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The degenerate scales of BIEs for conduction in piecewise homogeneous domains

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ABSTRACT

The degenerate scale issue for 2D-boundary integral equations and boundary element methods has been already investigated for many cases when the properties of the medium are homogeneous. We address here the problem of several subdomains with different properties. Then, the domain decomposition into homogeneous subdomains gives rise to a system of BIEs. If there are n subdomains, (n bounded subdomains in the case of an interior problem, and $n - 1$ bounded subdomains and one unbounded subdomain in the case of an exterior problem) there are n degenerate scales in both cases. For an interior problem, the n degenerate scales are the degenerate scales of the n subdomains. For an exterior problem, there are $n - 1$ degenerate scales equal to the degenerate scales of the $n - 1$ bounded subdomains and one intrinsic degenerate scale linked to the solution of a specific boundary value problem. Some properties of this intrinsic degenerate scale are investigated by analytical and numerical methods. The case of a half-plane is also studied according to the boundary condition along the line bounding the half-plane.

1. Introduction

The loss of uniqueness for the solution of 2D Boundary Value Problems (BVP) is well known in the case of conduction in homogeneous domains, i.e. Laplace equation. In the present work, we consider the case where the domain is divided into several homogeneous subdomains when solving problems by Boundary Element Method (BEM). The relevant Boundary Integral Equation (BIE) is turned into a system of BIEs. This domain splitting can be made necessary because the properties of the media are not homogeneous but only piecewise homogeneous. However, this splitting can also be useful for numerical reasons; it leads to a blocked sparse matrix. We can refer for example to [1–6]. The case of continuously variable properties has been also investigated by different authors introducing an additional field function [7], using local integral equations [8]; some special cases of functionally graded media with a simple algebraic formula have been considered by Carslaw and Jaeger [9] and Sutradhar et al. [10] (conductivity $k = k_0 x^n$, $k = k_0(1 + \alpha x)$ or $k = k_0 e^{\alpha x}$).

For 2D problems on homogeneous domains, there can be one or several degenerate scales (depending of the physical problem under consideration), for which the BIE has more than one solution leading to numerical flaws in BEM [11]. This issue is intensively investigated since early works dealing only with mathematical aspects of the BIE [12]. The problem has been investigated for Laplace problem, plane elasticity

problem [13–16], biharmonic problem [17,18]. For Laplace Dirichlet problem, the degenerate scale is linked to the logarithmic capacity (or the transfinite diameter) of the potential theory [19–21]. Different numerical methods have been suggested to find the degenerate scale: a one trial method from the normal scale has been suggested [22], a similar method appears for plane elasticity in [13], a Newton iterative method for finding the zero of a determinant [23], null field BIE [24]. Recently, the cases of boundary conditions other than Dirichlet condition have been investigated [25,26] using a generalized eigenvalue solver to find the numerical value of the degenerate scale factor.

The issue of degenerate scales in the case of a domain split into several subdomains, each one having homogeneous properties, seems to have not been investigated yet. We focus on the conduction problem with Dirichlet boundary condition and the direct BIE method; we address both interior and exterior problems. After the present introductory section, Section 2 deals with the interior problem which is the simplest. The rest of the paper is devoted to the exterior problem. Section 3 gives the number of degenerate scales: n for n subdomains; $n - 1$ scales which are associated to each bounded subdomain and one intrinsic scale which depends only on the overall boundary value problem and not on the splitting into subdomains. Then, Section 4 gives some properties of this intrinsic degenerate scale for $n = 2$. The use of conformal mapping in Section 5 will allow us to apply the closed form results found for concentric circles to some other cases. Section 6 deals with the problem in

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a half-plane; the conclusion depends on the type of boundary condition on the line bounding the half-plane. In Section 7, we consider the case of two half-planes with different conductivities. In such a case, it is possible to find the Green function and to write only one BIE. In Section 8, a numerical method is shown to evaluate the degenerate scales and different numerical results are given. In Section 9, we sum up our results and state some possible extensions of the present work.

2. Interior problem

2.1. Background for an homogeneous domain

It is well known that, due to the presence of a logarithmic term in the Green function, the simple layer boundary operator is not invertible for a specific scale of the problem called the degenerate scale.

It means that, starting from a specific boundary Γ there exists an homothetic boundary $\rho\Gamma$, such that

$$\int_{\rho\Gamma} q(y)G(x, y)dS_y = 0, \quad (1)$$

with $G(x, y) = -1/2\pi \ln(|x - y|)$, can have an infinity of non null solutions $q(y)$ for a specific value of ρ , called the degenerate scale factor. Then, $\rho\Gamma$ is at a degenerate scale.

It is less known that the degenerate scale factor can be found by solving the following problem: find the couple $(q(x), \omega)$ complying with the system of equations:

$$\begin{cases} \int_{\Gamma} q(y)G(x, y)dS_y + \omega = 0, & (2a) \\ \int_{\Gamma} q(y)dS_y = 1; & (2b) \end{cases}$$

It has been shown that this problem has a unique solution $(q(x), \omega)$ (see for example [27]). The value of ω is related to the degenerate scale factor. Indeed, we consider the homothetic boundary $\rho\Gamma$ with the similarity ratio ρ , the function $q_\rho(y) = q(y/\rho)$ and we have

$$\int_{\rho\Gamma} q_\rho(y)G(x, y)dS_y = \int_{\Gamma} q(y)\left(G(x, y) - \frac{1}{2\pi} \ln(\rho)\right) \rho dS_y = -\rho\left(\omega + \frac{1}{2\pi} \ln(\rho)\right). \quad (3)$$

Hence the equation $\int_{\rho\Gamma} q_\rho(y)G(x, y)dS_y = 0$ has a non null solution for the boundary $\rho\Gamma$ with $\rho = e^{-2\pi\omega}$. This boundary $\rho\Gamma$ is therefore at its degenerate scale and the similarity ratio ρ is the degenerate scale factor of Γ .

The advantage of the problem in (q, ω) is to search for a specific factor by the solution of a well posed problem which has always a unique solution. Besides, the numerical solution of this problem is the simplest way to produce the degenerate scale factor.

We recall that for mixed boundary condition or Robin boundary condition the degenerate scale have been found to be the same as for Dirichlet condition in the case of the interior homogeneous problem [25, 26]. For the sake of simplicity, we focus hereafter on the case of Dirichlet boundary condition.

2.2. The problem on an heterogeneous domain

Each subdomain has a boundary Γ_b , which is a simple curve. The common boundary between Ω_i and Ω_j is denoted by $\Gamma_{ij} = \Gamma_i \cup \Gamma_j$. We denote by Ω_e the infinite part of the plane outside $\cup \Omega_i$ and its boundary is denoted by Γ_e . The common boundary of Ω_i and Ω_e is denoted by Γ_{ie} (Fig. 1).

We consider the following interior BVP of finding the functions v_i defined in each Ω_i :

$$\begin{cases} \Delta v_i(x) = 0, x \in \Omega_i; i \in \{1, \dots, n\}; & (4a) \\ v_i(x) = 0, x \in \Gamma_e \cup \Gamma_i; & (4b) \\ v_i(x) = v_j(x), x \in \Gamma_i \cup \Gamma_j; i, j \in \{1, \dots, n\}; & (4c) \\ k_i \frac{\partial v_i}{\partial n_i}(x) + k_j \frac{\partial v_j}{\partial n_j}(x) = 0; i, j \in \{1, \dots, n\}. & (4d) \end{cases} \quad (\text{BVP}_i)$$

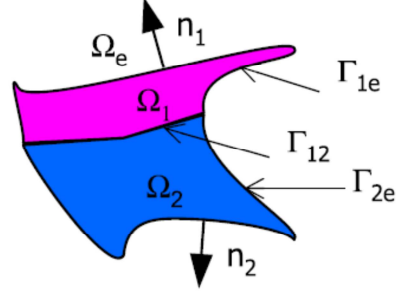


Fig. 1. Notations for the case of 2 subdomains.

We limit ourselves to the physically sound case of $k_i > 0$. (Without this assumption (for example if $k_1, k_2 < 0$) there are serious mathematical difficulties; for example the uniqueness of the solution cannot be deduced from Eq. 17.) We will also consider the following auxiliary problem:

$$\begin{cases} \Delta v_i(x) = 0, x \in \Omega_i; i \in \{1, \dots, n\}; & (5a) \\ v_i(x) = 0, x \in \Gamma_e \cup \Gamma_i; & (5b) \\ v_i(x) = v_j(x), x \in \Gamma_i \cup \Gamma_j; i, j \in \{1, \dots, n\}; & (5c) \\ k_i \frac{\partial v_i}{\partial n_i}(x) + k_j \frac{\partial v_j}{\partial n_j}(x) = \sigma_{ij}(x); i, j \in \{1, \dots, n\}. & (5d) \end{cases} \quad (\text{BVP}_i^*)$$

Following [3] we write the integral equation for each (connected) subdomain Ω_i : it results in n boundary integral equations (BIE):

$$\int_{\Gamma_i} (v_i(y) - v_i(x)) \frac{\partial G}{\partial n}(x, y) - \frac{q_i}{k_i} G(x, y) dS_y = 0 \quad x \in \Gamma_i; i = 1, \dots, n. \quad (6)$$

Finally, we get the following system with n_i the exterior normal to Γ_b , $u_i = v_i$ and $q_i = k_i \frac{\partial v_i}{\partial n_i}$ defined on Γ_i :

$$\begin{cases} \int_{\Gamma_i} (u_i(y) - u_i(x)) \frac{\partial G}{\partial n_i}(x, y) - \frac{q_i(y)}{k_i} G(x, y) dS_y = 0, & (7a) \\ x \in \Gamma_i, i \in \{1, \dots, n\}; & (7a) \\ u_i(x) = 0, x \in \Gamma_e \cup \Gamma_i, i \in \{1, \dots, n\}; & (7b) \\ u_i(x) = u_j(x), x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}; & (7c) \\ q_i(x) + q_j(x) = 0, x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}. & (7d) \end{cases} \quad (\text{BIE}_i)$$

We say that the system BIE_i is at a degenerate scale if it has a non null solution.

2.3. Necessary condition for a degenerate scale

We assume that the system BIE_i is at a degenerate scale and we prove that at least one of the Ω_i is at a degenerate scale.

Let us assume that none of the Ω_i is at the degenerate scale. Then, there is a solution v_i in each Ω_i of the Dirichlet problem defined by $\Delta v_i(x) = 0, x \in \Omega_i, v_i(x) = u_i(x), x \in \Gamma_i$. Then, $(v_i, \frac{\partial v_i}{\partial n_i})$ satisfies Eq. (7a) and, as the simple layer boundary integral operator is invertible (no degenerate scale) we have $q_i = k_i \frac{\partial v_i}{\partial n_i}$. If one of the u_i, q_i is not null, then at least one of the v_i is not null.

As a consequence, v_i should be a non null solution of the BVP_i and there is no such solution, the related domain being not at a degenerate scale. This is easy to prove by integrating by parts the quantity

$$\sum_{i=1, \dots, n} k_i \int_{\Omega_i} v_i \Delta v_i dS_y. \quad (8)$$

More details are given for the case of the exterior problem in Section 3.3.

Finally, we conclude that if BIE_i is at a degenerate scale then, at least one of the subdomain is at a degenerate scale.

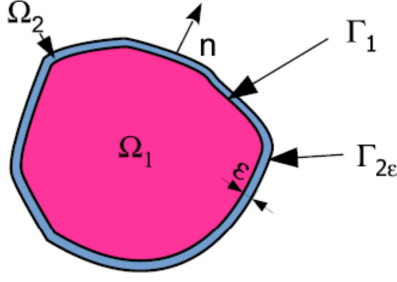


Fig. 2. Combination of BEM domains, one included in the other: interior problem.

2.4. Sufficient condition for a degenerate scale

We show now that if one of the component boundaries is at a degenerate scale, then the overall problem on the heterogeneous domain is also at a degenerate scale.

