)$ of coprotoperads.

Notation 5.5. We note $\mathfrak{S}\text{-mod}_k^{\text{gr}}$, the category $\text{Func}(\text{Fin}^{\text{op}}, \text{Ch}_k^{\text{gr}})$, where Ch_k^{gr} is the category of chain complexes with an \mathbb{N} -grading called the *weight*. All the previous constructions extend naturally to graded \mathfrak{S} -modules. Remark that this grading does not imply Koszul signs: the symmetry of the monoidal category Ch_k^{gr} is given, for C and D , two weight graded chain complexes, by

$$\begin{aligned} \tau_{C,D} : C \otimes D &\longrightarrow D \otimes C \\ c \otimes d &\longmapsto (-1)^{|c||d|} d \otimes c \end{aligned} ,$$

where, for c , an homogeneous element of C , $|c|$ is its homological degree.

Definition 5.6 ((Connected) Weight graded protoperad/coprotoperad). A protoperad (resp. coprotoperad) \mathcal{P} is *weight graded* if \mathcal{P} is a monoid (resp. comonoid) in the category $\mathfrak{S}\text{-mod}_k^{\text{red,gr}}$ for the product \boxtimes_c . We denote this grading by $\mathcal{P} = \bigoplus_{i \in \mathbb{N}} \mathcal{P}^{[i]}$. We say that a weight graded (co)protoperad $\mathcal{P} = \bigoplus_{i \in \mathbb{N}} \mathcal{P}^{[i]}$ is *connected* if $\mathcal{P}^{[0]} \cong I_\boxtimes$.

5.1. Partial compositions. One can describe (cf. [LV12, Sect. 5.3.4]) the operad structure on a \mathfrak{S} -module \mathcal{O} just by giving the partial compositions maps $\circ_s : \mathcal{O}(S) \otimes \mathcal{O}(R) \rightarrow \mathcal{O}(R \amalg S \setminus \{s\})$. We have a similar property for protoperads

Definition 5.7 (Partial compositions). Let P be a reduced \mathfrak{S} -module equipped with a morphism of \mathfrak{S} -modules $\epsilon : I_\boxtimes \hookrightarrow P$. Let M, N and S be three non-empty finite sets, with two diagrams of injections as follows:

$$\varphi := (i : M \hookrightarrow S \hookleftarrow N : j) \text{ and } \varphi^{\text{op}} := (j : N \hookrightarrow S \hookleftarrow M : i)$$

and such that

$$(5) \quad \begin{cases} \text{im}(i) \cup \text{im}(j) &= S \\ \text{im}(i) \cap \text{im}(j) &\neq \emptyset \end{cases} .$$

We say that P has a *partial composition system* if, for all diagrams φ , we have a morphism of chain complexes

$$\circ_\varphi : P(M) \otimes P(N) \longrightarrow P(S) ,$$

compatible with the action of $\text{Aut}(S)$, i.e. for all $\sigma \in \text{Aut}(S)$ with

$$\sigma \cdot \varphi := (i' : M' := \sigma(i(M)) \hookrightarrow S \hookleftarrow \sigma(j(N)) =: N' : j')$$

we have the commutative diagram

$$(6) \quad \begin{array}{ccc} P(M) \otimes P(N) & \xrightarrow[\cong]{P(\sigma|_M) \otimes P(\sigma|_N)} & P(M') \otimes P(N') \\ \circ_\varphi \downarrow & & \downarrow \circ_{\sigma \cdot \varphi} \\ P(S) & \xrightarrow{P(\sigma)} & P(S) ; \end{array}$$

and which satisfies the following compatibility properties, for all commutative diagram of injections

$$\begin{array}{ccccc}
 M & & N & & U \\
 & \searrow & & \swarrow & \\
 & \varphi & & \psi & \\
 & & R & & S \\
 & & & \searrow & \\
 & & & \xi & \\
 & & & & T
 \end{array}$$

with $\xi_L := M \rightarrow T \leftarrow S$ and $\xi_R := R \rightarrow T \leftarrow U$, such that the four pairs of arrows φ, ψ, ξ_L and ξ_R satisfy the Equation (5). The partial composition satisfies the three associativity axioms:

Axiom H: $\left(\begin{array}{c} \square \\ \square \end{array} \right)$

$$\begin{array}{ccc}
 P(M) \otimes P(N) \otimes P(U) & \xrightarrow{1 \otimes \overset{\circ}{\psi}} & P(M) \otimes P(S) \\
 \overset{\circ}{\varphi} \otimes 1 \downarrow & & \downarrow \overset{\circ}{\xi_L} \\
 P(R) \otimes P(U) & \xrightarrow{\overset{\circ}{\xi_R}} & P(T) ;
 \end{array}$$

Axiom V: $\left(\begin{array}{c} \square \\ \square \end{array} \right)$

$$\begin{array}{ccc}
 P(N) \otimes P(M) \otimes P(U) & \xrightarrow{(\overset{\circ}{\psi} \otimes 1)(1 \otimes \tau)} & P(S) \otimes P(M) \\
 \overset{\circ}{\varphi} \otimes 1 \downarrow & & \downarrow \overset{\circ}{\xi_L^{\text{op}}} \\
 P(R) \otimes P(U) & \xrightarrow{\overset{\circ}{\xi_R}} & P(T) ;
 \end{array}$$

Axiom Λ : $\left(\begin{array}{c} \square \\ \square \end{array} \right)$

$$\begin{array}{ccc}
 P(M) \otimes P(U) \otimes P(N) & \xrightarrow{1 \otimes \overset{\circ}{\psi} \otimes 1} & P(M) \otimes P(S) \\
 (1 \otimes \overset{\circ}{\varphi})(\tau \otimes 1) \downarrow & & \downarrow \overset{\circ}{\xi_L} \\
 P(U) \otimes P(R) & \xrightarrow{\overset{\circ}{\xi_R^{\text{op}}}} & P(T) .
 \end{array}$$

The partial products also satisfy the following unital property for all diagrams of the form

$$\iota := (i : \{*\} \hookrightarrow M \xleftarrow{\cong} M : \text{id}),$$

we have commutative diagrams

$$\begin{array}{ccc}
 I_{\boxtimes}(\{*\}) \otimes P(M) & \xrightarrow{\epsilon \otimes \text{id}} & P(\{*\}) \otimes P(M) \\
 & \searrow \cong & \downarrow \overset{\circ}{\iota} \\
 & & P(M)
 \end{array}$$

and

$$\begin{array}{ccc}
 P(M) \otimes I_{\boxtimes}(\{*\}) & \xrightarrow{\text{id} \otimes \epsilon} & P(M) \otimes P(\{*\}) \\
 & \searrow \cong & \downarrow \overset{\circ}{i} \\
 & & P(M)
 \end{array} .$$

Proposition 5.8. *A protoperad \mathcal{P} has canonically a partial compositions system. Conversely, a partial compositions system on a \mathfrak{S} -module P canonically extends to a protoperad structure.*

Proof. Let (P, μ) be a monoid in the category $(\mathfrak{S}\text{-mod}_k^{\text{red}}, \boxtimes_c, I_{\boxtimes})$. By the grading of \boxtimes_c which is implied by the analyticity of \boxtimes_c (cf. Lemma 2.9), the restriction of the product μ to $(P \boxtimes_c P)^{(2)^P}$ gives us directly all the partial compositions \circ_{φ} and the associativity and the unit of the product imply all diagrams of the definition hold.

Conversely, let P be a reduced \mathfrak{S} -module, with an injection of \mathfrak{S} -modules $I_{\boxtimes} \hookrightarrow P$ and a partial composition system. By the associativity of partial compositions \circ_{φ} for P , we define, for all $K \in \mathcal{Y}_m^{\text{or}}(S)$, $L \in \mathcal{Y}_n^{\text{or}}(S)$ with $\mathcal{K}(K, L) = \{S\}$, a morphism

$$\tilde{\mu}_{K,L} : \bigotimes_{i=1}^m P(K_i) \otimes \bigotimes_{j=1}^n P(L_j) \longrightarrow P(S).$$

The compatibility of the partial compositions with the action of the automorphism group of the target (cf. Equation (6)) implies that the following morphism

$$\sum_{(\sigma, \tau) \in \mathfrak{S}_m \times \mathfrak{S}_n} \tilde{\mu}_{K,L} : \bigoplus_{(\sigma, \tau) \in \mathfrak{S}_m \times \mathfrak{S}_n} \bigotimes_{i=1}^m P(K_{\sigma(i)}) \otimes \bigotimes_{j=1}^n P(L_{\tau(j)}) \longrightarrow P(S)$$

passes to the quotient

$$\mu_S : \bigoplus_{(K,L) \in \mathcal{X}^{\text{conn}}(S)} \bigotimes_{\alpha \in A} P(K_{\alpha}) \otimes \bigotimes_{\beta \in B} P(L_{\beta}) \longrightarrow P(S),$$

which gives us a natural transformation $\mu : P \boxtimes_c P \longrightarrow P$. This natural transformation makes P a unital associative monoid in \boxtimes_c , because the partial products satisfy the associativity and unital axioms. \square

Valette's works [Val09] gives us the construction of the free (co)monoid in a abelian monoidal category as $(\mathfrak{S}\text{-mod}_k^{\text{red}}, \boxtimes_c, I_{\boxtimes})$. We briefly review this construction in the next section.

5.2. Free monoid in abelian monoidal categories. We briefly recall the construction of the free monoid $\mathcal{F}(-)$ by Vallette in [Val03] for general abelian monoidal category and [Val09].

Let (\mathbf{A}, \odot, I) be an abelian monoidal category such that, for all objects A in \mathbf{A} , the endofunctors of \mathbf{A} R_A and L_A , given by $R_A(M) := M \odot A$ and $L_A(M) := A \odot M$ for all object $M \in \mathbf{A}$, preserve reflexive coequalizers and sequential colimits (cf. [Val09]). Fix an object V in the category \mathbf{A} . We consider the augmented object $V_+ := I \oplus V$ and we denote by $\eta_V : I \hookrightarrow V_+$, the injection of I in V_+ ; $\epsilon_V : V_+ \rightarrow I$, the projection of V_+ to I ; $i_V : V \rightarrow V_+$, the injection of V into V_+ :

$$\begin{array}{ccc} I & \xleftarrow{\eta_V} & I \oplus V & \xleftarrow{i_V} & V \\ & \searrow & \downarrow \epsilon_V & & \\ & = & I & & \end{array} .$$

For all natural numbers n , we denote $V_n := (V_+)^{\odot n}$, and $\lambda_A : I \odot A \xrightarrow{\cong} A$ and $\rho_A : A \odot I \xrightarrow{\cong} A$, the structure isomorphisms of the monoidal category (\mathbf{A}, \odot, I) .

