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► **To cite this version:**

Pierre Gilles Lemarié-Rieusset. Interpolation, extrapolation, Morrey spaces and local energy control for the Navier–Stokes equations.. Banach Center Publications, 2019, 119, pp.279-294. 10.4064/bc119-16 . hal-01981330

**HAL Id: hal-01981330**

**<https://hal.science/hal-01981330v1>**

Submitted on 15 Jan 2019

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# Interpolation, extrapolation, Morrey spaces and local energy control for the Navier–Stokes equations

Pierre Gilles Lemarié–Rieusset\*

## Abstract

Barker recently proved new weak-strong uniqueness results for the Navier–Stokes equations based on a criterion involving Besov spaces and a proof through interpolation between Besov–Hölder spaces and  $L^2$ . We improve slightly his results by considering Besov–Morrey spaces and interpolation between Besov–Morrey spaces and  $L^2_{\text{loc}}$ .

**Keywords :** Navier–Stokes equations, Morrey spaces, Besov spaces, uniformly locally square integrable functions, weak-strong uniqueness

**AMS classification :** 35K55, 35Q30, 76D05.

## 1 The Navier–Stokes equations

Let  $\vec{u}_0$  a divergence-free vector field on  $\mathbb{R}^3$ . We shall consider weak solutions to the Cauchy initial value problem for the Navier–Stokes equations which satisfy energy estimates.

The differential Navier–Stokes equations read as

$$\partial_t \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u} = \Delta \vec{u} - \vec{\nabla} p$$

$$\operatorname{div} \vec{u} = 0$$

$$\vec{u}(0, \cdot) = \vec{u}_0$$

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Under reasonable assumptions, the problem is equivalent to the following integro-differential problem :

$$\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u}, \vec{u})(t, x)$$

where

$$B(\vec{u}, \vec{v}) = \int_0^t e^{(t-s)\Delta}\mathbb{P} \operatorname{div} (\vec{u} \otimes \vec{v}) ds \quad (1)$$

and  $\mathbb{P}$  is the Leray projection operator. (See [LR 2, LR 6] for details).

## Weak Leray solutions for the Navier–Stokes equations

When  $\vec{u}_0 \in L^2$ , Leray proved existence of solutions  $\vec{u}$  on  $(0, +\infty) \times \mathbb{R}^3$  such that :

- $\vec{u} \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$
- $\lim_{t \rightarrow 0^+} \|\vec{u}(t, \cdot) - \vec{u}_0\|_2 = 0$
- we have the Leray energy inequality

$$\|\vec{u}(t, \cdot)\|_2^2 + 2 \int_0^t \|\vec{\nabla} \otimes \vec{u}\|_2^2 ds \leq \|\vec{u}_0\|_2^2 \quad (2)$$

Such solutions are weak solutions : the derivatives in the Navier–Stokes solutions are taken in the sense of distributions. Those solutions (that satisfy the energy inequality (2)) are called *Leray weak solutions*.

When  $\vec{u}_0 \in L^2$ , Leray’s proof of existence of solutions [Le] is based on mollification, energy estimates and compactness arguments :

- he solves

$$\partial_t \vec{u}_\epsilon + (\varphi_\epsilon * \vec{u}_\epsilon) \cdot \vec{\nabla} \vec{u}_\epsilon = \Delta \vec{u}_\epsilon - \vec{\nabla} p_\epsilon$$

with  $\operatorname{div} \vec{u}_\epsilon = 0$  and  $\vec{u}_\epsilon(0, \cdot) = \vec{u}_0$ . Here,  $\varphi \in \mathcal{D}$ ,  $\int \varphi dx = 1$  and  $\varphi_\epsilon(x) = \frac{1}{\epsilon^3} \varphi(\frac{x}{\epsilon})$ .

- the solution holds on an interval  $(0, T_\epsilon)$  where  $T_\epsilon$  depends on  $\epsilon$  and on  $\|\vec{u}_0\|_2$  and we have the equality

$$\|\vec{u}_\epsilon(t, \cdot)\|_2^2 + 2 \int_0^t \|\vec{\nabla} \otimes \vec{u}_\epsilon\|_2^2 ds = \|\vec{u}_0\|_2^2$$

- the solution is then global; moreover by Rellich theorem, we find a subsequence that converges strongly in  $(L_t^2 L_x^2)_{loc}$  to a Leray solution  $\vec{u}$

Such solutions (i. e. obtained by this mollification/extraction process) will be called in the following *restricted Leray weak solutions*.

Restricted Leray solutions satisfy the Leray energy inequality which takes into account the energy on the whole space. But they enjoy as well a pointwise inequality property : for a non-negative locally finite measure  $\mu$  we have

$$\partial_t(|\vec{u}|^2) + 2|\vec{\nabla} \otimes \vec{u}|^2 = \Delta(|\vec{u}|^2) - \operatorname{div}((2p + |\vec{u}|^2)\vec{u}) - \mu \quad (3)$$

Leray solutions that enjoy the pointwise energy inequality are called *suitable Leray solutions* [CKN].

## Local weak Leray solutions

The pointwise energy inequality allows one [LR 1, LR 2] to develop a theory of weak solutions with infinite energy. Consider  $\vec{u}_0$  a divergence-free vector field that is uniformly locally square integrable :

$$\sup_{x_0 \in \mathbb{R}^3} \int_{|x-x_0|<1} |\vec{u}_0(x)|^2 dx < +\infty.$$

A *local Leray solution* on  $(0, T) \times \mathbb{R}^3$  is a solution such that

- $\vec{u} \in L_t^\infty(L_{\text{uloc}}^2) \cap (L_t^2 \dot{H}_x^1)_{\text{uloc}}$
- for all compact subset  $K$  of  $\mathbb{R}^3$ ,  $\lim_{t \rightarrow 0^+} \int_K |\vec{u}(t, \cdot) - \vec{u}_0|^2 dx = 0$
- we have the pointwise energy inequality (3).

Local in time existence of restricted local Leray solutions has been proved for a positive  $T$  that depends only on  $\|\vec{u}_0\|_{L_{\text{uloc}}^2}$  (see section 8).

## 2 The Prodi–Serrin criterion for weak-strong uniqueness

Based on a compactness criterion, the proof of existence of Leray solutions does not provide any clue on the would-be uniqueness of the solution to the Cauchy initial value problem.

A classical case of uniqueness of Leray weak solutions is Serrin’s criterion for weak-strong uniqueness [Pr, Se]. If  $\vec{u}_0 \in L^2$  and if the Navier-Stokes equations has a solution  $\vec{u}$  on  $(0, T)$  such that

$$\vec{u} \in X_T = L_t^p L_x^q \text{ with } \frac{2}{p} + \frac{3}{q} = 1 \text{ and } 2 \leq p < +\infty$$

then if  $\vec{v}$  is a Leray solution we have  $\vec{u} = \vec{v}$  on  $(0, T)$ .

The proof of the criterion is based on the fact that if  $\vec{v}$  is a Leray solution and if  $\vec{u}$  is the mild solution with  $\|\vec{u}\|_{X_T} < +\infty$ , then the difference  $\vec{w} = \vec{u} - \vec{v}$  satisfies a Gronwall estimate :

$$\|\vec{w}(t, \cdot)\|_2^2 + 2 \int_0^t \|\vec{\nabla} \otimes \vec{w}\|_2^2 ds \leq 2 \int_0^t \left| \int \vec{u} \cdot (\vec{w} \cdot \vec{\nabla} \vec{w}) dx \right| ds.$$

We have (for  $\frac{2}{p} + \frac{3}{q} = 1$ )

$$\|\vec{u} \otimes \vec{w}\|_2 \leq C \|\vec{u}\|_q \|\vec{w}\|_2^{\frac{2}{p}} \|\vec{\nabla} \otimes \vec{w}\|_2^{\frac{3}{q}}$$

so that

$$\left| \int \vec{u} \cdot (\vec{w} \cdot \vec{\nabla} \vec{w}) dx \right| \leq \|\vec{u}\|_q \|\vec{w}\|_2^{\frac{2}{p}} \|\vec{\nabla} \otimes \vec{w}\|_2^{1+\frac{3}{q}} = \|\vec{u}\|_q (\|\vec{w}\|_2^2)^{\frac{1}{p}} (\|\vec{\nabla} \otimes \vec{w}\|_2^2)^{1-\frac{1}{p}}.$$

Thus, we find

$$2 \int_0^t \left| \int \vec{u} \cdot (\vec{w} \cdot \vec{\nabla} \vec{w}) dx \right| ds \leq C \int_0^t \|\vec{u}\|_q^p \|\vec{w}\|_2^2 ds + \int_0^t \|\vec{\nabla} \otimes \vec{w}\|_2^2 ds.$$

The facts that  $\|\vec{w}(0, \cdot)\|_2^2 = 0$  and  $\|\vec{w}(t, \cdot)\|_2^2 \leq C \int_0^t \|\vec{u}\|_q^p \|\vec{w}\|_2^2 ds$  then gives  $\vec{w} = 0$  on  $(0, T)$ .