We assume that one of the boundaries is at its degenerate scale. Without loss of generality, we can assume that this boundary is Γ_1 . Then there is $\sigma_1 \neq 0$ such that $\int_{\Gamma_1} \sigma_1(y)G(x, y)dS_y = 0$ for $x \in \Gamma_1$. We consider the following problem BVP_1^* , with the same conditions as the original problem BVP_1 except the transmission condition on Γ_1 :

$$k_1 \frac{\partial v_1}{\partial n_1}(x) + k_i \frac{\partial v_i}{\partial n_i}(x) = \sigma_1(x), \quad x \in \Gamma_1 \cap \Gamma_i, \quad i \neq 1. \quad (9)$$

Hsiao et al. [28] proved that there is always one solution to the corresponding Eq. (4) if all subdomains are strong Lipschitz. This result can also be extended to the case of $\sigma_i \neq 0$ in Eq. (9).

Then there is a function v such that $u_i = v_i$ and $q_i = (1/k_i)(\partial v_i / \partial n_i)$ on Γ_i satisfy the conditions (7) except (7a) for $i \neq 1$ which is replaced by $q_1 + q_j = \sigma_1$ for $i = 1$.

We choose:

$$u_i(x) = v_i(x), \quad x \in \Gamma_i; \quad (10)$$

$$q_1(x) = k_1 \frac{\partial v_1}{\partial n_1}(x) - \sigma_1, \quad q_i(x) = k_i \frac{\partial v_i}{\partial n_i}(x) \quad x \in \Gamma_i, \quad i \neq 1. \quad (11)$$

Then we have $q_1(x) = -q_i(x), x \in \Gamma_1 \cap \Gamma_i$ and also Eq. (7a) for Γ_1 since

$$\int_{\Gamma_1} \sigma_1(y)G(x, y)dS_y = 0, \quad x \in \Gamma_1. \quad (12)$$

So, we have found a solution $(u_i, q_i, i \in \{1 \dots n\})$ satisfying all the conditions of BIE_i . This solution is non null as σ_1 is non null and then, at least one $\partial v_i / \partial n_i \neq 0$ and then at least one $q_i \neq 0$.

We conclude that BIE_i is at a degenerate scale. Finally, we conclude that there are n degenerate scales for BIE_i which are equal to the degenerate scales of $\Omega_i, i \in \{1 \dots n\}$. Obviously, some of the related degenerate scale factors can be equal between themselves, which can lead to a lesser number of degenerate scale factors, for example when searching these factors by using a numerical method.

Remark: It is worthwhile noticing that the degenerate scales of the overall domain are the degenerate scales of the different subdomains but the associated solutions of the BIE are different from the case of one subdomain alone.

2.5. Comparison with Robin condition

We consider 2 boundaries $\Gamma_1, \Gamma_{2\epsilon}$ distant from ϵ (Fig. 2). The distance between Γ_1 and $\Gamma_{2\epsilon}$ is equal to $\epsilon \rightarrow 0$, and the ratio $k = k_2/k_1$ is assumed to be equal to ϵ/t . We assume homogeneous Dirichlet condition $v_2 = 0$ on $\Gamma_{2\epsilon}$. At the boundary Γ_1 , we have $v \approx \epsilon \partial v / \partial n_2$. Then using

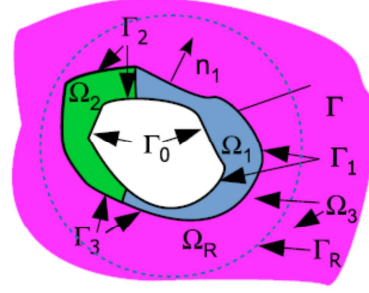


Fig. 3. Notations for the exterior problem: case of 3 subdomains ($n = 3$).

$\partial v / \partial n_2 = -1/k(\partial v / \partial n_1) = t \partial v / \partial n_1$, we deduce that $\partial v / \partial n_1 + tv = 0$. That means that v has a Robin type boundary condition on Γ_1 . Then, it is possible to have an approximation of the Robin problem with a 2 subdomains model. The degenerate scales are those of Γ_1 and $\Gamma_{2\epsilon}$. In the case of the interior problem and for the 2 subdomains model, there are two degenerate scales. Both tend to the degenerate scale for Γ_1 with Dirichlet boundary condition, that corresponds to the case of Robin boundary condition (see [26]).

3. Degenerate scales for the exterior problem

3.1. Preliminaries

In that section, we consider an exterior problem with n subdomains including only one infinite subdomain Ω_n which is the exterior of a bounded simple curve (Fig. 3).

The main objective of this section is to show that the associated system of BIEs has n degenerate scales (some of them may be equal). Amongst them, $n - 1$ are equal to the $n - 1$ degenerate scales of the different bounded subdomains and the last one is associated to a non null solution of a BVP coupling all subdomains.

3.2. Formulation of BVPs on an exterior heterogeneous domain

From a general point of view, an exterior problem behaves very differently, compared to interior problems. Indeed, as seen in the case of Robin or mixed boundary conditions for one domain, the degenerate scale for an exterior problem is different from the one related to Dirichlet BC. In the case of an interior problem with several subdomains, the degenerate scales correspond to the solution of BIEs related to a specific BVP, as seen in the previous section. Some important properties of the BVP related to an exterior problems can be obtained from those corresponding to interior problems by using the method of inversive geometry. So, in a first step, the method of inversive geometry is introduced. Next, the exterior boundary value problems related to possible degenerate scales are introduced and finally, the problem of degenerate scale for the exterior heterogeneous problem is studied.

3.2.1. Relation between exterior problem and inverse problem by inversive geometry

It is possible to derive some properties of the exterior problems from those of interior problems for conduction equation by using the method of inversive geometry [29–31]. Applied to harmonic functions the method is known as Kelvin transformation [32].

We consider the complex inversion $I: z \rightarrow 1/z$. It transforms a function $f(r, \theta)$ into the function $F(R, \Theta) = f(1/R, -\Theta)$. It is known that the inversion transforms a harmonic function into a harmonic function [29,30]. It can be checked readily by using the expression of Laplacian in polar coordinates. The complex inversion is also a conformal mapping that let the magnitude and the sign of angles invariant [33]. We can check

directly that the gradient is multiplied by $-1/R^2$ when performing the inversion:

$$\nabla F(R, \Theta) = \frac{\partial F}{\partial R} e_R + \frac{1}{R} \frac{\partial F}{\partial \Theta} e_\Theta \quad (13)$$

$$= -\frac{1}{R^2} \frac{\partial f}{\partial r} (1/R, -\Theta) e_R - \frac{1}{R} \frac{\partial f}{\partial \theta} (1/R, -\Theta) e_\Theta \quad (14)$$

$$= -\frac{1}{R^2} \left(\frac{\partial f}{\partial r} (1/R, -\Theta) e_R + \frac{1}{1/R} \frac{\partial f}{\partial \theta} (1/R, -\Theta) e_\Theta \right). \quad (15)$$

We deduce that $|\nabla F(R, \Theta)| = (1/R^2)|(\nabla f(1/R, -\Theta))|$ and as the angles are invariant: $\partial F/\partial N = \nabla F \cdot N = (1/R^2)\nabla f \cdot n = 1/R^2 \partial f/\partial n$ (if the normal n is oriented from Ω_i to Ω_j , the normal N is oriented from $I(\Omega_i)$ to $I(\Omega_j)$). So, a solution of BVP_i^{*} can be transformed into a solution of an exterior problem if the inversion center is within the considered domain, with $\sigma_{i,j}$ changed into $1/R^2 \sigma_{i,j}$.

The case of an exterior problem is different, because in this case, the boundary sources provide a field at infinity varying as $\text{Aln}(r)$. We consider the interior problem transformed of the exterior problem by the inversion of the center O which is in Ω_0 . Then we consider the solution of the interior problem corresponding to the boundary values obtained by inversion of the exterior problem with a negative unit source added in O . The inversion of this solution gives a solution of the original exterior problem with $U(R) = -1/2\pi \ln(R) + \omega + O(1/R)$ at infinity, which has a convenient behaviour at infinity.

Thanks to the complex inversion, the existence results for the solution of interior problems can be extended to exterior problems.

3.2.2. A BVP on an heterogeneous domain with continuity between subdomains

From a general point of view, the degenerate scale is related to the BIE formulation of a given BVP. So, it is important to consider a formulation of a BVP in an infinite domain. As recalled in Section 2, it is possible to transform the search of the degenerate scale into the search of a factor ω by using a well-posed problem.

In a first step, we will therefore consider the following BVP: to find the function v and the constant ω satisfying:

$$\left\{ \begin{array}{l} \Delta v_i(x) = 0, x \in \Omega_i, i \in \{1, \dots, n\}; \\ v_i(x) = 0, x \in \Gamma_0 \cup \Gamma_i; \end{array} \right. \quad (16a)$$

$$\left\{ \begin{array}{l} v_i(x) = v_j(x), x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}; \\ k_i \frac{\partial v_i}{\partial n_i}(x) + k_j \frac{\partial v_j}{\partial n_j}(x) = 0, i, j \in \{1, \dots, n\}; \end{array} \right. \quad (16b)$$

$$\left\{ \begin{array}{l} v(x) = -\frac{1}{2\pi} \ln(r) + \omega + O(r^{-1}), r \rightarrow \infty; \\ \frac{\partial v(x)}{\partial r} = -\frac{1}{2\pi r} + O(r^{-2}), r \rightarrow \infty. \end{array} \right. \quad (16c)$$

$$\left\{ \begin{array}{l} \Delta v_i(x) = 0, x \in \Omega_i, i \in \{1, \dots, n-1\}; \\ u_n(x) + \int_{\Gamma_n} (u_n(y) - u_n(x)) \frac{\partial G}{\partial n} - \frac{q_n}{k_n} G dS_y = 0, x \in \Gamma_n; \\ u_i(x) = 0, x \in \Gamma_0 \cup \Gamma_i; \\ u_i(x) = u_j(x), x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}; \\ q_i(x) + q_j(x) = 0, x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}. \end{array} \right. \quad (16d)$$

$$\left\{ \begin{array}{l} \Delta v_i(x) = 0, x \in \Omega_i, i \in \{1, \dots, n-1\}; \\ u_n(x) + \int_{\Gamma_n} (u_n(y) - u_n(x)) \frac{\partial G}{\partial n} - \frac{q_n}{k_n} G dS_y = 0, x \in \Gamma_n; \\ u_i(x) = 0, x \in \Gamma_0 \cup \Gamma_i; \\ u_i(x) = u_j(x), x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}; \\ q_i(x) + q_j(x) = 0, x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}. \end{array} \right. \quad (16e)$$

If there is only one domain, this BVP has been introduced in [20] where it is proved that it has a unique solution and that the corresponding BIE is at the degenerate scale iff $\omega = 0$. It must be emphasized that this formulation replaces the search of a degenerate scale factor by the solution of a well posed BVP. In a first step, we will study the uniqueness of the solution of this BVP.

The uniqueness of the solution in the heterogeneous case can be shown with the following standard argument. Consider the difference v between two solutions of BVP_e and the domain bounded by a sufficiently large circle Γ_R . We write the following sum of integrals and we use an integration by parts and the divergence theorem:

$$\begin{aligned} & \sum_{i=1}^{n-1} k_i \int_{\Omega_i} |\nabla v_i|^2 + k_n \int_{\Omega_n} |\nabla v_n|^2 \\ &= \sum_{i=1}^{n-1} k_i \int_{\Omega_i} \nabla \cdot (v_i \nabla v_i) + k_n \int_{\Omega_n} \nabla \cdot (v_n \nabla v_n) - \left(\sum_{i=1}^{n-1} k_i \int_{\Omega_i} v_i \Delta v_i + k_n \int_{\Omega_n} v_n \Delta v_n \right) \\ &= \sum_{i=1}^n k_i \int_{\Gamma_i} v_i \frac{\partial v_i}{\partial n_i} + k_n \int_{\Gamma_n} v_n \frac{\partial v_n}{\partial r}. \end{aligned} \quad (17)$$

Due to Eqs. (16a)–(16c), the right hand member of the above equation reduces to: $\int_{\Gamma_R} v_n \frac{\partial v_n}{\partial r} = O(1/R)$ and then its limit when $R \rightarrow \infty$ is null and the solution v is constant and null because of the boundary condition on Γ_0 . If there is one disk included in Γ_0 , then after a scaling of the problem we can assume that the radius of this disk is larger than 1. Then after geometric inversion, the domain of the corresponding interior problem is included in a disk of radius less than 1. Then, as it has been seen previously, this problem has a (unique) solution (the additional source does not affect the uniqueness of the solution as can be seen by considering the difference of two solutions with the same source) and by another inversion and rescaling we find the solution of BVP_e.