Proof. Let V be an object of \mathbf{C} . By hypothesis, the functor G commutes with colimits, in particular, with coproducts: so we have

$$G(V_+) \cong G(V) \oplus G(I_{\mathbf{C}}) \cong G(V) \oplus I_{\mathbf{D}} = G(V)_+$$

and, as G is monoidal, we have that, for all integers n , $G(V_n) \cong G(V)_n$. Since G preserves cokernels, for all objects A and B in the category \mathbf{A} , we have

$$G(R_{A,B}) \cong R_{G(A),G(B)}$$

and, for all integers n , $G(\widetilde{V}_n) \cong \widetilde{G(V)}_n$. Finally, as G commutes with colimits, we have

$$G(\mathcal{F}_{\mathbf{C}}(V)) \cong \mathcal{F}_{\mathbf{D}}(G(V)).$$

□

5.3. A first description of the free protoperad. We use the results of the previous section to describe the free protoperad $\mathcal{F}(V)$ over a \mathfrak{S} -module V . This first is not very explicit, in the sense that the underlying combinatorics is hidden by the formalism. Remark 5.19 and Section 1.2 will remedy this problem.

To a partition K of a finite set S , i.e. K is an element of $\mathcal{Y}(S)$ (see Section 1.3), we associate the non-ordered set $\Gamma(K)$ which labelled this partition: $S \cong \coprod_{\alpha \in \Gamma(K)} K_{\alpha}$. Recall that the functor $\mathbb{S}(-)$ is non-unitary and satisfies the exponential property (cf. Equation (4)): then, for two \mathfrak{S} -modules V_1 and V_2 , we have the following isomorphism of \mathfrak{S} -modules:

$$\mathbb{S}(V_1 \oplus V_2) \cong \mathbb{S}(V_1) \oplus \mathbb{S}(V_2) \oplus \mathbb{S}(V_1) \otimes^{\text{conc}} \mathbb{S}(V_2).$$

Notation 5.12. Let V be a \mathfrak{S} -module, we denote by the exponent $(-)_V$, the weight-grading by the number of terms V .

The functor $\mathbb{S}(-)$ is split analytic (cf. [Val03, Val09] for the definition of split analytic functor), so that, for three \mathfrak{S} -modules V_1 , V_2 and V_3 , we have the weight-bigrading:

$$\begin{aligned} \mathbb{S}(V_1 \oplus V_2) \square \mathbb{S}(V_3) &\cong \mathbb{S}V_1 \square \mathbb{S}V_3 \oplus \mathbb{S}V_2 \square \mathbb{S}V_3 \oplus (\mathbb{S}V_1 \otimes^{\text{conc}} \mathbb{S}V_2) \square \mathbb{S}V_3 \\ &\cong \bigoplus_{i,j \in \mathbb{N}^*} \mathbb{S}^j V_1 \square \mathbb{S}V_3 \oplus \mathbb{S}^j V_2 \square \mathbb{S}V_3 \oplus (\mathbb{S}^i V_1 \otimes^{\text{conc}} \mathbb{S}^j V_2) \square \mathbb{S}V_3 \\ &\stackrel{\text{not}}{=} \bigoplus_{i,j \in \mathbb{N}^*} (\mathbb{S}(V_1 \oplus V_2) \square \mathbb{S}(V_3))^{(i)_{V_1}, (j)_{V_2}} \end{aligned}$$

by the bi-additivity of the bifunctors $-\square-$ and $-\otimes^{\text{conc}}-$.

This bigrading induces, via the injection $(V_1 \oplus V_2) \boxtimes_c V_3 \hookrightarrow \mathbb{S}(V_1 \oplus V_2) \square \mathbb{S}(V_3)$ (cf. Proposition 2.11), the bigrading by weight of V_1 and V_2 on $(V_1 \oplus V_2) \boxtimes_c V_3$ which is denoted by

$$(V_1 \oplus V_2) \boxtimes_c V_3 =: \bigoplus_{i,j \in \mathbb{N}^*} ((V_1 \oplus V_2) \boxtimes_c V_3)^{(i)_{V_1}, (j)_{V_2}}.$$

By the symmetry of the product \boxtimes_c , we also have the bigrading

$$V_3 \boxtimes_c (V_1 \oplus V_2) =: \bigoplus_{i,j \in \mathbb{N}^*} (V_3 \boxtimes_c (V_1 \oplus V_2))^{(i)_{V_1}, (j)_{V_2}},$$

and we denote $((V_1 \oplus V_2) \boxtimes_c V_3)^{(j)_{V_2}} := \bigoplus_{i \in \mathbb{N}^*} ((V_1 \oplus V_2) \boxtimes_c V_3)^{(i)_{V_1}, (j)_{V_2}}$.

Remark 5.13. These gradings are natural: they arise from the split analytic property of the bifunctor $-\boxtimes_c-$.

Remark 5.14. In [MV09a] and [Val03], for two \mathfrak{S} -bimodules \mathcal{M} and \mathcal{P} , the weight grading on \mathcal{M} is denoted by

$$\underbrace{\mathcal{M} \boxtimes_c \mathcal{P}}_r.$$

As the functor Ind is monoidal and commutes with the direct sum, we have

$$\text{Ind}(((V_1 \oplus V_2) \boxtimes_c V_3)^{(r)v_2}) = (\text{Ind}(V_1) \oplus \underbrace{\text{Ind}(V_2)}_r) \boxtimes_c \text{Ind}(V_3).$$

As we have seen in Section 5.2, the construction of the free monoid generated by a \mathfrak{S} -module V is based on the formal addition of the unit I_{\boxtimes} to V . We consider the \mathfrak{S} -module $V_+ = V \oplus I_{\boxtimes}$, so that $V_+(S) = V(S)$ for $|S| \neq 1$ and $V_+(\{*\}) = V(\{*\}) \oplus k$. We also need the weight bigrading of the \mathfrak{S} -module $V_+ \boxtimes_c W_+$ given by the weight grading on V and the weight grading on W . For all finite sets S , this bigrading allows us to write the product $V_+ \boxtimes_c W_+(S)$ as a direct sum of terms with i copies of V and j copies of W . More precisely, the \mathfrak{S} -module $\mathbb{S}(V_+) \square \mathbb{S}(W_+)$ is bigraded by weights in V and W , and, via the injection

$$V_+ \boxtimes_c W_+ \hookrightarrow \mathbb{S}(V_+ \boxtimes_c W_+) \underset{(2.11)}{\cong} \mathbb{S}(V_+) \square \mathbb{S}(W_+),$$

the \mathfrak{S} -module $V_+ \boxtimes_c W_+$ naturally inherits a weight bigrading in V and W . To express $V_+ \boxtimes_c W_+(S)$ as a sum of terms indexed by the bigrading, we require the following notation.

Notation 5.15. Recall that, for all non-empty set S and for all pairs (I, J) in $\mathcal{Y}_2^{\text{or}}(S)$, by definition, we have that I and J are also non-empty; we note:

$$\mathcal{Y}_2^{\text{or},+}(S) \stackrel{\text{not}}{:=} \mathcal{Y}_2^{\text{or}}(S) \cup \{(S, \emptyset), (\emptyset, S)\}.$$

For a non-ordered partition $K \in \mathcal{Y}_n(S)$ with n terms, we want to distinguish the components of V and these with the unit I_{\boxtimes} in $\bigotimes_{\alpha \in \Gamma(K)} V_+(K_\alpha)$ (where $\Gamma(K)$ is the non-ordered set labeling the partition K). So, we introduce the following functor:

$$\mathfrak{Q}_{(R_1^K, R_2^K)}^V : \widetilde{\Gamma(K)} \longrightarrow \text{Ch}_k,$$

where $\widetilde{\Gamma(K)}$ is the discrete category on the set $\Gamma(K)$. For each (R_1^K, R_2^K) in $\mathcal{Y}_2^{\text{or},+}(\Gamma(K))$, the functor $\mathfrak{Q}_{(R_1^K, R_2^K)}^V$ associates, for all K_α with α in $\Gamma(K)$, a chain complex as follows:

$$\mathfrak{Q}_{(R_1^K, R_2^K)}^V(K_\alpha) := \begin{cases} V(K_\alpha) & \text{if } \alpha \in R_1^K \\ I_{\boxtimes}(K_\alpha) & \text{if } \alpha \in R_2^K \end{cases}.$$

Remark that, for α in R_2^K such that $|K_\alpha| \geq 2$, the complex $\mathfrak{Q}_{(R_1^K, R_2^K)}^V(K_\alpha)$ is zero.

So we decompose $V_+ \boxtimes_c W_+$ as follow: for a finite set S , we have

$$\begin{aligned} V_+ \boxtimes_c W_+(S) &= ((V \oplus I_{\boxtimes}) \boxtimes_c (W \oplus I_{\boxtimes}))(S) \\ &\stackrel{2.8}{=} \bigoplus_{(K,L) \in \mathcal{X}^{\text{conn}}(S)} \bigotimes_{\alpha \in \Gamma(K)} (V \oplus I_{\boxtimes})(K_\alpha) \otimes \bigotimes_{\beta \in \Gamma(L)} (W \oplus I_{\boxtimes})(L_\beta) \\ &\cong \bigoplus_{\substack{(K,L) \in \mathcal{X}^{\text{conn}}(S) \\ (R_1^K, R_2^K) \in \mathcal{Y}_2^{\text{or},+}(\Gamma(K)) \\ (R_1^L, R_2^L) \in \mathcal{Y}_2^{\text{or},+}(\Gamma(L))}} \bigotimes_{\alpha \in \Gamma(K)} \Omega_{(R_1^K, R_2^K)}^V(K_\alpha) \otimes \bigotimes_{\beta \in \Gamma(L)} \Omega_{(R_1^L, R_2^L)}^W(L_\beta). \end{aligned}$$

We collect terms by the number of copies of V and W , then the $\text{Aut}(S)$ -module $V_+ \boxtimes_c W_+(S)$ is isomorphic to

$$\begin{aligned} &\bigoplus_{\substack{(r,s) \in \mathbb{N}^2 \\ (K,L) \in \mathcal{X}^{\text{conn}}(S)}} \bigoplus_{\substack{(R_1^K, R_2^K) \in \mathcal{Y}_2^{\text{or},+}(\Gamma(K)) \\ (R_1^L, R_2^L) \in \mathcal{Y}_2^{\text{or},+}(\Gamma(L)) \\ |R_1^K| = r, |R_1^L| = s}} \bigotimes_{\alpha \in \Gamma(K)} \Omega_{(R_1^K, R_2^K)}^V(K_\alpha) \otimes \bigotimes_{\beta \in \Gamma(L)} \Omega_{(R_1^L, R_2^L)}^W(L_\beta) \\ &\cong I_{\boxtimes}(S) \oplus W(S) \oplus V(S) \\ &\oplus \bigoplus_{\substack{(r,s) \in (\mathbb{N}^*)^2 \\ (K,L) \in \mathcal{X}^{\text{conn}}(S)}} \bigoplus_{\substack{((R_1^K, R_2^K), (R_1^L, R_2^L)) \in \Xi_{K,L} \\ |R_1^K| = r, |R_1^L| = s}} \bigotimes_{\alpha \in R_1^K} V(K_\alpha) \otimes \bigotimes_{\beta \in R_1^L} W(L_\beta) \end{aligned}$$

where

$$\Xi_{K,L} := \left\{ \left((R_1^K, R_2^K), (R_1^L, R_2^L) \right) \in \begin{array}{l} \left(\mathcal{Y}_2^{\text{or}}(\Gamma(K)) \cup (\Gamma(K), \emptyset) \right) \\ \times \left(\mathcal{Y}_2^{\text{or}}(\Gamma(L)) \cup (\Gamma(L), \emptyset) \right) \end{array} \left| \begin{array}{l} \forall \beta \in R_2^K, |K_\beta| = 1, \\ \forall \beta \in R_2^L, |L_\beta| = 1 \end{array} \right. \right\},$$

which gives us the bigrading. This last isomorphism is given by distinguishing terms arising from the injections $I_{\boxtimes} \boxtimes_c W \hookrightarrow V_+ \boxtimes_c W_c$, $V \boxtimes_c I_{\boxtimes} \hookrightarrow V_+ \boxtimes_c W_c$ and $I_{\boxtimes} \boxtimes_c I_{\boxtimes} \hookrightarrow V_+ \boxtimes_c W_c$. This gives the following.