We may comment a little further in the case  $2 < p < +\infty$ . In that case, the bilinear operator  $B$  (given by (1)) is bounded on  $X_T = L_t^p L_x^q$ . Thus, we find that the existence of  $T > 0$  and of a solution in  $L^p L^q$  with  $\frac{2}{p} + \frac{3}{q}$  is equivalent to the existence of  $T'$  such that  $e^{t\Delta} \vec{u}_0 \in L^p L^q$  on  $(0, T')$  (and on  $(0, +\infty)$ , since  $\vec{u}_0 \in L^2$ ). Using the thermic characterization of Besov spaces, we can see that this is equivalent with

$$\vec{u}_0 \in \dot{B}_{q,p}^{-1+\frac{3}{q}}.$$

Thus, the initial value is not only in  $L^2$ , but it must belong as well to a Besov space with a better regularity than provided by the embedding

$$L^2 \subset \dot{B}_{q,2}^{-\frac{3}{2}+\frac{3}{q}} \subset \dot{B}_{q,p}^{-\frac{3}{2}+\frac{3}{q}}.$$

### 3 The Koch and Tataru theorem and T. Barker's question

We may now wonder how to generalize the Prodi–Serrin criterion. It means : given  $\vec{u}_0 \in L^2$  and weak Leray solutions associated to  $\vec{u}_0$ , find a space  $\mathbb{X}$

(as large as possible) such that if moreover  $\vec{u}_0 \in \mathbb{X}$  then we have a solution  $\vec{u} \in X_T$  for some space  $X_T$  of functions on  $(0, T) \times \mathbb{R}^3$  and such that the existence of a solution in  $X_T$  implies that any other weak Leray solution is equal to this solution  $\vec{u}$  for  $0 < t < T$ .

## The space $\text{BMO}^{-1}$

First of all, we precise which kind of space  $\mathbb{X}$  we are going to study. The idea is to look at an initial value which generates a solution in some uniqueness class (where uniqueness holds for small solutions). The setting where to construct such solutions is the setting of mild solutions, as introduced by Kato [Ka] : mild solutions are constructed by the Banach contraction principle.

Due to the symmetries of the equations (if  $\vec{u}$  is a solution for initial value  $\vec{u}_0$ , then  $\lambda\vec{u}(\lambda^2 t, \lambda(x - x_0))$  is a solution for the initial value  $\lambda\vec{u}_0(\lambda(x - x_0))$ ), we look for spaces with norms invariant through the transforms  $\vec{u}_0(\cdot) \mapsto \lambda\vec{u}(\lambda(\cdot - x_0))$  (for  $\lambda > 0$ ). Moreover, in order to be able to define

$$B(\vec{u}, \vec{v}) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} (\vec{u} \otimes \vec{v}) ds$$

at least for  $\vec{u} = \vec{v} = e^{t\Delta}\vec{u}_0$  (first step of the Picard iteration to find a fixed-point to  $\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u}, \vec{u})$ ), we ask that  $\int_{[0,1] \times B(0,1)} |e^{s\Delta}\vec{u}_0(y)|^2 ds dy < +\infty$ .

Thus, we are lead to introduce the space  $\mathbb{X}$  of distributions  $v$  such that

- $\sup_{t>0} \sqrt{t} \|e^{t\Delta} v\|_\infty < +\infty$
- $\sup_{0<t, x_0 \in \mathbb{R}^3} t^{-3/2} \int_0^t \int_{B(x_0, \sqrt{t})} |e^{s\Delta} v(y)|^2 dy ds)^{1/2}$

This space  $\mathbb{X}$  has been identified by Koch and Tataru [KocT] : this is the Triebel–Lizorkin space  $F_{\infty,2}^{-1}$ , or equivalently the space  $\text{BMO}^{-1} = \sqrt{-\Delta} \text{BMO}$ . Moreover, they proved the following theorem :

**Theorem 1** *For  $0 < T \leq \infty$ , define*

$$\|\vec{u}\|_{X_T} = \sup_{0<t<T} \sqrt{t} \|\vec{u}(t, \cdot)\|_\infty + \sup_{0<t<T, x_0 \in \mathbb{R}^3} (t^{-3/2} \int_0^t \int_{B(x_0, \sqrt{t})} |\vec{u}(s, y)|^2 dy ds)^{1/2}$$

*There exists  $C_0$  (which does not depend on  $T$ ) such that if  $T \in (0, +\infty]$ , if  $\vec{u}$  and  $\vec{v}$  are defined on  $(0, T) \times \mathbb{R}^3$  and if*

$$B(\vec{u}, \vec{v}) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div} (\vec{u} \otimes \vec{v}) ds$$

then

$$\|B(\vec{u}, \vec{v})\|_{X_T} \leq C_0 \|\vec{u}\|_{X_T} \|\vec{v}\|_{X_T}.$$

**Corollary 1** *If  $\|e^{t\Delta}\vec{u}_0\|_{X_T} < \frac{1}{4C_0}$ , then the integral Navier–Stokes equations have a solution on  $(0, T)$  such that  $\|\vec{u}\|_{X_T} \leq 2\|e^{t\Delta}\vec{u}_0\|_{X_T}$ .*

*This is the unique solution such that  $\|\vec{u}\|_{X_T} \leq \frac{1}{2C_0}$ .*

A special case of initial data that leads to a solution in some  $X_T$  is given by the subspace  $\text{VMO}^{-1}$  of  $\text{BMO}^{-1}$ .

**Definition 1**  $\text{VMO}^{-1}$  is the closure of compactly supported functions in  $\text{BMO}^{-1}$ .

If  $\vec{u}_0 \in \text{VMO}^{-1}$ , then  $\lim_{T \rightarrow 0} \|e^{t\Delta}\vec{u}_0\|_{X_T} = 0$ . Remark that we have the embedding  $\dot{B}_{q,p}^{-1+\frac{3}{q}} \subset \text{VMO}^{-1}$  for  $2 < p < +\infty$  and  $\frac{2}{p} + \frac{3}{q} = 1$ . As a matter of fact, we may consider  $\text{VMO}^{-1}$  as a limit case for the scale of spaces  $\dot{B}_{q,p}^{-1+\frac{3}{q}}$ . Thus, Barker [Ba] raised the following question :

**Question 1** *If  $\vec{u}_0$  belongs to  $L^2 \cap \text{VMO}^{-1}$ , does there exist a positive time  $T$  such that every weak Leray solution of the Cauchy problem for the Navier–Stokes equations with  $\vec{u}_0$  as initial value coincide with the mild solution in  $X_T$  ?*

If  $\vec{u}_0 \in L^2 \cap \text{VMO}^{-1}$  and if  $\|e^{t\Delta}\vec{u}_0\|_{X_T} \leq \frac{1}{4C_0}$ , then if  $\vec{u}$  is a restricted Leray solution of the Navier–Stokes solutions with initial value  $\vec{u}_0$ , then  $\|\vec{v}\|_{X_T} \leq 2\|e^{t\Delta}\vec{u}_0\|_{X_T}$ . In particular, we have uniqueness of *restricted Leray solutions* on  $(0, T)$ .

As a matter of fact, this proof of local uniqueness of restricted weak Leray solutions holds for a slightly more general class :

**Definition 2**  $\text{BMO}_0^{-1}$  is the space of distributions  $u_0$  in  $\text{BMO}^{-1}$  such that

$$\lim_{T \rightarrow 0} \|e^{t\Delta}u_0\|_{X_T} = 0.$$

*In the following, we will focus on the hypothesis  $\vec{u}_0 \in L^2 \cap \text{BMO}_0^{-1}$  and on the issue of uniqueness for Leray solutions.*

## 4 The limiting case

Up to now, we don't know how to prove local uniqueness of the Leray solutions when the initial value  $\vec{u}_0$  belongs to  $L^2 \cap \text{BMO}_0^{-1}$ . What we know for sure is that the mild solution  $\vec{u}$  in  $X_T$  belongs to  $L^\infty((\epsilon, T) \times \mathbb{R}^3)$  for every positive  $\epsilon \in (0, T)$ . Moreover,  $\vec{u}$  is a weak Leray solution and for every other Leray solution  $\vec{v}$  and for  $\epsilon > 0$ , we have

$$\partial_t(\vec{u} \cdot \vec{v}) = \vec{u} \cdot \partial_t \vec{v} + \partial_t \vec{u} \cdot \vec{v}$$

which gives for  $0 < \epsilon < t < T$

$$\begin{aligned} \int \vec{u}(t, x) \cdot \vec{v}(t, x) dx &= \int \vec{u}(\epsilon, x) \cdot \vec{v}(\epsilon, x) dx - 2 \int_\epsilon^t \int \vec{\nabla} \otimes \vec{u} \cdot \vec{\nabla} \otimes \vec{v} dx ds \\ &\quad - \int_\epsilon^t \int \vec{u} \cdot (\vec{v} \cdot \vec{\nabla} \vec{v}) + \vec{v} \cdot (\vec{u} \cdot \vec{\nabla} \vec{u}) dx ds \end{aligned}$$

