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Interpolation, extrapolation, Morrey spaces and local energy control for the Navier–Stokes equations

Pierre Gilles Lemarié–Rieusset*

Abstract

Barker recently proved new weak-strong uniqueness results for the Navier–Stokes equations based on a criterion involving Besov spaces and a proof through interpolation between Besov–Hölder spaces and L^2 . We improve slightly his results by considering Besov–Morrey spaces and interpolation between Besov–Morrey spaces and L^2_{loc} .

Keywords : Navier–Stokes equations, Morrey spaces, Besov spaces, uniformly locally square integrable functions, weak-strong uniqueness

AMS classification : 35K55, 35Q30, 76D05.

1 The Navier–Stokes equations

Let \vec{u}_0 a divergence-free vector field on \mathbb{R}^3 . We shall consider weak solutions to the Cauchy initial value problem for the Navier–Stokes equations which satisfy energy estimates.

The differential Navier–Stokes equations read as

$$\partial_t \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u} = \Delta \vec{u} - \vec{\nabla} p$$

$$\operatorname{div} \vec{u} = 0$$

$$\vec{u}(0, \cdot) = \vec{u}_0$$

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Under reasonable assumptions, the problem is equivalent to the following integro-differential problem :

$$\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u}, \vec{u})(t, x)$$

where

$$B(\vec{u}, \vec{v}) = \int_0^t e^{(t-s)\Delta}\mathbb{P} \operatorname{div} (\vec{u} \otimes \vec{v}) ds \quad (1)$$

and \mathbb{P} is the Leray projection operator. (See [LR 2, LR 6] for details).

Weak Leray solutions for the Navier–Stokes equations

When $\vec{u}_0 \in L^2$, Leray proved existence of solutions \vec{u} on $(0, +\infty) \times \mathbb{R}^3$ such that :

- $\vec{u} \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$
- $\lim_{t \rightarrow 0^+} \|\vec{u}(t, \cdot) - \vec{u}_0\|_2 = 0$
- we have the Leray energy inequality

$$\|\vec{u}(t, \cdot)\|_2^2 + 2 \int_0^t \|\vec{\nabla} \otimes \vec{u}\|_2^2 ds \leq \|\vec{u}_0\|_2^2 \quad (2)$$

Such solutions are weak solutions : the derivatives in the Navier–Stokes solutions are taken in the sense of distributions. Those solutions (that satisfy the energy inequality (2)) are called *Leray weak solutions*.

When $\vec{u}_0 \in L^2$, Leray's proof of existence of solutions [Le] is based on mollification, energy estimates and compactness arguments :

- he solves

$$\partial_t \vec{u}_\epsilon + (\varphi_\epsilon * \vec{u}_\epsilon) \cdot \vec{\nabla} \vec{u}_\epsilon = \Delta \vec{u}_\epsilon - \vec{\nabla} p_\epsilon$$

with $\operatorname{div} \vec{u}_\epsilon = 0$ and $\vec{u}_\epsilon(0, \cdot) = \vec{u}_0$. Here, $\varphi \in \mathcal{D}$, $\int \varphi dx = 1$ and $\varphi_\epsilon(x) = \frac{1}{\epsilon^3} \varphi(\frac{x}{\epsilon})$.

- the solution holds on an interval $(0, T_\epsilon)$ where T_ϵ depends on ϵ and on $\|\vec{u}_0\|_2$ and we have the equality

$$\|\vec{u}_\epsilon(t, \cdot)\|_2^2 + 2 \int_0^t \|\vec{\nabla} \otimes \vec{u}_\epsilon\|_2^2 ds = \|\vec{u}_0\|_2^2$$

- the solution is then global; moreover by Rellich theorem, we find a subsequence that converges strongly in $(L_t^2 L_x^2)_{loc}$ to a Leray solution \vec{u}

Such solutions (i. e. obtained by this mollification/extraction process) will be called in the following *restricted Leray weak solutions*.

Restricted Leray solutions satisfy the Leray energy inequality which takes into account the energy on the whole space. But they enjoy as well a pointwise inequality property : for a non-negative locally finite measure μ we have

$$\partial_t(|\vec{u}|^2) + 2|\vec{\nabla} \otimes \vec{u}|^2 = \Delta(|\vec{u}|^2) - \operatorname{div}((2p + |\vec{u}|^2)\vec{u}) - \mu \quad (3)$$

Leray solutions that enjoy the pointwise energy inequality are called *suitable Leray solutions* [CKN].

Local weak Leray solutions

The pointwise energy inequality allows one [LR 1, LR 2] to develop a theory of weak solutions with infinite energy. Consider \vec{u}_0 a divergence-free vector field that is uniformly locally square integrable :

$$\sup_{x_0 \in \mathbb{R}^3} \int_{|x-x_0|<1} |\vec{u}_0(x)|^2 dx < +\infty.$$

A *local Leray solution* on $(0, T) \times \mathbb{R}^3$ is a solution such that

- $\vec{u} \in L_t^\infty(L_{\text{uloc}}^2) \cap (L_t^2 \dot{H}_x^1)_{\text{uloc}}$
- for all compact subset K of \mathbb{R}^3 , $\lim_{t \rightarrow 0^+} \int_K |\vec{u}(t, \cdot) - \vec{u}_0|^2 dx = 0$
- we have the pointwise energy inequality (3).

Local in time existence of restricted local Leray solutions has been proved for a positive T that depends only on $\|\vec{u}_0\|_{L_{\text{uloc}}^2}$ (see section 8).

2 The Prodi–Serrin criterion for weak-strong uniqueness

Based on a compactness criterion, the proof of existence of Leray solutions does not provide any clue on the would-be uniqueness of the solution to the Cauchy initial value problem.

A classical case of uniqueness of Leray weak solutions is Serrin’s criterion for weak-strong uniqueness [Pr, Se]. If $\vec{u}_0 \in L^2$ and if the Navier-Stokes equations has a solution \vec{u} on $(0, T)$ such that

$$\vec{u} \in X_T = L_t^p L_x^q \text{ with } \frac{2}{p} + \frac{3}{q} = 1 \text{ and } 2 \leq p < +\infty$$

then if \vec{v} is a Leray solution we have $\vec{u} = \vec{v}$ on $(0, T)$.

The proof of the criterion is based on the fact that if \vec{v} is a Leray solution and if \vec{u} is the mild solution with $\|\vec{u}\|_{X_T} < +\infty$, then the difference $\vec{w} = \vec{u} - \vec{v}$ satisfies a Gronwall estimate :

$$\|\vec{w}(t, \cdot)\|_2^2 + 2 \int_0^t \|\vec{\nabla} \otimes \vec{w}\|_2^2 ds \leq 2 \int_0^t \left| \int \vec{u} \cdot (\vec{w} \cdot \vec{\nabla} \vec{w}) dx \right| ds.$$

We have (for $\frac{2}{p} + \frac{3}{q} = 1$)

$$\|\vec{u} \otimes \vec{w}\|_2 \leq C \|\vec{u}\|_q \|\vec{w}\|_2^{\frac{2}{p}} \|\vec{\nabla} \otimes \vec{w}\|_2^{\frac{3}{q}}$$

so that

$$\left| \int \vec{u} \cdot (\vec{w} \cdot \vec{\nabla} \vec{w}) dx \right| \leq \|\vec{u}\|_q \|\vec{w}\|_2^{\frac{2}{p}} \|\vec{\nabla} \otimes \vec{w}\|_2^{1+\frac{3}{q}} = \|\vec{u}\|_q (\|\vec{w}\|_2^2)^{\frac{1}{p}} (\|\vec{\nabla} \otimes \vec{w}\|_2^2)^{1-\frac{1}{p}}.$$

Thus, we find

$$2 \int_0^t \left| \int \vec{u} \cdot (\vec{w} \cdot \vec{\nabla} \vec{w}) dx \right| ds \leq C \int_0^t \|\vec{u}\|_q^p \|\vec{w}\|_2^2 ds + \int_0^t \|\vec{\nabla} \otimes \vec{w}\|_2^2 ds.$$

The facts that $\|\vec{w}(0, \cdot)\|_2^2 = 0$ and $\|\vec{w}(t, \cdot)\|_2^2 \leq C \int_0^t \|\vec{u}\|_q^p \|\vec{w}\|_2^2 ds$ then gives $\vec{w} = 0$ on $(0, T)$.