3.2.3. A second auxiliary problem with given sources at the boundary

We will also consider the auxiliary problem BVP_e^{*}: find the function v and the constant ω satisfying:

$$\left\{ \begin{array}{l} \Delta v_i(x) = 0, x \in \Omega_i, i \in \{1, \dots, n\}; \\ v_i(x) = 0, x \in \Gamma_0 \cup \Gamma_i; \end{array} \right. \quad (18a)$$

$$\left\{ \begin{array}{l} v_i(x) = v_j(x), x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}; \\ k_i \frac{\partial v_i}{\partial n_i}(x) + k_j \frac{\partial v_j}{\partial n_j}(x) = \sigma_{i,j}(x), i, j \in \{1, \dots, n\}; \end{array} \right. \quad (18b)$$

$$\left\{ \begin{array}{l} v(x) = \omega + O(r^{-1}), r \rightarrow \infty; \\ \frac{\partial v(x)}{\partial r} = O(r^{-2}), r \rightarrow \infty. \end{array} \right. \quad (18c)$$

$$\left\{ \begin{array}{l} \Delta v_i(x) = 0, x \in \Omega_i, i \in \{1, \dots, n-1\}; \\ u_n(x) + \int_{\Gamma_n} (u_n(y) - u_n(x)) \frac{\partial G}{\partial n} - \frac{q_n}{k_n} G dS_y = 0, x \in \Gamma_n; \\ u_i(x) = 0, x \in \Gamma_0 \cup \Gamma_i; \\ u_i(x) = u_j(x), x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}; \\ q_i(x) + q_j(x) = 0, x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}. \end{array} \right. \quad (18d)$$

$$\left\{ \begin{array}{l} \Delta v_i(x) = 0, x \in \Omega_i, i \in \{1, \dots, n-1\}; \\ u_n(x) + \int_{\Gamma_n} (u_n(y) - u_n(x)) \frac{\partial G}{\partial n} - \frac{q_n}{k_n} G dS_y = 0, x \in \Gamma_n; \\ u_i(x) = 0, x \in \Gamma_0 \cup \Gamma_i; \\ u_i(x) = u_j(x), x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}; \\ q_i(x) + q_j(x) = 0, x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}. \end{array} \right. \quad (18e)$$

where $\sigma_{i,j}$ is given on the boundary.

The uniqueness of the solution can be proved in the same way as for BVP_e. The existence of this solution follows from Section 3.2.1.

3.3. Necessary condition for a degenerate scale if none of the bounded subdomains is at a degenerate scale

In this part, we show that in the case of an heterogeneous domain whose homogeneous components are not at a degenerate scale, it appears a degenerate scale which is related to the behavior of the solution of BVP_e at infinity.

We investigate the following system of BIEs and its possible degenerate scales:

$$\left\{ \begin{array}{l} \int_{\Gamma_i} (u_i(y) - u_i(x)) \frac{\partial G}{\partial n} - \frac{q_i}{k_i} G dS_y = 0, x \in \Gamma_i, i \in \{1, \dots, n-1\}; \\ u_n(x) + \int_{\Gamma_n} (u_n(y) - u_n(x)) \frac{\partial G}{\partial n} - \frac{q_n}{k_n} G dS_y = 0, x \in \Gamma_n; \\ u_i(x) = 0, x \in \Gamma_0 \cup \Gamma_i; \\ u_i(x) = u_j(x), x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}; \\ q_i(x) + q_j(x) = 0, x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}. \end{array} \right. \quad (19a)$$

$$\left\{ \begin{array}{l} \int_{\Gamma_i} (u_i(y) - u_i(x)) \frac{\partial G}{\partial n} - \frac{q_i}{k_i} G dS_y = 0, x \in \Gamma_i, i \in \{1, \dots, n-1\}; \\ u_n(x) + \int_{\Gamma_n} (u_n(y) - u_n(x)) \frac{\partial G}{\partial n} - \frac{q_n}{k_n} G dS_y = 0, x \in \Gamma_n; \\ u_i(x) = 0, x \in \Gamma_0 \cup \Gamma_i; \\ u_i(x) = u_j(x), x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}; \\ q_i(x) + q_j(x) = 0, x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}. \end{array} \right. \quad (19b)$$

$$\left\{ \begin{array}{l} \int_{\Gamma_i} (u_i(y) - u_i(x)) \frac{\partial G}{\partial n} - \frac{q_i}{k_i} G dS_y = 0, x \in \Gamma_i, i \in \{1, \dots, n-1\}; \\ u_n(x) + \int_{\Gamma_n} (u_n(y) - u_n(x)) \frac{\partial G}{\partial n} - \frac{q_n}{k_n} G dS_y = 0, x \in \Gamma_n; \\ u_i(x) = 0, x \in \Gamma_0 \cup \Gamma_i; \\ u_i(x) = u_j(x), x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}; \\ q_i(x) + q_j(x) = 0, x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}. \end{array} \right. \quad (19c)$$

$$\left\{ \begin{array}{l} \int_{\Gamma_i} (u_i(y) - u_i(x)) \frac{\partial G}{\partial n} - \frac{q_i}{k_i} G dS_y = 0, x \in \Gamma_i, i \in \{1, \dots, n-1\}; \\ u_n(x) + \int_{\Gamma_n} (u_n(y) - u_n(x)) \frac{\partial G}{\partial n} - \frac{q_n}{k_n} G dS_y = 0, x \in \Gamma_n; \\ u_i(x) = 0, x \in \Gamma_0 \cup \Gamma_i; \\ u_i(x) = u_j(x), x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}; \\ q_i(x) + q_j(x) = 0, x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}. \end{array} \right. \quad (19d)$$

$$\left\{ \begin{array}{l} \int_{\Gamma_i} (u_i(y) - u_i(x)) \frac{\partial G}{\partial n} - \frac{q_i}{k_i} G dS_y = 0, x \in \Gamma_i, i \in \{1, \dots, n-1\}; \\ u_n(x) + \int_{\Gamma_n} (u_n(y) - u_n(x)) \frac{\partial G}{\partial n} - \frac{q_n}{k_n} G dS_y = 0, x \in \Gamma_n; \\ u_i(x) = 0, x \in \Gamma_0 \cup \Gamma_i; \\ u_i(x) = u_j(x), x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}; \\ q_i(x) + q_j(x) = 0, x \in \Gamma_i \cup \Gamma_j, i, j \in \{1, \dots, n\}. \end{array} \right. \quad (19e)$$

As previously, there is a solution v_i in each bounded subdomain Ω_i of the Dirichlet problem defined by $\Delta v_i(x) = 0, x \in \Omega_i, v_i(x) = u_i(x), x \in \Gamma_i$. Then, $v_i, \partial v_i/\partial n$ satisfies Eq. (19a). If none of the bounded subdomains are at the degenerate scale, we have $q_i = k_i \frac{\partial v_i}{\partial n_i}$.

We must consider now the unbounded subdomain Ω_n . There is a solution of the Dirichlet problem defined by: $\Delta v_n(x) = 0, x \in \Omega_n; v_n(x) = u_n(x), x \in \Gamma_n; v_n(x) = -(1/2\pi) \int_{\Gamma_n} q_n/k_n dS_y \ln(r) + \omega + O(1/r)$. Then v_n satisfies the following Green formula [20],

$$v_n(x) + \int_{\Gamma_n} \left[(v_n(y) - v_n(x)) \frac{\partial G}{\partial n} - \frac{\partial v_n}{\partial n} G \right] dS_y = \omega, x \in \Gamma_n \cup \Omega_n; \quad (20)$$

We deduce by comparing (20) and (19b), that:

$$\int_{\Gamma_n} \left(\frac{q_n}{k_n} - \frac{\partial v_n}{\partial n} \right) G dS_y = \omega, x \in \Gamma_n \cup \Omega_n; \quad (21)$$

Then it is known (e.g. [34]) that there is a unique solution σ_n to the following problem: find $\sigma_n, C \in \mathbb{R}$ such that $\int_{\Gamma_n} \sigma_n G dS_y = C, \int_{\Gamma_n} \sigma_n dS_y =$

1 and that if $\int_{\Gamma_n} f G dS_y$ is constant then $f = \alpha \sigma_n$. So we can write

$$\left(\frac{q_n}{k_n} - \frac{\partial v_n}{\partial n} \right) = \alpha \sigma_n. \quad (22)$$

But considering the asymptotic behavior of (20), we have

$$\int_{\Gamma_n} \frac{q_n}{k_n} dS_y = \int_{\Gamma_n} \frac{\partial v_n}{\partial n} dS_y \quad (23)$$

and we conclude that $\alpha = 0$ in (22) and finally we have $\frac{q_n}{k_n} = \frac{\partial v_n}{\partial n}$. Then comparing (19b) and (20) we find that $\omega = 0$; that means that we have found up to a multiplicative factor a solution of BVP_e with $\omega = 0$. This solution is non null if the solution u of BIE_e is non null.

So the only possible degenerate scales are the degenerate scales of the bounded subdomains Ω_i and the scale for which BVP_e has a solution with $\omega = 0$. We call this last degenerate scale the *intrinsic degenerate scale* as it does not depend on the decomposition into subdomains: when all the homogeneous subdomains are split into several smaller subdomains, all the degenerate scales are changed except the intrinsic degenerate scale which remains unchanged as it depends only on the formulation of BVP_e and on the choice of the Green function. The other degenerate scales depend only on the geometry of the bounded subdomains (and not on the ratios k_i/k_j). By contrast with the intrinsic degenerate scale, these degenerate scales can be named geometric ones. For the interior problem, there are only geometric degenerate scales.

3.4. Sufficient conditions for BIE_e being at a degenerate scale

Having found the possible degenerate scales in the previous subsection, we show now that when one of these possible degenerate scales occurs, BIE_e is actually at a degenerate scale.

3.4.1. Intrinsic degenerate scale: sufficient condition when the solution of BVP_e is such that $\omega = 0$

We have found previously that, if no subdomain is at a degenerate scale, a necessary condition for a degenerate scale to appear in the problem related to the heterogeneous domain is that the related solution of BVP_e leads to $\omega = 0$. If this condition is not met, this condition can be recovered by performing a uniform scaling over all the subdomains.

So, having $\omega = 0$ and knowing the solution v of BVP_e, the restriction u of v to Γ is a solution of BIE_e by using the third Green identity. In the case of the equation corresponding to the unbounded subdomain, we consider as previously in Section 3.1 a circle Γ_R with $R \rightarrow \infty$. Then $\omega = 0$ ensures Eq. (19b).

3.4.2. Sufficient condition when a subdomain is at the degenerate scale

We assume that one bounded domain Ω_1 is at the degenerate scale. We assume also that BIE_e is not at the intrinsic degenerate scale and so that the solution of BVP_e is such that $\omega \neq 0$. We denote by σ_1 a non null solution of $\int_{\Gamma_1} \sigma_1(y) G(x, y) dS_y = 0, x \in \Gamma_1$.

Then we consider the solution v', ω' of the problem BVP_e^{*} with $\sigma_{1,j} = \sigma_1$.

It satisfies BIE_e except that:

- (19e) for $i = 1$ is replaced by $k_1 \partial v'_1 / \partial n_1 + k_j \partial v'_j / \partial n_j = \sigma_1$
- (19b) is not generally verified. Indeed ω' is generally different from 0 and in this case, the boundary integral equation for the infinite domain must contain a complementary term function of ω' as shown in [20].