From

$$\int \vec{u} \cdot (\vec{v} \cdot \vec{\nabla} \vec{v}) + \vec{v} \cdot (\vec{u} \cdot \vec{\nabla} \vec{u}) dx = \int \vec{u} \cdot (\vec{v} \cdot \vec{\nabla} (\vec{v} - \vec{u})) + (\vec{v} - \vec{u}) \cdot (\vec{u} \cdot \vec{\nabla} \vec{u}) dx = \int \vec{u} \cdot ((\vec{v} - \vec{u}) \cdot \vec{\nabla} (\vec{v} - \vec{u})) dx$$

and letting  $\epsilon$  go to 0, we get

$$\begin{aligned} \int \vec{u}(t, x) \cdot \vec{v}(t, x) dx &= \|\vec{u}_0\|_2^2 - 2 \int_0^t \int \vec{\nabla} \otimes \vec{u} \cdot \vec{\nabla} \otimes \vec{v} dx ds \\ &\quad - \lim_{\epsilon \rightarrow 0} \int_\epsilon^t \int \vec{u} \cdot ((\vec{v} - \vec{u}) \cdot \vec{\nabla} (\vec{v} - \vec{u})) dx ds \end{aligned}$$

and (letting  $\vec{v} = \vec{u}$ )

$$\|\vec{u}(t, \cdot)\|_2^2 = \|\vec{u}_0\|_2^2 - 2 \int_0^t \|\vec{\nabla} \otimes \vec{u}\|_2^2 ds.$$

Combining those two equalities with the Leray energy inequality for  $\vec{v}$

$$\|\vec{v}(t, \cdot)\|_2^2 + 2 \int_0^t \|\vec{\nabla} \otimes \vec{v}\|_2^2 ds \leq \|\vec{u}_0\|_2^2$$

we get the following inequality for  $\vec{w} = \vec{u} - \vec{v}$ :

$$\|\vec{w}(t, \cdot)\|_2^2 + 2 \int_0^t \|\vec{\nabla} \otimes \vec{w}\|_2^2 ds \leq 2 \int_0^T \left| \int \vec{u} \cdot (\vec{w} \cdot \vec{\nabla} \vec{w}) dx \right| ds. \quad (4)$$



As a matter of fact, the key ingredient in Prodi–Serrin’s criterion is the estimation of the integral

$$I(\vec{u}, \vec{w}) = \int_0^t \left| \int \vec{u}(\cdot, \vec{w} \cdot \vec{\nabla} \vec{w}) dx \right| ds$$

but, if  $\vec{w} = \vec{v} - \vec{u}$  with  $\vec{v}$  a Leray solution and  $\vec{u}$  the mild solution in  $X_T$ , we don’t even know whether  $I(\vec{u}, \vec{w})$  is finite.

In the limiting case of Prodi and Serrin, (for  $p = 2$  and  $q = +\infty$ ), we write

$$\|\vec{w}(t, \cdot)\|_2^2 + 2 \int_0^t \|\vec{\nabla} \otimes \vec{w}\|_2^2 ds \leq 2 \int_0^t \int_0^t \left| \int \vec{u}(\cdot, \vec{w} \cdot \vec{\nabla} \vec{w}) dx \right| ds \leq 2 \int_0^t \|\vec{u} \otimes \vec{w}\|_2 \|\vec{\nabla} \otimes \vec{w}\|_2 ds$$

and get

$$\|\vec{w}(t, \cdot)\|_2^2 \leq \int_0^t \|\vec{u}\|_\infty^2 \|\vec{w}\|_2^2 ds. \quad (5)$$

Of course, we may conclude under the assumption that  $\vec{u} \in L_t^2 L_x^\infty$ . Actually, we shall not be interested in measurability issues for functions with values in a non-separable space such as  $L^\infty$  (i.e. in Bochner measurability for instance), as we are dealing with locally integrable functions for the Lebesgue measure  $dt dx$  on  $(0, T) \times \mathbb{R}^3$ . Thus, for almost every  $t$  the quantity  $\|\vec{u}(t, \cdot)\|_\infty$  will be well-defined as a measurable function of  $t$ , and  $\vec{u} \in L_t^2 L_x^\infty$  will simply mean that  $\int_0^T \|\vec{u}(t, \cdot)\|_{L^\infty(dx)}^2 dt < +\infty$ . (See [LR 2] for details.)

If  $\vec{u} \in L_t^2 L_x^\infty$  on  $(0, T) \times \mathbb{R}^3$ , the Prodi–Serrin criterion proves that every weak Leray solution  $\vec{v}$  on  $(0, T)$  is equal to the mild solution  $\vec{u}$ . (This has even been extended to the case  $\vec{u} \in L_t^2 \text{BMO}_x$  by Kozono and Taniuchi [KozT]). But it is not easy to translate the condition that  $\vec{u} \in L_t^2 L_x^\infty$  into an equivalent assumption on  $\vec{u}_0$ . The problem comes from the fact that the bilinear operator  $B$  is not bounded on  $L_t^2 L_x^\infty$ .

On the other hand, if we only assume  $\vec{u}_0 \in L^2 \cap \text{BMO}_0^{-1}$ , we only know the inequality  $\|\vec{u}(t, \cdot)\|_\infty \leq \|\vec{u}\|_{X_T} \frac{o(1)}{\sqrt{t}}$ . We find an integrability issue near  $t = 0$ . To check that this is actually an issue, consider the following example : take  $\vec{w}$  a divergence-free vector field in the Schwartz class such that the Fourier transform of  $\vec{w}$  is compactly supported in the annulus  $1 < |\xi| < 2$ ; define

$$\vec{u}_0(x) = \sum_{j=0}^{+\infty} 2^j \frac{1}{\sqrt{1+j}} \vec{w}(2^j x);$$

we have  $\vec{u}_0 \in L^2$ ;  $\vec{u}_0 \in \dot{B}_{q,p}^{-1+\frac{3}{q}} \subset \dot{B}_{\infty,p}^{-1}$  (with  $2 < p < +\infty$  and  $\frac{2}{p} + \frac{3}{q} = 1$ ) but  $\vec{u}_0 \notin \dot{B}_{\infty,2}^{-1}$ ; as the bilinear operator  $B$  is bounded on  $L_t^p L_x^q \cap L_t^2 L_x^\infty$ , if we assume that the mild solution  $\vec{u} \in L_t^p L_x^q$  belongs to  $L_t^2 L_x^\infty$ , we would find that  $e^{t\Delta} \vec{u}_0 \in L_t^2 L_x^\infty$ , and thus  $\vec{u}_0 \in \dot{B}_{\infty,2}^{-1}$ ; thus, we have  $\int_0^T \|\vec{u}\|_\infty^2 dt = +\infty$ .

## 5 Barker's theorem

In this section, we shall sketch the proof of Barker [Ba], as we shall extend it in Section 7 to the case of Besov–Morrey spaces. The main idea in the recent paper of Barker is the following one : if we want to use only the inequality

$$\|\vec{u}(t, \cdot)\|_\infty \leq \|\vec{u}\|_{X_T} \frac{o(1)}{\sqrt{t}}$$

to deal with the Gronwall inequality (5), we need to assume more than  $\vec{w} \in L_t^\infty L_x^2$ . Indeed, we have the easy following lemma :

**Lemma 1** *Let  $\delta > 0$ . Let  $A$  and  $B$  be locally bounded non-negative measurable functions on  $(0, T]$  such that*

$$\lim_{t \rightarrow 0} tA(t) = 0 \text{ and } \sup_{0 < t < T} t^{-\delta} B(t) = 0.$$

*If we have moreover, for all  $t \in (0, T]$ ,*

$$B(t) \leq \int_0^t A(s) B(s) ds$$

*then  $B = 0$ .*

**Proof :** We have

$$B(t) \leq \frac{t^\delta}{\delta} \sup_{0 < s < t} sA(s) \sup_{0 < \sigma < t} \sigma^{-\delta} B(\sigma)$$

so that  $B = 0$  on  $(0, T_0]$  as long as  $\sup_{0 < s < T_0} sA(s) < \delta$ . For  $t > T_0$ , we then write  $B(t) \leq \sup_{T_0 < s < T} A(s) \int_{T_0}^t B(s) ds$  and we find  $B = 0$ .  $\diamond$

As  $\vec{w} = \vec{v} - \vec{u} = (\vec{v} - e^{t\Delta} \vec{u}_0) - (\vec{u} - e^{t\Delta} \vec{u}_0)$ , the extra information on  $\|\vec{w}\|_2$  will be provided by the following lemma :

**Lemma 2** *Let  $\vec{u}_0$  be a divergence-free vector field with  $\vec{u}_0 \in L^2$  and let  $\vec{v}$  be a weak Leray solution of the Navier–Stokes equations with initial value  $\vec{u}_0$ . If moreover*

$$\vec{u}_0 \in [L^2, B_{\infty, \infty}^{-\gamma}]_{\theta, \infty} \text{ for some } -1 < -\gamma < 0 \text{ and } 0 < \theta < 1$$

*then there exists  $\delta > 0$  such that*

$$\sup_{t > 0} t^{-\delta} \|\vec{v}(t, \cdot) - e^{t\Delta} \vec{u}_0\|_2 < +\infty.$$

**Proof :** Assume  $\vec{u}_0 \in [L^2, B_{\infty, \infty}^{-\gamma}]_{\theta, \infty}$ . For  $0 < \epsilon < 1$ , split  $\vec{u}_0$  in  $\vec{\alpha}_\epsilon + \vec{\beta}_\epsilon$  with

$$\|\vec{\alpha}_\epsilon\|_2 \leq C_0 \epsilon^\theta \text{ and } \|\vec{\beta}_\epsilon\|_{\dot{B}_{\infty, \infty}^{-\gamma}} \leq C_0 \epsilon^{\theta-1}$$

where  $C_0$  does not depend on  $\epsilon$  (but depends on  $\vec{u}_0$ ).