We may comment a little further in the case $2 < p < +\infty$. In that case, the bilinear operator B (given by (1)) is bounded on $X_T = L_t^p L_x^q$. Thus, we find that the existence of $T > 0$ and of a solution in $L^p L^q$ with $\frac{2}{p} + \frac{3}{q}$ is equivalent to the existence of T' such that $e^{t\Delta} \vec{u}_0 \in L^p L^q$ on $(0, T')$ (and on $(0, +\infty)$, since $\vec{u}_0 \in L^2$). Using the thermic characterization of Besov spaces, we can see that this is equivalent with

$$\vec{u}_0 \in \dot{B}_{q,p}^{-1+\frac{3}{q}}.$$

Thus, the initial value is not only in L^2 , but it must belong as well to a Besov space with a better regularity than provided by the embedding

$$L^2 \subset \dot{B}_{q,2}^{-\frac{3}{2}+\frac{3}{q}} \subset \dot{B}_{q,p}^{-\frac{3}{2}+\frac{3}{q}}.$$

3 The Koch and Tataru theorem and T. Barker's question

We may now wonder how to generalize the Prodi–Serrin criterion. It means : given $\vec{u}_0 \in L^2$ and weak Leray solutions associated to \vec{u}_0 , find a space X

(as large as possible) such that if moreover $\vec{u}_0 \in \mathbb{X}$ then we have a solution $\vec{u} \in X_T$ for some space X_T of functions on $(0, T) \times \mathbb{R}^3$ and such that the existence of a solution in X_T implies that any other weak Leray solution is equal to this solution \vec{u} for $0 < t < T$.

The space BMO^{-1}

First of all, we precise which kind of space \mathbb{X} we are going to study. The idea is to look at an initial value which generates a solution in some uniqueness class (where uniqueness holds for small solutions). The setting where to construct such solutions is the setting of mild solutions, as introduced by Kato [Ka] : mild solutions are constructed by the Banach contraction principle.

Due to the symmetries of the equations (if \vec{u} is a solution for initial value \vec{u}_0 , then $\lambda\vec{u}(\lambda^2 t, \lambda(x - x_0))$ is a solution for the initial value $\lambda\vec{u}_0(\lambda(x - x_0))$), we look for spaces with norms invariant through the transforms $\vec{u}_0(\cdot) \mapsto \lambda\vec{u}(\lambda(\cdot - x_0))$ (for $\lambda > 0$). Moreover, in order to be able to define

$$B(\vec{u}, \vec{v}) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} (\vec{u} \otimes \vec{v}) ds$$

at least for $\vec{u} = \vec{v} = e^{t\Delta}\vec{u}_0$ (first step of the Picard iteration to find a fixed-point to $\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u}, \vec{u})$), we ask that $\int_{[0,1] \times B(0,1)} |e^{s\Delta}\vec{u}_0(y)|^2 ds dy < +\infty$.

Thus, we are lead to introduce the space \mathbb{X} of distributions v such that

- $\sup_{t>0} \sqrt{t} \|e^{t\Delta} v\|_\infty < +\infty$
- $\sup_{0<t, x_0 \in \mathbb{R}^3} t^{-3/2} \int_0^t \int_{B(x_0, \sqrt{t})} |e^{s\Delta} v(y)|^2 dy ds)^{1/2}$

This space \mathbb{X} has been identified by Koch and Tataru [KocT] : this is the Triebel–Lizorkin space $F_{\infty,2}^{-1}$, or equivalently the space $\text{BMO}^{-1} = \sqrt{-\Delta} \text{BMO}$. Moreover, they proved the following theorem :

Theorem 1 *For $0 < T \leq \infty$, define*

$$\|\vec{u}\|_{X_T} = \sup_{0<t<T} \sqrt{t} \|\vec{u}(t, \cdot)\|_\infty + \sup_{0<t<T, x_0 \in \mathbb{R}^3} (t^{-3/2} \int_0^t \int_{B(x_0, \sqrt{t})} |\vec{u}(s, y)|^2 dy ds)^{1/2}$$

There exists C_0 (which does not depend on T) such that if $T \in (0, +\infty]$, if \vec{u} and \vec{v} are defined on $(0, T) \times \mathbb{R}^3$ and if

$$B(\vec{u}, \vec{v}) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} (\vec{u} \otimes \vec{v}) ds$$

then

$$\|B(\vec{u}, \vec{v})\|_{X_T} \leq C_0 \|\vec{u}\|_{X_T} \|\vec{v}\|_{X_T}.$$

Corollary 1 *If $\|e^{t\Delta}\vec{u}_0\|_{X_T} < \frac{1}{4C_0}$, then the integral Navier–Stokes equations have a solution on $(0, T)$ such that $\|\vec{u}\|_{X_T} \leq 2\|e^{t\Delta}\vec{u}_0\|_{X_T}$.*

This is the unique solution such that $\|\vec{u}\|_{X_T} \leq \frac{1}{2C_0}$.

A special case of initial data that leads to a solution in some X_T is given by the subspace VMO^{-1} of BMO^{-1} .

Definition 1 VMO^{-1} is the closure of compactly supported functions in BMO^{-1} .

If $\vec{u}_0 \in \text{VMO}^{-1}$, then $\lim_{T \rightarrow 0} \|e^{t\Delta}\vec{u}_0\|_{X_T} = 0$. Remark that we have the embedding $\dot{B}_{q,p}^{-1+\frac{3}{q}} \subset \text{VMO}^{-1}$ for $2 < p < +\infty$ and $\frac{2}{p} + \frac{3}{q} = 1$. As a matter of fact, we may consider VMO^{-1} as a limit case for the scale of spaces $\dot{B}_{q,p}^{-1+\frac{3}{q}}$. Thus, Barker [Ba] raised the following question :

Question 1 *If \vec{u}_0 belongs to $L^2 \cap \text{VMO}^{-1}$, does there exist a positive time T such that every weak Leray solution of the Cauchy problem for the Navier–Stokes equations with \vec{u}_0 as initial value coincide with the mild solution in X_T ?*

If $\vec{u}_0 \in L^2 \cap \text{VMO}^{-1}$ and if $\|e^{t\Delta}\vec{u}_0\|_{X_T} \leq \frac{1}{4C_0}$, then if \vec{u} is a restricted Leray solution of the Navier–Stokes equations with initial value \vec{u}_0 , then $\|\vec{v}\|_{X_T} \leq 2\|e^{t\Delta}\vec{u}_0\|_{X_T}$. In particular, we have uniqueness of *restricted Leray solutions* on $(0, T)$.

As a matter of fact, this proof of local uniqueness of restricted weak Leray solutions holds for a slightly more general class :

Definition 2 BMO_0^{-1} is the space of distributions u_0 in BMO^{-1} such that

$$\lim_{T \rightarrow 0} \|e^{t\Delta}u_0\|_{X_T} = 0.$$

In the following, we will focus on the hypothesis $\vec{u}_0 \in L^2 \cap \text{BMO}_0^{-1}$ and on the issue of uniqueness for Leray solutions.

4 The limiting case

Up to now, we don't know how to prove local uniqueness of the Leray solutions when the initial value \vec{u}_0 belongs to $L^2 \cap \text{BMO}_0^{-1}$. What we know for sure is that the mild solution \vec{u} in X_T belongs to $L^\infty((\epsilon, T) \times \mathbb{R}^3)$ for every positive $\epsilon \in (0, T)$. Moreover, \vec{u} is a weak Leray solution and for every other Leray solution \vec{v} and for $\epsilon > 0$, we have

$$\partial_t(\vec{u} \cdot \vec{v}) = \vec{u} \cdot \partial_t \vec{v} + \partial_t \vec{u} \cdot \vec{v}$$

which gives for $0 < \epsilon < t < T$

$$\begin{aligned} \int \vec{u}(t, x) \cdot \vec{v}(t, x) dx &= \int \vec{u}(\epsilon, x) \cdot \vec{v}(\epsilon, x) dx - 2 \int_\epsilon^t \int \vec{\nabla} \otimes \vec{u} \cdot \vec{\nabla} \otimes \vec{v} dx ds \\ &\quad - \int_\epsilon^t \int \vec{u} \cdot (\vec{v} \cdot \vec{\nabla} \vec{v}) + \vec{v} \cdot (\vec{u} \cdot \vec{\nabla} \vec{u}) dx ds \end{aligned}$$

