Non-stability of Paneitz–Branson type equations in arbitrary dimensions

Laurent Bakri, Jean-Baptiste Casteras

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Laurent Bakri *† Jean-Baptiste Casteras ‡§

Abstract

Let (M, g) be a compact riemannian manifold of dimension $n \geq 5$. We consider a Paneitz-Branson type equation with general coefficients

$$\Delta_g^2 u - div_g(A_g du) + hu = |u|^{2^*-2-\varepsilon} u \text{ on } M, \qquad \text{(E)}$$

where A_g is a smooth symmetric $(2,0)$-tensor, $h \in C^\infty(M)$, $2^* = \dfrac{2n}{n-4}$ and ε is a small positive parameter. Assuming that there exists a positive nondegenerate solution of (E) when $\varepsilon = 0$ and under suitable conditions, we construct solutions u_ε of type $(u_0 - BBl_\varepsilon)$ to (E) which blow up at one point of the manifold when ε tends to 0 for all dimensions $n \geq 5$.

Keywords: Paneitz-Branson type equations, blow up solutions, Liapunov-Schmidt reduction procedure.

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1 Introduction and statements of the results

The existence of conformally covariant operators and the study of their associated curvature invariants have attracted a lot of attention these last

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decades. In 1983, Paneitz in [22] introduced a fourth order operator, conformally covariant, on 4-dimensional manifold. A few years later, Branson [2] generalized this operator to all n-dimensional manifolds for $n \geq 5$. In analogy with the conformal laplacian, a curvature called the Q-curvature has been associated to this operator (see [4]). It turns out that this curvature appears in a lot of geometric and physic problems. For instance, it is worth pointing out that, in dimension 4, the integrand of the Chern-Gauss-Bonnet formula for the Euler characteristic is the Q-curvature (up to the conformally invariant Weyl's tensor). The Q-curvature also appears in the study of fully nonlinear equations involving the symmetric functions of the Schouten tensor and in the zeta function determinant. The Q-curvature is also linked to the scattering theory of conformally compact manifolds whose study was initiated by Fefferman and Graham. We refer to the articles of Branson and Gover [3], Chang [5], [6], Chang and Yang [7], and Gursky [13] (and the references therein) for more details and very interesting material on the geometric and physic aspects associated to the notion of Q-curvature.

In what follows we let (M, g) be a compact riemannian manifold of dimension $n \geq 5$. We will be interested in solutions $u \in C^{4,\theta}(M)$, $\theta \in (0,1)$, of the following equation

$$P_g u := \Delta_g^2 u - div_g(A_g du) + hu = |u|^{2^*-2}u, \tag{1.1}$$

where A_g is a smooth symmetric $(2,0)$-tensor, $h \in C^\infty(M)$ and $2^* = \dfrac{2n}{n-4}$. Following the terminology introduced in [9], the operator P_g has been referred to as a Paneitz-Branson type operator with general coefficients. When A_g is given by

$$A_g = A_{paneitz} := \frac{(n-2)^2+4}{2(n-1)(n-2)} R_g g - \frac{4}{n-2} Ric_g, \tag{1.2}$$

where R_g (resp. Ric_g) stands for the scalar curvature (resp. Ricci curvature) with respect to the metric g, and $h = \dfrac{n-4}{2} Q_g$ where Q_g is the Q-curvature with respect to the metric g which is defined by

$$Q_g = \frac{1}{2(n-1)} \Delta_g R_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R_g^2 - \frac{2}{(n-2)^2} |Ric_g|_g^2,$$

then P_g is the so-called Paneitz-Branson operator and equation (1.1) is referred to as the Paneitz-Branson equation. It is well known that the Paneitz operator is conformally invariant, i.e. if $\tilde{g} = \varphi^{\frac{4}{n-4}} g$ then, for all $u \in C^\infty(M)$, we have

$$P_g^n(u\varphi) = \varphi^{\frac{n+4}{n-4}} P_{\tilde{g}}^n(u).$$

We also point out that if (M,g) is Einstein ($Ric_g = \lambda g$, $\lambda \in \mathbb{R}$), then the Paneitz-Branson operator takes the form

$$P_g u = \Delta_g^2 u + b\Delta_g u + cu, \tag{1.3}$$

where $b = \dfrac{n^2-2n-4}{2(n-1)}\lambda$ and $c = \dfrac{n(n-4)(n^2-4)}{16(n-1)^2}\lambda$. More generally, when b and c are two real numbers, the operator P_g defined in (1.3) is referred to as a Paneitz-Branson type operator with constant coefficients. Existence, compactness and stability of solutions to (1.1) when P_g is a Paneitz-Branson type operator with constant coefficients, have been widely investigated this last decade (see for example [11, 14, 15, 24, 27] and the references therein). However, less is known for solutions of (1.1) in the case where P_g is a Paneitz-Branson type operator with general coefficients. Esposito and Robert [10] proved the existence of a non trivial solution to (1.1) under the hypothesis that $n \geq 8$ and $\min_M Tr_g(A_g - A_{paneitz}) < 0$. In [26], Sandeep studied the stability of equation (1.1) in the following sense : he considered sequences of positive solutions $(u_\alpha)_\alpha$ of

$$\Delta_g^2 u_\alpha - div_g(A_\alpha du_\alpha) + a_\alpha u_\alpha = u_\alpha^{2^*-1},\ u_\alpha \in C^{4,\theta},$$

where A_α are smooth $(2,0)$ symmetric tensors and a_α are smooth functions. Sandeep proved that if A_α converges in $C^1(M)$ to a smooth symmetric tensor A_g, a_α converges in $C^0(M)$ to a smooth positive function a and u_α converges weakly in $H^2(M)$ to a function u_0, then u_0 is nontrivial provided that $A_g - A_{paneitz}$ is either positive or negative definite (generalizing a result of [16]). Recently, Pistoia and Vaira [23] studied the stability of (1.1) when it is the Paneitz-Branson equation, namely they considered the following equation

$$\Delta_g^2 u - div_g((A_{paneitz} + \varepsilon B)du) + Q_g u = |u|^{2^*-2}u, \tag{1.4}$$

where ε is a small positive parameter and B is a smooth symmetric $(2,0)$ tensor. They proved that if (M,g) is not conformally flat, $n \geq 9$ and there exists $\xi_0 \in M$ a C^1 stable critical point (see below for the definition) of the function $\xi \to \dfrac{Tr_g B(\xi)}{|Weyl_g(\xi)|_g}$, such that $Tr_g B(\xi_0) > 0$, then equation (1.4) is not stable, i.e. there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, equation (1.4) admits a solution u_ε such that $u_\varepsilon(\xi_0) \underset{\varepsilon \to 0^+}{\longrightarrow} +\infty$.

The aim of this paper is to investigate the stability in the sense of Deng-Pistoia of (1.1). We say that (1.1) is stable if, for any sequences of real positive numbers $(\varepsilon_\alpha)_\alpha$ such that $\varepsilon_\alpha \underset{\alpha \to \infty}{\longrightarrow} 0$ and for any sequences of solutions $(u_\alpha)_\alpha \in C^{4,\theta}(M)$, $\theta \in (0,1)$, of

$$\Delta_g^2 u_\alpha - div_g(A_g du_\alpha) + hu_\alpha = |u_\alpha|^{2^*-2-\varepsilon_\alpha}u_\alpha, \tag{1.5}$$

bounded in $H^2(M)$, then up to a subsequence, u_α converges in $C^4(M)$ to some smooth function u solution of (1.1). Deng and Pistoia [8] proved that (1.1) is not stable if

a. $n \geq 7$, $Tr_g(A_g - A_{paneitz})$ is not constant and $\min_M Tr_g(A_g - A_{paneitz}) > 0$,

b. or $n \geq 8$ and $\xi_0 \in M$ a C^1 stable critical point of $Tr_g(A_g - A_{paneitz})$ such that $Tr_g(A_g - A_{paneitz})(\xi_0) > 0$.

Our main result shows that under suitable assumptions, equation (1.1) is not stable for any $n \geq 5$. In fact, inspired by the recent result of Robert and Vétois [25] on scalar curvature type equations, we investigate the existence of families $(u_\varepsilon)_\varepsilon \in C^{4,\theta}(M)$ of blow-up solutions to (1.5) of type $(u_0 - BBl_\varepsilon)$. Following the terminology of Robert and Vétois, we say that a blow-up sequence $(u_\varepsilon)_\varepsilon$ is of type $(u_0 - BBl_\varepsilon)$ if there exists $u_0 \in C^{4,\theta}(M)$ and a bubble $BBl_\varepsilon(x) = [n(n-4)(n^2-4)]^{\frac{n-4}{8}} \left(\frac{\mu_\varepsilon}{\mu_\varepsilon + d_g(x, x_\varepsilon)^2} \right)^{\frac{n-4}{2}}$, where $x, x_\varepsilon \in M$ and $\mu_\varepsilon \in \mathbb{R}^+$ is such that $\mu_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} 0$, such that

$$u_\varepsilon = u_0 - BBl_\varepsilon + o(1),$$

where $o(1) \underset{\varepsilon \to 0}{\longrightarrow} 0$. Before stating more precisely the results, we would like to recall that a solution of (1.5) is called nondegenerate if the kernel of the linearization of the equation is trivial (see (2.3)). Let $\phi \in C^1(M)$, we also recall that a critical point ξ_0 of ϕ is said C^1 stable if there exists an open neighborhood Ω of ξ_0 such that, for any point $\xi \in \bar{\Omega}$, there holds $\nabla_g \phi(\xi) = 0$ if and only if $\xi = \xi_0$ and such that the Brower degree $deg(\nabla_g \phi, \Omega, 0) \neq 0$. We obtain :

Theorem 1.1. *Let (M, g) be a compact riemannian manifold of dimension n, A_g and h be such that P_g is coercive. Let $u_0 \in C^{4,\theta}$, $\theta \in (0,1)$, be a positive nondegenerate solution of (1.1). Assume in addition that one of the following condition holds:*

a. $5 \leq n < 7$,

b. $8 \leq n \leq 13$ and there exists $\xi_0 \in M$ a C^1 stable critical point of

$$\begin{aligned} \varphi(\xi) = \frac{(n-1)}{(n-6)(n^2-4)} (Tr_g(A_g - A_{paneitz}))(\xi) \\ + \frac{2^n u_0(\xi) \omega_{n-1}}{(n+2)(n(n-4)(n^2-4))^{\frac{n-4}{8}} \omega_n} 1_{n=8}, \ \xi \in M, \end{aligned} \tag{1.6}$$

such that $\varphi(\xi_0) > 0$,

c. $n > 13$ *and* $\min_M Tr_g(A_g - A_{paneitz}) > 0$,

then, for any $\varepsilon > 0$, *there exists a solution* u_ε *of type* $u_0 - BBl_\varepsilon$ *to* (1.5). *In particular,* (1.5) *is not stable.*

Let us notice that in the geometric case i.e. when $A_g = A_{paneitz}$, the previous theorem only applies if $5 \leq n \leq 8$. However, with a small modification of the proof, we can construct a solution of type $u_0 - BBl_\varepsilon$ to (1.5) when $5 \leq n \leq 11$ and $A_g = A_{paneitz}$. More precisely, we prove the following result :

Theorem 1.2. *Let* (M, g) *be a compact riemannian manifold of dimension* n, A_g *and* h *be such that* P_g *is coercive. Let* $u_0 \in C^{4,\theta}$, $\theta \in (0, 1)$, *be a positive nondegenerate solution of* (1.1). *Assume that* $A_g = A_{paneitz}$. *Then, for any* $5 \leq n \leq 11$ *and any* $\varepsilon > 0$, *there exists a solution* u_ε *of type* $u_0 - BBl_\varepsilon$ *to* (1.5). *In particular,* (1.5) *is not stable.*

The proof of the theorems relies on a well known Lyapunov-Schmidt reduction procedure which permits to reduce the problem to a finite dimensional one for which we defined a reduced energy. The solutions to (1.5) will then be obtained as critical points of this reduced energy. We refer to [1] and the references therein for more information on the Lyapunov-Schmidt reduction procedure. We would like to emphasize that the proof of Theorem 1.1 is inspired by the previous work of Robert and Vétois [25]. Thus we will keep their notations. We also want to point out that we use without proof computations done in Deng and Pistoia [8] (for more details on these computations, see their paper). The plan of the paper is the following : in section 2 we introduce notations and perform the finite dimensional reduction. In section 3 we study the reduced problem and prove Theorem 1.1. The error estimate and the C^1 uniform asymptotic expansion of the reduced energy are done in the appendix.