Lemma 5.16. *The isomorphism of $\text{Aut}(S)$ -modules between $V_+ \boxtimes_c W_+(S)$ and*

$$\bigoplus_{(K,L) \in \mathcal{X}^{\text{conn}}(S)} \bigoplus_{\substack{(R_1^K, R_2^K) \in \mathcal{Y}_2^{\text{or},+}(\Gamma(K)) \\ (R_1^L, R_2^L) \in \mathcal{Y}_2^{\text{or},+}(\Gamma(L)) \\ |R_1^K| = r, |R_1^L| = s}} \bigotimes_{\alpha \in \Gamma(K)} \Omega_{(R_1^K, R_2^K)}^V(K_\alpha) \otimes \bigotimes_{\beta \in \Gamma(L)} \Omega_{(R_1^L, R_2^L)}^W(L_\beta)$$

gives the bigrading of $V_+ \boxtimes_c W_+(S) = \bigoplus_{(r,s) \in \mathbb{N}^2} (V_+ \boxtimes_c W_+)^{(r)v, (s)w}$ where, for integers r and s in \mathbb{N}^* , the term $(V_+ \boxtimes_c W_+)^{(r)v, (s)w}(S)$ is isomorphic to

$$\bigoplus_{(K,L) \in \mathcal{X}^{\text{conn}}(S)} \bigoplus_{\substack{((R_1^K, R_2^K), (R_1^L, R_2^L)) \in \Xi_{K,L} \\ |R_1^K| = r, |R_1^L| = s}} \bigotimes_{\alpha \in R_1^K} V(K_\alpha) \otimes \bigotimes_{\beta \in R_1^L} W(L_\beta)$$

with

$$\Xi_{K,L} := \left\{ \left((R_1^K, R_2^K), (R_1^L, R_2^L) \right) \in \begin{array}{l} \left(\mathcal{Y}_2^{\text{or}}(\Gamma(K)) \cup (\Gamma(K), \emptyset) \right) \\ \times \left(\mathcal{Y}_2^{\text{or}}(\Gamma(L)) \cup (\Gamma(L), \emptyset) \right) \end{array} \left| \begin{array}{l} \forall \beta \in R_2^K, |K_\beta| = 1, \\ \forall \beta \in R_2^L, |L_\beta| = 1 \end{array} \right. \right\},$$

$$\begin{aligned} \bigoplus_{s \in \mathbb{N}^*} (V_+ \boxtimes_c W_+)^{(0)v, (s)w}(S) &\cong (V_+ \boxtimes_c W_+)^{(0)v, (1)w}(S) = W(S); \\ \bigoplus_{r \in \mathbb{N}^*} (V_+ \boxtimes_c W_+)^{(r)v, (0)w}(S) &\cong (V_+ \boxtimes_c W_+)^{(1)v, (0)w}(S) = V(S) \text{ and} \\ (V_+ \boxtimes_c W_+)^{(0)v, (0)w}(S) &= I_{\boxtimes}(S). \end{aligned}$$

We describe $(V_+ \boxtimes_c W_+)^{(1)v, (1)w}$ explicitly.

Proposition 5.17 (Case of $(V_+ \boxtimes_c W_+)^{(1)v, (1)w}$). *Let V and W be two reduced \mathfrak{S} -modules. For all finite sets S , we have the following isomorphism:*

$$(V_+ \boxtimes_c W_+)^{(1)v, (1)w}(S) \cong \bigoplus_{\substack{K, L \subset S \\ K \cup L = S \\ K \cap L \neq \emptyset}} V(K) \otimes W(L).$$

Proof. The $\text{Aut}(S)$ -module $(V_+ \boxtimes_c W_+)^{(1)v, (1)w}(S)$ is isomorphic to

$$\begin{aligned} &\bigoplus_{(K, L) \in \mathcal{X}^{\text{conn}}(S)} \bigoplus_{\substack{(R_1^K, R_2^K) \in \mathcal{Y}_2^{\text{or}, +}(\Gamma(K)) \\ (R_1^L, R_2^L) \in \mathcal{Y}_2^{\text{or}, +}(\Gamma(L)) \\ |R_1^K| = 1, |R_1^L| = 1}} \bigotimes_{\alpha \in \Gamma(K)} \mathfrak{Q}_{(R_1^K, R_2^K)}^V(K_\alpha) \otimes \bigotimes_{\beta \in \Gamma(L)} \mathfrak{Q}_{(R_1^L, R_2^L)}^W(L_\beta) \\ &\cong \bigoplus_{\substack{(K, L) \in \mathcal{X}^{\text{conn}}(S) \\ \exists a \in A, b \in B \mid \forall \alpha \in \Gamma(K) \setminus \{a\}, \beta \in \Gamma(L) \setminus \{b\} \\ K_\alpha \cong \{*\} \cong L_\beta}} V(K_a) \otimes W(L_b) \end{aligned}$$

since, if there exists α in $A \setminus \{a\}$ such that $K_\alpha \not\cong \{*\}$, then $I_{\boxtimes}(K_\alpha) = 0$; likewise, for $B \setminus \{b\}$. Finally, we rewrite $(V_+ \boxtimes_c W_+)^{(1)v, (1)w}(S)$ as follows:

$$(V_+ \boxtimes_c W_+)^{(1)v, (1)w}(S) \cong \bigoplus_{\substack{K, L \subset S \\ K \cup L = S \\ K \cap L \neq \emptyset}} V(K) \otimes W(L).$$

□

When we take $V = W$, the bigrading of $V_+ \boxtimes_c W_+$ induces a weight grading of $V_+ \boxtimes_c V_+$: for a finite set S , we have $V_+ \boxtimes_c V_+(S)$, which is isomorphic to

$$\begin{aligned} &\bigoplus_{\substack{\rho \in \mathbb{N}^* \\ (K, L) \in \mathcal{X}^{\text{conn}}(S)}} \bigoplus_{\substack{(R_1^K, R_2^K) \in \mathcal{Y}_2^{\text{or}, +}(\Gamma(K)) \\ (R_1^L, R_2^L) \in \mathcal{Y}_2^{\text{or}, +}(\Gamma(L)) \\ |R_1^K| + |R_1^L| = \rho}} \bigotimes_{\alpha \in \Gamma(K)} \mathfrak{Q}_{(R_1^K, R_2^K)}^V(K_\alpha) \otimes \bigotimes_{\beta \in \Gamma(L)} \mathfrak{Q}_{(R_1^L, R_2^L)}^V(L_\beta) \\ &\cong \underbrace{I_{\boxtimes}(S) \oplus V(S) \otimes \bigotimes_S I_{\boxtimes}(\ast) \oplus \bigotimes_S I_{\boxtimes}(\ast) \otimes V(S)}_{\stackrel{\text{not}}{=} (V_+ \boxtimes_c V_+)^{(1)v}(S)} \\ &\oplus \underbrace{\bigoplus_{\rho \in \mathbb{N} \setminus \{0, 1\}} \bigoplus_{(K, L) \in \mathcal{X}^{\text{conn}}(S)} \bigoplus_{\substack{((R_1^K, R_2^K), (R_1^L, R_2^L)) \in \Xi_{K, L} \\ |R_1^K| + |R_1^L| = \rho}} \bigotimes_{\alpha \in R_1^K} V(K_\alpha) \otimes \bigotimes_{\beta \in R_1^L} V(L_\beta)}_{\stackrel{\text{not}}{=} (V_+ \boxtimes_c V_+)^{(\rho)v}(S)} \end{aligned}$$

where we recall that $\Xi_{K, L} := \{((R_1^K, R_2^K), (R_1^L, R_2^L)) \in (\mathcal{Y}_2^{\text{or}}(\Gamma(K)) \cup (\Gamma(K), \emptyset)) \times (\mathcal{Y}_2^{\text{or}}(\Gamma(L)) \cup (\Gamma(L), \emptyset)) \mid \forall \beta \in R_2^K, |K_\beta| = 1, \forall \beta \in R_2^L, |L_\beta| = 1\}$. More generally,

the \mathfrak{S} -module $V_n := (V_+)^{\boxtimes_c n}$ is weight-graded in V ; for all finite set S , we have

$$\begin{aligned}
V_n(S) & \stackrel{def}{=} (V_+)^{\boxtimes_c n}(S) \\
& = \bigoplus_{\substack{(J^1, \dots, J^n) \in \mathcal{Y}^n(S) \\ \mathcal{K}_S^{n-1}(J^1, \dots, J^n) = S \\ ((R_1^{J^1}, R_2^{J^1}), \dots, (R_1^{J^n}, R_2^{J^n})) \\ \in \prod_{i \in [1, n]} \mathcal{Y}_2^{\text{or}, +}(\Gamma(J^i))}} \bigotimes_{\alpha \in \Gamma(J^1)} \mathfrak{Q}_{(R_1^{J^1}, R_2^{J^1})}^V(J_\alpha^{J^1}) \otimes \dots \otimes \bigotimes_{\alpha \in \Gamma(J^n)} \mathfrak{Q}_{(R_1^{J^n}, R_2^{J^n})}^V(J_\alpha^{J^n}) \\
& \cong \bigoplus_{\rho \in \mathbb{N}} \bigoplus_{\substack{(J^1, \dots, J^n) \in \mathcal{Y}^n(S) \\ \mathcal{K}_S^{n-1}(J^1, \dots, J^n) = S \\ ((R_1^{J^i}, R_2^{J^i}))_i \\ \in \prod_{i \in [1, n]} \mathcal{Y}_2^{\text{or}, +}(\Gamma(J^i)) \\ \sum_{i=1}^n |R_1^{J^i}| = \rho}} \bigotimes_{\alpha \in \Gamma(J^1)} \mathfrak{Q}_{(R_1^{J^1}, R_2^{J^1})}^V(J_\alpha^{J^1}) \otimes \dots \otimes \bigotimes_{\alpha \in \Gamma(J^n)} \mathfrak{Q}_{(R_1^{J^n}, R_2^{J^n})}^V(J_\alpha^{J^n})
\end{aligned}$$

and, if we note by $V_n^{(\rho)V}$ the following:

$$\bigoplus_{\substack{l \in [1, n] \\ \{r_1, \dots, r_l\} \subset [1, n] \\ (J^{r_1}, \dots, J^{r_l}) \in \mathcal{Y}^l(S) \\ \mathcal{K}_S^{l-1}(J^{r_1}, \dots, J^{r_l}) = S}} \bigoplus_{\substack{((R_1^{J^{r_1}}, R_2^{J^{r_1}}), \dots, (R_1^{J^{r_l}}, R_2^{J^{r_l}})) \in \Xi_{(J^{r_1}, \dots, J^{r_l})} \\ \sum_{i \in \{r_1, \dots, r_l\}} |R_1^{J^i}| = \rho}} \bigotimes_{\alpha \in R_1^{J^{r_1}}} V(J_\alpha^{r_1}) \otimes \dots \otimes \bigotimes_{\alpha \in R_1^{J^{r_l}}} V(J_\alpha^{r_l})$$

where the set $\Xi_{(J^1, \dots, J^n)}$ is the following

$$\left\{ \begin{array}{l} ((R_1^{J^1}, R_2^{J^1}), \dots, (R_1^{J^n}, R_2^{J^n})) \\ \in \prod_{i \in [1, n]} (\mathcal{Y}_2^{\text{or}}(\Gamma(J^i)) \cup (\Gamma(J^i), \emptyset)) \end{array} \middle| \forall i \in [1, n], \forall \beta \in R_2^{J^i}, |J_\beta^i| = 1 \right\},$$

then we have

$$V_n(S) \cong I_{\boxtimes}(S) \oplus V_n^{(\rho)V}.$$

The isomorphisms $\lambda_V : I_{\boxtimes} \boxtimes_c V \rightarrow$ and $\rho_V : V \boxtimes_c I_{\boxtimes} \rightarrow V$ preserve the grading because $V \boxtimes_c I_{\boxtimes} = (V \boxtimes_c I_{\boxtimes})^{(1)V}$, and preserve all the constructions which are in the construction of \tilde{V}_n (cf. Section 5.2), then the grading of V_n carries on \tilde{V}_n . Also, the injection $\tilde{V}_n \hookrightarrow \tilde{V}_{n+1}$ preserves the weight-grading, so, finally, the free monoid $\mathcal{F}(V)$ is weight-graded by the number of copies of V .