From

$$\int \vec{u} \cdot (\vec{v} \cdot \vec{\nabla} \vec{v}) + \vec{v} \cdot (\vec{u} \cdot \vec{\nabla} \vec{u}) dx = \int \vec{u} \cdot (\vec{v} \cdot \vec{\nabla} (\vec{v} - \vec{u})) + (\vec{v} - \vec{u}) \cdot (\vec{u} \cdot \vec{\nabla} \vec{u}) dx = \int \vec{u} \cdot ((\vec{v} - \vec{u}) \cdot \vec{\nabla} (\vec{v} - \vec{u})) dx$$

and letting ϵ go to 0, we get

$$\begin{aligned} \int \vec{u}(t, x) \cdot \vec{v}(t, x) dx &= \|\vec{u}_0\|_2^2 - 2 \int_0^t \int \vec{\nabla} \otimes \vec{u} \cdot \vec{\nabla} \otimes \vec{v} dx ds \\ &\quad - \lim_{\epsilon \rightarrow 0} \int_\epsilon^t \int \vec{u} \cdot ((\vec{v} - \vec{u}) \cdot \vec{\nabla} (\vec{v} - \vec{u})) dx ds \end{aligned}$$

and (letting $\vec{v} = \vec{u}$)

$$\|\vec{u}(t, \cdot)\|_2^2 = \|\vec{u}_0\|_2^2 - 2 \int_0^t \|\vec{\nabla} \otimes \vec{u}\|_2^2 ds.$$

Combining those two equalities with the Leray energy inequality for \vec{v}

$$\|\vec{v}(t, \cdot)\|_2^2 + 2 \int_0^t \|\vec{\nabla} \otimes \vec{v}\|_2^2 ds \leq \|\vec{u}_0\|_2^2$$

we get the following inequality for $\vec{w} = \vec{u} - \vec{v}$:

$$\|\vec{w}(t, \cdot)\|_2^2 + 2 \int_0^t \|\vec{\nabla} \otimes \vec{w}\|_2^2 ds \leq 2 \int_0^T \left| \int \vec{u} \cdot (\vec{w} \cdot \vec{\nabla} \vec{w}) dx \right| ds. \quad (4)$$

As a matter of fact, the key ingredient in Prodi–Serrin’s criterion is the estimation of the integral

$$I(\vec{u}, \vec{w}) = \int_0^t \left| \int \vec{u}(\cdot, \vec{w} \cdot \vec{\nabla} \vec{w}) dx \right| ds$$

but, if $\vec{w} = \vec{v} - \vec{u}$ with \vec{v} a Leray solution and \vec{u} the mild solution in X_T , we don’t even know whether $I(\vec{u}, \vec{w})$ is finite.

In the limiting case of Prodi and Serrin, (for $p = 2$ and $q = +\infty$), we write

$$\|\vec{w}(t, \cdot)\|_2^2 + 2 \int_0^t \|\vec{\nabla} \otimes \vec{w}\|_2^2 ds \leq 2 \int_0^t \int_0^t \left| \int \vec{u}(\cdot, \vec{w} \cdot \vec{\nabla} \vec{w}) dx \right| ds \leq 2 \int_0^t \|\vec{u} \otimes \vec{w}\|_2 \|\vec{\nabla} \otimes \vec{w}\|_2 ds$$

and get

$$\|\vec{w}(t, \cdot)\|_2^2 \leq \int_0^t \|\vec{u}\|_\infty^2 \|\vec{w}\|_2^2 ds. \quad (5)$$

Of course, we may conclude under the assumption that $\vec{u} \in L_t^2 L_x^\infty$. Actually, we shall not be interested in measurability issues for functions with values in a non-separable space such as L^∞ (i.e. in Bochner measurability for instance), as we are dealing with locally integrable functions for the Lebesgue measure $dt dx$ on $(0, T) \times \mathbb{R}^3$. Thus, for almost every t the quantity $\|\vec{u}(t, \cdot)\|_\infty$ will be well-defined as a measurable function of t , and $\vec{u} \in L_t^2 L_x^\infty$ will simply mean that $\int_0^T \|\vec{u}(t, \cdot)\|_{L^\infty(dx)}^2 dt < +\infty$. (See [LR 2] for details.)

If $\vec{u} \in L_t^2 L_x^\infty$ on $(0, T) \times \mathbb{R}^3$, the Prodi–Serrin criterion proves that every weak Leray solution \vec{v} on $(0, T)$ is equal to the mild solution \vec{u} . (This has even been extended to the case $\vec{u} \in L_t^2 \text{BMO}_x$ by Kozono and Taniuchi [KozT]). But it is not easy to translate the condition that $\vec{u} \in L_t^2 L_x^\infty$ into an equivalent assumption on \vec{u}_0 . The problem comes from the fact that the bilinear operator B is not bounded on $L_t^2 L_x^\infty$.

On the other hand, if we only assume $\vec{u}_0 \in L^2 \cap \text{BMO}_0^{-1}$, we only know the inequality $\|\vec{u}(t, \cdot)\|_\infty \leq \|\vec{u}\|_{X_T} \frac{o(1)}{\sqrt{t}}$. We find an integrability issue near $t = 0$. To check that this is actually an issue, consider the following example : take \vec{w} a divergence-free vector field in the Schwartz class such that the Fourier transform of \vec{w} is compactly supported in the annulus $1 < |\xi| < 2$; define

$$\vec{u}_0(x) = \sum_{j=0}^{+\infty} 2^j \frac{1}{\sqrt{1+j}} \vec{w}(2^j x);$$

we have $\vec{u}_0 \in L^2$; $\vec{u}_0 \in \dot{B}_{q,p}^{-1+\frac{3}{q}} \subset \dot{B}_{\infty,p}^{-1}$ (with $2 < p < +\infty$ and $\frac{2}{p} + \frac{3}{q} = 1$) but $\vec{u}_0 \notin \dot{B}_{\infty,2}^{-1}$; as the bilinear operator B is bounded on $L_t^p L_x^q \cap L_t^2 L_x^\infty$, if we assume that the mild solution $\vec{u} \in L_t^p L_x^q$ belongs to $L_t^2 L_x^\infty$, we would find that $e^{t\Delta} \vec{u}_0 \in L_t^2 L_x^\infty$, and thus $\vec{u}_0 \in \dot{B}_{\infty,2}^{-1}$; thus, we have $\int_0^T \|\vec{u}\|_\infty^2 dt = +\infty$.

5 Barker's theorem

In this section, we shall sketch the proof of Barker [Ba], as we shall extend it in Section 7 to the case of Besov–Morrey spaces. The main idea in the recent paper of Barker is the following one : if we want to use only the inequality

$$\|\vec{u}(t, \cdot)\|_\infty \leq \|\vec{u}\|_{X_T} \frac{o(1)}{\sqrt{t}}$$

to deal with the Gronwall inequality (5), we need to assume more than $\vec{w} \in L_t^\infty L_x^2$. Indeed, we have the easy following lemma :

Lemma 1 *Let $\delta > 0$. Let A and B be locally bounded non-negative measurable functions on $(0, T]$ such that*

$$\lim_{t \rightarrow 0} tA(t) = 0 \text{ and } \sup_{0 < t < T} t^{-\delta} B(t) = 0.$$

If we have moreover, for all $t \in (0, T]$,

$$B(t) \leq \int_0^t A(s) B(s) ds$$

then $B = 0$.