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2 Finite dimensional reduction.

Let $(\xi_\alpha)_\alpha$ be a sequence of points of M. In all the following, we will suppose up to extracting a subsequence that, for α large enough, all the points ξ_α belong to a small open set Ω of M in which there exists a smooth orthogonal frame. Thus, we will identify the tangent spaces $T_\xi M$ with $\mathbb{R}^n$ for all $\xi \in \Omega$. We recall that we suppose that P_g is coercive.

In all the following, we will denote by $\langle .,. \rangle_{P_g}$, the scalar product, for $u, v \in H^2(M)$,

$$\langle u, v \rangle_{P_g} = \int_M \Delta_g u \Delta_g v dV + \int_M A_g(\nabla_g u, \nabla_g v) dV + \int_M huv dV,$$

where here and in the following dV stands for the volume element with respect to the metric g, and $\|.\|_{P_g}$, for the associated norm which is then equivalent to the standard norm of $H^2(M)$. We denote by $i^* : L^{\frac{2n}{n+4}}(M) \to H^2(M)$ the adjoint operator of the embedding $i : H^2(M) \to L^{\frac{2n}{n-4}}(M)$, i.e. for all $\varphi \in L^{\frac{2n}{n+4}}(M)$, the function $u = i^*(\varphi) \in H^2(M)$ is the unique solution of $\Delta_g^2 u - div_g(A_g du) + hu = \varphi$. Using this notation, we see that equation (1.5) can be rewritten as, for $u \in H^2(M)$,

$$u = i^*(f_\varepsilon(u)),$$

where $f_\varepsilon(u) = |u|^{2^*-2-\varepsilon}u$. Before proceeding we recall some basic facts. It is well known (see [18]) that all solutions $u \in H^2(\mathbb{R}^n)$ of the equation

$$\Delta_{eucl}^2 u = u^{2^*-1} = u^{\frac{n+4}{n-4}} \ in \ \mathbb{R}^n$$

are given by

$$U_{\delta,y}(x) = \delta^{\frac{4-n}{2}} U(\frac{x-y}{\delta}), \ \delta > 0, \ y \in \mathbb{R}^n$$

where

$$U(x) = [n(n-4)(n^2-4)]^{\frac{n-4}{8}} \left(\frac{1}{1+|x|^2}\right)^{\frac{n-4}{2}} = \alpha_n \left(\frac{1}{1+|x|^2}\right)^{\frac{n-4}{2}}. \quad (2.1)$$

It is also well known (see [19]) that all solutions $v \in H^2(\mathbb{R}^n)$ of

$$\Delta_{eucl}^2 v = (2^*-1)U^{2^*-2}v$$

are linear combinations of

$$V_0(x) = \alpha_n \frac{n-4}{2} \frac{|x|^2-1}{(1+|x|^2)^{\frac{n-2}{2}}}$$

and

$$V_i(x) = \alpha_n (n-4) \frac{x_i}{(1+|x|^2)^{\frac{n-2}{2}}}, \ i = 1, \dots, n.$$

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth cutoff function such that $0 \leq \chi \leq 1$, $\chi(x) = 1$ if $x \in [-\frac{r_0}{2}, \frac{r_0}{2}]$ and $\chi(x) = 0$ if $x \in \mathbb{R} \backslash (-r_0, r_0)$. We define, for any real δ strictly positive, $\xi \in M$ and $x \in M$,

$$W_{\delta,\xi}(x) = \chi(d_g(x,\xi)) \delta^{\frac{4-n}{2}} U(\delta^{-1} \exp_\xi^{-1}(x)),$$

where $d_g(x,\xi)$ stands for the distance from x to ξ with respect to the metric g and $\exp_\xi$ is the exponential map with respect to the metric g. We also define, for any real δ strictly positive, $\xi \in M$ and $x \in M$,

$$Z_{\delta,\xi}(x) = \chi(d_g(x,\xi))\delta^{\frac{n-4}{2}} \frac{d(x,\xi)^2 - \delta^2}{(\delta^2 + d(x,\xi)^2)^{\frac{n-2}{2}}},$$

and, for $\omega \in T_\xi M$,

$$Z_{\delta,\xi,\omega}(x) = \chi(d_g(x,\xi))\delta^{\frac{n-2}{2}} \frac{\left\langle \exp_\xi^{-1} x, \omega \right\rangle_g}{(\delta^2 + d(x,\xi)^2)^{\frac{n-2}{2}}}.$$

We denote by $\Pi_{\delta,\xi}$ respectively $\Pi^\perp_{\delta,\xi}$ the projection of $H^2(M)$ onto

$$K_{\delta,\xi} = \text{span}\left\{Z_{\delta,\xi}, (Z_{\delta,\xi,e_i})_{i=1..n}\right\}$$

respectively

$$K^\perp_{\delta,\xi} = \left\{\phi \in H^2(M) / \left\langle \phi, Z_{\delta,\xi} \right\rangle_{P_g} = 0 \text{ and } \left\langle \phi, Z_{\delta,\xi,\omega} \right\rangle_{P_g} = 0,\ \forall \omega \in T_\xi M\right\}. \tag{2.2}$$

We recall that a solution u_0 of (1.5) is nondegenerate if the linearization of the equation has trivial kernel, that is

$$K = \left\{\varphi \in C^{4,\theta}(M) / P_g\varphi = (2^* - 1)|u_0|^{2^*-2}\varphi\right\} = \{0\}. \tag{2.3}$$

We are looking for solution u to (1.5) of the form

$$u = u_0 - W_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon} + \phi_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon},$$

where u_0 is a nondegenerate positive solution of (1.5), $\phi_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon} \in K^\perp_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}$ and

$$\delta_\varepsilon(t_\varepsilon) = \begin{cases} \sqrt{t_\varepsilon \varepsilon} \text{ if } n \geq 8 \\ (t_\varepsilon \varepsilon)^{\frac{2}{n-4}} \text{ if } 5 \leq n \leq 8 \end{cases},\ t_\varepsilon > 0. \tag{2.4}$$

It is easy to see that equation (1.5) is equivalent to the following system

$$\Pi_{\delta_\varepsilon(t),\xi}(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi} - i^*(f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}))) = 0, \tag{2.5}$$

and

$$\Pi^\perp_{\delta_\varepsilon(t),\xi}(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi} - i^*(f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}))) = 0. \tag{2.6}$$

We begin by solving (2.6).

Proposition 2.1. *Let $u_0 \in C^{4,\theta}(M)$ be a nondegenerate positive solution of (1.5). Given two real numbers $a < b$, there exists a positive constant $C_{a,b}$ such that for ε small, for any $t \in [a,b]$ and any $\xi \in M$, there exists a unique function $\phi_{\delta_\varepsilon(t),\xi} \in K^\perp_{\delta_\varepsilon(t),\xi}$ which solves equation (2.6) and satisfies*

$$\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_{P_g} \leq C_{a,b}\varepsilon|\ln\varepsilon|. \tag{2.7}$$

Moreover, $\phi_{\delta_\varepsilon(t),\xi}$ is continuously differentiable with respect to t and ξ.

In order to prove the previous proposition, we set, for ε small, for any positive real number δ and $\xi \in M$, the map $L_{\varepsilon,\delta,\xi} : K^\perp_{\delta,\varepsilon} \to K^\perp_{\delta,\varepsilon}$ defined by, for $\phi \in K^\perp_{\delta,\varepsilon}$,

$$L_{\varepsilon,\delta,\xi}(\phi) = \Pi^\perp_{\delta,\xi}(\phi - i^*(f'_\varepsilon(u_0 - W_{\delta,\xi})\phi)).$$

We will first prove that this map is inversible for δ and ε small.

Lemma 2.1. *There exists a positive constant $C_{a,b}$ such that for ε small, for any $t \in [a,b]$, any $\xi \in M$ and any $\phi \in K^\perp_{\delta,\varepsilon}$, we have*

$$\left\|L_{\varepsilon_\alpha,\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}(\phi)\right\|_{P_g} \geq C_{a,b}\left\|\phi\right\|_{P_g}.$$

Proof. Assume by contradiction that there exist two sequences of positive real numbers $(\varepsilon_\alpha)_\alpha$ and $(t_\alpha)_\alpha$ such that $\varepsilon_\alpha \underset{\alpha\to+\infty}{\longrightarrow} 0$ and $a \leq t_\alpha \leq b$, a sequence of points $(\xi_\alpha))_\alpha$ of M and a sequence of functions $(\phi_\alpha)_\alpha$ such that

$$\phi_\alpha \in K^\perp_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha},\ \left\|\phi_\alpha\right\|_{P_g} = 1 \ \text{ and } \ \left\|L_{\varepsilon_\alpha,\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}(\phi_\alpha)\right\|_{P_g} \underset{\alpha\to\infty}{\longrightarrow} 0. \tag{2.8}$$

To simplify notations, we set $L_\alpha = L_{\varepsilon_\alpha,\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}$, $W_\alpha = W_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}$, $Z_{0,\alpha} = Z_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}$ and $Z_{i,\alpha} = Z_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha,e_i}$ for $i = 1,\dots,n$ where e_i is the i-th vector in the canonical basis of $\mathbb{R}^n$. By definition of L_α, there exist real numbers $\lambda_{i,\alpha}$, $i = 0,\dots,n$ such that

$$\phi_\alpha - i^*(f'_{\varepsilon_\alpha}(u_0 - W_\alpha)\phi_\alpha) - L_\alpha(\phi_\alpha) = \sum_{i=0}^n \lambda_{i,\alpha}Z_{i,\alpha}. \tag{2.9}$$