In the specific case $\omega' = 0$, (19b) is satisfied, and by choosing $q_1 = k_1 \partial v'_1 / \partial n_1 - \sigma_1, q_i = k_i \partial v'_i / \partial n_i, i \neq 1$, we find a non null solution of BIE_e and we conclude that BIE_e is at a degenerate scale.

If $\omega' \neq 0$, we consider the solution (v, ω) of BVP_e. We have assumed $\omega \neq 0$ (BIE_e is not at the intrinsic degenerate scale), then it is possible to consider $w = v' - (\omega'/\omega)v$. The restriction of this function to

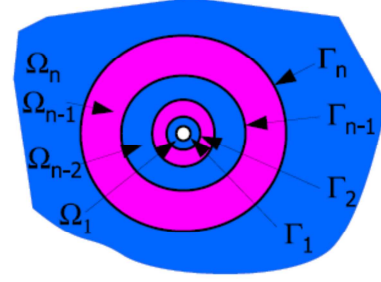


Fig. 4. The case of n circles.

the boundaries satisfies all the equations of BIE_e, except the transmission condition involving the boundary Γ_1 . For this equation, we choose $q_1 = k_1 \partial w_1 / \partial n - \sigma_1$.

This solution of BIE_e is non null. If $w = 0$ on Γ_1 then $w = 0$ in the bounded subdomain Ω_1 and $\partial w / \partial n = 0$ on Γ_1 , then $q_1 \neq 0$. We finally conclude that BIE_e is at a degenerate scale.

3.5. Conclusion: number of degenerate scales for the exterior problem

Finally, we have found all the degenerate scales of BIE_e. There are n degenerate scales but some of them can be equal between them. There are n degenerate scales which are the degenerate scales of the $n - 1$ bounded subdomains and an intrinsic degenerate scale corresponding to the non null solution of BVP_e. This intrinsic degenerate scale depends only on BVP_e and on the choice of the Green function G and not on the division into different subdomains.

3.6. An example: case of n concentric circles

Let us find the degenerate scale corresponding to a solution of BVP_e for n concentric circles (Fig. 4). We write the possible solution in Ω_{n-1} : $v_{n-1} = A_{n-1} \ln(r) + B_{n-1}$. Then writing the conditions at the interface, we get the following relation:

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} \frac{k_{n-1}}{k_n} & 0 \\ k_{n-1} \ln(R_n) \left(\frac{1}{k_{n-1}} - \frac{1}{k_n} \right) & 1 \end{pmatrix} \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix}. \quad (24)$$

Then, we deduce by recursion:

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} \frac{k_1}{k_n} & 0 \\ k_1 \sum_{i=2}^n \ln(R_i) \left(\frac{1}{k_{i-1}} - \frac{1}{k_i} \right) & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}. \quad (25)$$

The system is at a degenerate scale if $B_n = 0, A_n \neq 0, A_1 \ln(R_1) + B_1 = 0$. We get:

$$-\ln R_1 \frac{1}{k_1} + \sum_{i=2}^n \ln(R_i) \left(\frac{1}{k_{i-1}} - \frac{1}{k_i} \right) = 0. \quad (26)$$

The intrinsic degenerate scale factor ρ is given by:

$$\ln(\rho) = k_n \left(-\ln R_1 \frac{1}{k_1} + \sum_{i=2}^n \ln(R_i) \left(\frac{1}{k_{i-1}} - \frac{1}{k_i} \right) \right). \quad (27)$$

If all k_i are equal, then the condition (27) becomes $R_1 = 1$.

Eq. (27) can also be written in the following way:

$$\ln(\rho) = k_n \left(\sum_{i=1}^{n-1} \frac{1}{k_i} \ln \left(\frac{R_{i+1}}{R_i} \right) \right) - \ln R_n. \quad (28)$$

On this example, it can be checked that the intrinsic degenerate scale depends on the ratios k_i/k_j , though the other degenerate scales do not

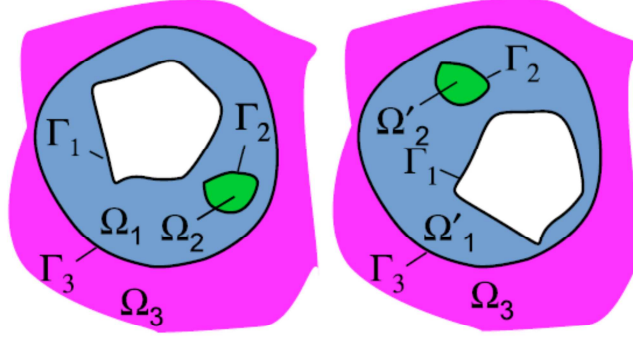


Fig. 5. Case when the degenerate scale associated to each subdomain is unchanged.

depend on these ratios. If one subdomain is included in another subdomain, the degenerate scale associated to each subdomain remains constant if the inner boundary is moved inside the larger subdomain (see Fig. 5). Numerical results show that this is not true for the intrinsic degenerate scale (see Section 8.2.2).

Let us check now that $R_i = 1, i \in \{2, \dots, n\}$ are the other degenerate scales.

We assume that $R_l = 1$ with $1 < l < n$ and that the problem is not at its intrinsic degenerate scale. Then we look for the function denoted by w in Section 3.4.2.

The condition on the normal derivatives on Γ_l gives: $k_{l-1}A_{l-1} - k_l A_l = 1$, so we have:

$$\begin{pmatrix} A_l \\ B_l \end{pmatrix} = \begin{pmatrix} \frac{k_{l-1}}{k_l} & 0 \\ k_{l-1} \ln(R_l) \left(\frac{1}{k_{l-1}} - \frac{1}{k_l} \right) & 1 \end{pmatrix} \begin{pmatrix} A_{l-1} \\ B_{l-1} \end{pmatrix} + \begin{pmatrix} -\frac{1}{k_l} \\ 0 \end{pmatrix} \quad (29)$$

$$= \begin{pmatrix} \frac{k_1}{k_l} & 0 \\ k_1 \sum_{i=2}^l \ln(R_i) \left(\frac{1}{k_{i-1}} - \frac{1}{k_i} \right) & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{k_l} \\ 0 \end{pmatrix} \quad (30)$$

We then can deduce:

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} \frac{k_1}{k_n} & 0 \\ k_1 \sum_{i=2}^n \ln(R_i) \left(\frac{1}{k_{i-1}} - \frac{1}{k_i} \right) & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} \quad (31)$$

$$= \begin{pmatrix} \frac{k_1}{k_n} & 0 \\ k_1 \sum_{i=2}^n \ln(R_i) \left(\frac{1}{k_{i-1}} - \frac{1}{k_i} \right) & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{k_n} \\ 0 \end{pmatrix} \quad (32)$$

The boundary condition gives $B_1 = -\ln(R_1)A_1$. The radiation condition gives $B_n = 0$. We deduce:

$$\left(k_1 \sum_{i=2}^n \ln(R_i) \left(\frac{1}{k_{i-1}} - \frac{1}{k_i} \right) - \ln(R_1) \right) A_1 = \sum_{i=2}^n \ln(R_i) \left(\frac{1}{k_{i-1}} - \frac{1}{k_i} \right) \quad (33)$$

The coefficient of A_1 in Eq. (33) appears in Eq. (27): if the problem is not at its intrinsic degenerate scale, the coefficient is non null and it is possible to find A_1 , and then, all coefficients A_i, B_i . From these coefficients, we deduce the functions u_i, q_i on Γ_i which satisfy BIE_e.

Table 1

Variation of the degenerate scale factor as a function of k_2 for two concentric circles.

| k_2 | $k_2 = 0$ | $k_2 = k_1$ | $k_2 = +\infty$ |
|--------|-----------|-------------|-----------------|
| ρ | $1/R_2$ | $1/R_1$ | $+\infty$ |

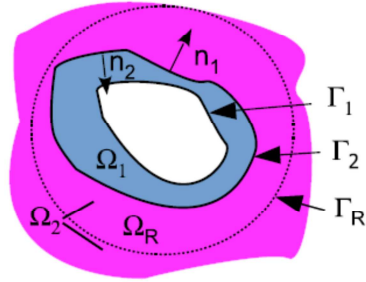


Fig. 6. Combination of BEM domains, one included in the other.

Remark 1. If $n=2$, Eq. (27) shows that if R_1, k_2, k_1 remain constant and $R_2 \rho'$, then $\rho' \nearrow$ if $k_2 > k_1$ and $\rho' \searrow$ if $k_2 < k_1$. A proof for simple curves (not only concentric circles) is given in Section 4.3.

Remark 2. If $n=2$, Eq. (28) shows that if R_1, R_2, k_1 remain constant and $k_2 \rho'$, then $\rho' \nearrow$. The variations of the degenerate scale factor ρ is given in Table 1.

The more general case of 2 simple curves is addressed in Section 4.2.

Remark 3. As $k_i > 0$ and $R_i > R_{i-1}$, it can be seen from Eq. (28) that $\ln(\rho) > -\ln(R_n)$; if $R_n < 1$, then the problem cannot be at a degenerate scale. If all $k_i, (i \neq n)$ remain constant, and $k_n \rightarrow 0$ then $\ln(\rho) \rightarrow \ln(R_n)$.

4. Study of the intrinsic degenerate scale for a case of 2 subdomains

We investigate the case of Fig. 6 where the conductivity has two different values in two areas. The boundary condition is applied on Γ_1 and the boundary Γ_1 is strictly inside the domain interior to Γ_2 . Therefore, there is no intersection between Γ_1 and the unbounded domain Ω_2 .

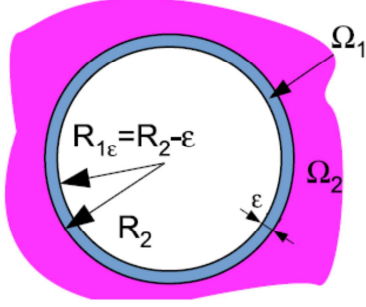


Fig. 7. Comparison with Robin condition for the case of two concentric circles.

4.1. An example: the comparison with Robin condition in the case of circles

In this section, we show that a heterogeneous domain comprising two concentric circles where the radii are close to each other can be approximated by a homogeneous problem using a Robin boundary condition. We consider here the case of 2 concentric circles (Fig. 7).

We assume that the $R_{1\epsilon} = R_2 - \epsilon$ with $\epsilon \ll R_2$. Considering the 2 subdomains problem, we can write Eq. (27): $\ln(\rho) = (k_2/k_1) \ln(R_2/R_1) - \ln(R_2)$. The problem is at its intrinsic degenerate scale if $\rho = 1$. That is if $(k_2/k_1) \ln(1 + \epsilon/R_{1\epsilon}) \approx (k_2/k_1)\epsilon/R_2 = \ln(R_2)$ or $R_2 \ln(R_2) = (k_2/k_1)\epsilon$. We have also: $u(R_2) \approx \epsilon k \partial u_2 / \partial r$. So at the circle $r = R_2$, we have the Robin type condition $\partial u / \partial r = tu$ with $t = 1/((k_2/k_1)\epsilon)$. The Robin type problem in Ω_2 is at its degenerate scale if $t R_2 \ln(R_2) = 1$ [25]. Then, we check that the degenerate scale for Robin condition with parameter t is equal to the limit of the intrinsic degenerate scale of the 2 domains problem when $\epsilon \rightarrow 0$ with $k_2/k_1 \epsilon = t$, t being constant.