Proposition 5.18 (First description of the free protoperad $\mathcal{F}(V)$). *Let S be a finite set and V be a reduced \mathfrak{S} -module. We have the isomorphism of chain complexes:*

$$\mathcal{F}(V)(S) \cong I_{\boxtimes}(S) \oplus \bigoplus_{n \in \mathbb{N}^*} \bigoplus_{\substack{(J^1, \dots, J^n) \in \mathcal{Y}^n(S) \\ \mathcal{K}_S^{n-1}(J^1, \dots, J^n) = S}} \bigoplus_{\Xi_{(J^1, \dots, J^n)} \alpha \in R_1^{J^1}} \bigotimes V(J_\alpha^{J^1}) \otimes \dots \otimes \bigotimes_{\alpha \in R_1^{J^n}} V(J_\alpha^{J^n}),$$

where

$$\tilde{\Xi}_{(J^1, \dots, J^n)} := \left\{ \begin{array}{l} ((R_1^{J^1}, R_2^{J^1}), \dots, (R_1^{J^n}, R_2^{J^n})) \\ \in \prod_{i \in [1, n]} (\mathcal{Y}_2^{\text{or}}(\Gamma(J^i)) \cup (\Gamma(J^i), \emptyset)) \end{array} \middle| \begin{array}{l} \forall i \in [1, n-1], \forall \beta \in R_1^{J^{i+1}}, \\ J_\beta^{i+1} \cap \prod_{\alpha \in R_1^{J^i}} J_\alpha^i \neq \emptyset \\ \forall i \in [1, n], \forall \beta \in R_2^{J^i}, |J_\beta^i| = 1 \end{array} \right\}.$$

Moreover, the free protoperad is weight-graded:

$$\mathcal{F}(V) \cong \bigoplus_{\rho \in \mathbb{N}} \mathcal{F}^{(\rho)}(V)$$

with $\mathcal{F}^{(0)}(V) = I_{\boxtimes}(V)$ and, for all integers ρ in \mathbb{N}^* , the ρ -weighted part $\mathcal{F}^{(\rho)}(V)(S)$ is isomorphic to:

$$\bigoplus_{\substack{n \in \mathbb{N}^* \\ \mathcal{K}_S^{n-1}(J^1, \dots, J^n) = S}} \bigoplus_{(J^1, \dots, J^n) \in \mathcal{Y}^n(S)} \bigoplus_{\substack{\Xi_{(J^1, \dots, J^n)} \\ \sum_{i=1}^n |R_1^{J^i}| = \rho}} \bigotimes_{\alpha \in R_1^{J^1}} V(J_\alpha^1) \otimes \dots \otimes \bigotimes_{\alpha \in R_1^{J^n}} V(J_\alpha^n).$$

Proof. This isomorphism corresponds to the choice of a representative for the quotient $V_n \twoheadrightarrow \tilde{V}_n$, as we will see below. We define the morphism $\tau_V : V \rightarrow V_2$ as the following composition:

$$V \xrightarrow{\lambda_V^{-1} + \rho_V^{-1}} I_{\boxtimes} \boxtimes_c V \oplus V \boxtimes_c I_{\boxtimes} \xrightarrow{\eta \boxtimes_c i_V - i_V \boxtimes_c \eta} (I_{\boxtimes} \oplus V) \boxtimes_c (I_{\boxtimes} \oplus V) =: V_2$$

=:\tau

with $\eta : I_{\boxtimes} \hookrightarrow I_{\boxtimes} \oplus V$, which appears in the definition of $R_{A,B}$ for all reduced \mathfrak{S} -modules A and B : the \mathfrak{S} -module $R_{A,B}$ is defined as the image of the composition:

$$A \boxtimes_c (V \oplus V_2) \boxtimes_c B \xrightarrow{\Xi_{LA,B}} A \boxtimes_c (V \oplus V_2) \boxtimes_c B \xrightarrow{A \boxtimes_c (\tau + \text{id}_{V_2}) \boxtimes_c B} A \boxtimes_c V_2 \boxtimes_c B.$$

We also define the \mathfrak{S} -module \tilde{V}_n as the cokernel of the morphism

$$\bigoplus_{i=0}^{n-2} R_{V_i, V_{n-i-2}} \longrightarrow V_n.$$

The quotient \tilde{V}_n corresponds to the identification of the images of morphisms $(\text{id}_{V_i} \boxtimes_c \eta \boxtimes_c i_V \boxtimes_c \text{id}_{V_{n-i-2}}) \circ \iota_{V_i, V_{n-i-2}}$ and $(\text{id}_{V_i} \boxtimes_c i_V \boxtimes_c \eta \boxtimes_c \text{id}_{V_{n-i-2}}) \circ \iota_{V_i, V_{n-i-2}}$, for all $i \in \llbracket 1, n \rrbracket$, in V_n . We choose to identify each class of \tilde{V}_n with an element of the image of $\sum_{i \in \llbracket 0, n-2 \rrbracket} (\text{id}_{V_i} \boxtimes_c \eta \boxtimes_c i_V \boxtimes_c \text{id}_{V_{n-i-2}}) \circ \iota_{V_i, V_{n-i-2}}$. Then, for all finite sets S , we have

$$\begin{aligned} \tilde{V}_n(S) \cong & \bigoplus_{\substack{h \in \llbracket 1, n \rrbracket \\ \mathcal{K}_S^{h-1}(J^1, \dots, J^h) = S}} \bigoplus_{(J^1, \dots, J^h) \in \mathcal{Y}^h(S)} \bigoplus_{\substack{\Xi_{(J^1, \dots, J^h)} \\ \alpha \in R_1^{J^1}}} \bigotimes_{\alpha \in R_1^{J^1}} V(J_\alpha^1) \otimes \dots \\ & \otimes \bigotimes_{\alpha \in R_1^{J^h}} V(J_\alpha^h) \otimes \underbrace{\bigotimes_S I(*) \otimes \dots \otimes \bigotimes_S I(*)}_{n-h \text{ terms}}, \end{aligned}$$

where

$$\tilde{\Xi}_{(J^1, \dots, J^h)} := \left\{ \begin{array}{l} ((R_1^{J^1}, R_2^{J^1}), \dots, (R_1^{J^h}, R_2^{J^h})) \\ \in \prod_{i \in \llbracket 1, h \rrbracket} (\mathcal{Y}_2^{\text{gr}}(\Gamma(J^i)) \cup (\Gamma(J^i), \emptyset)) \end{array} \middle| \begin{array}{l} \forall i \in \llbracket 1, h-1 \rrbracket, \forall \beta \in R_1^{J^{i+1}}, \\ J_\beta^{i+1} \cap \prod_{\alpha \in R_1^{J^i}} J_\alpha^i \neq \emptyset; \\ \forall i \in \llbracket 1, h \rrbracket, \forall \beta \in R_2^{J^i}, |J_\beta^i| = 1 \end{array} \right\}.$$

Then, for all integers $n > 1$, we have

$$\tilde{V}_n(S) \cong \tilde{V}_{n-1}(S) \oplus \bigoplus_{\substack{(J^1, \dots, J^n) \in \mathcal{Y}^n(S) \\ \mathcal{K}_S^{n-1}(J^1, \dots, J^n) = S}} \bigoplus_{\substack{\Xi_{(J^1, \dots, J^n)} \\ \alpha \in R_1^{J^1}}} \bigotimes_{\alpha \in R_1^{J^1}} V(J_\alpha^1) \otimes \dots \otimes \bigotimes_{\alpha \in R_1^{J^n}} V(J_\alpha^n),$$

which exactly describes the injections $\tilde{V}_{n-1} \hookrightarrow \tilde{V}_n$. \square

Remark 5.19 (Description of $\mathcal{F}^{(2)}(V)$). We have the isomorphism

$$(V_+ \boxtimes_c V_+)^{(2)v} \cong (V_+ \boxtimes_c V_+)^{(1)v, (1)v},$$

then, by Proposition 5.17 and Proposition 5.18, we have an explicit description of the sub- \mathfrak{S} -module of weight 2 of the free protoperad:

$$\mathcal{F}^{(2)}(V) \cong \bigoplus_{\substack{K, L \subset S \\ K \cup L = S \\ K \cap L \neq \emptyset}} V(K) \otimes V(L).$$

By the definition of the free monoid in $(\mathfrak{S}\text{-mod}_k^{\text{red}}, \boxtimes_c, I_{\boxtimes})$, the following proposition and corollary follow from Proposition 5.11 and [Val03, Prop. 28]. Thereby, the functor Ind sends a protoperad defined by generators \mathcal{G} and relations \mathcal{R} to a properad defined by generators $\text{Ind}(\mathcal{G})$ and relations $\text{Ind}(\mathcal{R})$

Proposition 5.20. (1) *The functor Ind commutes with the functors \mathcal{F} and \mathcal{F}^{Val} , i.e.*

$$\mathcal{F}^{\text{Val}}(\text{Ind}(-)) \cong \text{Ind}(\mathcal{F}(-)),$$

where \mathcal{F}^{Val} is the functor of free properad (see [Val03, Val07]).

(2) *Let V be a reduced \mathfrak{S} -module and R be a sub- \mathfrak{S} -module of the free monoid $\mathcal{F}(V)$. Then, we have the isomorphism:*

$$\text{Ind}(\mathcal{F}(V)/\langle R \rangle) \cong \mathcal{F}(\text{Ind}(V)) / \langle \text{Ind}(R) \rangle.$$

Proof. This follows from Theorem 4.16, Proposition 4.4, [Val03, Prop. 28] and the construction of the free monoid \square

We have seen that the description of the free protoperad is rather complicated, so we prefer a combinatorial description generalizing the identification of Remark 5.19. This is the purpose of the next subsection.