We have a solution  $\vec{U}_\epsilon$  for the Navier-Stokes equations with initial value  $\vec{\beta}_\epsilon$  such that  $\|\vec{U}_\epsilon(t, \cdot)\|_\infty \leq C_1 t^{-\frac{\gamma}{2}} \epsilon^{\theta-1}$  on an interval  $(0, T_\epsilon)$  with  $T_\epsilon^{\frac{1-\gamma}{2}} \epsilon^{\theta-1} = C_2$ . Moreover  $\vec{\beta}_\epsilon = \vec{u}_0 - \vec{\alpha}_\epsilon \in L^2$  and we find that  $\sup_{0 < t < T_\epsilon} \|\vec{U}_\epsilon\|_2 \leq C_3$  and that  $\|\vec{U}_\epsilon - e^{t\Delta} \vec{\beta}_\epsilon\|_2 \leq C_4 t^{(1-\gamma)/2} \epsilon^{\theta-1}$ .

Since  $\vec{U}_\epsilon$  is in  $L_t^2 L_x^\infty$  on every bounded interval, we get

$$\begin{aligned} \int \vec{U}_\epsilon(t, x) \cdot \vec{v}(t, x) dx &= \int \vec{\beta}_\epsilon(x) \cdot \vec{u}_0(x) dx - 2 \int_0^t \int \vec{\nabla} \otimes \vec{U}_\epsilon \cdot \vec{\nabla} \otimes \vec{v} dx ds \\ &\quad - \int_0^t \int \vec{U}_\epsilon \cdot ((\vec{v} - \vec{U}_\epsilon) \cdot \vec{\nabla} (\vec{v} - \vec{U}_\epsilon)) dx ds \end{aligned}$$

and

$$\|\vec{U}_\epsilon(t, \cdot)\|_2^2 = \|\vec{\beta}_\epsilon\|_2^2 - 2 \int_0^t \|\vec{\nabla} \otimes \vec{U}_\epsilon\|_2^2 ds.$$

Combining those two equalities with the Leray energy inequality for  $\vec{v}$

$$\|\vec{v}(t, \cdot)\|_2^2 + 2 \int_0^t \|\vec{\nabla} \otimes \vec{v}\|_2^2 ds \leq \|\vec{u}_0\|_2^2$$

we get the following inequality for  $\vec{W}_\epsilon = \vec{v} - \vec{U}_\epsilon$ :

$$\|\vec{W}_\epsilon(t, \cdot)\|_2^2 + 2 \int_0^t \|\vec{\nabla} \otimes \vec{W}_\epsilon\|_2^2 ds \leq \|\vec{\alpha}_\epsilon\|_2^2 + 2 \int_0^T \left| \int \vec{U}_\epsilon \cdot (\vec{W}_\epsilon \cdot \vec{\nabla} \vec{W}_\epsilon) dx \right| ds.$$

Thus, we get

$$\|\vec{W}_\epsilon(t, \cdot)\|_2^2 \leq C_0^2 \epsilon^{2\theta} + C_1^2 \epsilon^{2(\theta-1)} \int_0^t s^{-\gamma} \|\vec{W}_\epsilon(s, \cdot)\|_2^2 ds$$

so that

$$\|\vec{W}_\epsilon(t, \cdot)\|_2^2 \leq C_0^2 \epsilon^{2\theta} e^{C_1^2 \epsilon^{2(\theta-1)} \frac{t^{1-\gamma}}{1-\gamma}}.$$

Now, for  $\tau < 1$ , take  $\epsilon = \tau^\mu$  with  $\frac{1-\gamma}{2} + \mu(\theta-1) > 0$ . We find that, for  $0 < t < T_\epsilon$  with

$$T_\epsilon = C_2^{\frac{2}{1-\gamma}} \epsilon^{\frac{2(1-\theta)}{1-\gamma}} = C_2^{\frac{2}{1-\gamma}} \tau^{\mu \frac{2(1-\theta)}{1-\gamma}} \text{ [ where } \mu \frac{2(1-\theta)}{1-\gamma} < 1 \text{ ],}$$

we have the inequality

$$\begin{aligned}\|\vec{v} - e^{t\Delta}\vec{u}_0\|_2 &\leq \|\alpha_\epsilon\|_2 + \|\vec{W}_\epsilon\|_2 + \|\vec{U}_\epsilon - e^{t\Delta}\vec{\beta}_\epsilon\|_2 \\ &\leq C_0\epsilon^\theta \left(1 + e^{\frac{C_1^2}{2(1-\gamma)}\frac{t^{1-\gamma}}{T_\epsilon^{1-\gamma}}}\right) + C_4 t^{\frac{1-\gamma}{2}} \epsilon^{\theta-1}\end{aligned}$$

If  $\tau$  is small enough, we have  $\tau < T_\epsilon$  and we find

$$\|\vec{v}(\tau, \cdot) - e^{\tau\Delta}\vec{u}_0\|_2 \leq C_0\tau^{\mu\theta} \left(1 + e^{\frac{C_1^2}{2(1-\gamma)}}\right) + C_4\tau^{\frac{1-\gamma}{2} + \mu(\theta-1)}$$

The lemma is proved.  $\diamond$  Barker's theorem then reads as :

**Theorem 2** *Let  $\vec{u}_0$  be a divergence-free vector field with  $\vec{u}_0 \in L^2$  and let  $\vec{v}$  be a weak Leray solution of the Navier–Stokes equations with initial value  $\vec{u}_0$ . If moreover*

$$\vec{u}_0 \in \text{BMO}_0^{-1} \cap \dot{B}_{q,\infty}^{-s} \text{ with } 3 < q < +\infty \text{ and } -s > -1 + \frac{2}{q}$$

*then there exists  $T > 0$  such that if  $\vec{v}$  is a Leray solution and if  $\vec{u}$  is the mild solution with  $\|\vec{u}\|_{X_T} < +\infty$ , then  $\vec{u} = \vec{v}$  on  $(0, T)$ .*

**Remark** We have the embeddings  $L^2 \subset \dot{B}_{q,2}^{-\frac{3}{2} + \frac{3}{q}} \subset \dot{B}_{q,\infty}^{-\frac{3}{2} + \frac{3}{q}}$ , so that the information conveyed by the hypothesis  $\vec{u}_0 \in \dot{B}_{q,\infty}^{-s}$  is interesting only for the high frequencies of  $\vec{u}_0$ . Moreover the embeddings  $L^2 = \dot{B}_{2,2}^0$  and  $\text{BMO}^{-1} \subset \dot{B}_{\infty,\infty}^{-1}$  gives that

$$L^2 \cap \text{BMO}_0^{-1} \subset \dot{B}_{q,q}^{-1 + \frac{2}{q}} \subset \dot{B}_{q,\infty}^{-1 + \frac{2}{q}}.$$

Thus, the information conveyed by the hypothesis  $\vec{u}_0 \in \dot{B}_{q,\infty}^{-s}$  is not contained in the assumption  $\vec{u}_0 \in L^2 \cap \text{BMO}_0^{-1}$ . Finally, if  $-s > -1 + \frac{3}{q} = \frac{2}{p}$ , then we have

$$L^2 \cap \text{BMO}_0^{-1} \cap \dot{B}_{q,\infty}^{-s} \subset \dot{B}_{q,q}^{-1 + \frac{2}{q}} \cap \dot{B}_{q,\infty}^{-s} \subset \dot{B}_{q,p}^{-1 + \frac{3}{q}}.$$

Thus, the theorem is interesting only in the range  $-1 + \frac{2}{q} < -s \leq -1 + \frac{3}{q}$ , which corresponds to the gap between  $L^2 \cap \text{BMO}_0^{-1}$ . (where local uniqueness is conjectured to hold) and  $\dot{B}_{q,p}^{-1 + \frac{3}{q}}$  (for which the Prodi–Serrin criterion shows that local uniqueness holds).

A further remark is that we have the embedding  $\dot{B}_{q,\infty}^{-1 + \frac{3}{q}} \subset \text{BMO}^{-1}$ , so that we have a Prodi–Serrin criterion with  $\vec{u} \in L_t^p L_x^q$  (with  $\frac{2}{p} + \frac{3}{q} = 1$ ) replaced with

$$\sup_{0 < t < T} t^{\frac{1}{p}} \|\vec{u}\|_q < +\infty \text{ and } \lim_{t \rightarrow 0} t^{\frac{1}{p}} \|\vec{u}(t, \cdot)\|_q = 0.$$

**Proof :**

The first step is the use of interpolation inequalities in order to be able to check that  $\vec{u}_0$  fulfills the assumptions of Lemma 2.