Proof : We have

$$B(t) \leq \frac{t^\delta}{\delta} \sup_{0 < s < t} sA(s) \sup_{0 < \sigma < t} \sigma^{-\delta} B(\sigma)$$

so that $B = 0$ on $(0, T_0]$ as long as $\sup_{0 < s < T_0} sA(s) < \delta$. For $t > T_0$, we then write $B(t) \leq \sup_{T_0 < s < T} A(s) \int_{T_0}^t B(s) ds$ and we find $B = 0$. \diamond

As $\vec{w} = \vec{v} - \vec{u} = (\vec{v} - e^{t\Delta} \vec{u}_0) - (\vec{u} - e^{t\Delta} \vec{u}_0)$, the extra information on $\|\vec{w}\|_2$ will be provided by the following lemma :

Lemma 2 *Let \vec{u}_0 be a divergence-free vector field with $\vec{u}_0 \in L^2$ and let \vec{v} be a weak Leray solution of the Navier–Stokes equations with initial value \vec{u}_0 . If moreover*

$$\vec{u}_0 \in [L^2, B_{\infty, \infty}^{-\gamma}]_{\theta, \infty} \text{ for some } -1 < -\gamma < 0 \text{ and } 0 < \theta < 1$$

then there exists $\delta > 0$ such that

$$\sup_{t > 0} t^{-\delta} \|\vec{v}(t, \cdot) - e^{t\Delta} \vec{u}_0\|_2 < +\infty.$$

Proof : Assume $\vec{u}_0 \in [L^2, B_{\infty, \infty}^{-\gamma}]_{\theta, \infty}$. For $0 < \epsilon < 1$, split \vec{u}_0 in $\vec{\alpha}_\epsilon + \vec{\beta}_\epsilon$ with

$$\|\vec{\alpha}_\epsilon\|_2 \leq C_0 \epsilon^\theta \text{ and } \|\vec{\beta}_\epsilon\|_{\dot{B}_{\infty, \infty}^{-\gamma}} \leq C_0 \epsilon^{\theta-1}$$

where C_0 does not depend on ϵ (but depends on \vec{u}_0).

We have a solution \vec{U}_ϵ for the Navier-Stokes equations with initial value $\vec{\beta}_\epsilon$ such that $\|\vec{U}_\epsilon(t, \cdot)\|_\infty \leq C_1 t^{-\frac{\gamma}{2}} \epsilon^{\theta-1}$ on an interval $(0, T_\epsilon)$ with $T_\epsilon^{\frac{1-\gamma}{2}} \epsilon^{\theta-1} = C_2$. Moreover $\vec{\beta}_\epsilon = \vec{u}_0 - \vec{\alpha}_\epsilon \in L^2$ and we find that $\sup_{0 < t < T_\epsilon} \|\vec{U}_\epsilon\|_2 \leq C_3$ and that $\|\vec{U}_\epsilon - e^{t\Delta} \vec{\beta}_\epsilon\|_2 \leq C_4 t^{(1-\gamma)/2} \epsilon^{\theta-1}$.

Since \vec{U}_ϵ is in $L_t^2 L_x^\infty$ on every bounded interval, we get

$$\begin{aligned} \int \vec{U}_\epsilon(t, x) \cdot \vec{v}(t, x) dx &= \int \vec{\beta}_\epsilon(x) \cdot \vec{u}_0(x) dx - 2 \int_0^t \int \vec{\nabla} \otimes \vec{U}_\epsilon \cdot \vec{\nabla} \otimes \vec{v} dx ds \\ &\quad - \int_0^t \int \vec{U}_\epsilon \cdot ((\vec{v} - \vec{U}_\epsilon) \cdot \vec{\nabla} (\vec{v} - \vec{U}_\epsilon)) dx ds \end{aligned}$$

and

$$\|\vec{U}_\epsilon(t, \cdot)\|_2^2 = \|\vec{\beta}_\epsilon\|_2^2 - 2 \int_0^t \|\vec{\nabla} \otimes \vec{U}_\epsilon\|_2^2 ds.$$

Combining those two equalities with the Leray energy inequality for \vec{v}

$$\|\vec{v}(t, \cdot)\|_2^2 + 2 \int_0^t \|\vec{\nabla} \otimes \vec{v}\|_2^2 ds \leq \|\vec{u}_0\|_2^2$$

we get the following inequality for $\vec{W}_\epsilon = \vec{v} - \vec{U}_\epsilon$:

$$\|\vec{W}_\epsilon(t, \cdot)\|_2^2 + 2 \int_0^t \|\vec{\nabla} \otimes \vec{W}_\epsilon\|_2^2 ds \leq \|\vec{\alpha}_\epsilon\|_2^2 + 2 \int_0^T \left| \int \vec{U}_\epsilon \cdot (\vec{W}_\epsilon \cdot \vec{\nabla} \vec{W}_\epsilon) dx \right| ds.$$

Thus, we get

$$\|\vec{W}_\epsilon(t, \cdot)\|_2^2 \leq C_0^2 \epsilon^{2\theta} + C_1^2 \epsilon^{2(\theta-1)} \int_0^t s^{-\gamma} \|\vec{W}_\epsilon(s, \cdot)\|_2^2 ds$$

so that

$$\|\vec{W}_\epsilon(t, \cdot)\|_2^2 \leq C_0^2 \epsilon^{2\theta} e^{C_1^2 \epsilon^{2(\theta-1)} \frac{t^{1-\gamma}}{1-\gamma}}.$$

Now, for $\tau < 1$, take $\epsilon = \tau^\mu$ with $\frac{1-\gamma}{2} + \mu(\theta-1) > 0$. We find that, for $0 < t < T_\epsilon$ with

$$T_\epsilon = C_2^{\frac{2}{1-\gamma}} \epsilon^{\frac{2(1-\theta)}{1-\gamma}} = C_2^{\frac{2}{1-\gamma}} \tau^{\mu \frac{2(1-\theta)}{1-\gamma}} \text{ [where } \mu \frac{2(1-\theta)}{1-\gamma} < 1 \text{],}$$

we have the inequality

$$\begin{aligned}\|\vec{v} - e^{t\Delta}\vec{u}_0\|_2 &\leq \|\alpha_\epsilon\|_2 + \|\vec{W}_\epsilon\|_2 + \|\vec{U}_\epsilon - e^{t\Delta}\vec{\beta}_\epsilon\|_2 \\ &\leq C_0\epsilon^\theta(1 + e^{\frac{C_1^2}{2(1-\gamma)}\frac{t^{1-\gamma}}{T_\epsilon^{1-\gamma}}}) + C_4t^{\frac{1-\gamma}{2}}\epsilon^{\theta-1}\end{aligned}$$

If τ is small enough, we have $\tau < T_\epsilon$ and we find

$$\|\vec{v}(\tau, \cdot) - e^{\tau\Delta}\vec{u}_0\|_2 \leq C_0\tau^{\mu\theta}(1 + e^{\frac{C_1^2}{2(1-\gamma)}}) + C_4\tau^{\frac{1-\gamma}{2} + \mu(\theta-1)}$$

The lemma is proved. \diamond Barker's theorem then reads as :

Theorem 2 *Let \vec{u}_0 be a divergence-free vector field with $\vec{u}_0 \in L^2$ and let \vec{v} be a weak Leray solution of the Navier–Stokes equations with initial value \vec{u}_0 . If moreover*

$$\vec{u}_0 \in \text{BMO}_0^{-1} \cap \dot{B}_{q,\infty}^{-s} \text{ with } 3 < q < +\infty \text{ and } -s > -1 + \frac{2}{q}$$

then there exists $T > 0$ such that if \vec{v} is a Leray solution and if \vec{u} is the mild solution with $\|\vec{u}\|_{X_T} < +\infty$, then $\vec{u} = \vec{v}$ on $(0, T)$.

Remark We have the embeddings $L^2 \subset \dot{B}_{q,2}^{-\frac{3}{2} + \frac{3}{q}} \subset \dot{B}_{q,\infty}^{-\frac{3}{2} + \frac{3}{q}}$, so that the information conveyed by the hypothesis $\vec{u}_0 \in \dot{B}_{q,\infty}^{-s}$ is interesting only for the high frequencies of \vec{u}_0 . Moreover the embeddings $L^2 = \dot{B}_{2,2}^0$ and $\text{BMO}^{-1} \subset \dot{B}_{\infty,\infty}^{-1}$ gives that

$$L^2 \cap \text{BMO}_0^{-1} \subset \dot{B}_{q,q}^{-1 + \frac{2}{q}} \subset \dot{B}_{q,\infty}^{-1 + \frac{2}{q}}.$$

Thus, the information conveyed by the hypothesis $\vec{u}_0 \in \dot{B}_{q,\infty}^{-s}$ is not contained in the assumption $\vec{u}_0 \in L^2 \cap \text{BMO}_0^{-1}$. Finally, if $-s > -1 + \frac{3}{q} = \frac{2}{p}$, then we have

$$L^2 \cap \text{BMO}_0^{-1} \cap \dot{B}_{q,\infty}^{-s} \subset \dot{B}_{q,q}^{-1 + \frac{2}{q}} \cap \dot{B}_{q,\infty}^{-s} \subset \dot{B}_{q,p}^{-1 + \frac{3}{q}}.$$

Thus, the theorem is interesting only in the range $-1 + \frac{2}{q} < -s \leq -1 + \frac{3}{q}$, which corresponds to the gap between $L^2 \cap \text{BMO}_0^{-1}$. (where local uniqueness is conjectured to hold) and $\dot{B}_{q,p}^{-1 + \frac{3}{q}}$ (for which the Prodi–Serrin criterion shows that local uniqueness holds).