5.4. Combinatorial version of the free protoperad. We begin by defining the category $(\text{Fin}^{\text{op}})_{\mathcal{W}_n^{\text{or}}}$ which has for objects the pairs (S, x) with S , a finite set and x an element of $\mathcal{W}_n^{\text{or}}(S)$. A morphism from (S, x) to (T, y) is a morphism $\varphi : S \rightarrow T$ in the category Fin^{op} such that $\mathcal{W}_n^{\text{or}}(\varphi)(x) = y$. We have the two canonical functors:

- the functor $\text{For} : (\text{Fin}^{\text{op}})_{\mathcal{W}_n^{\text{or}}} \rightarrow (\text{Fin}^{\text{op}})^{\times n}$ is defined, for an object (S, X) , by $\text{For}_n(S, x) = \bar{x}$ with \bar{x} the n -tuple of sets underlying x and, on morphisms, by $\text{For}_n(\varphi) = \mathcal{W}_n^{\text{or}}(\varphi)$;
- the projection functor $\pi : (\text{Fin}^{\text{op}})_{\mathcal{W}_n} \rightarrow \text{Fin}^{\text{op}}$ defined by $\pi(S, x) = S$.

Fix a reduced \mathfrak{S} -module P : by the external product of \mathfrak{S} -modules, we define the functor

$$P^{\times n} : (\text{Fin}^{\text{op}})^{\times n} \longrightarrow \text{Ch}_k.$$

Then, we have the functors

$$\hat{P} : \begin{array}{ccc} (\text{Fin}^{\text{op}})_{\mathcal{W}_n^{\text{or}}} & \longrightarrow & \text{Ch}_k \\ (S, x) & \longmapsto & P^{\times n} \circ \text{For}_n(S, x) \end{array} ;$$

and

$$P\mathcal{W}_n^{\text{or}} : \begin{array}{ccc} \text{Fin}^{\text{op}} & \longrightarrow & \text{Ch}_k \\ S & \longmapsto & \bigoplus_{x \in \mathcal{W}_n^{\text{or}}(S)} P^{\times n} \circ \text{For}_n(S, x) \end{array} .$$

Finally, for all finite sets S , we have

$$P^{\mathcal{W}_n^{\text{or}}}(S) = \bigoplus_{((K_1, \dots, K_n), \leq) \in \mathcal{W}_n^{\text{or}}(S)} P(K_1) \otimes \dots \otimes P(K_n).$$

The free action of \mathfrak{S}_n on $P^{\times n}$ acts on $P^{\mathcal{W}_n^{\text{or}}}$, by the symmetry of $(\text{Ch}_k, \otimes, \sigma)$: then, we obtain the functor

$$\begin{aligned} (P^{\mathcal{W}_n^{\text{or}}})_{\mathfrak{S}_n}(S) &= \left(\bigoplus_{((K_1, \dots, K_n), \leq) \in \mathcal{W}_n^{\text{or}}(S)} P(K_1) \otimes \dots \otimes P(K_n) \right)_{\mathfrak{S}_n} \\ &\cong \bigoplus_{(\{K_\alpha\}_{\alpha \in A}, \leq) \in \mathcal{W}_n(S)} \left(\bigoplus_{(K_1, \dots, K_n) \in \{K_\alpha\}_{\alpha \in A}} P(K_1) \otimes \dots \otimes P(K_n) \right)_{\mathfrak{S}_n} \end{aligned}$$

which we denote as follows:

$$P^{\mathcal{W}_n}(S) \stackrel{\text{not.}}{:=} \bigoplus_{(\{K_\alpha\}_{\alpha \in A}, \leq) \in \mathcal{W}_n(S)} \bigotimes_{\alpha \in A} P(K_\alpha).$$

The construction works for the functor of connected walls $\mathcal{W}_n^{\text{conn,ord}}$, which gives us the functor

$$P^{\mathcal{W}_n^{\text{conn}}}(S) := \bigoplus_{(\{K_\alpha\}_{\alpha \in A}, \leq) \in \mathcal{W}_n^{\text{conn}}(S)} \bigotimes_{\alpha \in A} P(K_\alpha).$$

We define the partial composition product. Let φ be a diagram of injections $i : S \hookrightarrow R \hookleftarrow T : j$ with $\text{im}(i) \cap \text{im}(j) \neq \emptyset$ and $\text{im}(i) \cup \text{im}(j) = R$. We have, by Proposition 1.8, the morphism

$$\begin{aligned} P^{\mathcal{W}_m^{\text{or}}}(S) \otimes P^{\mathcal{W}_n^{\text{or}}}(T) &= \left(\bigoplus_{((K_1, \dots, K_m), \leq_K) \in \mathcal{W}_m^{\text{or}}(S)} P(K_1) \otimes \dots \otimes P(K_m) \right) \\ &\quad \otimes \left(\bigoplus_{((L_1, \dots, L_n), \leq_L) \in \mathcal{W}_n^{\text{or}}(T)} P(L_1) \otimes \dots \otimes P(L_n) \right) \\ &\cong \bigoplus_{\substack{((K_1, \dots, K_m), \leq_K) \in \mathcal{W}_m^{\text{or}}(S) \\ ((L_1, \dots, L_n), \leq_L) \in \mathcal{W}_n^{\text{or}}(T)}} P(K_1) \otimes \dots \otimes P(K_m) \otimes P(L_1) \otimes \dots \otimes P(L_n) \\ &\rightarrow \bigoplus_{\substack{(i(K_1), \dots, i(K_m), \\ j(L_1), \dots, j(L_n), \leq_{i(K)}^{j(L)}) \\ \in \mathcal{W}_{m+n}^{\text{or}}(R)}} P(K_1) \otimes \dots \otimes P(K_m) \otimes P(L_1) \otimes \dots \otimes P(L_n), \end{aligned}$$

where $\leq_{i(K)}^{j(L)}$ is defined as follows: we have on $\cup_a i(K_a)$ (resp. $\cup_b j(L_b)$) the partial order $\leq_{i(K)}$ (resp. $\leq_{j(L)}$) induced by that of K (resp. L), i.e. $i(K_a) \leq_{i(K)}^{j(L)} i(K_b)$ if $K_a \leq_K K_b$ (resp. $j(L_a) \leq_{i(K)}^{j(L)} j(L_b)$ if $L_a \leq_L L_b$) which gives to

$(K_1, \dots, K_m, L_1, \dots, L_n)$, the order $\leqslant_i^{j(L)}$, by Lemma 1.3. Thus we have the morphism

$$P^{\mathcal{W}_m^{\text{or}}}(S) \otimes P^{\mathcal{W}_n^{\text{or}}}(T) \longrightarrow P^{\mathcal{W}_{m+n}}(R) = \bigoplus_{(\{H_\alpha\}_{\alpha \in A, \leqslant H}) \in \mathcal{W}_{m+n}(R)} \bigotimes_{\alpha \in A} P(H_\alpha)$$

which factorizes through $P^{\mathcal{W}_m}(S) \otimes P^{\mathcal{W}_n}(T)$, giving the partial composition product

$$\circlearrowleft_\varphi : P^{\mathcal{W}_m}(S) \otimes P^{\mathcal{W}_n}(T) \longrightarrow P^{\mathcal{W}_{m+n}}(R).$$

As $\text{im}(i) \cap \text{im}(j) \neq \emptyset$, this partial composition product can be restricted to the connected version: we have the partial composition product

$$\circlearrowleft_\varphi : P^{\mathcal{W}_m^{\text{conn}}}(S) \otimes P^{\mathcal{W}_n^{\text{conn}}}(T) \longrightarrow P^{\mathcal{W}_{m+n}^{\text{conn}}}(R).$$

These partial products make $P^{\mathcal{W}^{\text{conn}}} := \prod_n P^{\mathcal{W}_n^{\text{conn}}}$, a protoperad by Proposition 5.8.

Theorem 5.21 (Description of the free protoperad). *Let V be a reduced \mathfrak{S} -module and ρ be an integer in \mathbb{N}^* . We have, for all finite sets S , the isomorphism of (right) $\text{Aut}(S)$ -modules*

$$\mathcal{F}^\rho(V)(S) \cong \bigoplus_{(\{K_\alpha\}_{\alpha \in A, \leqslant}) \in \mathcal{W}_\rho^{\text{conn}}(S)} \bigotimes_{\alpha \in A} V(K_\alpha).$$

Proof. By Proposition 5.18, for all ρ in \mathbb{N}^* , we have the isomorphism

$$\mathcal{F}^{(\rho)}(V)(S) \cong \bigoplus_{n \in \mathbb{N}^*} \bigoplus_{\substack{(J^1, \dots, J^n) \in \mathcal{Y}^n(S) \\ \mathcal{K}_S^{n-1}(J^1, \dots, J^n) = S}} \bigoplus_{\substack{\tilde{\Xi}_{(J^1, \dots, J^n)} \\ \sum_{i=1}^n |R_1^{J^i}| = \rho}} \bigotimes_{\alpha \in R_1^{J^1}} V(J_\alpha^1) \otimes \dots \otimes \bigotimes_{\alpha \in R_1^{J^n}} V(J_\alpha^n).$$

with

$$\tilde{\Xi}_{(J^1, \dots, J^n)} := \left\{ \begin{array}{l} ((R_1^{J^1}, R_2^{J^1}), \dots, (R_1^{J^n}, R_2^{J^n})) \\ \in \prod_{i \in [1, n]} (\mathcal{Y}_2^{\text{or}}(\Gamma(J^i)) \cup (\Gamma(J^i), \emptyset)) \end{array} \middle| \begin{array}{l} \forall i \in [1, n-1], \forall \beta \in R_1^{J^{i+1}}, \\ J_\beta^{i+1} \cap \prod_{\alpha \in R_1^{J^i}} J_\alpha^i \neq \emptyset \\ \forall i \in [1, n], \forall \beta \in R_2^{J^i}, |J_\beta^i| = 1 \end{array} \right\}.$$

Let (J^1, \dots, J^n) in $\mathcal{Y}^n(S)$ such that $\mathcal{K}_S^{n-1}(J^1, \dots, J^n) = S$ and $\tilde{\Xi}_{(J^1, \dots, J^n)} \neq \emptyset$; we associate to (J^1, \dots, J^n) the wall W in $\mathcal{W}_\rho^{\text{conn}}(S)$ with sets $\{J_{\alpha^i}^i \subset S \mid i \in [1, n], \alpha^i \in R_1^{J^i}\}$ and the partial order induced by the relations $J_\alpha^i < J_\beta^j$ if $J_\alpha^i \cap J_\beta^j \neq \emptyset$, $\alpha \in R_1^{J^i}$, $\beta \in R_1^{J^j}$ and $i < j$. So we have the following morphism of $\text{Aut}(S)$ -modules:

$$\Phi : \mathcal{F}^\rho(V)(S) \longrightarrow \bigoplus_{(\{W_\alpha\}_{\alpha \in A, \leqslant}) \in \mathcal{W}_\rho^{\text{conn}}(S)} \bigotimes_{\alpha \in A} V(W_\alpha).$$