- Since  $\vec{u}_0 \in L^2 = \dot{B}_{2,2}^0$  and  $\vec{u}_0 \in \text{BMO}^{-1} \subset \dot{B}_{\infty,\infty}^{-1}$ , we have  $\vec{u}_0 \in \dot{B}_{q,q}^{-1+\frac{2}{q}}$ .
- Since  $\vec{u}_0 \in \dot{B}_{q,q}^{-1+\frac{2}{q}} \cap \dot{B}_{q,\infty}^{-s}$ , we have for  $-1 + \frac{2}{q} < -\sigma < -s$ ,  $\vec{u}_0 \in \dot{B}_{q,1}^{-\sigma}$ .
- The Besov space  $\dot{B}_{q,1}^{-\sigma}$  is embedded in the Sobolev space  $\dot{W}^{-\sigma,q}$ .
- For  $q < r < \infty$ , we have  $\dot{W}^{-\sigma,q} = [L^2, \dot{W}^{-\delta,r}]_{[\theta]}$  with  $\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{r}$  and  $-\sigma = -\theta\delta$  (complex interpolation)
- Since  $\dot{W}^{-\delta,r} \subset \dot{B}_{\infty,\infty}^{-1+\frac{3}{r}}$ , we have  $[L^2, \dot{W}^{-\delta,r}]_{[\theta]} \subset [L^2, \dot{B}_{\infty,\infty}^{-\delta-\frac{3}{r}}]_{\theta,\infty}$  with

$$-\delta = -\frac{s}{\theta} = -s \frac{\frac{1}{2} - \frac{1}{r}}{\frac{1}{2} - \frac{1}{q}} = -\frac{(-s)}{-1 + \frac{2}{q}} + O\left(\frac{1}{r}\right)$$

so that  $-\delta - \frac{3}{r} > -1$  for  $r$  large enough

We may now end the proof : recall that if  $\vec{v}$  is a Leray solution and if  $\vec{u}$  is the mild solution with  $\|\vec{u}\|_{X_T} < +\infty$ , then the difference  $\vec{w} = \vec{u} - \vec{v}$  satisfies a Gronwall estimate :

$$\|\vec{w}(t, \cdot)\|_2^2 \leq \int_0^t \|\vec{u}\|_\infty^2 \|\vec{w}\|_2^2 ds.$$

By Lemma 2, we have  $\|\vec{v}(t, \cdot) - e^{t\Delta}\vec{u}_0\|_2 = O(t^\delta)$  and  $\|\vec{u}(t, \cdot) - e^{t\Delta}\vec{u}_0\|_2 = O(t^\delta)$  for some positive  $\delta$ . On the other hand, we know that  $\|\vec{u}(t, \cdot)\|_\infty = o(\frac{1}{\sqrt{t}})$ . Using Lemma 1, we find that  $\vec{w} = 0$ , and  $\vec{v} = \vec{u}$ .  $\diamond$

## 6 The Prodi–Serrin criterion for Besov–Morrey spaces

Morrey spaces provide a natural tool for extending the Prodi–Serrin criterion.

**Definition 3** For  $1 < r \leq q < +\infty$ , we define the Morrey space  $\dot{M}^{r,q}$  as the space of Lebesgue measurable functions  $f$  on  $\mathbb{R}^3$  such that

$$\sup_{R>0, x_0 \in \mathbb{R}^3} R^{\frac{3}{q} - \frac{3}{r}} \left( \int_{B(x_0, R)} |f(x)|^r dx \right)^{1/r} = \|f\|_{\dot{M}^{r,q}} < +\infty.$$

Similarly, the space  $\dot{M}^{1,q}$  is the space of locally finite Borelian (signed) measure  $\mu$  such that

$$\sup_{R>0, x_0 \in \mathbb{R}^3} R^{\frac{3}{q} - 3} \left( \int_{B(x_0, R)} d|\mu| \right) = \|\mu\|_{\dot{M}^{1,q}} < +\infty.$$

*Remark :* For absolutely continuous measures  $d\mu = f dx$  with  $f \in L^1_{\text{loc}}$ , we have

$$\|\mu\|_{\dot{M}^{1,q}} = \sup_{R>0, x_0 \in \mathbb{R}^3} R^{\frac{3}{q}-3} \left( \int_{B(x_0, R)} |f(x)| dx \right).$$

The key inequality in the proof of the Prodi–Serrin criterion was the inequality (for all  $w \in H^1$ )

$$\|uw\|_2 \leq C \|u\|_q \|w\|_2^{\frac{2}{p}} \|\vec{\nabla} w\|_2^{\frac{3}{q}}$$

with  $\frac{2}{p} + \frac{3}{q} = 1$  and  $3 < q < +\infty$ . If we want to replace this inequality by a more general inequality

$$\|uw\|_2 \leq N(u) \|w\|_2^{\frac{2}{p}} \|\vec{\nabla} w\|_2^{\frac{3}{q}}$$

(again with  $\frac{2}{p} + \frac{3}{q} = 1$  and  $3 < q < +\infty$ ), then we proved in [LR 3] that the existence of a finite  $N(u)$  is equivalent to the fact that  $u \in \dot{M}^{2,q}$ , and moreover that  $N(u) \approx \|u\|_{\dot{M}^{2,q}}$ .

This leads to the following easy extension of the Prodi–Serrin criterion :

**Theorem 3** *If  $\vec{u}_0 \in L^2$  and if the Navier–Stokes equations has a solution  $\vec{u}$  such that*

$$\vec{u} \in L^p_t \dot{M}^{2,q}_x \text{ with } \frac{2}{p} + \frac{3}{q} = 1 \text{ and } 3 < q < +\infty$$

*then if  $\vec{v}$  is a Leray solution we have  $\vec{u} = \vec{v}$  on  $(0, T)$ .*

If  $3 < q < +\infty$ , the existence of  $T > 0$  and of a solution in  $L^p \dot{M}^{2,q}$  with  $\frac{2}{p} + \frac{3}{q} = 1$  is equivalent to the existence of  $T'$  such that  $e^{t\Delta} \vec{u}_0 \in L^p \dot{M}^{2,q}$  on  $(0, T')$  (and on  $(0, +\infty)$ , since  $\vec{u}_0 \in L^2$ ), thus with

$$\vec{u}_0 \in \dot{B}^{-1+\frac{3}{q}}_{\dot{M}^{2,q,p}}.$$

This Besov–Morrey space has been introduced in 1994. by Kozono and Yamazaki [KoY]. It is easy to check that, for  $2 < p < +\infty$  and  $\frac{2}{p} + \frac{3}{q} = 1$ , we have the inequality

$$\|e^{t\Delta} u_0\|_{X_T} \leq C_q \left( \int_0^T \|e^{t\Delta} u_0\|_{\dot{M}^{2,q}}^p dt \right)^{1:p}$$

so that we have the embedding  $\dot{B}^{-1+\frac{3}{q}}_{\dot{M}^{2,q,p}} \subset \text{BMO}_0^{-1}$  for  $2 < p < +\infty$  and  $\frac{2}{p} + \frac{3}{q} = 1$ .

## 7 Barker's theorem and Besov-Morrey spaces

We shall extend Barker's theorem.

**Theorem 4** *If*

- $\vec{u}_0 \in L^2 \cap \text{BMO}_0^{-1}$
- $3 < q < +\infty$ ,  $-s > -1 + \frac{2}{q}$  and  $\vec{u}_0 \in \dot{B}_{M^{1,q},\infty}^{-s}$

*then there exists  $T > 0$  such that if  $\vec{v}$  is a suitable Leray solution and if  $\vec{u}$  is the mild solution with  $\|\vec{u}\|_{X_T} < +\infty$ , then  $\vec{u} = \vec{v}$  on  $(0, T)$ .*

**Proof :** As we shall see, the proof is very similar to Barker's proof for Theorem 2 [Ba]. However, we shall meet some technical issues.