A further remark is that we have the embedding $\dot{B}_{q,\infty}^{-1 + \frac{3}{q}} \subset \text{BMO}^{-1}$, so that we have a Prodi–Serrin criterion with $\vec{u} \in L_t^p L_x^q$ (with $\frac{2}{p} + \frac{3}{q} = 1$) replaced with

$$\sup_{0 < t < T} t^{\frac{1}{p}} \|\vec{u}\|_q < +\infty \text{ and } \lim_{t \rightarrow 0} t^{\frac{1}{p}} \|\vec{u}(t, \cdot)\|_q = 0.$$

Proof :

The first step is the use of interpolation inequalities in order to be able to check that \vec{u}_0 fulfills the assumptions of Lemma 2.

- Since $\vec{u}_0 \in L^2 = \dot{B}_{2,2}^0$ and $\vec{u}_0 \in \text{BMO}^{-1} \subset \dot{B}_{\infty,\infty}^{-1}$, we have $\vec{u}_0 \in \dot{B}_{q,q}^{-1+\frac{2}{q}}$.
- Since $\vec{u}_0 \in \dot{B}_{q,q}^{-1+\frac{2}{q}} \cap \dot{B}_{q,\infty}^{-s}$, we have for $-1 + \frac{2}{q} < -\sigma < -s$, $\vec{u}_0 \in \dot{B}_{q,1}^{-\sigma}$.
- The Besov space $\dot{B}_{q,1}^{-\sigma}$ is embedded in the Sobolev space $\dot{W}^{-\sigma,q}$.
- For $q < r < \infty$, we have $\dot{W}^{-\sigma,q} = [L^2, \dot{W}^{-\delta,r}]_{[\theta]}$ with $\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{r}$ and $-s = -\theta\delta$ (complex interpolation)
- Since $\dot{W}^{-\delta,r} \subset \dot{B}_{\infty,\infty}^{-1+\frac{3}{r}}$, we have $[L^2, \dot{W}^{-\delta,r}]_{[\theta]} \subset [L^2, \dot{B}_{\infty,\infty}^{-\delta-\frac{3}{r}}]_{\theta,\infty}$ with

$$-\delta = -\frac{s}{\theta} = -s \frac{\frac{1}{2} - \frac{1}{r}}{\frac{1}{2} - \frac{1}{q}} = -\frac{(-s)}{-1 + \frac{2}{q}} + O\left(\frac{1}{r}\right)$$

so that $-\delta - \frac{3}{r} > -1$ for r large enough

We may now end the proof : recall that if \vec{v} is a Leray solution and if \vec{u} is the mild solution with $\|\vec{u}\|_{X_T} < +\infty$, then the difference $\vec{w} = \vec{u} - \vec{v}$ satisfies a Gronwall estimate :

$$\|\vec{w}(t, \cdot)\|_2^2 \leq \int_0^t \|\vec{u}\|_\infty^2 \|\vec{w}\|_2^2 ds.$$

By Lemma 2, we have $\|\vec{v}(t, \cdot) - e^{t\Delta}\vec{u}_0\|_2 = O(t^\delta)$ and $\|\vec{u}(t, \cdot) - e^{t\Delta}\vec{u}_0\|_2 = O(t^\delta)$ for some positive δ . On the other hand, we know that $\|\vec{u}(t, \cdot)\|_\infty = o(\frac{1}{\sqrt{t}})$. Using Lemma 1, we find that $\vec{w} = 0$, and $\vec{v} = \vec{u}$. \diamond

6 The Prodi–Serrin criterion for Besov–Morrey spaces

Morrey spaces provide a natural tool for extending the Prodi–Serrin criterion.

Definition 3 For $1 < r \leq q < +\infty$, we define the Morrey space $\dot{M}^{r,q}$ as the space of Lebesgue measurable functions f on \mathbb{R}^3 such that

$$\sup_{R>0, x_0 \in \mathbb{R}^3} R^{\frac{3}{q} - \frac{3}{r}} \left(\int_{B(x_0, R)} |f(x)|^r dx \right)^{1/r} = \|f\|_{\dot{M}^{r,q}} < +\infty.$$

Similarly, the space $\dot{M}^{1,q}$ is the space of locally finite Borelian (signed) measure μ such that

$$\sup_{R>0, x_0 \in \mathbb{R}^3} R^{\frac{3}{q} - 3} \left(\int_{B(x_0, R)} d|\mu| \right) = \|\mu\|_{\dot{M}^{1,q}} < +\infty.$$

Remark : For absolutely continuous measures $d\mu = f dx$ with $f \in L^1_{\text{loc}}$, we have

$$\|\mu\|_{\dot{M}^{1,q}} = \sup_{R>0, x_0 \in \mathbb{R}^3} R^{\frac{3}{q}-3} \left(\int_{B(x_0, R)} |f(x)| dx \right).$$

The key inequality in the proof of the Prodi–Serrin criterion was the inequality (for all $w \in H^1$)

$$\|uw\|_2 \leq C \|u\|_q \|w\|_2^{\frac{2}{p}} \|\vec{\nabla} w\|_2^{\frac{3}{q}}$$

with $\frac{2}{p} + \frac{3}{q} = 1$ and $3 < q < +\infty$. If we want to replace this inequality by a more general inequality

$$\|uw\|_2 \leq N(u) \|w\|_2^{\frac{2}{p}} \|\vec{\nabla} w\|_2^{\frac{3}{q}}$$

(again with $\frac{2}{p} + \frac{3}{q} = 1$ and $3 < q < +\infty$), then we proved in [LR 3] that the existence of a finite $N(u)$ is equivalent to the fact that $u \in \dot{M}^{2,q}$, and moreover that $N(u) \approx \|u\|_{\dot{M}^{2,q}}$.

This leads to the following easy extension of the Prodi–Serrin criterion :

Theorem 3 *If $\vec{u}_0 \in L^2$ and if the Navier–Stokes equations has a solution \vec{u} such that*

$$\vec{u} \in L^p_t \dot{M}^{2,q}_x \text{ with } \frac{2}{p} + \frac{3}{q} = 1 \text{ and } 3 < q < +\infty$$

then if \vec{v} is a Leray solution we have $\vec{u} = \vec{v}$ on $(0, T)$.

If $3 < q < +\infty$, the existence of $T > 0$ and of a solution in $L^p \dot{M}^{2,q}$ with $\frac{2}{p} + \frac{3}{q} = 1$ is equivalent to the existence of T' such that $e^{t\Delta} \vec{u}_0 \in L^p \dot{M}^{2,q}$ on $(0, T')$ (and on $(0, +\infty)$, since $\vec{u}_0 \in L^2$), thus with

$$\vec{u}_0 \in \dot{B}^{-1+\frac{3}{q}}_{\dot{M}^{2,q,p}}.$$

This Besov–Morrey space has been introduced in 1994. by Kozono and Yamazaki [KoY]. It is easy to check that, for $2 < p < +\infty$ and $\frac{2}{p} + \frac{3}{q} = 1$, we have the inequality

$$\|e^{t\Delta} u_0\|_{X_T} \leq C_q \left(\int_0^T \|e^{t\Delta} u_0\|_{\dot{M}^{2,q}}^p dt \right)^{1:p}$$

so that we have the embedding $\dot{B}^{-1+\frac{3}{q}}_{\dot{M}^{2,q,p}} \subset \text{BMO}_0^{-1}$ for $2 < p < +\infty$ and $\frac{2}{p} + \frac{3}{q} = 1$.

7 Barker's theorem and Besov-Morrey spaces

We shall extend Barker's theorem.

