HAL
open science

Carleman Estimates for the Schrödinger Operator. Applications to Quantitative Uniqueness

Laurent Bakri

To cite this version:

Laurent Bakri. Carleman Estimates for the Schrödinger Operator. Applications to Quantitative Uniqueness. Communications in Partial Differential Equations, 2013, 38 (1), pp.69-91. 10.1080/03605302.2012.736912 . hal-01981180

HAL Id: hal-01981180

https://hal.science/hal-01981180v1

Submitted on 14 Jan 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Carleman estimates for the Schrödinger operator. Applications to quantitative uniqueness

Bakri Laurent

E-mail : laurent.bakri@gmail.com

Abstract

On a closed manifold, we give a quantitative Carleman estimate on the Schrödinger operator. We then deduce quantitative uniqueness results for solutions to the Schrödinger equation using doubling estimates. Finally we investigate the sharpness of this results with respect to the electric potential.

1 Introduction and statement of the results

Let (M, g) be an closed, connected, n-dimensional smooth Riemannian manifold, Δ the (negative) Laplace operator on M, W a bounded function on M and V a bounded vector field on M. Let u be a non trivial solution to

$$\Delta u + Wu + g(V, \nabla u) = 0. \tag{1.1}$$

We consider the possible vanishing order, depending on W and V, of u in any point. In the case that W is a constant and $V = 0$, *i.e.* when dealing with the eigenfunctions of the Laplacian, it is a well known result of H. Donnelly and C. Fefferman [5] that the vanishing order is everywhere bounded by $c\sqrt{\lambda}$, with c a constant depending only on M. When W is a $\mathcal{C}^1$ function ($V = 0$), the author has shown in [2] that the vanishing order is bounded by $C_1\sqrt{\|W\|_{\mathcal{C}^1}}+C_2$, where $\|W\|_{\mathcal{C}^1} = \sup_M |W|+\sup_M |\nabla W|$ and the norm $|\nabla W|$ is taken with respect to the metric g. Both results were based on Carleman inequalities. In this paper we give a L^2 Carleman estimate (theorem 2.5) on the operator P defined by

$$Pu = \Delta u + Wu + g(V, \nabla u) \tag{1.2}$$

which allows to derive quantitative uniqueness results for solutions to (1.1). Recall that L^2 Carleman estimates are a priori estimates of the following form :

$$\|e^{\tau\varphi}Pu\|_{L^2} \geq C\|e^{\tau\varphi}u\|_{L^2},$$

for a great enough parameter $\tau \geq \tau_0$, where the function φ satisfies some convexity properties with respect to the operator P (see [9, 10]). One of the key ideas of Donnelly and Fefferman [5] is to carefully investigate how the parameter τ depends on the eigenvalue λ (*i.e.* $\tau \geq C_1\sqrt{\lambda}+C_2$). In the same spirit, we will use that a Carleman estimate is valid (theorem 2.2) for

$$\tau \geq C(1 + \|W\|_\infty^{\frac{2}{3}} + \|V\|_\infty^2) \tag{1.3}$$

This will allows us to derive the following quantitative uniqueness results:

Theorem 1.1. *There exists a non-negative constant C depending only on M, such that, for any solution u to* (1.1) *and for any point x_0 in M, the vanishing order of any non-zero solution to* (1.1) *is everywhere less than*

$$C(1 + \|W\|_\infty^{\frac{2}{3}} + \|V\|_\infty^2).$$

In the case $V = 0$ this result was shown by C. E. Kenig [13]. The theorem 1.1 is shown via a stronger doubling inequality. We note here that the potentials W, V and the solutions u may take complex values (*i.e.* W is a complex valued function on M and V is a section of the complexified tangent bundle $TM \otimes \mathbb{C}$) . This seems to be the first algebraic bound in term of $\|V\|_\infty$. We don't know if this result is sharp with respect to the magnetic potential V. However theorem 1.1 is sharp in the following sense, considering $V = 0$ and complex valued solutions of (1.1), the exponent 2/3 on $\|W\|_\infty$ is the lowest one can obtain in the upper bound on the vanishing order of solutions. Here we show

Theorem 1.2. *There exists a constant C such that, if $N > 0$ is an arbitrary great number, there exists a function $W \in L^\infty(\mathbb{S}^2, \mathbb{C})$ with $N \geq C\|W\|_\infty^{\frac{2}{3}}$ and a solution $u \in \mathcal{C}^2(\mathbb{S}^2, \mathbb{C})$ to* (1.1) *which vanishes with order N in P. Moreover W can be chosen of compact support with*

$$\mathrm{supp}(W) \subset \left(\mathbb{S}^2 \setminus \{P, Q\}\right)$$

where the points P, Q are antipodal and such that $\mathrm{supp}(W)$ *can be chosen of arbitrary small measure.*

The fact that 2/3 is the optimal exponent has already been shown in [13], based on Meshkov construction [21] (see section 4). However, our example, still based on Meshkov construction, is more precise, since we show that on $\mathbb{S}^2$ the sharp exponent can be reached with a compactly support potential W outside two antipodal points. For the real case, the upper bound

$$C(1 + \|W\|_\infty^{\frac{1}{2}} + \|V\|_\infty)$$

is expected (see e.g. [13, 17]). However this seems to be a difficult problem which, to the knowledge author's, has not been solved yet. As mentioned in [3], since Carleman estimates do not distinguish between the real and complex case, it seems difficult to obtain such a result with this method.