We sketch the proof :

- $\vec{u}_0 \in L^2 = \dot{B}_{2,2}^0$  and  $\vec{u}_0 \in \text{BMO}^{-1} \subset \dot{B}_{\infty,\infty}^{-1}$ , thus  $\vec{u}_0 \in \dot{B}_{q,q}^{-1+\frac{2}{q}}$
- $\vec{u}_0 \in \dot{B}_{q,q}^{-1+\frac{2}{q}} \cap \dot{B}_{M^{1,q},\infty}^{-s}$  thus  $\vec{u}_0 \in \dot{B}_{M^{p,q},\infty}^{-\sigma}$  for  $1 < p < q$ ,  $\frac{1}{p} = 1 - \theta + \frac{\theta}{q}$  and  $-\sigma = -s(1 - \theta) + \theta(-1 + \frac{2}{q}) > -1 + \frac{2}{q}$ . We shall take  $p > 2$ .
- as  $p < q$ , we have  $\dot{B}_{q,q}^{-1+\frac{2}{q}} \subset \dot{B}_{q,p}^{1+\frac{2}{q}}$ . Thus, for  $-1 + \frac{2}{q} < -\gamma < -\sigma$ ,

$$\dot{B}_{q,q}^{-1+\frac{2}{q}} \cap \dot{B}_{M^{p,q},\infty}^{-\sigma} \subset \dot{B}_{M^{p,q},1}^{-\gamma} \subset \dot{W}^{-\gamma, M^{p,q}}.$$

We now encounter our first problem. We can no longer write  $\dot{W}^{-\gamma, M^{p,q}}$  as a subspace of an interpolate space between  $L^2$  and  $\dot{B}_{\infty,\infty}^{-1+\delta}$ . More precisely, let us assume  $\dot{B}_{M^{p,q},1}^{-\gamma} \subset [L^2, \dot{B}_{\infty,\infty}^{-1+\delta}]_{\theta,\infty}$ ; by homogeneity of the norms, we must have  $-\gamma - \frac{3}{q} = -(1 - \theta)\frac{3}{2} - \theta(1 - \delta)$ . We have

$$[L^2, \dot{B}_{\infty,\infty}^{-1+\delta}]_{\theta,\infty} \subset \dot{B}_{L^{r,\infty},\infty}^{-\theta(1-\delta)}$$

with  $r = \frac{2}{1-\theta}$ . In particular, for  $u \in [L^2, \dot{B}_{\infty,\infty}^{-1+\delta}]_{\theta,\infty}$ , we have that  $e^\Delta e^{t\partial_3^2} u$  goes to 0 in  $\mathcal{S}'$  when  $t$  goes to  $+\infty$ . But if  $3p \leq 2q$ , if  $u$  depends only on  $(x_1, x_2)$  and not on  $x_3$ , and if  $u \in \dot{B}_{M^{p,\frac{2q}{3}},1}^{-\gamma}(\mathbb{R}^2)$ , then  $u \in \dot{B}_{M^{p,q},1}^{-\gamma}(\mathbb{R}^3)$  and  $e^\Delta e^{t\partial_3^2} u = e^\Delta u$ . Thus, we have a contradiction.

We better use complex interpolation and write that

$$\dot{W}^{-\gamma, M^{p,q}} = [\dot{M}^{2, \frac{2}{p}q}, \dot{W}^{-\rho, M^{r, \frac{r}{p}q}}]_{[\theta]}$$

for  $r > p$ ,  $\frac{1-\theta}{2} + \frac{\theta}{r} = \frac{1}{p}$ ,  $\gamma = \theta\rho$ . (For interpolation of Morrey spaces, see [LR 4, LR 5])

Then, we remark that  $\dot{M}^{2, \frac{2}{p}q} \subset L^2_{\text{uloc}}$  and write that

$$[\dot{M}^{2, \frac{2}{p}q}, \dot{W}^{-\rho, \dot{M}^{r, \frac{r}{p}q}}][\theta] \subset [\dot{M}^{2, \frac{2}{p}q}, \dot{W}^{-\rho, \dot{M}^{r, \frac{r}{p}q}}]_{\theta, \infty} \subset [L^2_{\text{uloc}}, \dot{B}^{-\rho - \frac{3p}{r}q}]_{\theta, \infty}$$

In order to finish the proof, we thus need to use the machinery of energy control for suitable local Leray solutions [LR 2, LR 6]. This will be done in the following sections, and we shall finish the proof in Section 10  $\diamond$

## 8 Weak local Leray solutions

We recall basic results for local weak Leray solutions. We endow  $L^2_{\text{uloc}}$  with the norm

$$\|u\|_{L^2_{\text{uloc}}} = \sup_{k \in \mathbb{Z}^3} \|u\varphi_0(x - k)\|_2,$$

where  $\varphi_0$  is a non-negative function in  $\mathcal{D}$ , supported in a ball  $B(0, R_0)$  and such that  $\sum_{k \in \mathbb{Z}^3} \varphi_0(x - k) = 1$ .

When  $\vec{u}_0 \in L^2_{\text{uloc}}$ , proof of existence of solutions for the Navier–Stokes equations is based on mollification, energy estimates and compactness arguments (for details, see [LR 6], section 14.1) :

- we solve

$$\partial_t \vec{u}_\epsilon + (\varphi_\epsilon * \vec{u}_\epsilon) \cdot \vec{\nabla} \vec{u}_\epsilon = \Delta \vec{u}_\epsilon - \vec{\nabla} p_\epsilon$$

with  $\text{div } \vec{u}_\epsilon = 0$  and  $\vec{u}_\epsilon(0, \cdot) = \vec{u}_0$ . Here,  $\varphi \in \mathcal{D}$ ,  $\int \varphi dx = 1$  and  $\varphi_\epsilon(x) = \frac{1}{\epsilon^3} \varphi(\frac{x}{\epsilon})$ . Here  $\vec{\nabla} p_\epsilon$  is given by the Leray projection :

$$\vec{\nabla} p_\epsilon = -(\varphi_\epsilon * \vec{u}_\epsilon) \cdot \vec{\nabla} \vec{u}_\epsilon + \mathbb{P} \text{div} ((\varphi_\epsilon * \vec{u}_\epsilon) \otimes \vec{u}_\epsilon).$$

- the solution holds at least on an interval  $(0, T_\epsilon)$  where  $T_\epsilon$  depends on  $\epsilon$  and on  $\|\vec{u}_0\|_{L^2_{\text{uloc}}}$  ( $T_\epsilon = \min(1, C_0 \frac{\epsilon^{3/2}}{\|\vec{u}_0\|_{L^2_{\text{uloc}}}^2}$ ). Moreover, we have the inequalities, for  $k \in \mathbb{Z}^3$ ,

$$\begin{aligned} & \int \varphi_0(x - k) |\vec{u}_\epsilon(t, x)|^2 dx + 2 \int_0^t \int \varphi_0(x - k) |\vec{\nabla} \otimes \vec{u}_\epsilon(s, x)|^2 dx ds \\ & \leq \int \varphi_0(x - k) |\vec{u}_0(t, x)|^2 dx + C_1 \int_0^t \int_{|x-k| \leq R_0} |\vec{u}_\epsilon(s, x)|^2 dx ds \\ & \quad + C_2 \int_0^t \int_{|x-k| > 5R_0} \frac{1}{|x - k|^4} |\vec{u}_\epsilon(s, x)|^3 dx ds \\ & \quad + C_2 \int_0^t \int_{|x-k| < 5R_0} |\vec{u}_\epsilon(s, x)|^3 dx ds \end{aligned}$$



- defining

$$\alpha_\epsilon(t) = \|\vec{u}_\epsilon(t, \cdot)\|_{L^2_{\text{uloc}}}$$

and

$$\beta_\epsilon(t) = \sup_{k \in \mathbb{Z}^3} \left( \int_0^t \int \varphi_0(x-k) |\vec{\nabla} \otimes \vec{u}_\epsilon(s, x)|^2 dx ds \right)^{1/2},$$

we get the inequality

$$\begin{aligned} & \int \varphi_0(x-k) |\vec{u}_\epsilon(t, x)|^2 dx + \int_0^t \int \varphi_0(x-k) |\vec{\nabla} \otimes \vec{u}_\epsilon(s, x)|^2 dx ds \\ & \leq \alpha_\epsilon(0)^2 + \frac{1}{2} \beta_\epsilon(t)^2 + C_3 \int_0^t \alpha_\epsilon(s)^2 ds + C_3 \int_0^t \alpha_\epsilon(s)^6 ds \end{aligned}$$

so that

$$\beta_\epsilon(t)^2 \leq 2\alpha_\epsilon(0)^2 + 2C_3 \int_0^t \alpha_\epsilon(s)^2 ds + 2C_3 \int_0^t \alpha_\epsilon(s)^6 ds$$

and finally

$$\alpha_\epsilon(t)^2 \leq 2\alpha_\epsilon(0)^2 + 2C_3 \int_0^t \alpha_\epsilon(s)^2 ds + 2C_3 \int_0^t \alpha_\epsilon(s)^6 ds$$

Thus, as long as  $8C_3 t < 1$  and  $128 C_3 t \|\vec{u}_0\|_{L^2_{\text{uloc}}}^4 < 1$ , we find that  $\alpha_\epsilon(t) \leq 2\|\vec{u}_0\|_{L^2_{\text{uloc}}}$  and  $\beta_\epsilon(t) \leq 2\|\vec{u}_0\|_{L^2_{\text{uloc}}}$ .