Theorem 4 *If*

- $\vec{u}_0 \in L^2 \cap \text{BMO}_0^{-1}$
- $3 < q < +\infty$, $-s > -1 + \frac{2}{q}$ and $\vec{u}_0 \in \dot{B}_{M^{1,q},\infty}^{-s}$

then there exists $T > 0$ such that if \vec{v} is a suitable Leray solution and if \vec{u} is the mild solution with $\|\vec{u}\|_{X_T} < +\infty$, then $\vec{u} = \vec{v}$ on $(0, T)$.

Proof : As we shall see, the proof is very similar to Barker's proof for Theorem 2 [Ba]. However, we shall meet some technical issues.

We sketch the proof :

- $\vec{u}_0 \in L^2 = \dot{B}_{2,2}^0$ and $\vec{u}_0 \in \text{BMO}^{-1} \subset \dot{B}_{\infty,\infty}^{-1}$, thus $\vec{u}_0 \in \dot{B}_{q,q}^{-1+\frac{2}{q}}$
- $\vec{u}_0 \in \dot{B}_{q,q}^{-1+\frac{2}{q}} \cap \dot{B}_{M^{1,q},\infty}^{-s}$ thus $\vec{u}_0 \in \dot{B}_{M^{p,q},\infty}^{-\sigma}$ for $1 < p < q$, $\frac{1}{p} = 1 - \theta + \frac{\theta}{q}$ and $-\sigma = -s(1 - \theta) + \theta(-1 + \frac{2}{q}) > -1 + \frac{2}{q}$. We shall take $p > 2$.
- as $p < q$, we have $\dot{B}_{q,q}^{-1+\frac{2}{q}} \subset \dot{B}_{q,p}^{1+\frac{2}{q}}$. Thus, for $-1 + \frac{2}{q} < -\gamma < -\sigma$,

$$\dot{B}_{q,q}^{-1+\frac{2}{q}} \cap \dot{B}_{M^{p,q},\infty}^{-\sigma} \subset \dot{B}_{M^{p,q},1}^{-\gamma} \subset \dot{W}^{-\gamma, M^{p,q}}.$$

We now encounter our first problem. We can no longer write $\dot{W}^{-\gamma, M^{p,q}}$ as a subspace of an interpolate space between L^2 and $\dot{B}_{\infty,\infty}^{-1+\delta}$. More precisely, let us assume $\dot{B}_{M^{p,q},1}^{-\gamma} \subset [L^2, \dot{B}_{\infty,\infty}^{-1+\delta}]_{\theta,\infty}$; by homogeneity of the norms, we must have $-\gamma - \frac{3}{q} = -(1 - \theta)\frac{3}{2} - \theta(1 - \delta)$. We have

$$[L^2, \dot{B}_{\infty,\infty}^{-1+\delta}]_{\theta,\infty} \subset \dot{B}_{L^{r,\infty},\infty}^{-\theta(1-\delta)}$$

with $r = \frac{2}{1-\theta}$. In particular, for $u \in [L^2, \dot{B}_{\infty,\infty}^{-1+\delta}]_{\theta,\infty}$, we have that $e^\Delta e^{t\partial_3^2} u$ goes to 0 in \mathcal{S}' when t goes to $+\infty$. But if $3p \leq 2q$, if u depends only on (x_1, x_2) and not on x_3 , and if $u \in \dot{B}_{M^{p,\frac{2q}{3}},1}^{-\gamma}(\mathbb{R}^2)$, then $u \in \dot{B}_{M^{p,q},1}^{-\gamma}(\mathbb{R}^3)$ and $e^\Delta e^{t\partial_3^2} u = e^\Delta u$. Thus, we have a contradiction.

We better use complex interpolation and write that

$$\dot{W}^{-\gamma, M^{p,q}} = [\dot{M}^{2, \frac{2}{p}q}, \dot{W}^{-\rho, M^{r, \frac{r}{p}q}}]_{[\theta]}$$

for $r > p$, $\frac{1-\theta}{2} + \frac{\theta}{r} = \frac{1}{p}$, $\gamma = \theta\rho$. (For interpolation of Morrey spaces, see [LR 4, LR 5])

Then, we remark that $\dot{M}^{2, \frac{2}{p}q} \subset L^2_{\text{uloc}}$ and write that

$$[\dot{M}^{2, \frac{2}{p}q}, \dot{W}^{-\rho, \dot{M}^{r, \frac{r}{p}q}}][\theta] \subset [\dot{M}^{2, \frac{2}{p}q}, \dot{W}^{-\rho, \dot{M}^{r, \frac{r}{p}q}}]_{\theta, \infty} \subset [L^2_{\text{uloc}}, \dot{B}^{-\rho - \frac{3p}{r}q}]_{\theta, \infty}$$

In order to finish the proof, we thus need to use the machinery of energy control for suitable local Leray solutions [LR 2, LR 6]. This will be done in the following sections, and we shall finish the proof in Section 10 \diamond

8 Weak local Leray solutions

We recall basic results for local weak Leray solutions. We endow L^2_{uloc} with the norm

$$\|u\|_{L^2_{\text{uloc}}} = \sup_{k \in \mathbb{Z}^3} \|u\varphi_0(x-k)\|_2,$$

where φ_0 is a non-negative function in \mathcal{D} , supported in a ball $B(0, R_0)$ and such that $\sum_{k \in \mathbb{Z}^3} \varphi_0(x-k) = 1$.

When $\vec{u}_0 \in L^2_{\text{uloc}}$, proof of existence of solutions for the Navier–Stokes equations is based on mollification, energy estimates and compactness arguments (for details, see [LR 6], section 14.1) :

- we solve

$$\partial_t \vec{u}_\epsilon + (\varphi_\epsilon * \vec{u}_\epsilon) \cdot \vec{\nabla} \vec{u}_\epsilon = \Delta \vec{u}_\epsilon - \vec{\nabla} p_\epsilon$$

with $\text{div } \vec{u}_\epsilon = 0$ and $\vec{u}_\epsilon(0, \cdot) = \vec{u}_0$. Here, $\varphi \in \mathcal{D}$, $\int \varphi dx = 1$ and $\varphi_\epsilon(x) = \frac{1}{\epsilon^3} \varphi(\frac{x}{\epsilon})$. Here $\vec{\nabla} p_\epsilon$ is given by the Leray projection :

$$\vec{\nabla} p_\epsilon = -(\varphi_\epsilon * \vec{u}_\epsilon) \cdot \vec{\nabla} \vec{u}_\epsilon + \mathbb{P} \text{div} ((\varphi_\epsilon * \vec{u}_\epsilon) \otimes \vec{u}_\epsilon).$$

- the solution holds at least on an interval $(0, T_\epsilon)$ where T_ϵ depends on ϵ and on $\|\vec{u}_0\|_{L^2_{\text{uloc}}}$ ($T_\epsilon = \min(1, C_0 \frac{\epsilon^{3/2}}{\|\vec{u}_0\|_{L^2_{\text{uloc}}}^2}$). Moreover, we have the inequalities, for $k \in \mathbb{Z}^3$,

$$\begin{aligned} & \int \varphi_0(x-k) |\vec{u}_\epsilon(t, x)|^2 dx + 2 \int_0^t \int \varphi_0(x-k) |\vec{\nabla} \otimes \vec{u}_\epsilon(s, x)|^2 dx ds \\ & \leq \int \varphi_0(x-k) |\vec{u}_0(t, x)|^2 dx + C_1 \int_0^t \int_{|x-k| \leq R_0} |\vec{u}_\epsilon(s, x)|^2 dx ds \\ & \quad + C_2 \int_0^t \int_{|x-k| > 5R_0} \frac{1}{|x-k|^4} |\vec{u}_\epsilon(s, x)|^3 dx ds \\ & \quad + C_2 \int_0^t \int_{|x-k| < 5R_0} |\vec{u}_\epsilon(s, x)|^3 dx ds \end{aligned}$$