Conversely, to a connected wall $W = (\{W_\alpha \mid \alpha \in A\}, \leqslant)$ with ρ bricks, i.e. W is in $\mathcal{W}_\rho^{\text{conn}}(S)$, with $\max_{\alpha \in A} (\mathfrak{h}(W_\alpha)) = n$ (where $\mathfrak{h} : W \rightarrow \mathbb{N} \cup \{\infty\}$ is the height in the poset W , see Section 1.1), we associate an element $(J^1, \dots, J^n) \in \mathcal{Y}^n(S)$ such that $\tilde{\Xi}_{(J^1, \dots, J^n)} \neq \emptyset$ as follows. We construct partitions J^i as the sets $\{W_{\alpha^i} \in W \mid \mathfrak{h}(W_\alpha) = i\}$, extended to a partition by singletons: so we have

$$J^i := \{W_{\alpha^i} \in W \mid \mathfrak{h}(W_\alpha) = i\} \amalg \{\{s\} \mid s \notin \amalg_{\alpha^i} W_{\alpha^i}\} = \{J_\beta^i \mid \beta \in B = \Gamma(J^i)\}$$

and the decomposition $(R_1^{J^i}, R_2^{J^i}) \in \mathcal{Y}_2^{\text{or}}(\Gamma(J^i) \cup \Gamma(J^i), \emptyset)$ is given by the definition of J^i :

$$\beta \in \begin{cases} R_1^{J^i} & \text{if } J_\beta^i \in \{W_{\alpha^i} \in W \mid \mathfrak{h}(W_\alpha) = i\}, \\ R_2^{J^i} & \text{otherwise.} \end{cases}$$

The connectedness of the wall W implies that the element (J^1, \dots, J^n) also satisfies the property of connectedness

$$\mathcal{K}_S^{n-1}(J^1, \dots, J^n) = S.$$

Finally, we have the following morphism of $\text{Aut}(S)$ -modules :

$$\Psi : \bigoplus_{\substack{(\{W_\alpha\}_{\alpha \in A, \leq}) \\ \in \mathcal{W}_\rho^{\text{conn}}(S)}} \bigotimes_{\alpha \in A} V(W_\alpha) \longrightarrow \mathcal{F}^\rho(V)(S)$$

which satisfies $\Phi \circ \Psi = \text{id}$ and $\Psi \circ \Phi = \text{id}$. \square

6. COLOURS ON WALLS

In this section, we associate to a wall W in $\mathcal{W}^{\text{conn}}$, a chain complex, called the *colouring complex*: it is determined by the combinatorics of the wall W . This is a combinatorial introduction to some results of [Ler18]: the colouring complex of a connected wall W over S encodes a part of the differential of the bar construction of a free protoperad (see [Ler18, Sect. 2]).

6.1. Coloured bricks. In this subsection, we define the notion of a colouring of a wall: throughout this section, we consider S , a non-empty finite set.

Definition 6.1 (Colouring). Let (W, \leq_W) in $\mathcal{W}(S)$ be a wall over S . A (connected) C -colouring of W is a surjective morphism of sets $\varphi : W \rightarrow C$, where C is called *the set of colours*, satisfying the following assertions:

- (1) the binary relation \leq_φ induced on C by the partial order of W , defined, for all c_1, c_2 in C , by

$$c_1 \leq_\varphi c_2 \text{ if } \exists k_1 \in \varphi^{-1}(c_1), k_2 \in \varphi^{-1}(c_2) \text{ such that } k_1 \leq_W k_2;$$

is a partial order;

- (2) the fibers of φ are connected, i.e. for each colour c in C , the set $\varphi^{-1}(c)$ belongs to $\mathcal{W}^{\text{conn}}(S_c)$ with $S_c := \bigcup_{W_\alpha \in \varphi^{-1}(c)} W_\alpha$.

We denote by $\text{Succ}(\varphi)$ or $\text{Succ}(C)$ the set of successive colours. Two colouring $\varphi : W \rightarrow C$ and $\psi : W \rightarrow D$ of a wall W are isomorphic if there exists an isomorphism of posets $\Phi : (C, \leq_\varphi) \rightarrow (D, \leq_\psi)$ such that the following diagram commutes:

$$\begin{array}{ccc} & W & \\ \varphi \swarrow & & \searrow \psi \\ C & \xrightarrow[\Phi]{\cong} & D \end{array}$$

We denote by $\text{Col}(W)$ the set of isomorphism classes of colourings of W :

$$\text{Col}(W) := \{\varphi : W \rightarrow C \mid \varphi \text{ a colouring}\} / \cong,$$

which is graded by the number of colours:

$$\text{Col}_\bullet(W) = \coprod_{n \in \mathbb{N}^*} \text{Col}_n(W) \text{ with } \text{Col}_n(W) := \{\varphi \in \text{Col}(W) \mid |\varphi(W)| = n\}.$$

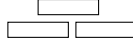
Remark 6.2. (1) As any colouring $\varphi : W \rightarrow C$ is a surjective map, for any colour c in C , the set of C -coloured bricks is non empty, i.e. $\varphi^{-1}(c) \neq \emptyset$. Furthermore, as W is a wall over S , i.e. $W \in \mathcal{W}(S)$, we have $\bigcup_{c \in C} S_c = S$ with

$$S_c := \bigcup_{W_\alpha \in \varphi^{-1}(c)} W_\alpha.$$

- (2) $\text{Col}_{|W|}(W)$ is reduced to a unique element.
- (3) If $n > |W|$, then $\text{Col}_n(W) = \emptyset$, as in example 6.3.
- (4) Let W be a non-connected wall over S . The decomposition into connected parts (cf. proposition 1.13) of $W = W^1 \amalg \dots \amalg W^l$ implies that we have the following graded product:

$$\text{Col}_\bullet(W) = \prod_{i=1}^l \text{Col}_\bullet(W^i).$$

Example 6.3 (Top-colouring). For any wall W , the identity morphism $W \rightarrow W$ defines the *top-colouring*, in which each brick of W has a different colour. For example, we represent the top-colouring of the following wall



by the coloured diagram:



Example 6.4 (Bot-colouring). For any connected wall, the projection to a point $W \rightarrow \{*\}$ defines a colouring called the *bot-colouring*, denoted by bot_W , which colours all the bricks of W the same colour; for example, we represent diagrammatically the bot-colouring of the previous wall by:



If $W \in \mathcal{W}(S)$ is a non connected wall over S , then, by the proposition 1.13, we have its decomposition in connected component $W^1 \amalg \dots \amalg W^n$, and the *bot-colouring* of W , also denoted by bot_W , is given by $\text{bot}_W = \text{bot}_{W^1} \amalg \dots \amalg \text{bot}_{W^n}$; for example:



Non-example 6.5. We consider the wall $W = \{W_a, W_b, W_c, W_d\}$ in $\mathcal{W}^{\text{conn}}(\llbracket 1, 4 \rrbracket)$ over $S = \llbracket 1, 4 \rrbracket$ with

$$W_a = \{1, 2\}, W_b = \{3, 4\}, W_c = \{2, 3\} \text{ and } W_d = \{1, 4\}$$

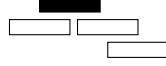
and the partial order given by $W_a < W_c$, $W_a < W_d$, $W_b < W_c$ and $W_b < W_d$. We consider the surjective map $f : W \rightarrow \{w, b\}$ which maps W_a and W_c to b and W_b and W_d to w : we diagrammatically represent f by



However, f does not define a colouring of W , because the binary relation \leq_f induced by the order of W is not a partial order. For the same reason, the following coloured diagram:



is not the diagram of a colouring. The diagram



is not the diagram of a colouring, because the white sub-wall is not connected.

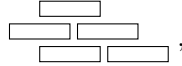
Lemma 6.6. *Let S be a non-empty finite set, W in $\mathcal{W}(S)$, a wall over S and $\varphi : W \rightarrow C$, a colouring of W with $\text{Succ}(C) \neq \emptyset$ (which implies $|C| > 1$). For any pair $(c_1 < c_2) \in \text{Succ}(C)$ of successive colours, the composition*

$$\begin{array}{ccc} W & \xrightarrow{\varphi} & C \\ \exists! \tilde{\varphi} \searrow & & \downarrow \pi_{c_1}^{c_2} \\ & & C / c_1 \sim c_2 \end{array}$$

where $\pi_{c_1}^{c_2}$ identified the two colours c_1 and c_2 , define a colouring $\tilde{\varphi}$ of W .

Proof. By Proposition 1.2: □

Example 6.7. We consider the wall W with five bricks represented by



and the colouring $\varphi : W \rightarrow \{w, b, g\}$ (with w for white, g for gray and b for black), diagrammatically represented by:



with $g \leq b \leq w$. As g and b are two successive colours, φ induces a colouring $\tilde{\varphi} : W \rightarrow C / g \sim b$ with two colours, which is diagrammatically represented by:



The colouring φ induces another colouring $\hat{\varphi} : W \rightarrow C / b \sim w$ represented by:



On the other hand, as G and W are not successive, the map $\tilde{\varphi} : W \rightarrow C / g \sim w$ does not define a colouring of W : the diagram



is not a colouring diagram.

Lemma 6.8. *Let S be a non-empty finite set, W in $\mathcal{W}(S)$ a wall (resp. W in $\mathcal{W}^{\text{conn}}(S)$ a connected wall) over S and $\varphi : W \rightarrow C$ a colouring of W . We note \sim_φ , the equivalence relation of W induced by φ , i.e. for k and l , two elements of W , we have $k \sim_\varphi l$ if $\varphi(k) = \varphi(l)$. Then W / \sim_φ is a wall (resp. W / \sim_φ is a connected wall) over S .*

Proof. By Proposition 1.2 and the definition of a colouring. □

6.2. The colouring complex.

Definition/Proposition 6.9. Consider $(S, <_S)$, a finite *totally* ordered set: for a wall $(W, <_W)$ in $\mathcal{W}(S)$ over S , we can extend as follows the partial order $<_W$ to a total order \prec_W on W , induced by that of S . For W_a and W_b in W , we have $W_a \prec_W W_b$, if:

- $W_a \cap W_b \neq \emptyset$ and $W_a <_W W_b$ (because $W_a \cap W_b \neq \emptyset$ implies that W_a and W_b are comparable for $<_W$);
- $W_a \cap W_b = \emptyset$ and $\mathfrak{h}(W_a) <_{\mathbb{N}} \mathfrak{h}(W_b)$ with $\mathfrak{h}(W_\alpha)$ the height of the brick W_α in the wall W (cf. Section 1.1);
- $W_a \cap W_b = \emptyset$, $\mathfrak{h}(W_a) = \mathfrak{h}(W_b)$ and $\min(W_a) <_S \min(W_b)$.