- the solution is then defined on  $(0, \min(\frac{1}{8C_3}, \frac{1}{128 C_3 \|\vec{u}_0\|_{L^2_{\text{uloc}}}^4}))$  and controlled independently from  $\epsilon$ . By Rellich theorem, we find a subsequence that converges strongly in  $(L^2_t L^2_x)_{\text{loc}}$  to a suitable local Leray solution  $\vec{u}$

An important point is the following one : assume moreover that  $\vec{u}_0 \in \text{BMO}_0^{-1}$  and that  $\|e^{t\Delta} \vec{u}_0\|_{X_T} < \frac{1}{4C_0}$  (where  $C_0$  is the constant of Theorem 1) then  $\vec{u}_\epsilon$  is defined at least on  $(0, T)$  and  $\|\vec{u}_\epsilon\|_{X_T} \leq 2\|e^{t\Delta} \vec{u}_0\|_{X_T}$ . As  $T$  does not depend on  $\epsilon$ , we see that the local Leray solution  $\vec{u}$  satisfies  $\vec{u} \in X_S$  with  $S = \min(T, \frac{1}{8C_3}, \frac{1}{128 C_3 \|\vec{u}_0\|_{L^2_{\text{uloc}}}^4})$  and  $\|\vec{u}\|_{X_S} \leq 2\|e^{t\Delta} \vec{u}_0\|_{X_T}$ .

Similarly, if we assume that  $\vec{u}_0 \in \dot{B}_{\infty, \infty}^{-\gamma}$  with  $-1 < -\gamma < 0$ , then  $\vec{u}_\epsilon$  is defined at least on  $(0, T)$  where  $T = C \|\vec{u}_0\|_{\dot{B}_{\infty, \infty}^{-\gamma}}^{\frac{2}{1-\gamma}}$  and  $\sup_{0 < t < T} t^{\frac{\gamma}{2}} \|\vec{u}(t, \cdot)\|_\infty \leq 2 \sup_{0 < t < T} t^{\frac{\gamma}{2}} \|e^{t\Delta} \vec{u}_0\|_\infty$ . As  $T$  does not depend on  $\epsilon$ , we see that the local Leray solution  $\vec{u}$  satisfies the inequality  $\sup_{0 < t < S} t^{\frac{\gamma}{2}} \|\vec{u}(t, \cdot)\|_\infty < +\infty$  where  $S = \min(T, \frac{1}{8C_3}, \frac{1}{128 C_3 \|\vec{u}_0\|_{L^2_{\text{uloc}}}^4})$ .

## 9 Comparison of local weak Leray solutions

If  $\vec{u}$  and  $\vec{v}$  are two local weak Leray solutions, on  $(0, T)$  with initial values  $\vec{u}_0$  and  $\vec{v}_0$ , we would like to be able to estimate  $\vec{u}(t, \cdot) - \vec{v}(t, \cdot)$  from the estimation of  $\vec{u}_0 - \vec{v}_0$ . This can be done only when at least one of the solutions is regular enough. We shall assume that  $\vec{u} \in L_t^2 L_x^\infty$ . We sketch the computations described in [LR 6], section 14.4.

Define  $\vec{w} = \vec{u} - \vec{v}$ ,

$$\alpha(t) = \|\vec{w}(t, \cdot)\|_{L_{\text{uloc}}^2}$$

and

$$\beta(t) = \sup_{k \in \mathbb{Z}^3} \left( \int_0^t \int \varphi_0(x-k) |\vec{\nabla} \otimes \vec{w}(s, x)|^2 dx ds \right)^{1/2},$$

Using the suitability of  $\vec{v}$  and the regularity of  $\vec{v}$ , we find (for  $0 < t < \min(1, T)$ )

$$\begin{aligned} & \int \varphi_0(x-k) |\vec{w}(t, x)|^2 dx + \int_0^t \int \varphi_0(x-k) |\vec{\nabla} \otimes \vec{w}(s, x)|^2 dx ds \\ & \leq \alpha(0)^2 + \frac{1}{2} \beta(t)^2 + C_1 \int_0^t \alpha(s)^2 ds + C_2 \int_0^t \alpha(s)^6 ds + C_3 \int_0^t \|\vec{u}(s, \cdot)\|_\infty^2 \alpha(s)^2 ds \end{aligned}$$

where the constants  $C_i$  do not depend on  $T$ ,  $\vec{u}$ , nor on  $\vec{v}$ . Finally, we find

$$\begin{aligned} \alpha(t)^2 & \leq 2\alpha(0)^2 + 2(C_1 + C_3(\|\vec{u}\|_{L_t^\infty L_{\text{uloc}}^2} + \|\vec{v}\|_{L_t^\infty L_{\text{uloc}}^2})^4) \int_0^t \alpha(s)^2 ds \\ & \quad + 2C_3 \int_0^t \|\vec{u}(s, \cdot)\|_\infty^2 \alpha(s)^2 ds. \end{aligned} \tag{6}$$

We have the same estimate even if  $\vec{u}$  is not integrable near  $t = 0$ . Let us only assume that  $\vec{u} \in L_t^2 L_x^\infty$  on every  $(\epsilon, T)$  with  $\epsilon > 0$ . Considering a time  $t_0 > 0$  which a Lebesgue point for the functions  $t \mapsto \int \varphi(x-k) |\vec{u}(t, x)|^2 dx$  and  $t \mapsto \int \varphi(x-k) |\vec{w}(t, x)|^2 dx$ , we have that  $\vec{u}$  and  $\vec{v}$  are local weak Leray solutions on  $(t_0, T)$  with initial values  $\vec{u}(t_0, \cdot)$  and  $\vec{v}(t_0, \cdot)$ . Thus, we shall find that, for  $t > t_0$

$$\begin{aligned} \int \varphi_0(x-k) |\vec{w}(t, x)|^2 dx & \leq 2 \int \varphi_0(x-k) |\vec{w}(t_0, x)|^2 dx \\ & \quad + 2(C_1 + C_3(\|\vec{u}\|_{L_t^\infty L_{\text{uloc}}^2} + \|\vec{v}\|_{L_t^\infty L_{\text{uloc}}^2})^4) \int_{t_0}^t \alpha(s)^2 ds \\ & \quad + 2C_3 \int_{t_0}^t \|\vec{u}(s, \cdot)\|_\infty^2 \alpha(s)^2 ds. \end{aligned} \tag{7}$$

It is then enough to let  $t_0$  go to 0 and then take the supremum with respect to  $k$ .

## 10 Proof of Theorem 4

We may now finish the proof. We consider two solutions of the Navier–Stokes equations with initial value  $\vec{u}_0 \in L^2 \cap \text{BMO}_0^{-1} \cap \dot{B}_{M^{1,q},\infty}^{-s}$  with  $-s > -1 + \frac{2}{q}$  (and  $3 < q < +\infty$ ) : we assume that  $\vec{v}$  is a suitable Leray solution and  $\vec{u}$  is the mild solution in  $X_T$ .

As  $\vec{v}$  is suitable,  $\vec{v}$  is a local Leray solution as well and we may estimate the  $L_{\text{uloc}}^2$  of  $\vec{u} - \vec{w}$  : defining  $B(t) = \|\vec{u}(t, \cdot) - \vec{v}(t, \cdot)\|_{L_{\text{uloc}}^2}^2$  and

$$A(t) = 2(C_1 + C_3(\|\vec{u}\|_{L_t^\infty L_{\text{uloc}}^2} + \|\vec{v}\|_{L_t^\infty L_{\text{uloc}}^2})^4) + 2C_3\|\vec{u}(t, \cdot)\|_\infty^2,$$

we get

$$B(t) \leq \int_0^t A(s)B(s) ds.$$

As  $\lim_{t \rightarrow 0} tA(t) = 0$ , we shall try to apply Lemma 1. Thus, we shall use interpolation estimates to search for a control of  $B(t)$  as  $O(t^{-\delta})$ , in the spirit of Lemma 2:.

Recall that we have introduced the following numbers :

- $p$  such that  $2 < p < \frac{2q}{3}$
- $1 - \theta$  the barycentric coordinate of  $\frac{1}{p}$  in  $[\frac{1}{q}, 1]$  :  $\frac{1}{p} = (1 - \theta) + \theta\frac{1}{q}$
- $-\sigma$  the corresponding point in  $[-1 + \frac{2}{q}, -s]$  :  $-\sigma = (1 - \theta)(-s) + \theta(-1 + \frac{2}{q})$
- $-\gamma$  such that  $-1 + \frac{2}{q} < -\gamma < -\sigma$
- $r$  such that  $p < r < +\infty$
- $1 - \eta$  the barycentric coordinate of  $\frac{1}{p}$  in the segment  $[\frac{1}{r}, \frac{1}{2}]$  :  $\frac{1}{p} = \frac{1-\eta}{2} + \frac{\eta}{r}$
- $-\rho$  the corresponding point in  $[-\gamma, 0]$  :  $-\rho = \eta(-\gamma)$

We have the following embeddings :

- $B_{2,2}^0 \cap \text{BMO}^{-1} \subset \dot{B}_{q,q}^{-1+\frac{2}{q}}$
- $\dot{B}_{q,q}^{-1+\frac{2}{q}} \cap \dot{B}_{M^{1,q},\infty}^{-s} \subset \dot{B}_{M^{p,q},\infty}^{-\sigma}$