Standard computations give

$$\langle Z_{i,\alpha}, Z_{j,\alpha}\rangle_{P_g} \underset{\alpha\to\infty}{\longrightarrow} \left\|\Delta_{eucl}V_i\right\|^2_{L^2(\mathbb{R}^n)}\delta_{ij}, \tag{2.10}$$

where δ_{ij} stands for the Kronecker symbol. Therefore, taking the scalar product of (2.9) with $Z_{i,\alpha}$, using the previous limit and recalling that ϕ_α and $L_\alpha(\phi_\alpha)$ belong to $K^\perp_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}$, we deduce that

$$\int_M f'_{\varepsilon_\alpha}(u_0 - W_\alpha)\phi_\alpha Z_{i,\alpha}dV = -\lambda_{i,\alpha}\left\|\Delta_{eucl}V_i\right\|^2_{L^2(\mathbb{R}^n)} + \left(\sum_{i=0}^n |\lambda_{i,\alpha}|\right)o(1), \tag{2.11}$$

where, here and in the following, $o(1) \underset{\alpha\to+\infty}{\longrightarrow} 0$. It is easy to see using the definition of W_α and $Z_{i,\alpha}$ and a change of variables that, for α large enough,

$$\begin{aligned}
&\int_M f'_{\varepsilon_\alpha}(u_0 - W_\alpha)\phi_\alpha Z_{i,\alpha} dV \\
= &\int_M f'_{\varepsilon_\alpha}(W_\alpha)\phi_\alpha Z_{i,\alpha} dV + o(1) \qquad (2.12)\\
= &\ (2^* - 1 - \varepsilon_\alpha)\delta_{\varepsilon_\alpha}(t_\alpha)^{\varepsilon_\alpha \frac{n+4}{2}} \int_{\mathbb{R}^n} \chi_\alpha^{2^*-2-\varepsilon_\alpha} U^{2^*-2-\varepsilon_\alpha} V_i \tilde{\phi}_\alpha dV_{\tilde{g}_\alpha} + o(1),
\end{aligned}$$

where $\chi_\alpha = \chi(\delta_{\varepsilon_\alpha}(t_\alpha)|x|)$, $\tilde{\phi}_\alpha(x) = \delta_{\varepsilon_\alpha}(t_\alpha)^{\frac{n-4}{2}} \chi_\alpha \phi_\alpha(\exp_{\xi_\alpha}(\delta_{\varepsilon_\alpha}(t_\alpha)x))$ and $\tilde{g}_\alpha(x) = \exp^*_{\xi_\alpha} g(\delta_{\varepsilon_\alpha}(t_\alpha)x)$. Since $(\phi_\alpha)_\alpha$ is bounded in $H^2(M)$, passing to a subsequence if necessary, we can assume that $(\tilde{\phi}_\alpha)_\alpha$ converges weakly to a function $\tilde{\phi} \in H^2(\mathbb{R}^n)$. Letting $\alpha \to +\infty$ in (2.12), we deduce that

$$\int_M f'_{\varepsilon_\alpha}(u_0 - W_\alpha)\phi_\alpha Z_{i,\alpha} dV \underset{\alpha\to\infty}{\longrightarrow} (2^* - 1)\int_{\mathbb{R}^n} U^{2^*-2} V_i \tilde{\phi} dV_{g_{eucl}} = 0, \qquad (2.13)$$

where we used that V_i is solution of $\Delta^2_{eucl} V_i = \dfrac{n+4}{n-4} U^{2^*-2} V_i$ in $\mathbb{R}^n$ and $\phi_\alpha \in K^\perp_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}$ to obtain the last equality. Therefore, from (2.11) and (2.13), we have

$$\lambda_{i,\alpha} = o(1) + o(\sum_{i=0}^{n} |\lambda_{i,\alpha}|).$$

From (2.9), this implies

$$\phi_\alpha - i^*(f'_{\varepsilon_\alpha}(u_0 - W_\alpha)\phi_\alpha) - L_\alpha(\phi_\alpha) \underset{\alpha\to\infty}{\longrightarrow} 0.$$

Since by assumption $\left\|L_{\varepsilon_\alpha,\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}(\phi_\alpha)\right\|_{P_g} \underset{\alpha\to\infty}{\longrightarrow} 0$, we finally obtain that

$$\left\|\phi_\alpha - i^*(f'_{\varepsilon_\alpha}(u_0 - W_\alpha)\phi_\alpha)\right\|_{P_g} \underset{\alpha\to\infty}{\longrightarrow} 0. \qquad (2.14)$$

Since $(\phi_\alpha)_\alpha$ is bounded in $H^2(M)$, up to taking a subsequence, we can assume that ϕ_α converges weakly in $H^2(M)$ to a function $\phi \in H^2(M)$. Then, using (2.14), we get, for any $\varphi \in H^2(M)$,

$$\begin{aligned}
\left|\langle\varphi, \phi_\alpha\rangle_{P_g} - \int_M f'_{\varepsilon_\alpha}(u_0 - W_\alpha)\varphi\phi_\alpha dV\right| &= \left|\left\langle\varphi, \phi_\alpha - i^*(f'_{\varepsilon_\alpha}(u_0 - W_\alpha)\phi_\alpha)\right\rangle_{P_g}\right| \\
&\le \|\varphi\|_{P_g} \left\|\phi_\alpha - i^*(f'_{\varepsilon_\alpha}(u_0 - W_\alpha)\phi_\alpha)\right\|_{P_g} \\
&= o(\|\varphi\|_{P_g}). \qquad (2.15)
\end{aligned}$$

We deduce from this that ϕ is a weak solution of $P_g\phi = (2^*-1)u_0^{2^*-2}\phi$. Since u_0 is a nondegenerate solution of (1.5), we obtain that $\phi = 0$. Therefore, $\phi_\alpha \underset{\alpha\to\infty}{\rightharpoonup} 0$ weakly in $H^2(M)$. Now we will show that $\tilde{\phi}_\alpha \underset{\alpha\to\infty}{\rightharpoonup} 0$ weakly in $H^2(\mathbb{R}^n)$. Let $\tilde{\varphi}$ be a smooth function with compact support in $\mathbb{R}^n$, we will use (2.15) with, for $x \in M$,

$$\varphi(x) = \chi(d_{g_{\xi_\alpha}}(x,\xi_\alpha))\delta_{\varepsilon_\alpha}(t_\alpha)^{\frac{4-n}{2}}\tilde{\varphi}(\delta_{\varepsilon_\alpha}(t_\alpha)^{-1}\exp_{\xi_\alpha}^{-1}(x)).$$

Thus, applying (2.15) to the previous φ and using a change of variable, we have,

$$\begin{aligned} & \int_{\mathbb{R}^n} \Delta_{\tilde{g}_\alpha}\tilde{\phi}_\alpha\Delta_{\tilde{g}_\alpha}\tilde{\varphi}dV_{\tilde{g}_\alpha} + \delta_{\varepsilon_\alpha}(t_\alpha)^2\int_{\mathbb{R}^n} A_{\tilde{g}_\alpha}(\nabla_{\tilde{g}_\alpha}\tilde{\phi}_\alpha,\nabla_{\tilde{g}_\alpha}\tilde{\varphi})dV_{\tilde{g}_\alpha} \\ + \;& \delta_{\varepsilon_\alpha}(t_\alpha)^4\int_{\mathbb{R}^n} h(\exp_{\xi_\alpha}(\delta_{\varepsilon_\alpha}(t_\alpha)x))\tilde{\phi}_\alpha\tilde{\varphi}dV_{\tilde{g}_\alpha} \qquad (2.16) \\ = \;& \delta_{\varepsilon_\alpha}(t_\alpha)^4\int_{\mathbb{R}^n} f'_{\varepsilon_\alpha}(u_{0,\alpha} - W_\alpha(\exp_{\xi_\alpha}(\delta_{\varepsilon_\alpha}(t_\alpha)x)))\tilde{\phi}_\alpha\varphi dV_{\tilde{g}_\alpha} + o(1), \end{aligned}$$

where $u_{0,\alpha}(.) = u_0(\exp_{\xi_\alpha}(\delta_{\varepsilon_\alpha}(t_\alpha).))$. Now it is easy to see that, letting $\alpha \to \infty$ in (2.16),

$$\int_{\mathbb{R}^n} \Delta_{eucl}\tilde{\phi}\Delta_{eucl}\tilde{\varphi}dV_{g_{eucl}} = (2^*-1)\int_{\mathbb{R}^n} U^{2^*-2}\tilde{\phi}\tilde{\varphi}dV_{g_{eucl}}.$$

Thus $\tilde{\phi}$ is a weak solution of $\Delta_{eucl}^2\tilde{\phi} = \dfrac{n+4}{n-4}U^{2^*-2}\tilde{\phi}$. So, from [19], we know that there exists $\lambda_i \in \mathbb{R}$, $i = 0,\dots,n$, such that $\tilde{\phi} = \sum_{i=0}^n \lambda_i V_i$. Since $\phi_\alpha \in K^\perp_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}$, using the same argument as in (2.13), we deduce that $\tilde{\phi} \equiv 0$. Using one more time (2.15) with $\varphi = \phi_\alpha$, a change of variables and since $\phi_\alpha \underset{\alpha\to\infty}{\rightharpoonup} 0$ weakly in $H^2(M)$ and $\tilde{\phi}_\alpha \underset{\alpha\to\infty}{\rightharpoonup} 0$ weakly in $H^2(\mathbb{R}^n)$, we get

$$\begin{aligned} \|\phi_\alpha\|_{P_g}^2 &= (2^*-1-\varepsilon_\alpha)\int_M |u_0 - W_\alpha|^{2^*-2-\varepsilon_\alpha}\phi_\alpha^2 dV + o(1) \\ &\le C\int_M \phi_\alpha^2 dV + C\int_M |W_\alpha|^{2^*-2-\varepsilon_\alpha}\phi_\alpha^2 dV + o(1) \\ &\le C\int_M \phi_\alpha^2 dV + C\int_M |U|^{2^*-2-\varepsilon_\alpha}\tilde{\phi}_\alpha^2 dV_{\tilde{g}_\alpha} + o(1) \underset{\alpha\to\infty}{\longrightarrow} 0. \end{aligned}$$

This yields to a contradiction with (2.8).