4.2. Effect of the variation of the ratio k_2/k_1

As shown previously, the intrinsic degenerate scale factor depends on the conductivity ratio. The aim of this subsection is to show that the degenerate scale depends monotonically on the ratio k_2/k_1 . We denote by v_a and v_b the solutions of the problem BVP_e associated with $k_a = k_{2a}/k_{1a}$, $k_b = k_{2b}/k_{1b}$ and $\kappa = k_b/k_a$ (Fig. 6). We apply the first Green identity to $(v_a - v_b)$ on Ω_R , we find:

$$\int_{\Omega_R} (v_a - v_b) \Delta (v_a - v_b) = 0$$

$$= \underbrace{\int_{\Gamma_2} (v_a - v_b) \frac{\partial (v_a - v_b)}{\partial n_2}}_{I_1} + \underbrace{\int_{\Gamma_R} (v_a - v_b) \frac{\partial (v_a - v_b)}{\partial n_2}}_{I_2} - \underbrace{\int_{\Omega_R} (\nabla (v_a - v_b))^2}_{I_3 \leq 0}. \quad (34)$$

It is easy to see that $I_2 \rightarrow 0$ as $R \rightarrow \infty$. We note $A_{a\beta} = \int_{\Gamma_2} v_a \frac{\partial v_a}{\partial n_2}$ and we can conclude that $I_1 \geq 0$ so that:

$$I_1 = A_{aa} - A_{ab} - A_{ba} + A_{bb} \geq 0; \quad (35)$$

We apply the first Green identity to $(\lambda v_a - v_b)$ on Ω_1 , λ being constant. We get

$$\int_{\Omega_1} (\lambda v_a - v_b) \Delta (\lambda v_a - v_b) = 0$$

$$= \underbrace{\int_{\Gamma_1} (\lambda v_a - v_b) \frac{\partial (\lambda v_a - v_b)}{\partial n_1}}_{I_4=0} + \underbrace{\int_{\Gamma_2} (\lambda v_a - v_b) \frac{\partial (\lambda v_a - v_b)}{\partial n_1}}_{I_5} - \underbrace{\int_{\Omega_1} (\nabla (\lambda v_a - v_b))^2}_{I_6 \leq 0}. \quad (36)$$

We conclude:

$$I_5 = -\lambda^2 k_a A_{aa} + \lambda k_b A_{ab} + \lambda k_a A_{ba} - k_b A_{bb} \geq 0. \quad (37)$$

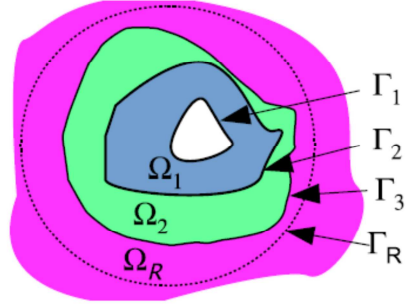


Fig. 8. Effect of the change of the boundary between the two BEM domains.

We choose $\lambda = \sqrt{\kappa}$ in (37) and combining the two inequalities (35) and (37), we get:

$$(\sqrt{\kappa} - 1)A_{ab} + \left(\frac{1}{\sqrt{\kappa}} - 1\right)A_{ba} \geq 0. \quad (38)$$

We apply the second Green identity to (v_a, v_b) on Ω_1 ,

$$\int_{\Omega_1} v_a \Delta (v_b) - v_b \Delta (v_a) = 0$$

$$= \underbrace{\int_{\Gamma_1} v_a \frac{\partial v_b}{\partial n_1} - v_b \frac{\partial v_a}{\partial n_1}}_{I_7=0} + \underbrace{\int_{\Gamma_2} v_a \frac{\partial v_b}{\partial n_1} - v_b \frac{\partial v_a}{\partial n_1}}_{I_8}. \quad (39)$$

We deduce that $I_8 = k_b A_{ab} - k_a A_{ba} = 0$, or:

$$A_{ba} = \kappa A_{ab}. \quad (40)$$

Substituting in (38) the value of A_{ba} , we get:

$$(-1 + 2\sqrt{\kappa} - \kappa)A_{ab} = -(1 - \sqrt{\kappa})^2 A_{ab} \geq 0, \quad (41)$$

and finally $A_{ab} \leq 0$.

We apply the second Green identity to (v_a, v_b) on Ω_R ,

$$\int_{\Omega_R} v_a \Delta (v_b) - v_b \Delta (v_a) = 0$$

$$= \underbrace{\int_{\Gamma_2} v_a \frac{\partial v_b}{\partial n_2} - v_b \frac{\partial v_a}{\partial n_2}}_{I_9} + \underbrace{\int_{\Gamma_R} v_a \frac{\partial v_b}{\partial r} - v_b \frac{\partial v_a}{\partial r}}_{I_{10}}. \quad (42)$$

We have: $I_9 = A_{ab} - A_{ba}$ and $\lim I_{10} = (\omega_b - \omega_a)$ when $R \rightarrow \infty$. So we deduce:

$$A_{ab} - A_{ba} + 2\pi(\omega_b - \omega_a) = 0. \quad (43)$$

Using (40), we get $(\omega_b - \omega_a) = (\kappa - 1)A_{ab}$. As $A_{ab} \leq 0$, we conclude:

$$\kappa = \frac{k_b}{k_a} > 1 \iff \omega_a \geq \omega_b \iff \rho_b = e^{-2\pi\omega_b} \geq \rho_a = e^{-2\pi\omega_a}. \quad (44)$$

Finally, it proves that the intrinsic degenerate scale factor ρ increases when k_2/k_1 increases.

4.3. Behavior when the domain near the internal boundary increases

The aim of this subsection is to study what happens when the sub-domain Ω_1 bounded by Γ_1 and Γ_2 grows and becomes Ω_2 bounded by Γ_1 and Γ_3 (Fig. 8). We show that the intrinsic degenerate scale factor increases when the bounded domain grows.

We use the same approach as in the next Section 4.2. We note v_a the solution of BVP_e with the boundaries Γ_1 and Γ_2 and v_b the solution of BVP_e with the boundaries Γ_1 and Γ_3 . We assume that the conductivity equals 1 in Ω_1 for v_a and v_b , equals 1 for v_b and equals k for v_a in Ω_2 and equals k outside Γ_R for both v_a and v_b .

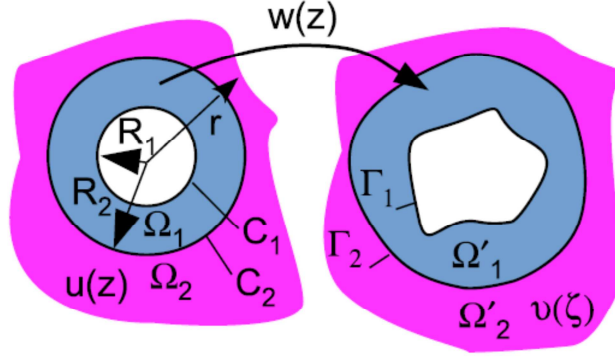


Fig. 9. Conformal mapping from a circle.

We denote:

$$A_{\alpha\beta ij} = \int_{\Gamma_i} v_\alpha \frac{\partial v_\beta}{\partial n_j}. \quad (45)$$

We apply the first Green identity to $(v_a - v_b)$ on Ω_1 , to $(\lambda v_a - v_b)$ on Ω_2 and to $(v_a - v_b)$ on Ω_R . We get successively:

$$A_{aa21} - A_{ab21} - A_{ba21} + A_{bb21} \geq 0; \quad (46)$$

$$\lambda^2 A_{aa22} - \lambda A_{ab22} - \lambda A_{ba22} + A_{bb22} + \lambda^2 A_{aa32} - \lambda A_{ab32} - \lambda A_{ba32} + A_{bb32} \geq 0; \quad (47)$$

$$A_{aa33} - A_{ab33} - A_{ba33} + A_{bb33} \geq 0. \quad (48)$$

We have:

$$\begin{aligned} A_{aa21} &= -k A_{aa22}; A_{ba21} = -k A_{ba22}; A_{ab21} = -A_{ab22}; A_{bb21} = -A_{bb22}; \\ A_{aa32} &= -A_{aa33}; A_{ba32} = -A_{ba33}; A_{ab32} = -k A_{aa33}; A_{bb32} = -k A_{bb33}. \end{aligned} \quad (49)$$

The equations become now:

$$-k A_{aa22} + A_{ab22} + k A_{ba22} - A_{bb22} \geq 0; \quad (50)$$

$$\lambda^2 A_{aa22} - \lambda A_{ab22} - \lambda A_{ba22} + A_{bb22} + \lambda^2 A_{aa32} - \lambda A_{ab32} - \lambda A_{ba32} + A_{bb32} \geq 0; \quad (51)$$

$$-k A_{aa32} + A_{ab32} + k A_{ba32} - A_{bb32} \geq 0. \quad (52)$$

We choose $\lambda = \sqrt{k}$. Adding the inequalities (50)–(52), we find:

$$(1 - \sqrt{k})(A_{aa22} - \sqrt{k}A_{ba22} + A_{ab32} - \sqrt{k}A_{ba32}) \geq 0. \quad (53)$$

We apply now the second Green identity to (v_a, v_b) on $\Omega_1, \Omega_2, \Omega_R$:

$$-A_{ab22} + k A_{ba22} = 0; \quad (54)$$

$$A_{ab22} - A_{ba22} + A_{ab32} - A_{ba32} = 0; \quad (55)$$

$$-\frac{1}{k} A_{ab32} + A_{ba32} + \omega_b - \omega_a = 0. \quad (56)$$

We assume that $k < 1$ (the conductivity is smaller in the unbounded subdomain). Then, we add the inequality (48) multiplied by $(1 + \sqrt{k})/(1 - \sqrt{k})$ (its sign does not change), the equality (54), the equality (55) multiplied by $-\sqrt{k}$ and the equality (56) by k . The final result

is $\omega_b - \omega_a \geq 0$ and we have the final conclusion for the degenerate scale factor: $\rho_a \geq \rho_b$. If $k > 1$, we have $\rho_b \geq \rho_a$.

It proves that the intrinsic degenerate scale factor increases when the bounded domain grows, for the same position of the internal boundary Γ_1 if $k > 1$.

5. Use of conformal mapping

Conformal mapping is powerful to provide solutions of problems involving 2D Laplace equation. It has been used many times to provide the degenerate scale factor in the homogeneous case or equivalently the capacity of the domain contained inside a given boundary Γ .

In the following, we consider its use for a conduction problem in an heterogeneous domain with two different values of the conductivity.

5.1. General case of two curves being the images by a conformal mapping of two concentric circles

As we know the value of the intrinsic degenerate scale factor for the case of concentric circles, it is natural to consider the case when the boundaries Γ_1 and Γ_2 are the images of two concentric circles C_1 and C_2 (Fig. 9) by a conformal mapping w which maps the outside of C_1 to the outside of Γ_1 . The conformal mapping is assumed to be $w(z) = z + a_1/z + \dots$ and the radius $R_1 = 1$. Then, Γ_1 is at the degenerate scale for the homogeneous conductivity problem.

We consider the solution defined by $v_i = \Re(A_i \ln(z) + B_i)$, $z \in \Omega_i$ for the circle problem with $A_2 = -1/2\pi$ (Section 3.6). We are going to prove that the function defined by $\tilde{v}_i = \Re(v_i(w^{-1}(\zeta)))$, $\zeta \in \Omega'_i$ is the solution of the problem BVP_e for Ω'_1, Ω'_2 . It is seen easily that \tilde{v}_i are harmonic functions in Ω'_i . The continuity of the function $\tilde{v}_1(\zeta) = \tilde{v}_2(\zeta)$, $\zeta \in \Gamma_2$ is a direct consequence of the continuity of v at the boundary C_2 . We must check the transmission condition of $\partial \tilde{v} / \partial n$.

We have:

$$\partial \tilde{v}_i / \partial n(\zeta) = \Re(\partial v_i(w^{-1}(\zeta)) / \partial n) = \Re(e^{-\theta} A_i / (w^{-1}(\zeta) w'(w^{-1}(\zeta))), \quad (57)$$

with θ the angle between ζ and n . As $A_i \in \mathbb{R}$ and $k_1 A_1 = k_2 A_2$ we conclude that $k_1 \partial \tilde{v}_1 / \partial n = k_2 \partial \tilde{v}_2 / \partial n$. The degenerate scale is found by considering the asymptotic behavior of $\tilde{v}_2(r)$. For large ζ , we have $\tilde{v}_2(\zeta) = \Re(-1/2\pi \ln(w^{-1}(\zeta)) + B_2)$. We conclude, as $w^{-1}(\zeta) = \zeta + O(1/|\zeta|)$ for $|\zeta| \rightarrow \infty$, that $\tilde{v}_2(\zeta) = -1/2\pi \ln|\zeta| + B_2$. We conclude that the degenerate scale is the same for the problem with circles (C_1, C_2) and for the problem with $(\Gamma_1 = w(C_1), \Gamma_2 = w(C_2))$, the values of the conductivity k_i being the same in the two problems.