Let $\varphi : W \rightarrow C$ be a colouring of W , a wall over S . As the order $<_W$ induces a partial order $<_\varphi$ on C , by definition of a colouring, the total order \prec_W induces a total order on C denoted by \prec_φ :

$$c_1 \prec_\varphi c_2 \text{ if } \exists k_1 \in \varphi^{-1}(c_1), k_2 \in \varphi^{-1}(c_2) \text{ such that } k_1 \prec_W k_2;$$

Lemma 6.10. For a connected wall W in $\mathcal{W}^{\text{conn}}(S)$ and a colouring φ in $\text{Col}(W)$, the set of pairs of successive colours $\text{Succ}(\varphi)$ has a total order \prec_φ defined as follows: for $c = (c_1, c_2)$ and $d = (d_1, d_2)$, two elements of $\text{Succ}(\varphi)$, we have $c \prec_\varphi d$ if

$$\min_{\prec_W} \left(\varphi^{-1}(c_1) \cup \varphi^{-1}(c_2) \right) \prec_W \min_{\prec_W} \left(\varphi^{-1}(d_1) \cup \varphi^{-1}(d_2) \right).$$

By this lemma, we index the projections π_c^d by integers: if $(c < d) \in \text{Succ}(\varphi)$ is the i -th element (for the total order \prec_φ), we note $\partial_i := \pi_c^d$. Furthermore, we observe that $\partial_i \partial_j = \partial_{j-1} \partial_i$ for all $i <_{\mathbb{N}} j$, as in a semi-simplicial set. However, we will see (cf. Example 6.14) that the set of colouring of a wall W is *not* a semi-simplicial set, but we can still associate a chain complex to the poset of colourings of a wall.

Definition/Proposition 6.11. Let S be a finite totally ordered set. For a connected wall W over S , the \mathbb{Z} -linearization of the graded set $\text{Col}_\bullet(W)$ gives a chain complex, called the *colouring complex*, denoted by $C_\bullet^{\text{col}}(W)$, where the differential is given by

$$(W, \varphi) \xrightarrow{\partial^{\text{col}}} \sum_{(c < d) \in \text{Succ}(\varphi)} (-1)^\Lambda (W, \pi_c^d \circ \varphi)$$

with

$$\begin{aligned} \Lambda := & \# \{ x \in \varphi(W) \mid x \prec_\varphi d \text{ and } x \neq c \} \\ & + \# \{ x \in \varphi(W) \mid c \prec_\varphi x \prec_\varphi d \text{ and } (x < d) \in \text{Succ}(\varphi) \}. \end{aligned}$$

Remark 6.12. The condition $c \prec_\varphi x \prec_\varphi d$ implies $x \neq c$ and $x \neq d$: the inequalities are strict.

Proof. We need to prove that $\partial^{\text{col}} \circ \partial^{\text{col}} = 0$. We just need to understand what are the signs of terms $\pi_a^b \pi_c^d$ and $\pi_c^d \pi_a^b$ in $\partial^{\text{col}} \circ \partial^{\text{col}}$. We consider two pairs of successive colours $(a < b) \prec_\varphi (c < d)$ for φ , a colouring of W .

- We start with two pairs of successive colours $(a < b) \prec_\varphi (c < d)$ such that $b \neq d$. Then, in the composition $\partial^{\text{col}} \circ \partial^{\text{col}}$, we have the contribution

$$\begin{aligned}
 & (-1)^{\Lambda_1} \pi_a^b \pi_c^d + (-1)^{\Lambda_2} \pi_c^d \pi_a^b, \text{ with} \\
 \Lambda_1 = & \#\{x \in \varphi(W) \mid x \prec_\varphi d \text{ and } x \neq c\} \\
 & + \#\{x \in \varphi(W) \mid c \prec_\varphi x \prec_\varphi d \text{ and } (x < d) \in \text{Succ}(\varphi)\} \\
 & + \#\{x \in \pi_c^d \circ \varphi(W) \mid x \prec_{\pi_c^d \circ \varphi} b \text{ and } x \neq a\} \\
 & + \#\{x \in \pi_c^d \circ \varphi(W) \mid a \prec_{\pi_c^d \circ \varphi} x \prec_{\pi_c^d \circ \varphi} b \text{ and } (x < b) \in \text{Succ}(\pi_c^d \circ \varphi)\} \\
 = & \#\{x \in \varphi(W) \mid x \prec_\varphi d \text{ and } x \neq c\} \\
 & + \#\{x \in \varphi(W) \mid c \prec_\varphi x \prec_\varphi d \text{ and } (x < d) \in \text{Succ}(\varphi)\} \\
 & + \#\{x \in \varphi(W) \mid x \prec_\varphi b \text{ and } x \neq a\} \\
 & + \#\{x \in \varphi(W) \mid a \prec_\varphi x \prec_\varphi b \text{ and } (x < b) \in \text{Succ}(\varphi)\}
 \end{aligned}$$

and

$$\begin{aligned}
 \Lambda_2 = & \#\{x \in \varphi(W) \mid x \prec_\varphi b \text{ and } x \neq a\} \\
 & + \#\{x \in \varphi(W) \mid a \prec_\varphi x \prec_\varphi b \text{ and } (x < b) \in \text{Succ}(\varphi)\} \\
 & + \#\{x \in \pi_a^b \circ \varphi(W) \mid x \prec_{\pi_a^b \circ \varphi} d \text{ and } x \neq c\} \\
 & + \#\{x \in \pi_a^b \circ \varphi(W) \mid c \prec_{\pi_a^b \circ \varphi} x \prec_{\pi_a^b \circ \varphi} d \text{ and } (x < d) \in \text{Succ}(\pi_a^b \circ \varphi)\} \\
 = & \#\{x \in \varphi(W) \mid x \prec_\varphi b \text{ and } x \neq a\} \\
 & + \#\{x \in \varphi(W) \mid a \prec_\varphi x \prec_\varphi b \text{ and } (x < b) \in \text{Succ}(\varphi)\} \\
 & + \#\{x \in \varphi(W) \mid x \prec_\varphi d \text{ and } x \neq c\} - 1 \\
 & + \#\{x \in \varphi(W) \mid c \prec_\varphi x \prec_\varphi d \text{ and } (x < d) \in \text{Succ}(\varphi)\}
 \end{aligned}$$

so the contribution $(-1)^{\Lambda_1} \pi_a^b \pi_c^d + (-1)^{\Lambda_2} \pi_c^d \pi_a^b$ is null.

- We consider the case which we have $(a < c) \prec_\varphi (b < c)$. The contribution $(-1)^{\Lambda_1} \pi_a^c \pi_b^c + (-1)^{\Lambda_2} \pi_b^c \pi_a^c$ have the signs given by:

$$\begin{aligned}
 \Lambda_1 = & \#\{x \in \varphi(W) \mid x \prec_\varphi c \text{ and } x \neq b\} \\
 & + \#\{x \in \varphi(W) \mid b \prec_\varphi x \prec_\varphi c \text{ and } (x < c) \in \text{Succ}(\varphi)\} \\
 & + \#\{x \in \pi_b^c \circ \varphi(W) \mid x \prec_{\pi_b^c \circ \varphi} c \text{ and } x \neq a\} \\
 & + \#\{x \in \pi_b^c \circ \varphi(W) \mid a \prec_{\pi_b^c \circ \varphi} x \prec_{\pi_b^c \circ \varphi} c \text{ and } (x < c) \in \text{Succ}(\pi_b^c \circ \varphi)\} \\
 = & \#\{x \in \varphi(W) \mid x \prec_\varphi c \text{ and } x \neq b\} \\
 & + \#\{x \in \varphi(W) \mid b \prec_\varphi x \prec_\varphi c \text{ and } (x < c) \in \text{Succ}(\varphi)\} \\
 & + \#\{x \in \varphi(W) \mid x \prec_\varphi c \text{ and } x \neq a\} - 1 \\
 & + \#\{x \in \varphi(W) \mid a \prec_\varphi x \prec_\varphi c \text{ and } (x < c) \in \text{Succ}(\varphi)\} - 1
 \end{aligned}$$

and

$$\begin{aligned}
 \Lambda_2 = & \#\{x \in \varphi(W) \mid x \prec_\varphi c \text{ and } x \neq b\} - 1 \\
 & + \#\{x \in \varphi(W) \mid b \prec_\varphi x \prec_\varphi c \text{ and } (x < c) \in \text{Succ}(\varphi)\} \\
 & + \#\{x \in \varphi(W) \mid x \prec_\varphi c \text{ and } x \neq a\} \\
 & + \#\{x \in \varphi(W) \mid a \prec_\varphi x \prec_\varphi c \text{ and } (x < c) \in \text{Succ}(\varphi)\}
 \end{aligned}$$


so the contribution of $(-1)^{\Lambda_1} \pi_a^c \pi_b^c + (-1)^{\Lambda_2} \pi_b^c \pi_a^c$ is null.

Then, we have $\partial^{\text{col}} \circ \partial^{\text{col}} = 0$. □

Lemma 6.13. *If $W \in \mathcal{W}(S)$ is a non connected wall over a totally ordered S , and $W^1 \amalg \dots \amalg W^l$ is the decomposition in connected component of W , we have*

$$C_{\bullet}^{\text{Col}}(W) \cong \bigotimes_{i=1}^l C_{\bullet}^{\text{Col}}(W^i).$$

Proof. By Remark 6.2 4, and the fact that the functor of linearization preserves coproducts. \square

Example 6.14. Consider the colouring complex of the wall $W \in \mathcal{W}^{\text{conn}}(\llbracket 1, 4 \rrbracket)$ represented by 

- $\text{Col}_4(W)$ contains only the top-colouring of W :



the poset $\text{Succ}(\text{top})$, where we denote by \prec_{top} its total order, is given by:

$$\text{Succ}(\text{top}) = \left\{ \begin{array}{c} \text{grey} \text{ grey} \\ \text{black} \end{array} \prec_{\text{top}} \begin{array}{c} \text{grey} \\ \text{black} \end{array} \prec_{\text{top}} \begin{array}{c} \text{grey} \\ \text{white} \end{array} \right\}$$

where any diagram represent a pair of successive bricks. So, we have the following three arrows $\text{Col}_4(W) \xrightarrow{\partial_i} \text{Col}_3(W)$ represented by:

$$\begin{array}{c} \text{grey} \text{ grey} \\ \text{black} \end{array} \xrightarrow{\partial_1} \begin{array}{c} \text{black} \text{ grey} \\ \text{black} \end{array}; \begin{array}{c} \text{grey} \text{ grey} \\ \text{black} \end{array} \xrightarrow{\partial_2} \begin{array}{c} \text{grey} \\ \text{black} \end{array}; \begin{array}{c} \text{grey} \text{ grey} \\ \text{black} \end{array} \xrightarrow{\partial_3} \begin{array}{c} \text{grey} \text{ white} \\ \text{black} \end{array}.$$

- $\text{Col}_3(W)$ is the set of the following three colourings

$$\begin{array}{c} \text{black} \text{ grey} \\ \text{black} \end{array}, \begin{array}{c} \text{grey} \\ \text{black} \end{array} \text{ and } \begin{array}{c} \text{grey} \text{ white} \\ \text{black} \end{array}.$$

For each colouring in $\text{Col}_3(W)$, we have the set of successive colours:

$$\begin{aligned} \text{Succ} \left(\begin{array}{c} \text{black} \text{ grey} \\ \text{black} \end{array} \right) &= \left\{ \begin{array}{c} \text{black} \text{ grey} \\ \text{black} \end{array} \prec \begin{array}{c} \text{grey} \\ \text{white} \end{array} \right\}; \\ \text{Succ} \left(\begin{array}{c} \text{grey} \\ \text{black} \end{array} \right) &= \left\{ \begin{array}{c} \text{grey} \\ \text{black} \end{array} \prec \begin{array}{c} \text{black} \\ \text{white} \end{array} \right\}; \\ \text{Succ} \left(\begin{array}{c} \text{grey} \text{ white} \\ \text{black} \end{array} \right) &= \left\{ \begin{array}{c} \text{grey} \\ \text{black} \end{array} \prec \begin{array}{c} \text{white} \\ \text{white} \end{array} \right\}; \end{aligned}$$

which are given to us the maps from $\text{Col}_3(W)$ to $\text{Col}_2(W)$.