□

Proof of Proposition 2.1. It is easy to see that equation (2.6) is equivalent to

$$L_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi) = N_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi) + R_{\varepsilon,\delta_\varepsilon(t),\xi},$$

where

$$N_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi) = \Pi^{\perp}_{\delta_\varepsilon(t),\xi}(i^*(f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi)) - f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) - f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})\phi),$$

and

$$R_{\varepsilon,\delta_\varepsilon(t),\xi} = \Pi^{\perp}_{\delta_\varepsilon(t),\xi}(i^*_\varepsilon(f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})) - u_0 + W_{\delta_\varepsilon(t),\xi}).$$

Let $T_{\varepsilon,\delta_\varepsilon(t),\xi} : K^{\perp}_{\delta_{\varepsilon\alpha}(t_\alpha),\xi_\alpha} \to K^{\perp}_{\delta_{\varepsilon\alpha}(t_\alpha),\xi_\alpha}$ be the application defined by

$$T_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi) = L^{-1}_{\varepsilon,\delta_\varepsilon(t),\xi}(N_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi) + R_{\varepsilon,\delta_\varepsilon(t),\xi}),$$

and

$$B_{\varepsilon,\delta_\varepsilon(t),\xi}(\gamma) = \left\{\phi \in K^{\perp}_{\delta_{\varepsilon\alpha}(t_\alpha),\xi_\alpha} |\, \|\phi\|_{P_g} \leq \gamma \left\|R_{\varepsilon,\delta_\varepsilon(t),\xi}\right\|_{P_g}\right\},$$

where γ is a positive constant which will be chosen later in order to apply the fixed point theorem for $T_{\varepsilon,\delta_\varepsilon(t),\xi}$ restricted to $B_{\varepsilon,\delta_\varepsilon(t),\xi}(\gamma)$. Since, from Lemma 2.1, the map $L_{\varepsilon,\delta_\varepsilon(t),\xi}$ is inversible and has a continuous inverse, we have

$$\left\|T_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi)\right\|_{P_g} \leq C(\left\|N_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi)\right\|_{P_g} + \left\|R_{\varepsilon,\delta_\varepsilon(t),\xi}\right\|_{P_g}), \tag{2.17}$$

and

$$\left\|T_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi_1) - T_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi_2)\right\|_{P_g} \leq C \left\|N_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi_1) - N_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi_2)\right\|_{P_g}. \tag{2.18}$$

Since $i^* : L^{\frac{2n}{n+4}}(M) \to H^2(M)$ is continuous, we get

$$\left\|N_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi)\right\|_{P_g} \leq C \left\|f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi)) - f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) - f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})\phi\right\|_{L^{\frac{2n}{n+4}}},$$

where, here and in the following, $\|.\|_{L^p} = \|.\|_{L^p(M)}$, $p \in \mathbb{R}^+$. Using the mean value theorem, Hölder and Sobolev inequalities, we have, for $\tau \in (0,1)$,

$$\begin{aligned}\left\|N_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi)\right\|_{P_g} &\leq C \left\|\left[f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \tau\phi) - f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})\right](\phi)\right\|_{L^{\frac{2n}{n+4}}} \\ &\leq C \left\|f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \tau\phi) - f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})\right\|_{L^{\frac{n}{4}}} \|\phi\|_{L^{2^*}}.\end{aligned}$$

We will use here and through the paper the following easy consequences of Taylor's expansion [17, lemma 2.2], for all $\alpha > 0$, $\beta \in \mathbb{R}$,

$$||\alpha + \beta|^\theta - \alpha^\theta| \leq \begin{cases} C_\theta \min\left\{|\beta|^\theta, \alpha^{\theta-1}|\beta|\right\} \text{ if } 0 < \theta \leq 1, \\ C_\theta(\alpha^{\theta-1}|\beta| + |\beta|^\theta) \text{ if } \theta > 1, \end{cases} \tag{2.19}$$

and

$$||\alpha+\beta|^\theta(\alpha+\beta)-\alpha^{\theta+1}-(1+\theta)\alpha^\theta\beta| \le \begin{cases} C_\theta \min\left\{|\beta|^{\theta+1}, \alpha^{\theta-1}|\beta|^2\right\} \text{ if } \theta<1, \\ C_\theta \max\{|\beta|^{\theta+1}, \alpha^{\theta-1}|\beta|^2\} \text{ if } \theta \ge 1. \end{cases} \tag{2.20}$$

Thus, we obtain

$$\left\|N_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi)\right\|_{P_g} \le \begin{cases} C\left\|\phi\right\|_{P_g}^{2^*-1-\varepsilon} \text{ if } n \ge 12, \\ C(\|u_0 - W\|_{L^{2^*}}^{2^*-3-\varepsilon}\left\|\phi\right\|_{P_g}^2 + \left\|\phi\right\|_{P_g}^{2^*-1-\varepsilon}) \text{ if } 5 \le n < 12. \end{cases} \tag{2.21}$$

From the mean value theorem, Hölder and Sobolev inequalities, and (2.19), we also get, for some $\tau \in (0,1)$,

$$\begin{aligned} &\left\|N_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi_1) - N_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi_2)\right\|_{P_g} \\ &\quad \le C\left\|f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_1) - f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_2)\right. \\ &\qquad\qquad \left. - f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})(\phi_1 - \phi_2)\right\|_{L^{\frac{2n}{n+4}}} \\ &\quad \le C\left\|\left[f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \tau\phi_2 + (1-\tau)\phi_1)\right.\right. \\ &\qquad\qquad \left.\left. - f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})\right](\phi_1 - \phi_2)\right\|_{L^{\frac{2n}{n+4}}} \\ &\quad \le C\left\|f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \tau\phi_2 + (1-\tau)\phi_1) - f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})\right\|_{L^{\frac{n}{4}}} \\ &\qquad\qquad \times \left\|\phi_1 - \phi_2\right\|_{L^{2^*}} \\ &\quad \le \begin{cases} C(\left\|\phi_1\right\|_{P_g}^{2^*-2-\varepsilon} + \left\|\phi_2\right\|_{P_g}^{2^*-2-\varepsilon})\left\|\phi_1 - \phi_2\right\|_{P_g} & \text{if } n \ge 12, \\ C(\left\|u_0 - W_{\delta_\varepsilon(t),\xi}\right\|_{L^{2^*}(M)} + \left\|\phi_1\right\|_{P_g} + \left\|\phi_2\right\|_{P_g})^{2^*-3-\varepsilon} \\ \qquad \times(\left\|\phi_1\right\|_{P_g} + \left\|\phi_2\right\|_{P_g})\left\|\phi_1 - \phi_2\right\|_{P_g} & \text{if } 5 \le n < 12 \end{cases} \end{aligned} \tag{2.22}$$

Since $\left\|u_0 - W_{\delta_\varepsilon(t),\xi}\right\|_{L^{2^*}} = O(1)$, it follows from (2.17), (2.18), (2.21) and (2.22), that, for all $\phi,\ \phi_1,\ \phi_2 \in B_{\varepsilon,\delta_\varepsilon(t),\xi}(\gamma)$,

$$\left\|T_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi)\right\|_{P_g} \le \begin{cases} C(\gamma^{2^*-1-\varepsilon}\left\|R_{\varepsilon,\delta_\varepsilon(t),\xi}\right\|_{P_g}^{2^*-1-\varepsilon} + \left\|R_{\varepsilon,\delta_\varepsilon(t),\xi}\right\|_{P_g}) \text{ if } n \ge 12 \\ C(\gamma^2\left\|R_{\varepsilon,\delta_\varepsilon(t),\xi}\right\|_{P_g}^2 + \gamma^{2^*-1-\varepsilon}\left\|R_{\varepsilon,\delta_\varepsilon(t),\xi}\right\|_{P_g}^{2^*-1-\varepsilon} \\ \qquad + \left\|R_{\varepsilon,\delta_\varepsilon(t),\xi}\right\|_{P_g}) \qquad \text{if } 5 \le n < 12 \end{cases}$$

and

$$\left\|T_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi_1) - T_{\varepsilon,\delta_\varepsilon(t),\xi}(\phi_2)\right\|_{P_g} \le C\gamma^{2^*-2-\varepsilon}\left\|R_{\varepsilon,\delta_\varepsilon(t),\xi}\right\|_{P_g}^{2^*-2-\varepsilon}\left\|\phi_1 - \phi_2\right\|_{P_g},$$

where C stands for positive constants not depending on $\gamma,\ \varepsilon,\ \xi,\ t,\ \phi,\ \phi_1$ and ϕ_2. Thus from Lemma 5.1, if γ is fixed large enough, for ε small, for any $t \in [a,b]$ and any $\xi \in M$, $T_{\varepsilon,\delta_\varepsilon(t),\xi}$ is a contraction mapping from $B_{\varepsilon,\delta_\varepsilon(t),\xi}(\gamma)$

onto $B_{\varepsilon,\delta_\varepsilon(t),\xi}(\gamma)$. Therefore, using the fixed point theorem, there exists a function $\phi_{\delta_\varepsilon(t),\xi} \in K^\perp_{\delta_\varepsilon(t),\xi}$ which solves equation (2.6). Now, (2.7) follows from Lemma 5.1. The fact that $\phi_{\delta_\varepsilon(t),\xi}$ is continuously differentiable with respect to t and ξ is standard. □

3 The reduced problem.

For $\varepsilon > 0$ small enough, we defined the energy associated to (1.5) by, for $u \in H^2(M)$,

$$J_\varepsilon(u) = \frac{1}{2}\int_M (\Delta_g u)^2 + \frac{1}{2}\int_M A_g(\nabla_g u, \nabla_g u)dV + \frac{1}{2}\int_M hu^2 dV - \int_M F_\varepsilon(u)dV,$$

where $F_\varepsilon(u) = \displaystyle\int_0^u f_\varepsilon(s)ds$. We set $I_\varepsilon(t,\xi) = J_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi})$, $t \in \mathbb{R}^*_+$ and $\xi \in M$ where $\phi_{\delta_\varepsilon(t),\xi} \in K^\perp_{\delta_\varepsilon(t),\xi}$ is the function defined in Proposition 2.1. In the next proposition, we give the expansion of I_ε with respect to ε.

Proposition 3.1. *Let $u_0 \in C^{4,\theta}(M)$, $\theta \in (0,1)$ be a nondegenerate positive solution of (1.5). Then there exist constants $c_i(n,u_0)$, $i = 2,5$ depending on n and u_0 and $c_i(n)$, $i = 1,3,4$, depending on n such that*

$$I_\varepsilon(t,\xi) = c_5(n,u_0) + c_2(n,u_0)\varepsilon + c_3(n)\varepsilon\ln\varepsilon - c_4(n)\varepsilon\ln(t) + c_1(n)\varphi(\xi)\varepsilon t + o(\varepsilon) \tag{3.1}$$

as $\varepsilon \to 0$ C^0 uniformly with respect to t in compact subsets of $\mathbb{R}^*_+$ and with respect to $\xi \in M$ and C^1 uniformly if $8 \le n \le 13$. Moreover, we have that $c_4(n) > 0$, $c_1(n) = \dfrac{2}{n}K_n^{-\frac{n}{4}}$ and

$$\varphi(\xi) = \left(\frac{(n-1)}{(n-6)(n^2-4)}(Tr_g(A_g - A_{paneitz})(\xi)1_{n\ge 8} + \frac{2^n u_0(\xi)\omega_{n-1}}{(n+2)(n(n-4)(n^2-4))^{\frac{n-4}{8}}\omega_n}1_{n\le 8}\right),$$

where ω_n stands for the volume of $\mathbb{S}^n$ and K_n is the sharp constant for the embedding of $H^2(\mathbb{R}^n)$ into $L^{2^*}(\mathbb{R}^n)$ given by $K_n^{-1} = \dfrac{n(n-4)(n^2-4)\omega_n^{\frac{4}{n}}}{16}$.