- defining

$$\alpha_\epsilon(t) = \|\vec{u}_\epsilon(t, \cdot)\|_{L^2_{\text{uloc}}}$$

and

$$\beta_\epsilon(t) = \sup_{k \in \mathbb{Z}^3} \left(\int_0^t \int \varphi_0(x-k) |\vec{\nabla} \otimes \vec{u}_\epsilon(s, x)|^2 dx ds \right)^{1/2},$$

we get the inequality

$$\begin{aligned} & \int \varphi_0(x-k) |\vec{u}_\epsilon(t, x)|^2 dx + \int_0^t \int \varphi_0(x-k) |\vec{\nabla} \otimes \vec{u}_\epsilon(s, x)|^2 dx ds \\ & \leq \alpha_\epsilon(0)^2 + \frac{1}{2} \beta_\epsilon(t)^2 + C_3 \int_0^t \alpha_\epsilon(s)^2 ds + C_3 \int_0^t \alpha_\epsilon(s)^6 ds \end{aligned}$$

so that

$$\beta_\epsilon(t)^2 \leq 2\alpha_\epsilon(0)^2 + 2C_3 \int_0^t \alpha_\epsilon(s)^2 ds + 2C_3 \int_0^t \alpha_\epsilon(s)^6 ds$$

and finally

$$\alpha_\epsilon(t)^2 \leq 2\alpha_\epsilon(0)^2 + 2C_3 \int_0^t \alpha_\epsilon(s)^2 ds + 2C_3 \int_0^t \alpha_\epsilon(s)^6 ds$$

Thus, as long as $8C_3 t < 1$ and $128 C_3 t \|\vec{u}_0\|_{L^2_{\text{uloc}}}^4 < 1$, we find that $\alpha_\epsilon(t) \leq 2\|\vec{u}_0\|_{L^2_{\text{uloc}}}$ and $\beta_\epsilon(t) \leq 2\|\vec{u}_0\|_{L^2_{\text{uloc}}}$.

- the solution is then defined on $(0, \min(\frac{1}{8C_3}, \frac{1}{128 C_3 \|\vec{u}_0\|_{L^2_{\text{uloc}}}^4}))$ and controlled independently from ϵ . By Rellich theorem, we find a subsequence that converges strongly in $(L^2_t L^2_x)_{\text{loc}}$ to a suitable local Leray solution \vec{u}

An important point is the following one : assume moreover that $\vec{u}_0 \in \text{BMO}_0^{-1}$ and that $\|e^{t\Delta} \vec{u}_0\|_{X_T} < \frac{1}{4C_0}$ (where C_0 is the constant of Theorem 1) then \vec{u}_ϵ is defined at least on $(0, T)$ and $\|\vec{u}_\epsilon\|_{X_T} \leq 2\|e^{t\Delta} \vec{u}_0\|_{X_T}$. As T does not depend on ϵ , we see that the local Leray solution \vec{u} satisfies $\vec{u} \in X_S$ with $S = \min(T, \frac{1}{8C_3}, \frac{1}{128 C_3 \|\vec{u}_0\|_{L^2_{\text{uloc}}}^4})$ and $\|\vec{u}\|_{X_S} \leq 2\|e^{t\Delta} \vec{u}_0\|_{X_T}$.

Similarly, if we assume that $\vec{u}_0 \in \dot{B}_{\infty, \infty}^{-\gamma}$ with $-1 < -\gamma < 0$, then \vec{u}_ϵ is defined at least on $(0, T)$ where $T = C \|\vec{u}_0\|_{\dot{B}_{\infty, \infty}^{-\gamma}}^{\frac{2}{1-\gamma}}$ and $\sup_{0 < t < T} t^{\frac{\gamma}{2}} \|\vec{u}(t, \cdot)\|_\infty \leq 2 \sup_{0 < t < T} t^{\frac{\gamma}{2}} \|e^{t\Delta} \vec{u}_0\|_\infty$. As T does not depend on ϵ , we see that the local Leray solution \vec{u} satisfies the inequality $\sup_{0 < t < S} t^{\frac{\gamma}{2}} \|\vec{u}(t, \cdot)\|_\infty < +\infty$ where $S = \min(T, \frac{1}{8C_3}, \frac{1}{128 C_3 \|\vec{u}_0\|_{L^2_{\text{uloc}}}^4})$.

9 Comparison of local weak Leray solutions

If \vec{u} and \vec{v} are two local weak Leray solutions, on $(0, T)$ with initial values \vec{u}_0 and \vec{v}_0 , we would like to be able to estimate $\vec{u}(t, \cdot) - \vec{v}(t, \cdot)$ from the estimation of $\vec{u}_0 - \vec{v}_0$. This can be done only when at least one of the solutions is regular enough. We shall assume that $\vec{u} \in L_t^2 L_x^\infty$. We sketch the computations described in [LR 6], section 14.4.

Define $\vec{w} = \vec{u} - \vec{v}$,

$$\alpha(t) = \|\vec{w}(t, \cdot)\|_{L_{\text{uloc}}^2}$$

and

$$\beta(t) = \sup_{k \in \mathbb{Z}^3} \left(\int_0^t \int \varphi_0(x-k) |\vec{\nabla} \otimes \vec{w}(s, x)|^2 dx ds \right)^{1/2},$$

Using the suitability of \vec{v} and the regularity of \vec{v} , we find (for $0 < t < \min(1, T)$)

$$\begin{aligned} & \int \varphi_0(x-k) |\vec{w}(t, x)|^2 dx + \int_0^t \int \varphi_0(x-k) |\vec{\nabla} \otimes \vec{w}(s, x)|^2 dx ds \\ & \leq \alpha(0)^2 + \frac{1}{2} \beta(t)^2 + C_1 \int_0^t \alpha(s)^2 ds + C_2 \int_0^t \alpha(s)^6 ds + C_3 \int_0^t \|\vec{u}(s, \cdot)\|_\infty^2 \alpha(s)^2 ds \end{aligned}$$

where the constants C_i do not depend on T , \vec{u} , nor on \vec{v} . Finally, we find

$$\begin{aligned} \alpha(t)^2 & \leq 2\alpha(0)^2 + 2(C_1 + C_3(\|\vec{u}\|_{L_t^\infty L_{\text{uloc}}^2} + \|\vec{v}\|_{L_t^\infty L_{\text{uloc}}^2})^4) \int_0^t \alpha(s)^2 ds \\ & \quad + 2C_3 \int_0^t \|\vec{u}(s, \cdot)\|_\infty^2 \alpha(s)^2 ds. \end{aligned} \tag{6}$$

We have the same estimate even if \vec{u} is not integrable near $t = 0$. Let us only assume that $\vec{u} \in L_t^2 L_x^\infty$ on every (ϵ, T) with $\epsilon > 0$. Considering a time $t_0 > 0$ which a Lebesgue point for the functions $t \mapsto \int \varphi(x-k) |\vec{u}(t, x)|^2 dx$ and $t \mapsto \int \varphi(x-k) |\vec{w}(t, x)|^2 dx$, we have that \vec{u} and \vec{v} are local weak Leray solutions on (t_0, T) with initial values $\vec{u}(t_0, \cdot)$ and $\vec{v}(t_0, \cdot)$. Thus, we shall find that, for $t > t_0$

$$\begin{aligned} \int \varphi_0(x-k) |\vec{w}(t, x)|^2 dx & \leq 2 \int \varphi_0(x-k) |\vec{w}(t_0, x)|^2 dx \\ & \quad + 2(C_1 + C_3(\|\vec{u}\|_{L_t^\infty L_{\text{uloc}}^2} + \|\vec{v}\|_{L_t^\infty L_{\text{uloc}}^2})^4) \int_{t_0}^t \alpha(s)^2 ds \\ & \quad + 2C_3 \int_{t_0}^t \|\vec{u}(s, \cdot)\|_\infty^2 \alpha(s)^2 ds. \end{aligned} \tag{7}$$

It is then enough to let t_0 go to 0 and then take the supremum with respect to k .

10 Proof of Theorem 4

We may now finish the proof. We consider two solutions of the Navier–Stokes equations with initial value $\vec{u}_0 \in L^2 \cap \text{BMO}_0^{-1} \cap \dot{B}_{M^{1,q},\infty}^{-s}$ with $-s > -1 + \frac{2}{q}$ (and $3 < q < +\infty$) : we assume that \vec{v} is a suitable Leray solution and \vec{u} is the mild solution in X_T .

As \vec{v} is suitable, \vec{v} is a local Leray solution as well and we may estimate the L_{uloc}^2 of $\vec{u} - \vec{w}$: defining $B(t) = \|\vec{u}(t, \cdot) - \vec{v}(t, \cdot)\|_{L_{\text{uloc}}^2}^2$ and

$$A(t) = 2(C_1 + C_3(\|\vec{u}\|_{L_t^\infty L_{\text{uloc}}^2} + \|\vec{v}\|_{L_t^\infty L_{\text{uloc}}^2})^4) + 2C_3\|\vec{u}(t, \cdot)\|_\infty^2,$$

we get

$$B(t) \leq \int_0^t A(s)B(s) ds.$$

As $\lim_{t \rightarrow 0} tA(t) = 0$, we shall try to apply Lemma 1. Thus, we shall use interpolation estimates to search for a control of $B(t)$ as $O(t^{-\delta})$, in the spirit of Lemma 2:.