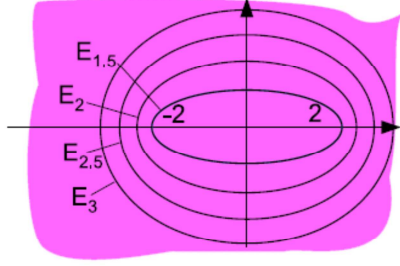


Fig. 10. The notch and some ellipses E_r .

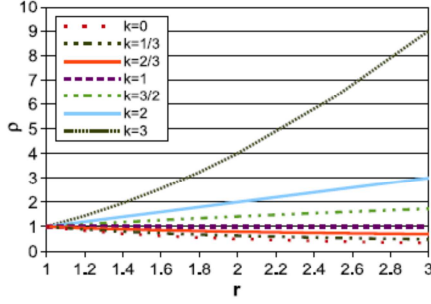


Fig. 11. Value of the degenerate scale factor ρ as a function of r .

5.2. Example: case of a notch

We consider the conformal mapping $z \rightarrow z + 1/z$. The image of the unit circle is the segment $[-2, 2]$ (Fig. 10). The images of the circles of radius r are ellipses E_r with semi-axes $(r + 1/r, r - 1/r)$, i.e. for an aspect ratio $(r^2 + 1)/(r^2 - 1)$. We consider the problem where Γ_1 is the segment $[-2, 2]$ and Γ_2 is the ellipse E_r . Fig. 11 gives the value of the degenerate scale factor as a function of r and for different values of k , the ratio of the conductivity in the two areas. Fig. 11 shows that if $k \neq 1$ then $\rho \neq 1$ for a fixed r , that is coherent with Section 4.2. When $r \nearrow$, then ρ increases if $k > 1$ and decreases if $k < 1$, as seen in Section 4.3.

6. Case of problems in a half-plane

In this section, the case of boundary value problems defined within a half-plane P^+ is studied. It is assumed that the boundary of the studied domain has no common point with the line bounding the half-plane. The BIEs corresponding to the bounded subdomains are the same as for a problem in the whole plane. The BIE corresponding to the unbounded subdomain is similar but the Green function must be changed [32,35]:

$$\bar{G}(x, y) = G(x, y) + \epsilon G(x, \bar{y}) \quad (58)$$

with $\epsilon = 1$ for the boundary condition $\frac{\partial u}{\partial n} = 0$ on the boundary line and $\epsilon = -1$ for the boundary condition $u = 0$ and \bar{y} is the symmetric of y along the line D (Fig. 12).

We need only to consider exterior problems: indeed, if the boundary has no common point with the bounding line, interior problems involve only the usual Green functions of the plane and not the one related to the half-plane. As for the main study of the exterior problems in the plane, we consider one unbounded domain Ω_e and $(n-1)$ bounded domains Ω_i . Fig. 12 displays the geometry of the problem in the upper half-plane (domains Ω_e, Ω_i). It is known that the Green function for the half-plane is obtained by considering also the points obtained by symmetry with respect to the boundary line. It is therefore useful to consider the domains obtained by such a symmetry: they are denoted $\bar{\Omega}_e, \bar{\Omega}_i$ within Fig. 12. The symmetrized domains will be of prime importance in the following.

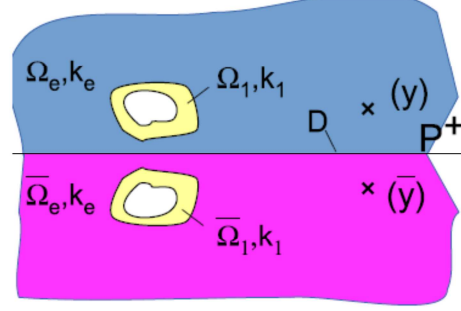


Fig. 12. Problem in the half-plane and its symmetrized counterpart in the plane.

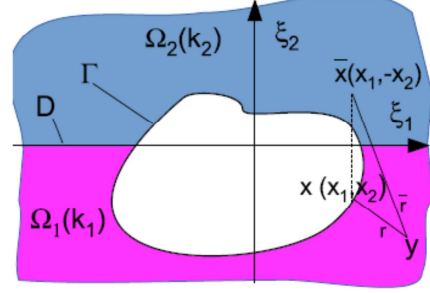


Fig. 13. Case of two half-planes with conductivities k_1, k_2 .

6.1. Earlier results for a homogeneous half-plane

A preceding paper [35] has shown, in the homogeneous case, that there is one degenerate scale if the boundary condition on the line D bounding the half-plane is $\frac{\partial u}{\partial n} = 0$ and no degenerate scale if the boundary condition is $u = 0$. The methods of this earlier paper can be applied to the piecewise homogeneous problems under consideration. An important result of this earlier paper is that the degenerate scale for the half-plane with a flux condition on the boundary line is the same as the degenerate scale for the whole plane containing the original boundary and its symmetric with respect to the boundary line. This last problem will be called the “symmetrized problem” in the following.

We are considering again the symmetrized solutions. The (u, q) solution of the BIE in the half-plane P^+ can be completed in the whole plane by

$$\hat{u}(x) = \begin{cases} u(x), & x \in P^+ \\ \epsilon u(\bar{x}), & x \notin P^+ \end{cases}; \quad (59)$$

$$\hat{q}(x) = \begin{cases} q(x), & x \in P^+ \\ \epsilon q(\bar{x}), & x \notin P^+. \end{cases} \quad (60)$$

with $\epsilon = -1$ for the boundary condition $u = 0$ and $\epsilon = 1$ for the boundary condition $\frac{\partial u}{\partial n} = 0$.

6.2. The degenerate scales of a problem in the half-plane with $u = 0$ on the boundary line are also degenerate scales of the symmetrized problem in the whole plane

The (u, q) non null solution of the homogeneous BIEs can be completed by (\hat{u}, \hat{q}) (with $\epsilon = -1$). The BIEs are the same (eventually up to a symmetry) for the bounded subdomains in the half-plane and in the plane. For the unbounded domain, it can be checked directly that the

corresponding BIE for the whole plane is also satisfied (see [35]). Then, if a problem in the half-plane with $u = 0$ on the boundary line is at the degenerate scale, the symmetrized problem is also at the degenerate scale in the plane.

But it should be noticed that a non null solution of the BVP in the half-plane with $v = 0$ on the boundary line is not possible. By symmetrization, the solution in the whole plane is such that $\int_{\Gamma} \hat{q} = 0$ since $q(x) = -q(\bar{x})$ with asymptotic behavior $v(x) \rightarrow 0, x \rightarrow \infty$ and this leads to a null solution in the plane (uniqueness of the solution of BVP * , see Section 3.2.3) and then also in the half-plane.

6.3. The degenerate scales of a problem in the half-plane with $\frac{\partial u}{\partial n} = 0$ on the boundary line are also degenerate scales of the symmetrized problem in the whole plane

We obtain similar results for $\frac{\partial u}{\partial n} = 0$ on the boundary line. The (u_i, q_i) non null solution of the homogeneous BIE in the half-plane can be completed by $\hat{u}_i(x), \hat{q}_i(x)$ within the whole plane with $\epsilon = 1$ and these extended solutions are solutions of the BIEs in the whole plane.

So we can conclude that the degenerate scales for the two problems in the half-plane are all degenerate scales for the symmetrized problem in the plane.

6.4. Determination of the non null solutions of the BIEs of the half-plane from the solutions corresponding to the degenerate scales of the whole plane

We consider now the geometrically symmetrized problem. The corresponding BIE problem has $(2n - 1)$ degenerate scales. Due to the symmetry, the $2n - 2$ degenerate scales corresponding to the bounded subdomains are equal two by two. Then the procedure described in Section 3.4.2 can be applied to each bounded domains Ω_i and to $\bar{\Omega}_i$. We assume that Ω_i (and $\bar{\Omega}_i$) is at its degenerate scale. We denote by u^i, q^i and $\bar{u}^i = u^i(\bar{x}), \bar{q}^i = q^i(\bar{x})$ the corresponding non null solutions of the homogeneous BIEs. By considering $v^i(x) = u^i(x) + \epsilon \bar{u}^i(x)$ $r^i(x) = q^i(x) + \epsilon \bar{q}^i(x)$, we define a solution for the BIE system in the half-planes with $\epsilon = -1$ for boundary condition $u = 0$ or $\epsilon = 1$ for boundary condition $\frac{\partial u}{\partial n} = 0$. As in Section 3.4.2, it can be checked that $(v^i, r^i) \neq (0, 0)$. This defines $(n - 1)$ degenerate scales associated to the $n - 1$ bounded domains for the two cases of problems in the half-plane.

We consider now the intrinsic degenerate scale of the problem in the plane. Due to the symmetry, the solution is also symmetric (it is unique and the symmetrized solution is also associated to the intrinsic degenerate scale) then we have also $\frac{\partial u}{\partial n} = 0$ on D and conclude that this a degenerate scale in the half-plane with boundary condition $\frac{\partial u}{\partial n} = 0$. So with the condition $\frac{\partial u}{\partial n} = 0$, there is an intrinsic degenerate scale.

6.5. Conclusion

From Sections 6.2 and 6.3, we see that the sum of the numbers of the degenerate scales for the two problems in the half-plane with $n - 1$ bounded subdomains and 1 semi-infinite subdomain is at most equal to the number of degenerate scales of the symmetrized problem which is equal to $2n - 1$ taking into account their multiplicity.

From Section 6.4 we have found $(n - 1)$ geometric degenerate scales each of them equal to the degenerate scale of one of the bounded subdomains for the two cases of half-plane problems and one intrinsic degenerate scale for the problem with $\frac{\partial u}{\partial n} = 0$, this intrinsic degenerate scale being equal to the intrinsic degenerate scale of the symmetrized problem. The total number of found degenerate scales is $2n - 1$, taking into account their multiplicity.

Finally we have found all the degenerate scales for the two problems in the half-plane. These results generalize the results obtained for the homogeneous half-plane [35] which can be considered as the special case $n = 1$.

7. Case of two half-planes with different conductivities

Up to now, boundaries between the constitutive domains have been assumed as bounded. It is therefore interesting to see how the results obtained previously can be extended to the case where the internal boundaries are not bounded. We consider therefore the case of internal boundaries located along a line that separates the plane into two half-planes as in Fig. 12. There is an unbounded boundary, and as a consequence it is no longer possible to consider the logarithmic capacity of this boundary.

7.1. Case of two half-planes minus one bounded domain

In that case, the approach is different from the method used in the previous sections. Instead of writing several BIEs, it is possible to look for the fundamental solution $G(x, y)$ related to the whole plane with a different conductivity k in each half-plane such that:

$$\nabla_y \cdot (k(y) \nabla_y (G(x, y))) = -\delta(x - y), \quad (61)$$

and that the transmission conditions at the interface between the two half-planes are satisfied (see for the case of elasticity [36] and an early reference for electrostatics [37]). For a source point x in the domain Ω_1 , we consider:

$$G(x, y) = \begin{cases} -\frac{1}{2\pi k_1} \ln(r) + \frac{k_2 - k_1}{2\pi k_1(k_1 + k_2)} \ln(\bar{r}), & y \in \Omega_1; \\ -\frac{1}{\pi(k_1 + k_2)} \ln(r), & y \in \Omega_2. \end{cases} \quad (62)$$

If x is in the subdomain Ω_2 , the subscripts 1 and 2 must be exchanged in the above formula.