Proof. We begin by proving that

$$I_\varepsilon(t,\xi) = J_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) + o(\varepsilon), \tag{3.2}$$

as $\varepsilon \to 0$, uniformly with respect to t in compact subsets of $\mathbb{R}_+^*$ and points $\xi \in M$ (we will show in Lemma 5.2 that, when $8 \leq n \leq 13$, this estimate holds C^1 uniformly with respect to t and ξ). Indeed, we have

$$\begin{aligned} &I_\varepsilon(t,\xi) - J_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) \\ &\quad = \left\langle u_0 - W_{\delta_\varepsilon(t),\xi} - i^*(f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})), \phi_{\delta_\varepsilon(t),\xi} \right\rangle_{P_g} + O(\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_{P_g}^2) \end{aligned} \tag{3.3}$$

when $\varepsilon \to 0$. Using Lemma 5.1 and Proposition 2.1, we get

$$\begin{aligned} &\left\langle u_0 - W_{\delta_\varepsilon(t),\xi} - i^*(f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})), \phi_{\delta_\varepsilon(t),\xi} \right\rangle_{P_g} \\ &\qquad + O(\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_{P_g}^2) = O(\varepsilon^2 |\ln \varepsilon|^2) = o(\varepsilon). \end{aligned}$$

Now, the proposition is reduced to estimate $J_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})$. We will focus on C^0-estimates. The C^1-estimates can be obtained using the same argument as in Lemma 4.1 of [21]. Since u_0 is a solution of (1.5), we have

$$\begin{aligned} J_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) &= \frac{1}{2}\int_M u_0^{2^*} dV + \frac{1}{2}\int_M (\Delta_g W_{\delta_\varepsilon(t),\xi})^2 dV \\ &+ \frac{1}{2}\int_M A_g(\nabla_g W_{\delta_\varepsilon(t),\xi}, \nabla_g W_{\delta_\varepsilon(t),\xi}) dV + \frac{1}{2}\int_M h W_{\delta_\varepsilon(t),\xi}^2 dV \\ &- \int_M f_\varepsilon(u_0) W_{\delta_\varepsilon(t),\xi} dV - \int_M F_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) dV. \end{aligned}$$

Using a Taylor expansion with respect to ε, we get

$$\begin{aligned} &\frac{1}{2}\int_M u_0^{2^*} dV - \frac{1}{2^* - \varepsilon}\int_M u_0^{2^*-\varepsilon} dV \\ &= \frac{1}{2}\int_M u_0^{2^*} dV - \frac{1}{2^*}(1 + \frac{\varepsilon}{2^*})\int_M u_0^{2^*}(1 - \varepsilon \ln u_0) dV + O(\varepsilon^2) \\ &= (\frac{1}{2} - \frac{1}{2^*})\int_M u_0^{2^*} dV + \frac{\varepsilon}{2^*}\int_M u_0^{2^*}(\ln u_0 - \frac{1}{2^*}) dV + O(\varepsilon^2) \end{aligned}$$

Thus from the two previous equalities, we obtain

$$\begin{aligned} J_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) = (\frac{1}{2} - \frac{1}{2^*})\int_M u_0^{2^*} dV + \frac{\varepsilon}{2^*}\int_M u_0^{2^*}(\ln u_0 - \frac{1}{2^*}) dV \\ + I_{1,\varepsilon,t,\xi} + I_{2,\varepsilon,t,\xi} + I_{3,\varepsilon,t,\xi} + O(\varepsilon^2), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} I_{1,\varepsilon,t,\xi} = \frac{1}{2}\int_M (\Delta_g W_{\delta_\varepsilon(t),\xi})^2 dV + \frac{1}{2}\int_M A_g(\nabla_g W_{\delta_\varepsilon(t),\xi}, \nabla_g W_{\delta_\varepsilon(t),\xi}) dV \\ + \frac{1}{2}\int_M h W_{\delta_\varepsilon(t),\xi}^2 dV - \int_M F_\varepsilon(W_{\delta_\varepsilon(t),\xi}) dV, \end{aligned}$$

$$I_{2,\varepsilon,t,\xi} = \int_M f_\varepsilon(W_{\delta_\varepsilon(t),\xi})u_0 dV,$$

and

$$\begin{aligned} I_{3,\varepsilon,t,\xi} = -\int_M & F_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) - F_\varepsilon(u_0) - F_\varepsilon(W_{\delta_\varepsilon(t),\xi}) \\ & + f_\varepsilon(u_0)W_{\delta_\varepsilon(t),\xi} + f_\varepsilon(W_{\delta_\varepsilon(t),\xi})u_0 dV. \end{aligned} \tag{3.5}$$

We begin by estimating I_3. Using Taylor expansion (cf (2.20)) and rough estimations, we have

$$\begin{aligned} |I_{3,\varepsilon,t,\xi}| \leq & \left\| (F_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) - F_\varepsilon(W_{\delta_\varepsilon(t),\xi}) + f_\varepsilon(W_{\delta_\varepsilon(t),\xi})u_0) 1_{B(\sqrt{\delta_\varepsilon(t)})} \right\|_{L^1} \\ & + \left\| (F_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) - F_\varepsilon(u_0) + f_\varepsilon(u_0)W_{\delta_\varepsilon(t),\xi}) 1_{M\setminus B(\sqrt{\delta_\varepsilon(t)})} \right\|_{L^1} \\ & + \left\| F_\varepsilon(u_0) 1_{B(\sqrt{\delta_\varepsilon(t)})} \right\|_{L^1} + \left\| f_\varepsilon(u_0)W_{\delta_\varepsilon(t),\xi} 1_{B(\sqrt{\delta_\varepsilon(t)})} \right\|_{L^1} \\ & + \left\| F_\varepsilon(W_{\delta_\varepsilon(t),\xi}) 1_{M\setminus B(\sqrt{\delta_\varepsilon(t)})} \right\|_{L^1} + \left\| u_0 f_\varepsilon(W_{\delta_\varepsilon(t),\xi}) 1_{M\setminus B(\sqrt{\delta_\varepsilon(t)})} \right\|_{L^1} \\ \leq & \left\| u_0^2 W_{\delta_\varepsilon(t),\xi}^{2^*-2-\varepsilon} 1_{B(\sqrt{\delta_\varepsilon(t)})} \right\|_{L^1} + \left\| u_0^{2^*-2-\varepsilon} W_{\delta_\varepsilon(t),\xi}^2 1_{M\setminus B(\sqrt{\delta_\varepsilon(t)})} \right\|_{L^1} \\ & + \left\| F_\varepsilon(W_{\delta_\varepsilon(t),\xi}) 1_{M\setminus B(\sqrt{\delta_\varepsilon(t)})} \right\|_{L^1} + \left\| u_0 f_\varepsilon(W_{\delta_\varepsilon(t),\xi}) 1_{M\setminus B(\sqrt{\delta_\varepsilon(t)})} \right\|_{L^1} \\ & + \left\| F_\varepsilon(u_0) 1_{B(\sqrt{\delta_\varepsilon(t)})} \right\|_{L^1} + \left\| f_\varepsilon(u_0)W_{\delta_\varepsilon(t),\xi} 1_{B(\sqrt{\delta_\varepsilon(t)})} \right\|_{L^1} \\ \leq & C \left\| u_0^2 W_{\delta_\varepsilon(t),\xi}^{2^*-2-\varepsilon} 1_{B(\sqrt{\delta_\varepsilon(t)})} \right\|_{L^1} + C \left\| u_0^{2^*-2-\varepsilon} W_{\delta_\varepsilon(t),\xi}^2 1_{M\setminus B(\sqrt{\delta_\varepsilon(t)})} \right\|_{L^1} \\ & + O(\delta_\varepsilon(t)^{\frac{n}{2}}) \end{aligned}$$

Therefore estimating the last two terms and using the definition of δ, we obtain

$$|I_{3,\varepsilon,t,\xi}| \leq \begin{cases} O(\delta_\varepsilon(t)^{\frac{n}{2}}) = O(\varepsilon^{\frac{n}{4}}) = o(\varepsilon^2) \text{ if } n > 8 \\ O(\delta_\varepsilon(t)^4 |\ln \delta|) = O(\varepsilon^2 |\ln \varepsilon|) \text{ if } n = 8 \\ O(\delta_\varepsilon(t)^{n-4}) = O(\varepsilon^2) \text{ if } n < 8. \end{cases} \tag{3.6}$$

Now, let us estimate $I_{2,\varepsilon,t,\xi}$. We recall that the Cartan expansion of the metric gives

$$\sqrt{|g|}(x) = 1 - \frac{1}{6} Ric_{ij} x^i x^j - \frac{1}{12} \nabla_k Ric_{ij} x^i x^j x^k + O(|x|^4), \tag{3.7}$$

where $|g|$ stands for the determinant of the metric g in geodesic normal coordinates. Then, using a change of variables, Taylor expansion and by

symmetry, we have

$$\begin{aligned}
I_{2,\varepsilon,t,\xi} &= u_0(\xi)\omega_{n-1}\alpha_n^{\frac{n+4}{n-4}-\varepsilon}\delta_\varepsilon(t)^{\frac{n-4}{2}(1+\varepsilon)} \\
&\quad \times \int_0^{\frac{r_0}{2\delta_\varepsilon}(t)} \frac{r^{n-1}}{(1+r^2)^{\frac{n+4}{2}-\varepsilon\frac{n-4}{2}}}(1+O(\delta^2 r^2))dr \\
&\quad + O(\delta_\varepsilon(t)^{\frac{n}{2}} + \varepsilon^2|\ln\delta_\varepsilon(t)|) \\
&= \frac{2u_0(\xi)\omega_{n-1}\alpha_n^{\frac{n+4}{n-4}}\delta_\varepsilon(t)^{\frac{n-4}{2}}}{n(n+2)} + O(\delta_\varepsilon(t)^{\frac{n}{2}} + \varepsilon^2|\ln\delta_\varepsilon(t)|) \\
&= \frac{2^{n+1}u_0(\xi)K_n^{-\frac{n}{4}}\omega_{n-1}\delta_\varepsilon(t)^{\frac{n-4}{2}}}{n(n+2)\alpha_n\omega_n} + O(\delta_\varepsilon(t)^{\frac{n}{2}} + \varepsilon^2|\ln\delta_\varepsilon(t)|), \qquad (3.8)
\end{aligned}$$

where α_n is defined in (2.1). Finally, we use the computations of section 4 of [10] and the estimate (4.2) of [8] to estimate $I_{1,\varepsilon,t,\delta}$. We notice, using (3.7) and by symmetry, that the remaining in equation (4.2) of [8] (namely $o(\delta_\varepsilon(t)^2)$) is actually in $O(\delta_\varepsilon(t)^4)$. We thus have

$$\begin{aligned}
I_{1,\varepsilon,t,\delta} &= \frac{2}{n}K_n^{-\frac{n}{4}}\Bigg(1 - C_n\varepsilon - \frac{(n-4)^2}{8}\varepsilon\ln\delta \\
&\quad + \frac{(n-1)}{(n-6)(n^2-4)}(Tr_g(A_g - A_{paneitz})\delta_\varepsilon(t)^2 1_{n\geq 8}) \qquad (3.9) \\
&\qquad\qquad + o(\varepsilon) + O(\delta_\varepsilon(t)^4)\Bigg),
\end{aligned}$$

where

$$\begin{aligned}
C_n &= 2^{n-4}(n-4)^2\frac{\omega_{n-1}}{\omega_n}\int_0^\infty \frac{r^{\frac{n-2}{2}}\ln(1+r)}{(1+r)^n}dr \\
&\quad + \frac{(n-4)^2}{8(n-2)}(1 - \frac{1}{2}\ln\sqrt{n(n-4)(n^2-4)}). \qquad (3.10)
\end{aligned}$$

Thus, combining (3.4), (3.6), (3.8) and (3.9), we obtain

$$\begin{aligned}
J_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) &= (\frac{1}{2} - \frac{1}{2^*})\int_M u_0^{2^*} + \frac{\varepsilon}{2^*}\int_M u_0^{2^*}(\ln u_0 - \frac{1}{2^*})dV \\
&\quad + \frac{2}{n}K_n^{-\frac{n}{4}}\Bigg(1 - C_n\varepsilon - \frac{(n-4)^2}{8}\varepsilon\ln\delta_\varepsilon(t) \\
&\qquad + \frac{(n-1)}{(n-6)(n^2-4)}(Tr_g(A_g - A_{paneitz})\delta_\varepsilon(t)^2\Bigg) \\
&\quad + \frac{2^{n+1}u_0(\xi)K_n^{-\frac{n}{4}}\omega_{n-1}\delta_\varepsilon(t)^{\frac{n-4}{2}}}{n(n+2)\alpha_n\omega_n} + o(\varepsilon). \qquad (3.11)
\end{aligned}$$

The lemma follows from (3.2) and (3.11). □

The next proposition shows that, in order to construct a solution to (1.5), we only need to find a critical point for the reduced energy I_ε.