- $\dot{B}_{q,q}^{-1+\frac{2}{q}} \cap \dot{B}_{\dot{M}^{p,q},\infty}^{-\sigma} \subset \dot{B}_{\dot{M}^{p,q},1}^{-\gamma} \subset \dot{W}^{-\gamma,\dot{M}^{p,q}}$ .
- $W^{-\gamma,\dot{M}^{p,q}} = [\dot{M}^{2,\frac{2}{p}q}, \dot{W}^{-\rho,\dot{M}^{r,\frac{r}{p}q}}][\eta]$
- $[\dot{M}^{2,\frac{2}{p}q}, \dot{W}^{-\rho,\dot{M}^{r,\frac{r}{p}q}}][\eta] \subset [\dot{M}^{2,\frac{2}{p}q}, \dot{W}^{-\rho,\dot{M}^{r,\frac{r}{p}q}}]_{\eta,\infty} \subset [L^2_{\text{uloc}}, \dot{B}_{\infty,\infty}^{-\rho-\frac{3p}{rq}}]_{\eta,\infty}$

If we take  $r$  very large, we have  $\eta = 1 - \frac{2}{p} + o(1)$  and

$$-\rho - \frac{3p}{rq} = (1 - \frac{2}{p})(-\gamma) + o(1) \in ] -1, 0[.$$

Thus far, we have seen that  $\vec{u}_0 \in [L^2_{\text{uloc}}, \dot{B}_{\infty,\infty}^{-\lambda}]_{\eta,\infty}$  for some  $\eta \in (0, 1)$  and some  $\lambda \in (0, 1)$ . We shall now estimate  $\vec{v} - e^{t\Delta}\vec{u}_0$  when  $\vec{v}$  is a weak local Leray solution. For  $0 < \epsilon < 1$ , split  $\vec{u}_0$  in  $\vec{\alpha}_\epsilon + \vec{\beta}_\epsilon$  with

$$\|\vec{\alpha}_\epsilon\|_{L^2_{\text{uloc}}} \leq C_0\epsilon^\eta \text{ and } \|\vec{\beta}_\epsilon\|_{\dot{B}_{\infty,\infty}^{-\lambda}} \leq C_0\epsilon^{\eta-1}$$

where  $C_0$  does not depend on  $\epsilon$  (but depends on  $\vec{u}_0$ ).

As  $\beta_\epsilon = \vec{u}_0 - \alpha_\epsilon$ , we have  $\|\vec{\beta}_\epsilon\|_{L^2_{\text{uloc}}} \leq \|\vec{u}_0\|_{L^2_{\text{uloc}}} + C_0$  (an estimation which does not depend on  $\epsilon$ ) and we know that we have a (restricted) weak Leray solution of the Navier–Stokes equations  $\vec{U}_\epsilon$  with initial value  $\vec{\beta}_\epsilon$  such that  $\sup_{0 < t < T_0} \|\vec{U}_\epsilon(t, \cdot)\|_{L^2_{\text{uloc}}} \leq C_1$ , where  $T_0$  and  $C_1$  depends only on  $\|\vec{u}_0\|_{L^2_{\text{uloc}}}$  (and not on  $\epsilon$ ).

As  $\beta_\epsilon \in \dot{B}_{\infty,\infty}^{-\lambda}$ , we have as well that  $\|\vec{U}_\epsilon(t, \cdot)\|_\infty \leq C_2 t^{-\frac{\lambda}{2}} \epsilon^{\eta-1}$  on an interval  $(0, T_\epsilon)$  with  $T_\epsilon^{\frac{1-\lambda}{2}} \epsilon^{\eta-1} = C_3$ . It is then easy to check that  $\|\vec{U}_\epsilon - e^{t\Delta}\vec{\beta}_\epsilon\|_{L^2_{\text{loc}}} \leq C_4 t^{(1-\lambda)/2} \epsilon^{\eta-1}$ .

Using our results on comparison of suitable local Leray solutions, we find that we get the following inequality for  $\vec{W}_\epsilon = \vec{v} - \vec{U}_\epsilon$  and  $A_\epsilon(t) = \sup_{k \in \mathbb{Z}^3} \int \varphi_0(x - k) |\vec{W}_\epsilon(t, \cdot)|^2 dx$  :

$$A_\epsilon(t)^2 \leq 2\|\alpha_\epsilon\|_{L^2_{\text{uloc}}}^2 + C_5 \int_0^t A_\epsilon(s)^2 ds + C_6 \int_0^t \|\vec{u}(s, \cdot)\|_\infty^2 A_\epsilon(s)^2 ds. \quad (8)$$

Thus, we get

$$A_\epsilon(t) \leq C_7 \epsilon^{2\eta} + C_8 \int_0^t A_\epsilon(s)^2 ds + C_9 \epsilon^{2(\eta-1)} \int_0^t s^{-\lambda} A_\epsilon(s)^2 ds.$$

so that

$$A_\epsilon(t) \leq C_7 \epsilon^{2\eta} e^{C_8 t} e^{C_9 \epsilon^{2(\eta-1)} \frac{t^{1-\lambda}}{1-\lambda}}.$$

Now, for  $\tau < 1$ , take  $\epsilon = \tau^\mu$  with  $\frac{1-\lambda}{2} + \mu(\eta - 1) > 0$ . We find that, for  $0 < t < \min(T_\epsilon, T_0)$  with

$$T_\epsilon = C_3^{\frac{2}{1-\lambda}} \epsilon^{\frac{2(1-\eta)}{1-\lambda}} = C_3^{\frac{2}{1-\lambda}} \tau^{\mu \frac{2(1-\eta)}{1-\lambda}} \quad \left[ \text{where } \mu \frac{2(1-\eta)}{1-\lambda} < 1 \right],$$

we have the inequality

$$\begin{aligned} \|\vec{v} - e^{t\Delta} \vec{u}_0\|_{L^2_{\text{uloc}}} &\leq \|\alpha_\epsilon\|_2 + \|\vec{W}_\epsilon\|_2 + \|\vec{U}_\epsilon - e^{t\Delta} \vec{\beta}_\epsilon\|_2 \\ &\leq C_{10} \epsilon^\eta (1 + e^{C_{11} \frac{t^{1-\lambda}}{T_\epsilon^{1-\lambda}}}) + C_4 t^{\frac{1-\lambda}{2}} \epsilon^{\eta-1} \end{aligned}$$

If  $\tau$  is small enough, we have  $\tau < T_\epsilon$  and we find

$$\|\vec{v}(\tau, \cdot) - e^{\tau\Delta} \vec{u}_0\|_2 \leq C_0 \tau^{\mu\eta} (1 + e^{\frac{C_1^2}{2(1-\lambda)}}) + C_4 \tau^{\frac{1-\lambda}{2} + \mu(\eta-1)}$$

The theorem is proved.

## References

- [Ba] T. Barker, *Uniqueness Results for Weak Leray–Hopf Solutions of the Navier–Stokes System with Initial Values in Critical Spaces*, J. Math. Fluid Mech. 20 (2018), 133–160.
- [CKN] L. Caffarelli, R. Kohn and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier–Stokes equations*, Comm. Pure Appl. Math. 35 (1982), 771–831.
- [Ka] T. Kato, *Nonlinear evolution equations in Banach spaces*, in : Proceedings of the Symposium on Applied Mathematics 17 (1965), 50–67.
- [KocT] H. Koch and D. Tataru, *Well-posedness for the Navier–Stokes equations*, Adv. Math. 157 (2001), 22–35.
- [KozT] H. Kozono and Y. Taniuchi, *Bilinear estimates in BMO and Navier–Stokes equations*, Math.Z. 157 (2000), 173–194.
- [KoY] H. Kozono and M. Yamazaki, *Semilinear heat equations and the Navier–Stokes equations with distributions in new function spaces as initial data*, Comm. Partial Differential Equations 19 (1994), 959–1014.
- [LR 1] P.G. Lemarié–Rieusset, *Solutions faibles d’énergie infinie pour les équations de Navier–Stokes dans  $\mathbb{R}^3$* , C. R. Acad. Sci. Paris, Serie I. 328 (1999), 1133–1138.

- [LR 2] P.G. Lemarié-Rieusset, *Recent developments in the Navier–Stokes problem*, CRC Press, 2002.
- [LR 3] P.G. Lemarié–Rieusset, *The Navier–Stokes equations in the critical Morrey–Campanato space*, Rev. Mat. Iberoamericana 23 (2007), 897–930.
- [LR 4] P.G. Lemarié–Rieusset, *Multipliers and Morrey spaces*, Potential Anal. 38 (2013), 741–752.
- [LR 5] P.G. Lemarié–Rieusset, *Erratum to “Multipliers and Morrey spaces”*, Potential Anal. 41 (2014), 13591362.
- [LR 6] P.G. Lemarié–Rieusset, *The Navier–Stokes problem in the 21st century*, Chapman & Hall/CRC, (2016).
- [Le] J. Leray, *Essai sur le mouvement d’un fluide visqueux emplissant l’espace*, Acta Math. 63 (1934), 193–248.
- [Pr] G. Prodi, *Un teorema di unicità per le equazioni di Navier–Stokes*, Ann. Mat. Pura Appl. 48 (1959), 173–182.
- [Se] J. Serrin, *The initial value problem for the Navier–Stokes equations*, in : Nonlinear Problems (Proc. Sympos., Madison, Wis., 1962), Univ. of Wisconsin Press, Madison, Wis., 1963.