It can be checked easily that $\text{div}_y(k(y) \text{grad}_y(G(x, y))) = -\delta(x - y)$. At the boundary D between the two half-planes, we have $r = \bar{r}$ in (62) and the continuity of $G(x, y)$ is easily checked. The last condition is on the normal derivative on D. We must have the continuity of $k(y_2) \partial G(x, y) / \partial y_2$ at $y_2 = 0$. This is also an easy verification to do, using $\partial(\ln(r)) / \partial y_2 = -x_2 / r^{3/2}$ and $\partial(\ln(\bar{r})) / \partial y_2 = x_2 / r^{3/2}$ for $y_2 = 0$. We have also the symmetry: $G(x, y) = G(y, x)$.

For $k_1 = k_2$, the standard value of G for the plane is recovered. For $k_2 = 0$ the solution in the half-plane Ω_1 is recovered (for the standard boundary condition $q = 0$ at $x_2 = 0$). For $k_2 = \infty$, the solution in the half-plane Ω_2 is recovered for the less usual boundary condition $u = 0$ [38]. It should be noted that the behavior at infinity is for all cases $x \in \Omega_i$ and $y \in \Omega_j$:

$$G(x, y) = -\frac{1}{\pi(k_1 + k_2)} \ln(r) + O\left(\frac{1}{r}\right) \quad r \rightarrow \infty. \quad (63)$$

7.2. Study of the BIE

With this specific Green function we can build the usual direct regularized BIE for the exterior problem [38]. For the sake of simplicity we assume that there are no internal sources. We write the BIE:

$$\int_{\Gamma} [u(y) - u(x)] H(x, y) - q(y) G(x, y) dS_y = 0, \quad (64)$$

with $H(x, y) = k(y) \partial G(x, y) / \partial n_y$. As with standard Green function for the plane, there is a non null solution of the BIE with homogeneous Dirichlet condition if:

$$\int_{\Gamma} q(y) G(x, y) dS_y = 0. \quad (65)$$

7.3. Study of a special case

We consider the case when $\Gamma \subset \Omega_1$ and when the diameter $d \gg a$ with d the distance of Γ to D and a the diameter of Γ (Fig. 14).

The logarithmic capacity of Γ is denoted by C. For $k_2 = 0$, the Green function is the same as for the half-plane with Neumann boundary condition on line D, then the degenerate scale corresponds to the logarithmic

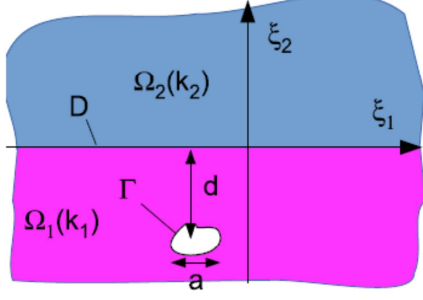


Fig. 14. Case of two half-planes with Γ "far" from D .

Table 2
Variation of the degenerate scale factor as a function of k_2 for two half-planes and Γ 'far' from D .

| k_2 | $k_2 = 0$ | $k_2 = k_1$ | $k_2 = +\infty$ |
|--------|-----------------------|-------------|-----------------|
| ρ | $\approx 1/\sqrt{dC}$ | $1/C$ | $+\infty$ |

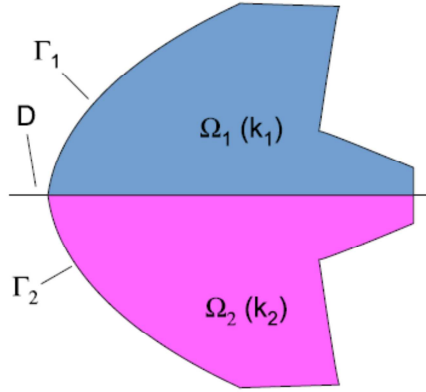


Fig. 15. Study of a symmetric boundary.

capacity of the union of Γ and its symmetric along D . If the diameter a of Γ is far much smaller than the distance d between Γ and D , then the logarithmic capacity of the union of Γ and its symmetric is $\approx \sqrt{dC}$ and therefore the degenerate scale factor is $\approx 1/\sqrt{dC}$ [35]. If $k_2 = k_1$, the plane is homogeneous and the scale is degenerate if $C = 1$, then we have $\rho = 1/C$. If $k_2 \rightarrow +\infty$, then the Green function tends to the Green function of the half-plane with Dirichlet condition on the line bounding the half-plane and there is no finite degenerate scale factor [35], and this suggests $\rho \rightarrow +\infty$. These conclusions are summarized in Table 2.

These results are similar to those of Table 1.

7.4. Interior problem: symmetric case

Contrarily to the exterior problem, the interior problem with symmetry has not the same feature: the degenerate scale is the same as for an homogeneous domain having the same shape.

We prove this property by considering the case of an interior problem which is along the line D bounding the areas with different conductivities, with a symmetry between partial subdomains (Fig. 15).

Then the degenerate scale of the corresponding BIE is the usual degenerate scale for a homogeneous problem. Let us check that by choosing: $q_1 = k_1\sigma$ on Γ_1 , $q_2 = k_2\sigma$ on Γ_2 , with $\int_{\Gamma} \ln|x-y|\sigma(y)dS_y = 0$, this distribution of normal flux gives a non null solution to the BIE for the heterogeneous problem with Dirichlet boundary condition.

The BIE is at its degenerate scale if $\int_{\Gamma} G(x,y)q(y)dS_y = 0$. For $x \in \Gamma_1$ this condition writes out:

$$\int_{\Gamma} G(x,y)q(y)dS_y = 0 \quad (66)$$

$$= \int_{\Gamma_1} G(x,y)q(y)dS_y + \int_{\Gamma_2} G(x,y)q(y)dS_y \quad (67)$$

$$= \underbrace{\int_{\Gamma_1} -\frac{1}{2\pi k_1} \ln(r)q(y)dS_y}_{I_1} + \underbrace{\int_{\Gamma_1} \frac{k_2 - k_1}{2\pi k_1(k_1 + k_2)} \ln(\bar{r})q(y)dS_y}_{I_2} + \underbrace{\int_{\Gamma_2} -\frac{1}{\pi(k_1 + k_2)} \ln(r)q(y)dS_y}_{I_3} \quad (68)$$

The integral I_2 can be turned into an integral on Γ_2 :

$$I_2 = \underbrace{\int_{\Gamma_2} \frac{k_2 - k_1}{2\pi k_1(k_1 + k_2)} \ln(r)q(\bar{y})dS_y}_{I_4} \quad (69)$$

Because of the symmetry, we have $\sigma(y) = \sigma(\bar{y})$ and we can replace $q(\bar{y})$ by $k_1\sigma(y)$ in I_4 . Then, we have $I_3 + I_4 = \int_{\Gamma_2} -\frac{1}{2\pi} \ln(r)\sigma(y)dS_y$ and we finally conclude $\int_{\Gamma} G(x,y)q(y)dS_y = \int_{\Gamma} G(x,y)\sigma(y)dS_y = 0$.

This result is interesting, because it shows that the use of the Green function accounting for the heterogeneity leads to a degenerate scale which is different from the ones obtained by discretizing both boundaries and using the classical Green's function. As seen before, these last ones correspond to the degenerate scales of both subdomains.

8. Computation of the intrinsic degenerate scale factor by BEM and numerical examples

In a first step, the methods using different Green functions are described before to present several numerical examples.

8.1. Methods of numerical computation of the degenerate scales by using BEM

8.1.1. Case of the full plane

The numerical computation of the degenerate scales will be described in the case of two subdomains. Let us therefore consider two subdomains $\Omega_{(l)}$ ($l = 1, 2$). The BEM formulation on each subdomain can be expressed in the form:

$$[H_{(l)}][u] = \frac{1}{k_l} [G_{(l)}][q]$$

for $l = 1, 2$. Column vectors $[u]$ and $[q]$ contain the value of the potential and normal flux at the nodes located on the boundary. $[G_{(l)}]$ and $[H_{(l)}]$ are interaction matrices computed from Green's function (for a unit value of conductivity) and normal derivatives of Green's function over boundary elements. Each matrix can be split into one part related to $\Omega_{(l)}$ and one related to the interface between the two subdomains, leading to:

$$\frac{1}{k_1} \{ [G_{(1,1)}][q_1] + [G_{(1,2)}][q_2] \} - [H_{(1,1)}][u_1] = [H_{(1,1)}][u_1]$$

for $\Omega_{(1)}$, where $[G_{(l,l)}]$ denotes the part of matrix $[G_{(l)}]$ containing the column vectors related to the interface nodes and to:

$$[H_{(2,1)}][u_1] + [H_{(2,2)}][u_2] = -\frac{1}{k_2} [G_{(2,1)}][q_1] + \frac{1}{k_2} [G_{(2,2)}][q_2]$$

for $\Omega_{(2)}$. For a Dirichlet boundary condition on the external boundary, this leads to:

$$[K] \begin{bmatrix} q_1 \\ q_I \\ -q_2 \\ u_I \end{bmatrix} = \begin{bmatrix} \frac{1}{k_1} G_{(1,1)} & \frac{1}{k_1} G_{(1,I)} & 0 & -H_{(1,I)} \\ 0 & \frac{1}{k_2} G_{(2,I)} & \frac{1}{k_2} G_{(2,2)} & H_{(2,I)} \end{bmatrix} \begin{bmatrix} q_1 \\ q_I \\ -q_2 \\ u_I \end{bmatrix} = 0$$

Let us denote by $[G_{0(I)}]$ the interaction matrix built for a given boundary $\Gamma_{(I)}$, the interaction matrix $[G_{\rho(I)}]$ built for a scaled boundary $\rho\Gamma_{(I)}$ can be written as

$$[G_{\rho(I)}] = [G_{0(I)}] + C[B_{(I)}] \quad (70)$$

where $[B_{(I)}]$ is built by using only an integration of the interpolation functions over the elements (see details in Ref. [26]) and $C = \ln(\rho)$. Using the same notation for the decomposition of matrix $[B_{(I)}]$ leads to:

$$[K_{(0)}] + C[L] = 0 \quad (71)$$

where $[K_{(0)}]$ is $[K]$ being built by using $[G_{0(I)}]$ and $[L]$ is given by:

$$[L] = \begin{bmatrix} \frac{1}{k_1} B_{(1,1)} & \frac{1}{k_1} B_{(1,I)} & 0 & 0 \\ 0 & \frac{1}{k_2} B_{(2,I)} & \frac{1}{k_2} B_{(2,2)} & 0 \end{bmatrix}$$

It shows that if the heterogeneous domain is at a degenerate scale, C is the generalized eigenvalue μ_p of $([K_{(0)}], -[L])$. Then, the degenerate scale factors are given by $\rho_p = e^{\mu_p}$.

In the following, all computations are performed by using Matlab software. The interaction matrices are built by using constant elements.

8.1.2. Case of the half-plane

The BEM formulation of the half-plane is effected by introducing the Green function for the half-plane:

$$\tilde{G}(x, y) = G(x, y) + \epsilon G(\bar{x}, y) \quad (72)$$

where $\epsilon = 1$ for the condition of null flux over the boundary line and $\epsilon = -1$ for the condition of null displacement. \bar{x} is the symmetric of x with respect to the line bounding the halfplane and $G(x, y)$ is the Green function for the full plane. As a consequence the matrix $[\tilde{G}_{(\epsilon)}]$ of the outer part can be written as:

$$[\tilde{G}_{(\epsilon)}] = [G_{(\epsilon)}] + \epsilon[\tilde{G}_{(\epsilon)}] \quad (73)$$

where $[\tilde{G}_{(\epsilon)}]$ is built by using $G(\bar{x}, y)$. When introducing the scaling, each part of \tilde{G} is affected by the scaling and therefore:

$$[\tilde{B}_{(\epsilon)}] = [B_{(\epsilon)}] + \epsilon[\tilde{B}_{(\epsilon)}] = (1 + \epsilon)[B_{(\epsilon)}] \quad (74)$$

because $[\tilde{B}_{(\epsilon)}] = [B_{(\epsilon)}]$.

Using $[\tilde{G}_{(\epsilon)}]$ and $[\tilde{B}_{(\epsilon)}]$ allows us to build the matrices K and L to find the degenerate scales, similarly to the case of the full plane. It is worthwhile noticing that $[\tilde{B}_{(\epsilon)}]$ is null for the case of null displacement on the bounding line ($\epsilon = -1$). This explains why there is no degenerate scale for the homogeneous problem in this case and why there is no intrinsic degenerate scale in the heterogeneous case.