Proposition 3.2. *Given two positive real numbers $a < b$, for ε small, if $(t_\varepsilon, \xi_\varepsilon) \in (a,b) \times M$ is a critical point of I_ε, then the function $u_0 - W_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon} + \phi_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}$ is a solution of (1.5).*

Proof. Let $(\xi_\alpha)_\alpha$ be a sequence of points of M and suppose that $(t_\alpha)_\alpha$ and $(\varepsilon_\alpha)_\alpha$ are two sequences of real numbers such that $\varepsilon_\alpha \underset{\alpha\to\infty}{\longrightarrow} 0$, $a \leq t_\alpha \leq b$ and (t_α, ξ_α) is a critical point of I_{ε_α} for all $\alpha \in \mathbb{N}$. To simplify notations, we set, for $i = 1, \dots, n$,

$$Z_{0,\alpha} = Z_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha} \text{ and } Z_{i,\alpha} = Z_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha,e_i}.$$

Since $\phi_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}$ is a solution of (2.6) by Proposition 2.1, there exist real numbers $\lambda_{i,\alpha}$, $i = 0, \dots, n$ such that

$$DJ_{\varepsilon_\alpha}(u_0 - W_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha} + \phi_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}) = \sum_{i=0}^{n} \lambda_{i,\alpha} \left\langle Z_{i,\alpha}, . \right\rangle_{P_g}. \tag{3.12}$$

Using the previous equality, we see that

$$\frac{\partial I_{\varepsilon_\alpha}}{\partial t}(t_\alpha, \xi_\alpha) = \sum_{i=0}^{n} \lambda_{i,\alpha} \left\langle Z_{i,\alpha}, \frac{\partial}{\partial t}(-W_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha} + \phi_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}) \right\rangle_{P_g}. \tag{3.13}$$

A simple computation gives

$$\frac{\partial}{\partial t}(W_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha})|_{t=t_\alpha} = \frac{\tilde{C}_n}{t_\alpha} Z_{0,\alpha}, \tag{3.14}$$

where $\tilde{C}_n = \alpha_n$ if $n < 8$ and $\tilde{C}_n = \dfrac{\alpha_n(n-4)}{4}$ if $n \geq 8$ (see (2.1) for the definition of α_n). Taking the derivative of $\left\langle Z_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}, \phi_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha} \right\rangle_{P_g} = 0$ with respect to t, we obtain

$$\left\langle \frac{\partial}{\partial t} Z_{\delta_{\varepsilon_\alpha}(t_\alpha)|_{t=t_\alpha},\xi_\alpha}, \phi_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha} \right\rangle_{P_g} = - \left\langle Z_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}, \frac{\partial}{\partial t} \phi_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}|_{t=t_\alpha} \right\rangle_{P_g}. \tag{3.15}$$

Since a straight forward computation gives $\left\| \dfrac{\partial}{\partial t} Z_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}|_{t=t_\alpha} \right\|_{P_g} = O(1)$, from (2.7), (3.13), (3.14) and (3.15), we deduce that

$$\frac{\partial I_{\varepsilon_\alpha}}{\partial t}(t_\alpha, \xi_\alpha) = -\frac{\tilde{C}_n}{t_\alpha} \lambda_{0,\alpha} \left\| \Delta_{eucl} V_0 \right\|^2_{L^2(\mathbb{R}^n)} + o(\sum_{i=0}^{n} \lambda_{i,\alpha}), \tag{3.16}$$

where $o(1) \underset{\alpha\to+\infty}{\longrightarrow} 0$. Arguing the same way and noting that

$$\frac{\partial}{\partial y_i}(W_{\delta_{\varepsilon_\alpha}(t_\alpha),\exp_{\xi_\alpha}(y)})|_{y=0} = \frac{\alpha_n(n-4)}{\delta_{\varepsilon_\alpha}(t_\alpha)}Z_{i,\alpha} + R_{i,\alpha}$$

where $R_{i,\alpha} \underset{\alpha\to+\infty}{\longrightarrow} 0$ in $H^2(M)$, and

$$\left\| \frac{\partial}{\partial y_i} Z_{j,\delta_{\varepsilon_\alpha}(t_\alpha),\exp_{\xi_\alpha}(y)}|_{y=0} \right\|_{P_g} = O(\frac{1}{\delta_{\varepsilon_\alpha}(t_\alpha)}),$$

we obtain

$$\delta_{\varepsilon_\alpha}(t_\alpha)\frac{\partial I_{\varepsilon_\alpha}}{\partial y_i}(t_\alpha,\exp_{\xi_\alpha}(y))|_{y=0} = -\lambda_{i,\alpha}\left\|\Delta_{eucl}V_i\right\|^2_{L^2(\mathbb{R}^n)} + o(\sum_{i=0}^{n}\lambda_{i,\alpha}). \quad (3.17)$$

Therefore, from (3.12), (3.16) and (3.17), it follows that if (t_α,ξ_α) is a critical point of I_{ε_α} then $u_0 - W_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha} + \phi_{\delta_{\varepsilon_\alpha}(t_\alpha),\xi_\alpha}$ is a solution of (1.5). □

We are now in position to prove the theorems.

4 Proof of the theorems.

We begin by proving Theorem 1.1.

Proof of Theorem 1.1. We set $G : \mathbb{R}^*_+ \times M \to \mathbb{R}$ the function defined by

$$G(t,\xi) = -c_4(n)\ln t + c_1(n)\varphi(\xi)t,$$

where $c_4(n)$, $c_1(n)$ and $\varphi(\xi)$ are defined in (3.1). From Proposition 3.1, we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(I_\varepsilon(t,\xi) - c_5(n,u_0) - c_2(n,u_0)\varepsilon - c_3(n)\varepsilon\ln\varepsilon) = G(t,\xi), \quad (4.1)$$

C^1 uniformly with respect to $\xi \in M$ and t in compact subset of $\mathbb{R}^*_+$. We will consider two cases depending on the dimension of the manifold.

First case : $8 \le n \le 13$**.**

We argue as in [21]. Let ξ_0 be the C^1 stable critical point of φ such that $\varphi(\xi_0) > 0$ and set

$$t_0 = \frac{c_4(n)}{c_1(n)\varphi(\xi_0)} > 0.$$

Identifying the tangent space at ξ with $\mathbb{R}^n$ we define the map H from $[0,1]\times\mathbb{R}^+\times\mathbb{R}^n$into $\mathbb{R}^{n+1}$ by

$$H(s,t,\xi)=s\left(\frac{\partial G(t,\exp_\xi(y))}{\partial t},\frac{\partial G(t,\exp_\xi(y))}{\partial y_1}|_{y=0},\dots,\frac{\partial G(t,\exp_\xi(y))}{\partial y_n}|_{y=0}\right)$$
$$+(1-s)\left(t-t_0,\frac{\partial(\varphi\circ\exp_\xi(y))}{\partial y_1}|_{y=0},\dots,\frac{\partial(\varphi\circ\exp_\xi(y))}{\partial y_n}|_{y=0}\right).$$

By the invariance of the Brower degree via homotopy, we have that (t_0,ξ_0) is a C^1 stable critical point of G. From Proposition 3.1 and standard properties of the Brower degree (see *e.g.* [12]), there exists a couple $(t_\varepsilon,\xi_\varepsilon)$ of critical points of I_ε converging to (t_0,ξ_0).

Second case : $5\le n<8$ **and** $n>13$**.**

Since $c_4(n)$ and $c_1(n)$ are positive, we have

$$\lim_{t\to0^+}G(t,\xi)=\lim_{t\to\infty}G(t,\xi)=+\infty,$$

uniformly in $\xi\in M$. Therefore, from (4.1) we deduce that, for ε small enough,there exists a couple $(t_\varepsilon,\xi_\varepsilon)$ which is a minimum for the functional I_ε in $(a,b)\times M$ where a,b are positive constants not depending on ε. This implies from Proposition 3.2 that $u_0-W_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}-\phi_{\delta_\varepsilon(t_\varepsilon),\xi_\varepsilon}$ is a solution of (1.5). Thus Theorem 1.1 is established. □

Finally, we prove Theorem 1.2.

Proof of Theorem 1.2. The proof of Theorem 1.2 will follow closely the proof of Theorem 1.1 therefore we will only sketch it. We restrict ourselves to the case where $9\le n\le11$ (the case $5\le n\le8$ is contained in Theorem 1.1). The main difference is that here we will take $\delta_\varepsilon(t_\varepsilon)=(t_\varepsilon\varepsilon)^{\frac{2}{n-4}}$, for $9\le n\le11$. We will only point out the impact of this choice in the two key estimates, namely the estimate of $\phi_{\delta_\varepsilon(t),\xi}$ in Proposition 2.1 (given in Lemma 5.1) and the estimate of the reduced energy (see Proposition 3.1). Let us first consider the error estimate i.e. Lemma 5.1. With our new choice of $\delta(t_\varepsilon)$, it is immediate to check that the leading term in the expansion of Lemma 5.1 will be given by the term $\|f_0(W_{\delta_\varepsilon(t),\xi})-P(W_{\delta_\varepsilon(t),\xi})\|_{L^{\frac{2n}{n+4}}}$. This implies that Lemma 5.1 will rewrite as

$$\|i^*(f_\varepsilon(u_0-W_{\delta_\varepsilon(t),\xi}))-u_0+W_{\delta_\varepsilon(t),\xi}\|_{P_g}=0(\delta_\varepsilon(t)^2)=0(\varepsilon^{\frac{4}{n-4}}).\tag{4.2}$$

Therefore we deduce that

$$\|\phi_{\delta_\varepsilon(t),\xi}\|_{P_g}=0(\varepsilon^{\frac{4}{n-4}}),\tag{4.3}$$

where $\phi_{\delta_\varepsilon(t),\xi}$ is the function defined in Proposition 2.1. Now, let us consider the changes that occur in Proposition 3.1. Using (3.3), (4.2) and (4.3), we obtain that, for $9 \leq n \leq 11$,

$$I_\varepsilon(t,\xi) - J_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) = 0(\|\phi_{\delta_\varepsilon(t),\xi}\|^2_{P_g})0(\delta^4_\varepsilon(t)) = 0(\varepsilon^{\frac{8}{n-4}}) = o(\varepsilon).$$

Then, it only remains to compute $J_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})$. Being a bit careful with the different remainings apppearing in the proof of Proposition 3.1 and using that $A_g = A_{paneitz}$, we see that

$$\begin{aligned} J_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) = (\frac{1}{2} - \frac{1}{2^*}) \int_M u_0^{2^*} + \frac{\varepsilon}{2^*} \int_M u_0^{2^*} (\ln u_0 - \frac{1}{2^*}) dV \\ + \frac{2}{n} K_n^{-\frac{n}{4}} \left(1 - C_n \varepsilon - \frac{(n-4)}{4} \varepsilon \ln(t\varepsilon) \right) \\ + \frac{2^{n+1} u_0(\xi) K_n^{-\frac{n}{4}} \omega_{n-1} t\varepsilon}{n(n+2)\alpha_n \omega_n} + o(\varepsilon). \end{aligned}$$

Using this last estimate, we can argue exactly as in the case $5 \leq n < 8$ of the proof of Theorem 1.1. This concludes the proof of Theorem 1.2. □

5 Appendix.

In this section, we will give an estimate of the error $R_{\varepsilon,\delta_\varepsilon(t),\xi}$ (see Proposition 2.1) and complete the proof of Proposition 3.1 by showing that (3.2) holds C^1 uniformly with respect to t in compact subsets of $\mathbb{R}^*_+$ and $\xi \in M$ when $8 \leq n \leq 13$. Let us begin with the estimate of the error.