- $\text{Col}_2(W)$ contains the following colourings:

$$\begin{array}{c} \text{black} \text{ black} \\ \text{black} \end{array}, \begin{array}{c} \text{black} \\ \text{black} \end{array} \text{ and } \begin{array}{c} \text{grey} \text{ black} \\ \text{black} \end{array}.$$

As these colourings just have two colours, the sets of successive colours associated to them are reduced to only one element.

We consider the sub-complex $C_{\bullet}^{\text{Col}, k < l_1}(W)$ which is isomorphic to $C_{\bullet}^{\text{Col}}(W/k \sim l_1)$ which is acyclic by induction. So we consider the following short exact sequence:

$$0 \longrightarrow C_{\bullet}^{\text{Col}, k < l_1}(W) \longrightarrow C_{\bullet}^{\text{Col}}(W) \longrightarrow C_{\bullet}^{\text{Col}}(W)/C_{\bullet}^{\text{Col}, k < l_1}(W) \longrightarrow 0 ;$$

The chain complex $C_{\bullet}^{\text{Col}}(W)/C_{\bullet}^{\text{Col}, k < l_1}(W)$ is isomorphic to $C_{\bullet}^{\text{Col}}(W \setminus \{l_1\})$, so the term on the right hand side of the exact sequence is also acyclic, so the complex $C_{\bullet}^{\text{Col}}(W)$ too.

Now, we suppose that W does not have the exceptional shape Equation (8). We choose $(k < l) \in \text{Succ}(W)$ with $\mathfrak{h}(k) = 1$ and we consider the subcomplex $C_{\bullet}^{\text{Col}, (k < l)}(W) \subset C_{\bullet}^{\text{Col}}(W)$ of colourings φ of W such that $\varphi(k) = \varphi(l)$. We consider $W/k \sim l$, the poset with $(W \setminus \{k, l\}) \cup \{k \cup l\}$, its underlying set with the partial order induced by the W one: for $j \in W$ such that $j > k$ or $j > l$ (resp. $j < k$ or $j < l$) then we have $j > (k \cup l)$ (resp. $j < (k \cup l)$). By the definition of the differential of the colouring complex, we have the following isomorphism of chain complexes:

$$C_{\bullet}^{\text{Col}, k < l}(W) \cong C_{\bullet}^{\text{Col}}(W/k \sim l).$$

As $|W/k \sim l| = |W| - 1$, then $C_{\bullet}^{\text{Col}, k < l}(W)$ is acyclic by induction.

We denote by $\text{Succ}_W(k)$, the set $\{l \in W \mid (k < l) \in \text{Succ}(W)\}$ and we consider the chain complex

$$\sum_{l \in \text{Succ}_W(k)} C_{\bullet}^{\text{Col}, k < l}(W).$$

By Lemma 6.16 below, this complex is acyclic if, for all non-empty subsets $\widetilde{\text{Succ}}(k) \subset \text{Succ}_W(k)$, the complex

$$\bigcap_{l \in \widetilde{\text{Succ}}(k)} C_{\bullet}^{\text{Col}, k < l}(W)$$

is acyclic. Let $\widetilde{\text{Succ}}(k)$ be a non-empty subset of $\text{Succ}_W(k)$: we have the isomorphism of chain complexes

$$\bigcap_{l \in \widetilde{\text{Succ}}(k)} C_{\bullet}^{\text{Col}, k < l}(W) \cong C_{\bullet}^{\text{Col}, k < l}(W/\widetilde{\text{Succ}}(k)).$$

There are two cases: if $|W/\widetilde{\text{Succ}}(k)| = 1$, which means that $\widetilde{\text{Succ}}(k) = \text{Succ}_W(k)$ and so W has the exceptional shape as in Equation (8), which is excluded by the hypothesis. Otherwise $|W/\widetilde{\text{Succ}}(k)| > 1$: in this case, $C_{\bullet}^{\text{Col}, k < l}(W/\widetilde{\text{Succ}}(k))$ is acyclic by induction.

We consider the complex

$$C_{\bullet}^{\text{Col}}(W) / \left(\sum_{l \in \text{Succ}_W(k)} C_{\bullet}^{\text{Col}, k < l}(W) \right).$$

which is isomorphic to $C_{\bullet}^{\text{Col}}(W \setminus \{k\})$. Note that this complex is generally a \otimes -product of complexes because $W \setminus \{k\}$ is not necessarily connected. As we have the short exact sequence

$$0 \rightarrow \sum_{l \in \text{Succ}_W(k)} C_{\bullet}^{\text{Col}, k < l}(W) \rightarrow C_{\bullet}^{\text{Col}}(W) \rightarrow \frac{C_{\bullet}^{\text{Col}}(W)}{\sum_{l \in \text{Succ}_W(k)} C_{\bullet}^{\text{Col}, k < l}(W)} \rightarrow 0 ,$$

with the left and the right hand sides acyclic, the complex $C_{\bullet}^{\text{Col}}(W)$ is too. \square

- Lemma 6.16** (Algebraic Mayer-Vietoris). (1) Let A and B be acyclic chain complexes. If the complex $A \cap B$ is acyclic, then the complex $A + B$ is too.
- (2) More generally, let m be an integer in \mathbb{N}^* and $\{A_i\}_{i \in \llbracket 1, m \rrbracket}$ a sequence of m acyclic complexes. If, for all subsets $J \subset \llbracket 1, m \rrbracket$, the complex $\bigcap_{j \in J} A_j$ is acyclic, then the complex $\sum_{j=1}^m A_j$ is too.

Proof. (1) We have the square

$$\begin{array}{ccc} A \cap B & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & A + B \end{array} \quad \square$$

which induces the short exact sequence

$$0 \longrightarrow A \cap B \longrightarrow A \oplus B \longrightarrow A + B \longrightarrow 0.$$

We conclude the proof by the associated long exact sequence in homology and the additivity of the functor $H_\bullet(-)$.

- (2) We prove the result by induction on m . For $m = 2$, we have (1). Let m be an integer and suppose by induction that for all family A_1, \dots, A_m of m acyclic complexes, the complex $\bigcap_{j \in J} A_j$ is acyclic, then the complex $\sum_{j=1}^m A_j$ is acyclic. Let A_0, \dots, A_m be a family of $m + 1$ acyclic complexes such that, for all $J \subset \llbracket 0, m \rrbracket$, the complex $\bigcap_{j \in J} A_j$ is acyclic: so the complex $\sum_{j=1}^m A_j$ is acyclic, by induction. We have the following commutative square:

$$\begin{array}{ccc} \sum_{j=1}^m A_j \cap A_0 & \longrightarrow & A_0 \\ \downarrow & & \downarrow \\ \sum_{j=1}^m A_j & \longrightarrow & \sum_{j=0}^m A_j \end{array} \quad \square$$

with A_0 , $\sum_{j=1}^m A_j$ and $\sum_{j=1}^m A_j \cap A_0$ acyclic. We conclude by (1). □

REFERENCES

- [BCER12] Yuri Berest, Xiaojun Chen, Farkhod Eshmatov, and Ajay Ramadoss. Noncommutative Poisson structures, derived representation schemes and Calabi-Yau algebras. In *Mathematical aspects of quantization*, volume 583 of *Contemp. Math.*, pages 219–246. Amer. Math. Soc., Providence, RI, 2012. 1
- [BKR13] Yuri Berest, George Khachatryan, and Ajay Ramadoss. Derived representation schemes and cyclic homology. *Adv. Math.*, 245:625–689, 2013. 1
- [CMW16] Ricardo Campos, Sergei Merkulov, and Thomas Willwacher. The Frobenius properad is Koszul. *Duke Math. J.*, 165(15):2921–2989, 2016. 3
- [DK10] Vladimir Dotsenko and Anton Khoroshkin. Gröbner bases for operads. *Duke Math. J.*, 153(2):363–396, 2010. 3
- [Gan03] Wee Liang Gan. Koszul duality for dioperads. *Math. Res. Lett.*, 10(1):109–124, 2003. 2
- [Gin05] V. Ginzburg. Lectures on Noncommutative Geometry. *ArXiv Mathematics e-prints*, June 2005. 1
- [Hof10] Eric Hoffbeck. A Poincaré-Birkhoff-Witt criterion for Koszul operads. *Manuscripta Math.*, 131(1-2):87–110, 2010. 3
- [Ler17] Johan Leray. *Approche fonctorielle et combinatoire de la propétrade des algèbres double Poisson*. PhD thesis, 2017. 5

- [Ler18] Johan Leray. Protoperads II: Koszul duality and application. December 2018. 1, 3, 4, 5, 45
- [LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*, volume 346 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2012. 2, 3, 12, 13, 15, 32
- [Mar08] Martin Markl. Operads and PROPs. In *Handbook of algebra. Vol. 5*, volume 5 of *Handb. Algebr.*, pages 87–140. Elsevier/North-Holland, Amsterdam, 2008. 31
- [ML65] Saunders Mac Lane. Categorical algebra. *Bull. Amer. Math. Soc.*, 71:40–106, 1965. 2
- [MV09a] Sergei Merkulov and Bruno Vallette. Deformation theory of representations of prop(erad)s. I. *J. Reine Angew. Math.*, 634:51–106, 2009. 1, 2, 37
- [MV09b] Sergei Merkulov and Bruno Vallette. Deformation theory of representations of prop(erad)s. II. *J. Reine Angew. Math.*, 636:123–174, 2009. 1
- [Ser71] Jean-Pierre Serre. *Représentations linéaires des groupes finis*. Hermann, Paris, 1971. Deuxième édition, refondue. 26
- [Val03] Bruno Vallette. *Dualité de Koszul des PROPs*. Prépublication de l’Institut de Recherche Mathématique Avancée [Prepublication of the Institute of Advanced Mathematical Research], 2003/30. Université Louis Pasteur, Département de Mathématique, Institut de Recherche Mathématique Avancée, Strasbourg, 2003. Dissertation, Université de Strasbourg I, Strasbourg, 2003. 1, 2, 5, 15, 22, 23, 25, 34, 35, 36, 37, 42
- [Val07] Bruno Vallette. A Koszul duality for PROPs. *Trans. Amer. Math. Soc.*, 359(10):4865–4943, 2007. 1, 2, 3, 5, 15, 23, 42
- [Val09] Bruno Vallette. Free monoid in monoidal abelian categories. *Appl. Categ. Structures*, 17(1):43–61, 2009. 5, 16, 17, 34, 35, 36
- [VdB08a] Michel Van den Bergh. Double Poisson algebras. *Trans. Amer. Math. Soc.*, 360(11):5711–5769, 2008. 1
- [VdB08b] Michel Van den Bergh. Non-commutative quasi-Hamiltonian spaces. In *Poisson geometry in mathematics and physics*, volume 450 of *Contemp. Math.*, pages 273–299. Amer. Math. Soc., Providence, RI, 2008. 1
- [Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

LAGA, UNIVERSITÉ PARIS 13, 99 AVENUE JEAN BAPTISTE CLÉMENT 93430, VILLETANEUSE, FRANCE

E-mail address: leray@math.univ-paris13.fr