The formulation above allows us to compare the degenerate scale for the half-plane with the degenerate scale for the symmetrized problem. For example, in the homogeneous case, the degenerate scale for the half-plane is obtained by solving

$$\{[G] + [\tilde{G}]\} + C\{[B] + [\tilde{B}]\}[q] = 0. \quad (75)$$

By comparison, the degenerate scale for the symmetrized problem is given by:

$$[G^{sym}] \begin{bmatrix} q^0 \\ \bar{q} \end{bmatrix} = 0 \quad (76)$$

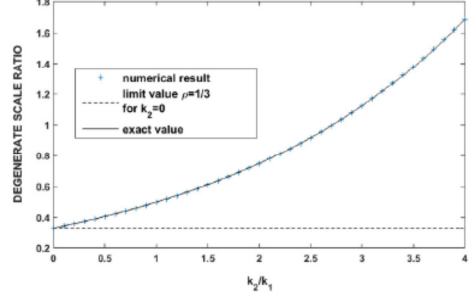


Fig. 16. Theoretical and numerical computations of the intrinsic degenerate scale factor as a function of the conductivity ratios k_2/k_1 for the case of Fig. 4 with two circles of radii 2 and 3.

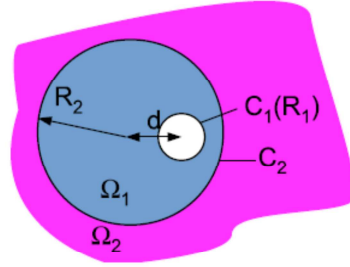


Fig. 17. Case of two non concentric circles.

where \bar{q} contains the fluxes at the symmetrized contour and

$$[G^{sym}] = \begin{bmatrix} G & \tilde{G} \\ \tilde{G} & G \end{bmatrix}. \quad (77)$$

One can see easily that if the half-plane is at a degenerate scale, the symmetrized contour is also at a degenerate scale by taking $q^0 = \bar{q}$.

8.1.3. Case of two half-planes with different conductivities

As shown in Section 7, the Green tensor for two half-planes with conductivities k_1 and k_2 are given by

$$\tilde{G}(x, y) = \{G(x, y) + \alpha G(\bar{x}, y)\} \quad (78)$$

where $\alpha = \frac{k_1 - k_2}{k_1 + k_2}$, in the case where x and y are within the same half-plane and with $k_1 = 1$. As a consequence, the related interaction matrix is given by:

$$[\tilde{G}] = \{[G] + \alpha[\tilde{G}]\} \quad (79)$$

and

$$[\tilde{B}] = (1 + \alpha)[B]. \quad (80)$$

The degenerate scale factors are then obtained from the generalized eigenvalues of $([\tilde{G}], -[\tilde{B}])$.

8.2. Numerical examples

From a general point of view, the numerical results confirm the nature of the results obtained theoretically in the previous sections: for interior problems, the numerical computations recovers the degenerate scale factors of the subdomains whose boundaries are discretized. For exterior problems, the computation in the case of two subdomains recovers the degenerate scale factor related to the finite subdomain with,

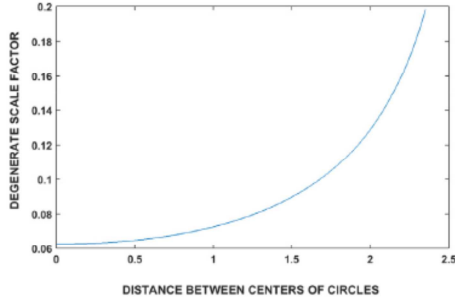


Fig. 18. Intrinsic degenerate scale factor for two circles of radii 1 and 4 as a function of the distance between the centers of the circles.

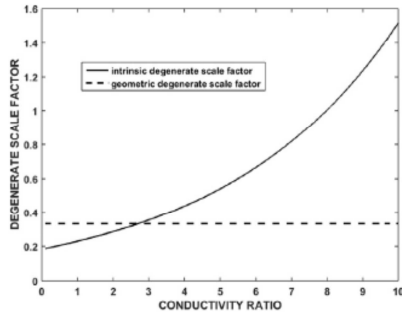


Fig. 19. Intrinsic degenerate scale factors for two circles of radii 1 and 3 within a half-plane with a null flux over the boundary line.

in addition, the “intrinsic” degenerate scale factor. Recovering the degenerate scale factor of the finite subdomain is of lesser interest. So, in the following, the examples are mainly focused on the intrinsic degenerate scale, a few examples being mentioned in the case of interior problems.

8.2.1. Comparison between theoretical and numerical computations of the intrinsic degenerate scale factor in an infinite plane

Fig. 16 displays the theoretical and numerical values of the intrinsic degenerate scale factor for the exterior problem corresponding to the case of Fig. 4 with two circles of radii 2 and 3 for a varying value of the conductivity ratio k_2/k_1 . The computation is effected by using 400

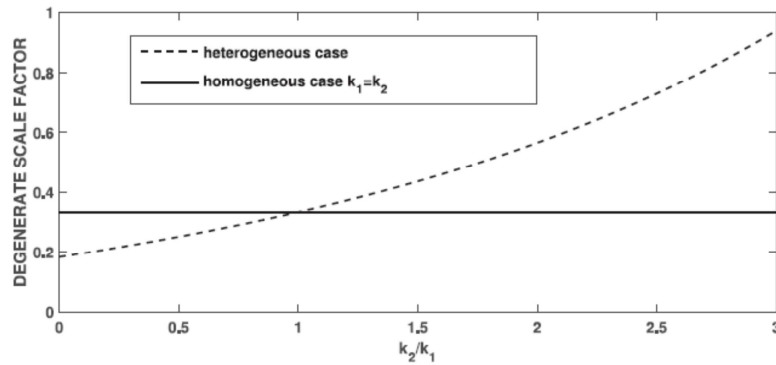


Fig. 20. Intrinsic degenerate scale factors for one circle of radius 3 within a half-plane with different conductivity ratios between upper and lower half-planes.

constant boundary elements for each of the two circles. In all cases, the numerical computation produces the intrinsic degenerate factor which depends on the conductivity ratio and a constant degenerate scale factor $\rho = 1/3$ corresponding to the domain comprised between the two circles.

The comparison with the theoretical results shows that the intrinsic degenerate scale factor is computed accurately in all cases. In the case of a null value of the conductivity of the outer domain ($r > 3$), the computation recovers the value $\rho = 1/3$ corresponding to the degenerate scale factor of the domain bounded by the two circles. If $k_1 = k_2$, we recover the homogeneous case and the corresponding degenerate scale factor $\rho = 1/2$.

A similar computation has been made for the heterogeneous interior problem with the same conductivities and the same radii of the circles. In this case, the conduction is taken into account inside the inner circle. The computations produce two degenerate scales: $\rho_1 = 1/3$, corresponding to the degenerate factor of the part between the two circles and $\rho_2 = 1/2$, corresponding to the inner circle.

8.2.2. Case of two non-concentric circles

Let us consider now the case of two non concentric circles C_1 and C_2 with again C_2 containing C_1 (Fig. 17). As shown in the previous sections, this problem displays two degenerate scale factors: one related to the interior domain for the part between C_1 and C_2 and one intrinsic degenerate scale. The first one does not depend on the relative position between the two circles. So, one may question if the intrinsic degenerate scale factor is also independent on the relative position.

Fig. 18 shows the values of the intrinsic degenerate scale factor for two non concentric circles of radii 1 and 4 as a function of the distance between the centers of the circles when the conductivity ratio is $k_2/k_1 = 2$.

It can be seen that the intrinsic degenerate scale factor increases as a function of the distance between the centers, from 0.062 for concentric circles up to 0.2 for a distance of 2.35 between the centers. It shows that contrarily to the case of the degenerate scale factors related to each partial domain, the intrinsic degenerate scale factor depends on the relative position between the circles.

8.2.3. Case of two concentric circles within a half-plane

The degenerate scale factors have been obtained for the exterior problem related to two concentric circles of radii $R_i = 2$ and $R_e = 3$ inside a half-plane, the distance between the centers of the circle and the boundary of the half-plane being equal to $1.5R_e$. The condition over the boundary of the half-plane is $q = 0$.

Fig. 19 displays the values of both degenerate scale factors as a function of the conductivity ratio. One can still observe the dependence of the intrinsic degenerate scale factor on the conductivity ratio, the geometric degenerate scale being constant, equal to $1/3$ for any conductiv-

ity ratio. Compared with the full plane case, one can observe that the intrinsic degenerate scale factor corresponding to small values of the conductivity ratio is no more equal to $1/3$. It is equal to the degenerate scale corresponding to the domain outside the outer circle in the half-plane, $\rho = 0.1827$ (see below).

A similar computation has been made in the case of the condition $u = 0$ over the boundary line. In this case, the computation produces only one constant geometric degenerate scale factor $\rho = 1/3$ and there is no more an intrinsic degenerate scale factor, as found in Section 6.

8.2.4. Case of two half-planes with different conductivities

The degenerate scale factors have been obtained in the case of the exterior problem of a circle of radius $R = 3$ inside a half-plane, the distance between the center of the circle and the boundary of the half-plane being equal to $1.5R$. Fig. 20 shows the degenerate scale factors for different conductivity ratios between lower and upper planes. For $k_2/k_1 = 0$, the computation gives the same value $\rho = 0.1827$ as for the previous case of one circle in the half-plane with a null flux at the boundary line. It can be found also from the logarithmic capacity of two circles of radii 3 with a distance of 9 between the centers [39]. For $k_2/k_1 = 1$, one obtains $\rho = 1/3$, corresponding to the case of the homogeneous case for the circle of radius 3. For $k_2/k_1 > 1$, ρ increases and becomes very large when k_2 tends to infinity, which corresponds to the case of $u = 0$ at the boundary, when there is no more degenerate scale. This is consistent with the theoretical considerations of the previous section.

9. Conclusion

We have considered the problem of the degenerate scale when the medium is piecewise homogeneous. The most convenient way to use BEM is then to split the plane into several homogeneous domains and to write a BIE for the boundary of each subdomain. This is possible if all boundaries are bounded. Then, if there are n bounded subdomains, there are n degenerate scales corresponding to each bounded subdomain for interior and exterior problems. For exterior problems, there is in addition an intrinsic degenerate scale linked to a non null solution of the BVP corresponding to the overall domain. This result has been proved theoretically and recovered by using a numerical computation of the degenerate scale factors. An analytical value of the intrinsic degenerate scale factor has been provided in the case of subdomains bounded by concentric circles. These results have been extended to the case where the internal boundaries are obtained from the circles by using conformal mapping. The case of an exterior problem in a half-plane with only one unbounded subdomain has been also considered: if the boundary condition on the line bounding the half-plane is $u = 0$, the degenerate scales are the degenerate scales of the bounded subdomains; if the boundary condition on the line bounding the half-plane is $\partial u / \partial n = 0$, the degenerate scales are the degenerate scales of the bounded subdomains and an intrinsic degenerate scale.

When it is possible to have an explicit Green function for the piecewise homogeneous problem, then the formulation of the problem needs only one BIE. We have been able to find some results in the case of domains within two half-planes, each having its own conductivity.

We focused on the conduction problem with Dirichlet boundary conditions but several other cases could be investigated.

The results obtained in this work can be extended to other kinds of boundary conditions. The necessary conditions of parts 2 and 3 can still be derived in the same way for Robin or mixed boundary condition as for Dirichlet one. The only possible degenerate scales are the ones of each subdomain and an intrinsic degenerate scale for the exterior problem. The sufficient conditions need the use of an existence theorem for the boundary value problems equivalent to BVP_i , BVP_e^* , BVP_e and BVP_e^* . The use of the inversion can reduce the problem of BVP_i^* to BVP_e^* as in Section 3. The extension of the results presented in this paper to the case of plane elasticity will be also considered in a future work.

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