Lemma 5.1. *Given two positive real numbers $a < b$, there exists a positive constant $C'_{a,b}$ such that for ε small, for any real number $t \in [a,b]$ and any point $\xi \in M$, there holds*

$$\left\| i^*(f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})) - u_0 + W_{\delta_\varepsilon(t),\xi} \right\|_{P_g} \leq C'_{a,b} \varepsilon |\ln \varepsilon|$$

Proof. All the estimates will be uniform in t, ξ and ε. Since i^* is continuous, we have

$$\begin{aligned} &\left\| i^*(f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})) - u_0 + W_{\delta_\varepsilon(t),\xi} \right\|_{P_g} \\ &\quad = O\left(\left\| (f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})) - P_g(u_0 - W_{\delta_\varepsilon(t),\xi}) \right\|_{L^{\frac{2n}{n+4}}} \right) \end{aligned} \tag{5.1}$$

where $f_\varepsilon(u) = |u|^{2^*-2-\varepsilon}u$. The triangular inequality yields to

$$\begin{aligned}
&\left\|i^*(f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})) - u_0 + W_{\delta_\varepsilon(t),\xi}\right\|_{P_g} \\
&\qquad \le C\left\|f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(u_0) + f_\varepsilon(W_{\delta_\varepsilon(t),\xi})\right\|_{L^{\frac{2n}{n+4}}} \\
&\qquad\quad + C\left\|f_\varepsilon(u_0) - P_g(u_0)\right\|_{L^{\frac{2n}{n+4}}} \\
&\qquad\quad + C\left\|f_\varepsilon(W_{\delta_\varepsilon(t),\xi}) - P_g(W_{\delta_\varepsilon(t),\xi})\right\|_{L^{\frac{2n}{n+4}}} \\
&\qquad \le C(I_1 + I_2 + I_3). \qquad (5.2)
\end{aligned}$$

We first estimate I_1. By triangular inequality we get

$$\begin{aligned}
I_1 \le\ &\left\|(f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) + f_\varepsilon(W_{\delta_\varepsilon(t),\xi}))1_{B_\xi(\sqrt{\delta_\varepsilon(t)})}\right\|_{L^{\frac{2n}{n+4}}} \\
&+ \left\|(f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(u_0))1_{M\setminus B_\xi(\sqrt{\delta_\varepsilon(t)})}\right\|_{L^{\frac{2n}{n+4}}} \\
&+ \left\|f_\varepsilon(W_{\delta_\varepsilon(t),\xi})1_{M\setminus B_\xi(\sqrt{\delta_\varepsilon(t)})}\right\|_{L^{\frac{2n}{n+4}}} + \left\|f_\varepsilon(u_0)1_{B_\xi(\sqrt{\delta_\varepsilon(t)})}\right\|_{L^{\frac{2n}{n+4}}}. \qquad (5.3)
\end{aligned}$$

From Taylor expansion (*e.g.* using (2.20)) and Young inequality, we obtain

$$\begin{aligned}
&\left\|(f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) + f_\varepsilon(W_{\delta_\varepsilon(t),\xi}))1_{B_\xi(\sqrt{\delta_\varepsilon(t)})}\right\|_{L^{\frac{2n}{n+4}}} \\
&\quad \le C\left\|u_0 W^{2^*-2-\varepsilon}_{\delta_\varepsilon(t),\xi}1_{B_\xi(\sqrt{\delta_\varepsilon(t)})}\right\|_{L^{\frac{2n}{n+4}}} + C\left\|u_0^{2^*-1-\varepsilon}1_{B_\xi(\sqrt{\delta_\varepsilon(t)})}\right\|_{L^{\frac{2n}{n+4}}},
\end{aligned}$$

as well as

$$\begin{aligned}
&\left\|(f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(u_0))1_{M\setminus B_\xi(\sqrt{\delta_\varepsilon(t)})}\right\|_{L^{\frac{2n}{n+4}}} \\
&\quad \le C\left\|u_0^{2^*-2-\varepsilon}W_{\delta_\varepsilon(t),\xi}1_{M\setminus B_\xi(\sqrt{\delta_\varepsilon(t)})}\right\|_{L^{\frac{2n}{n+4}}} + C\left\|W^{2^*-1-\varepsilon}_{\delta_\varepsilon(t),\xi}1_{M\setminus B_\xi(\sqrt{\delta_\varepsilon(t)})}\right\|_{L^{\frac{2n}{n+4}}}.
\end{aligned}$$

Using polar coordinates and a change of variables we deduce that:

$$I_1 = \begin{cases} O(\delta_\varepsilon^{\frac{n+4}{4}}(t)) = O(\varepsilon^{\frac{n+4}{8}}) & \text{if} \quad n > 12, \\ O(\delta_\varepsilon^4(t)|\ln\delta_\varepsilon(t)|^{\frac{2}{3}}) = O(\varepsilon^2|\ln\varepsilon|^{\frac{2}{3}}) & \text{if} \quad n = 12, \\ O(\delta_\varepsilon^{\frac{n-4}{2}}(t)) = O(\varepsilon) & \text{if} \quad n < 12. \end{cases}$$

Concerning I_2 we easily get from Taylor's expansion that

$$I_2 = \left\|f_\varepsilon(u_0) - f_0(u_0)\right\|_{L^{\frac{2n}{n+4}}} = O(\varepsilon).$$

We now estimate I_3. First we recall that with the help of the exponential map we can identify $B_\xi(R_0)$ with a neighborhood of the origin in $\mathbb{R}^n$. Therefore

with this chart we may define $\chi_{\xi,\delta_\varepsilon(t)}(.) := \chi(d(.\delta_\varepsilon(t),\xi))$. Using triangular inequality and a change of variables, we then get

$$\begin{aligned} I_3 \leq C\delta^{\frac{n-4}{2}\varepsilon} \left\| \chi^{2^*-1-\varepsilon}_{\xi,\delta_\varepsilon(t)}(U^{2^*-1-\varepsilon} - U^{2^*-1}) \right\|_{L^{\frac{2n}{n+4}}} \\ + C \left\| (\delta^{\frac{n-4}{2}\varepsilon}\chi^{2^*-1-\varepsilon}_{\xi,\delta_\varepsilon(t)} - \chi^{2^*-1}_{\xi,\delta_\varepsilon(t)})U^{2^*-1}_{\delta_\varepsilon(t),\xi} \right\|_{L^{\frac{2n}{n+4}}} \\ + \left\| f_0(W_{\delta_\varepsilon(t),\xi}) - P_g(W_{\delta_\varepsilon(t),\xi}) \right\|_{L^{\frac{2n}{n+4}}}. \end{aligned}$$

Following the computation in the proof of lemma 2.3 of [8] we obtain these three estimates:

$$\left\| \chi^{2^*-1-\varepsilon}_{\xi,\delta_\varepsilon(t)}(U^{2^*-1-\varepsilon} - U^{2^*-1}) \right\|_{L^{\frac{2n}{n+4}}} = O(\varepsilon),$$

$$\left\| (\delta_\varepsilon^{\frac{n-4}{2}}(t)\chi^{2^*-1-\varepsilon}_{\xi,\delta_\varepsilon(t)} - \chi^{2^*-1}_{\xi,\delta_\varepsilon(t)})U^{2^*-1} \right\|_{L^{\frac{2n}{n+4}}} = O(\varepsilon|\ln\delta_\varepsilon(t)|),$$

and

$$\left\| f_0(W_{\delta_\varepsilon(t),\xi}) - P_g(W_{\delta_\varepsilon(t),\xi}) \right\|_{L^{\frac{2n}{n+4}}} \leq C \begin{cases} \delta^2_\varepsilon(t) = O(\varepsilon) \text{ if } n > 8, \\ \delta^2_\varepsilon(t)|\ln\delta_\varepsilon(t)| = O(\varepsilon|\ln\varepsilon|) \text{ if } n = 8, \\ \delta_\varepsilon^{\frac{n-4}{2}}(t) = O(\varepsilon) \text{ if } n < 8. \end{cases}$$

This concludes the proof.

□

Finally, let us prove that (3.2) holds C^1 uniformly with respect to t in compact subsets of $\mathbb{R}^*_+$ and $\xi \in M$ when $8 \leq n \leq 13$.

Lemma 5.2. *If $8 \leq n \leq 13$, we have*

$$I_\varepsilon(t,\xi) = J_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) + o(\varepsilon)$$

C^1 uniformly with respect to t in compact subsets of $\mathbb{R}^*_+$ and $\xi \in M$.