Recall that we have introduced the following numbers :

- p such that $2 < p < \frac{2q}{3}$
- $1 - \theta$ the barycentric coordinate of $\frac{1}{p}$ in $[\frac{1}{q}, 1]$: $\frac{1}{p} = (1 - \theta) + \theta\frac{1}{q}$
- $-\sigma$ the corresponding point in $[-1 + \frac{2}{q}, -s]$: $-\sigma = (1 - \theta)(-s) + \theta(-1 + \frac{2}{q})$
- $-\gamma$ such that $-1 + \frac{2}{q} < -\gamma < -\sigma$
- r such that $p < r < +\infty$
- $1 - \eta$ the barycentric coordinate of $\frac{1}{p}$ in the segment $[\frac{1}{r}, \frac{1}{2}]$: $\frac{1}{p} = \frac{1-\eta}{2} + \frac{\eta}{r}$
- $-\rho$ the corresponding point in $[-\gamma, 0]$: $-\rho = \eta(-\gamma)$

We have the following embeddings :

- $B_{2,2}^0 \cap \text{BMO}^{-1} \subset \dot{B}_{q,q}^{-1+\frac{2}{q}}$
- $\dot{B}_{q,q}^{-1+\frac{2}{q}} \cap \dot{B}_{M^{1,q},\infty}^{-s} \subset \dot{B}_{M^{p,q},\infty}^{-\sigma}$

- $\dot{B}_{q,q}^{-1+\frac{2}{q}} \cap \dot{B}_{\dot{M}^{p,q},\infty}^{-\sigma} \subset \dot{B}_{\dot{M}^{p,q},1}^{-\gamma} \subset \dot{W}^{-\gamma,\dot{M}^{p,q}}$.
- $W^{-\gamma,\dot{M}^{p,q}} = [\dot{M}^{2,\frac{2}{p}q}, \dot{W}^{-\rho,\dot{M}^{r,\frac{r}{p}q}}][\eta]$
- $[\dot{M}^{2,\frac{2}{p}q}, \dot{W}^{-\rho,\dot{M}^{r,\frac{r}{p}q}}][\eta] \subset [\dot{M}^{2,\frac{2}{p}q}, \dot{W}^{-\rho,\dot{M}^{r,\frac{r}{p}q}}]_{\eta,\infty} \subset [L^2_{\text{uloc}}, \dot{B}_{\infty,\infty}^{-\rho-\frac{3p}{rq}}]_{\eta,\infty}$

If we take r very large, we have $\eta = 1 - \frac{2}{p} + o(1)$ and

$$-\rho - \frac{3p}{rq} = (1 - \frac{2}{p})(-\gamma) + o(1) \in] -1, 0[.$$

Thus far, we have seen that $\vec{u}_0 \in [L^2_{\text{uloc}}, \dot{B}_{\infty,\infty}^{-\lambda}]_{\eta,\infty}$ for some $\eta \in (0, 1)$ and some $\lambda \in (0, 1)$. We shall now estimate $\vec{v} - e^{t\Delta}\vec{u}_0$ when \vec{v} is a weak local Leray solution. For $0 < \epsilon < 1$, split \vec{u}_0 in $\vec{\alpha}_\epsilon + \vec{\beta}_\epsilon$ with

$$\|\vec{\alpha}_\epsilon\|_{L^2_{\text{uloc}}} \leq C_0\epsilon^\eta \text{ and } \|\vec{\beta}_\epsilon\|_{\dot{B}_{\infty,\infty}^{-\lambda}} \leq C_0\epsilon^{\eta-1}$$

where C_0 does not depend on ϵ (but depends on \vec{u}_0).

As $\beta_\epsilon = \vec{u}_0 - \alpha_\epsilon$, we have $\|\vec{\beta}_\epsilon\|_{L^2_{\text{uloc}}} \leq \|\vec{u}_0\|_{L^2_{\text{uloc}}} + C_0$ (an estimation which does not depend on ϵ) and we know that we have a (restricted) weak Leray solution of the Navier–Stokes equations \vec{U}_ϵ with initial value $\vec{\beta}_\epsilon$ such that $\sup_{0 < t < T_0} \|\vec{U}_\epsilon(t, \cdot)\|_{L^2_{\text{uloc}}} \leq C_1$, where T_0 and C_1 depends only on $\|\vec{u}_0\|_{L^2_{\text{uloc}}}$ (and not on ϵ).

As $\beta_\epsilon \in \dot{B}_{\infty,\infty}^{-\lambda}$, we have as well that $\|\vec{U}_\epsilon(t, \cdot)\|_\infty \leq C_2 t^{-\frac{\lambda}{2}} \epsilon^{\eta-1}$ on an interval $(0, T_\epsilon)$ with $T_\epsilon^{\frac{1-\lambda}{2}} \epsilon^{\eta-1} = C_3$. It is then easy to check that $\|\vec{U}_\epsilon - e^{t\Delta}\vec{\beta}_\epsilon\|_{L^2_{\text{loc}}} \leq C_4 t^{(1-\lambda)/2} \epsilon^{\eta-1}$.

Using our results on comparison of suitable local Leray solutions, we find that we get the following inequality for $\vec{W}_\epsilon = \vec{v} - \vec{U}_\epsilon$ and $A_\epsilon(t) = \sup_{k \in \mathbb{Z}^3} \int \varphi_0(x - k) |\vec{W}_\epsilon(t, \cdot x)|^2 dx$:

$$A_\epsilon(t)^2 \leq 2\|\alpha_\epsilon\|_{L^2_{\text{uloc}}}^2 + C_5 \int_0^t A_\epsilon(s)^2 ds + C_6 \int_0^t \|\vec{u}(s, \cdot)\|_\infty^2 A_\epsilon(s)^2 ds. \quad (8)$$

Thus, we get

$$A_\epsilon(t) \leq C_7 \epsilon^{2\eta} + C_8 \int_0^t A_\epsilon(s)^2 ds + C_9 \epsilon^{2(\eta-1)} \int_0^t s^{-\lambda} A_\epsilon(s)^2 ds.$$

so that

$$A_\epsilon(t) \leq C_7 \epsilon^{2\eta} e^{C_8 t} e^{C_9 \epsilon^{2(\eta-1)} \frac{t^{1-\lambda}}{1-\lambda}}.$$

Now, for $\tau < 1$, take $\epsilon = \tau^\mu$ with $\frac{1-\lambda}{2} + \mu(\eta - 1) > 0$. We find that, for $0 < t < \min(T_\epsilon, T_0)$ with

$$T_\epsilon = C_3^{\frac{2}{1-\lambda}} \epsilon^{\frac{2(1-\eta)}{1-\lambda}} = C_3^{\frac{2}{1-\lambda}} \tau^{\mu \frac{2(1-\eta)}{1-\lambda}} \quad \left[\text{where } \mu \frac{2(1-\eta)}{1-\lambda} < 1 \right],$$

we have the inequality

$$\begin{aligned} \|\vec{v} - e^{t\Delta} \vec{u}_0\|_{L^2_{\text{uloc}}} &\leq \|\alpha_\epsilon\|_2 + \|\vec{W}_\epsilon\|_2 + \|\vec{U}_\epsilon - e^{t\Delta} \vec{\beta}_\epsilon\|_2 \\ &\leq C_{10} \epsilon^\eta (1 + e^{C_{11} \frac{t^{1-\lambda}}{T_\epsilon^{1-\lambda}}}) + C_4 t^{\frac{1-\lambda}{2}} \epsilon^{\eta-1} \end{aligned}$$

If τ is small enough, we have $\tau < T_\epsilon$ and we find

$$\|\vec{v}(\tau, \cdot) - e^{\tau\Delta} \vec{u}_0\|_2 \leq C_0 \tau^{\mu\eta} (1 + e^{\frac{C_1^2}{2(1-\lambda)}}) + C_4 \tau^{\frac{1-\lambda}{2} + \mu(\eta-1)}$$

The theorem is proved.

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