Proof. To simplify notations, we set, for $i = 1, \dots, n$,

$$Z_0 = Z_{\delta_\varepsilon(t),\xi} \text{ and } Z_i = Z_{\delta_\varepsilon(t),\xi,e_i}.$$

We recall that

$$\frac{\partial}{\partial t}(W_{\delta_\varepsilon(t),\xi}) = \frac{\tilde{C}_n}{t} Z_0,$$

where $\tilde{C}_n = \dfrac{\alpha_n(n-4)}{4}$ (see (2.1) for the definition of α_n). Taking the derivative with respect to t to $I_\varepsilon(t,\xi) - J_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})$, we obtain

$$
\begin{aligned}
&\frac{\partial I_\varepsilon}{\partial t}(t,\xi) - \frac{\partial J_\varepsilon}{\partial t}(u_0 - W_{\delta(t),\xi}) \\
&= \int_M P_g(\phi_{\delta_\varepsilon(t),\xi})\frac{\partial}{\partial t}W_{\delta_\varepsilon(t),\xi}dV \\
&\quad - \int_M (f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}))\frac{\partial W_{\delta_\varepsilon(t),\xi}}{\partial t}dV \\
&\quad + DJ_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi})[\frac{\partial \phi_{\delta_\varepsilon(t),\xi}}{\partial t}] \\
&= \frac{\tilde{C}_n}{t}\left(\int_M (P_g(Z_0) - f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})Z_0)\phi_{\delta_\varepsilon(t),\xi}dV\right. \\
&\qquad\qquad - \int_M \big(f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) \\
&\qquad\qquad\qquad \left. - f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})\phi_{\delta_\varepsilon(t),\xi}\big)Z_0 dV\right) \\
&\quad + DJ_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi})[\frac{\partial \phi_{\delta_\varepsilon(t),\xi}}{\partial t}] \\
&= I_1 + I_2 + I_3, \qquad (5.4)
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \frac{\tilde{C}_n}{t}\int_M (P_g(Z_0) - f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})Z_0)\phi_{\delta_\varepsilon(t),\xi}dV, \qquad (5.5)\\
I_2 &= -\frac{\tilde{C}_n}{t}\int_M (f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) \\
&\qquad - f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})\phi_{\delta_\varepsilon(t),\xi})Z_0 dV, \qquad (5.6)\\
I_3 &= DJ_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi})[\frac{\partial \phi_{\delta_\varepsilon(t),\xi}}{\partial t}]. \qquad (5.7)
\end{aligned}
$$

In the same way, recalling that

$$
\frac{\partial}{\partial y_i}(W_{\delta_\varepsilon(t),\exp_\xi(y)})|_{y=0} = \frac{\alpha_n(n-4)}{\delta_\varepsilon(t)}Z_i + R_{\delta_\varepsilon(t),\xi},
$$

where $\left\|R_{\delta_\varepsilon(t),\xi}\right\|_{P_g} = O(\delta_\varepsilon(t)^2)$ (see (6.13) of [20]) and using (2.7), we find

$$\begin{aligned}
&\frac{\partial I_\varepsilon}{\partial y_i}(t,\exp_\xi(y))|_{y=0} - \frac{\partial J_\varepsilon}{\partial y_i}(u_0 - W_{\delta(t),\exp_\xi(y)})|_{y=0} \\
&= \frac{\alpha_n(n-4)}{\delta_\varepsilon(t)}\left(\int_M (P_g(Z_i) - f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})Z_i)\phi_{\delta_\varepsilon(t),\xi}dV\right. \\
&\qquad - \int_M \big(f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) \\
&\qquad\qquad \left. - f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})\phi_{\delta_\varepsilon(t),\xi}\big)Z_i dV\right) \\
&+ DJ_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi})[\frac{\partial \phi_{\delta_\varepsilon(t),\exp_\xi(y)}}{\partial y_i}]|_{y=0} \\
&+ O(\left\|R_{\delta_\varepsilon(t),\xi}\right\|_{P_g}\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_{P_g}) \\
&= I_4 + I_5 + I_6 + o(\varepsilon),
\end{aligned} \tag{5.8}$$

where

$$\begin{aligned}
I_4 &= \frac{\alpha_n(n-4)}{\delta_\varepsilon(t)}\int_M (P_g(Z_i) - f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})Z_i)\phi_{\delta_\varepsilon(t),\xi}dV, \\
I_5 &= -\frac{\alpha_n(n-4)}{\delta_\varepsilon(t)}\int_M (f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) \\
&\qquad\qquad - f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})\phi_{\delta_\varepsilon(t),\xi})Z_i dV, \\
I_6 &= DJ_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi})[\frac{\partial \phi_{\delta_\varepsilon(t),\exp_\xi(y)}}{\partial y_i}]|_{y=0}.
\end{aligned}$$

We begin by estimating the terms I_3 and I_6. We recall that

$$DJ_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi})[.] = \sum_{i=0}^n \lambda_i \left\langle Z_i, . \right\rangle_{P_g}.$$

Arguing the same way as in Proposition 3.2, we have

$$DJ(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi})[\frac{\partial \phi_{\delta_\varepsilon(t),\xi}}{\partial t}] = O\left(\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_{L^{\frac{2n}{n+4}}}\sum_{i=0}^n |\lambda_i|\right),$$

and

$$DJ_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi})[\frac{\partial \phi_{\delta_\varepsilon(t),\exp_\xi(y)}}{\partial y_i}]|_{y=0} = O\left(\frac{\left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_{L^{\frac{2n}{n+4}}}\sum_{i=0}^n |\lambda_i|}{\delta_\varepsilon(t)}\right).$$

We claim that $|\lambda_i| = O(\varepsilon \ln \varepsilon)$, for all $i = 0, \dots, n$. Using (2.10), to prove the claim, we just need to show that $DJ(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi})[Z_i] = O(\varepsilon \ln \varepsilon)$, for all $i = 0, \dots, n$. Since $\phi_{\delta_\varepsilon(t),\xi} \in K^{\perp}_{\delta_\varepsilon(t),\xi}$, using Hölder inequality, (2.7), Lemma 5.1 and rough estimates, we have

$$
\begin{aligned}
&DJ_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi})[Z^i] \\
&= \int_M P_g(u_0 - W_{\delta_\varepsilon(t),\xi}) Z_i dV - \int_M f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) Z_i dV \\
&= \int_M (P_g(u_0 - W_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})) Z_i dV \\
&\quad - \int_M (f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})) Z_i dV \\
&\le \left\| P_g(u_0 - W_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) \right\|_{L^{\frac{2n}{n+4}}} \|Z_i\|_{L^{2^*}} \\
&\quad + \left\| f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) \right\|_{L^{\frac{2n}{n+4}}} \|Z_i\|_{L^{2^*}} \\
&\le O(\left\| P_g(u_0 - W_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) \right\|_{L^{\frac{2n}{n+4}}}) \\
&\quad + O(\left\| \phi_{\delta_\varepsilon(t),\xi} \right\|_{L^{\frac{2n}{n-4}}} (\left\| W_{\delta_\varepsilon(t),\xi} \right\|^{2^*-2-\varepsilon}_{L^{\frac{2n}{n-4}}} + \left\| \phi_{\delta_\varepsilon(t),\xi} \right\|^{2^*-2-\varepsilon}_{L^{\frac{2n}{n-4}}})) \\
&\le O(\varepsilon \ln \varepsilon).
\end{aligned}
$$

Combining the previous estimates, we get

$$
DJ_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) \left[\frac{\partial \phi_{\delta_\varepsilon(t),\xi}}{\partial t} \right] = O(\varepsilon^2 (\ln \varepsilon)^2), \tag{5.9}
$$

and

$$
DJ_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) \left[\frac{\partial \phi_{\delta_\varepsilon(t),\exp_\xi(y)}}{\partial y_i} \right] |_{y=0} = O(\varepsilon^{\frac{3}{2}} (\ln \varepsilon)^2). \tag{5.10}
$$

Now let us estimate I_2 and I_5. Noticing that, if $8 \le n \le 13$,

$$
\left\| (u_0 - W_{\delta_\varepsilon(t),\xi})^{2^*-3-\varepsilon} Z_i \right\|_{L^{\frac{n}{4}}} = O(\varepsilon^{-\frac{1}{4}}),
$$

we obtain, using (2.19), for $i = 0, \dots, n$,

$$\begin{aligned}
&\int_M (f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi} + \phi_{\delta_\varepsilon(t),\xi}) - f_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) - f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})\phi_{\delta_\varepsilon(t),\xi})Z_i dV \\
&\leq C \begin{cases} \displaystyle\int_M (u_0 - W_{\delta_\varepsilon(t),\xi})^{2^*-3-\varepsilon}\phi^2_{\delta_\varepsilon(t),\xi} Z_i dV & \text{if } 12 \leq n \leq 13, \\ \displaystyle\int_M ((u_0 - W_{\delta_\varepsilon(t),\xi})^{2^*-3-\varepsilon}\phi^2_{\delta_\varepsilon(t),\xi} + \phi^{2^*-1-\varepsilon}_{\delta_\varepsilon(t),\xi} Z_i dV) & \text{if } 8 \leq n < 12, \end{cases} \\
&\leq C \begin{cases} \left\|(u_0 - W_{\delta_\varepsilon(t),\xi})^{2^*-3-\varepsilon} Z_i\right\|_{L^{\frac{n}{4}}} \left\|\phi_{\delta_\varepsilon(t),\xi}\right\|^2_{L^{\frac{2n}{n-4}}} & \text{if } 12 \leq n \leq 13, \\ \left\|\phi_{\delta_\varepsilon(t),\xi}\right\|^2_{L^{\frac{2n}{n-4}}} \left\|(u_0 - W_{\delta_\varepsilon(t),\xi})^{2^*-3-\varepsilon} Z_i\right\|_{L^{\frac{n}{4}}} & \\ \quad + \|Z_i\|_{L^{\frac{2n}{n-4}}} \left\|\phi_{\delta_\varepsilon(t),\xi}\right\|^{2^*-1-\varepsilon}_{L^{\frac{2n}{n-4}}} & \text{if } 8 \leq n < 12, \end{cases} \\
&\leq O(\varepsilon^{2-\frac{1}{4}}(\ln \varepsilon)^2)) \text{ when } 8 \leq n \leq 13. \qquad (5.11)
\end{aligned}$$

Finally, let us estimate I_1 and I_4. Since $\left\|P_g(Z_i) - f'_\varepsilon(W_{\delta_\varepsilon(t),\xi})Z_i\right\|_{L^{\frac{2n}{n+4}}} = O(\varepsilon \ln \varepsilon)$ (see [8], inequality (4.17)) and since, using rough estimates,

$$\left\|u_0^{2^*-2-\varepsilon} Z_i\right\|_{L^{\frac{2n}{n+4}}} + \left\|W^{2^*-3-\varepsilon}_{\delta_\varepsilon(t),\xi} Z_i\right\|_{L^{\frac{2n}{n+4}}} = O(\varepsilon \ln \varepsilon),$$

we obtain

$$\begin{aligned}
&\int_M (P_g(Z_i) - f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi})Z_i)\phi_{\delta_\varepsilon(t),\xi} dV \\
&\leq C \left(\left\|P_g(Z_i) - f'_\varepsilon(W_{\delta_\varepsilon(t),\xi})Z_i\right\|_{L^{\frac{2n}{n+4}}} \right. \\
&\qquad \left. + \left\|(f'_\varepsilon(u_0 - W_{\delta_\varepsilon(t),\xi}) - f'_\varepsilon(W_{\delta_\varepsilon(t),\xi}))Z_i\right\|_{L^{\frac{2n}{n+4}}} \right) \left\|\phi_{\delta_\varepsilon(t),\xi}\right\|_{L^{2^*}} \\
&\leq C\varepsilon \ln \varepsilon (\varepsilon \ln \varepsilon + \left\|u_0^{2^*-2-\varepsilon} Z_i\right\|_{L^{\frac{2n}{n+4}}} + \left\|W^{2^*-3-\varepsilon}_{\delta_\varepsilon(t),\xi} Z_i\right\|_{L^{\frac{2n}{n+4}}}) \\
&\leq O(\varepsilon^2 \ln \varepsilon^2). \qquad (5.12)
\end{aligned}$$

The lemma now follows from (5.4), (5.8), (5.9), (5.10), (5.11) and (5.12). □

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