Finally we should mention that I. Kukavica in [17] obtains quantitative results for the vanishing order of solutions to (1.1) when $V = 0$. He establishes the following upper bound :

$$C(\sup(W_-)^{\frac{1}{2}} + (\mathrm{osc}(W))^2 + 1).$$

The method of [17] was based on the frequency function [7, 19] when ours relies on Carleman estimates [5, 11, 13, 15, 16, 22, 23, ...]. These two methods are the principal way to obtain quantitative uniqueness results for solutions to partial differential equations. One should also notice that, since Lipschitz continuity is the natural assumption on the metric g such that strong unique continuation holds [1, 6, 14], Theorem 1.1 may be extended to manifolds with only Lipschitz metrics (and to manifolds with boundary [6]) but we will not be concerned which such refinements here.
This paper is organised as follows. In section 2 we derive from standard Carleman estimate on the Laplacian an estimate on P which is only true for great enough parameter $\tau \geq \tau_0$. Furthermore we state explicitly how τ_0 depends on the potentials V, W (1.3). Then we use this Carleman estimate with a special choice of weight functions to obtain in section 3 a three balls theorem and doubling inequalities on solutions of (1.1). Section 4 is devoted to prove theorem 1.2. We first construct an appropriate sequence (u_k, W_k) verifying $\Delta u_k + W_k u_k = 0$, on the two dimensional sphere. Then we show that the sharp upper bound $C_1\|W\|_\infty^{\frac{2}{3}} + C_2$ on the vanishing order can be obtained with W_k of small support in the neighbourhood of two antipodal points.

Notations

For a fixed point x_0 in M we will use the following standard notations:

- $r := r(x) = d(x, x_0)$ the Riemannian distance from x_0,

- $B_r := B_r(x_0)$ the geodesic ball centered at x_0 of radius r.
- $A_{r_1,r_2} := B_{r_2} \setminus B_{r_1}$.

Furthermore we denote

- v_g the volume form induced by g,
- $u \cdot v := g(u, v)$, the inner product between two tangent vectors with respect to the metric g,
- $g(V, \nabla u) := \partial_V u$ the directional derivative of u in direction V,
- $\|\cdot\|$ the L^2 norm on M and $\|\cdot\|_A$ the L^2 norm on the (measurable) set A. In case T is a vector field (or a tensor), $\|\cdot\|$ has to be understood as $\||T|_g\|$.
- c, C, c_i and C_i for $i = 1, 2, \cdots$ are generic constants which may depend on (M, g) and other quantities such as the weight functions in Carleman estimates (section 2.1), but not on the potentials or the solution u to (1.1). Their values can change from one line to another.

Acknowledgement: I would like to thank the anonymous referee for several helpful remarks that permit to significantly improve the presentation of the paper.

2 Carleman estimate

Recall that Carleman estimates are weighted integral inequalities with a weight function $e^{\tau\phi}$, where the function ϕ satisfies some convexity properties, see by example [9, 10, 12]. In this section we first state an L^2, singular weighted, Carleman inequality on the operator $u \mapsto \Delta u + V \cdot \nabla u + Wu$, for some class of weight functions satisfying convenient properties (see (2.1) and (2.2) below). Then we use a particular choice of weight functions which will allows us to derive a doubling estimate on solutions to (1.1) .

Let us first define the class of (singular) weight functions we will work with.
Let $f \ :]-\infty, T[\to \mathbb{R}$ be of class $\mathcal{C}^3$, and assume that there are constants $\mu_i > 0$, $i = 1, \cdots, 4$, such that :

$$\begin{array}{c} 0 < \mu_1 \leq f'(t) \leq \mu_2 \\ \mu_3 |f^{(3)}(t)| \leq -f''(t) \leq \mu_4 \quad , \ \forall t \in]-\infty, T[\\ \lim\limits_{t\to-\infty} -e^{-t} f''(t) = +\infty. \end{array} \tag{2.1}$$

One can check easily the following

Example 2.1. The functions defined by $f_\varepsilon(t) = t - e^{\varepsilon t}$, satisfy the conditions (2.1) provided $0 < \varepsilon < 1$ and T is a large negative number.

Finally we define our weight function as

$$\phi(x) = -f(\ln r(x)). \tag{2.2}$$

Now we can state the main result of this section:

Theorem 2.2. *There exist positive constants R_0, C, C_1, which depend only on M and f, such that, for any $x_0 \in M$, any $\delta \in (0, R_0)$, any $W \in L^\infty(M)$, any $V \in \Gamma(TM)$, any $u \in \mathcal{C}_0^\infty\left(B_{R_0}(x_0) \setminus B_\delta(x_0)\right)$ and any $\tau \geq C_1(1+\|W\|_\infty^{\frac{2}{3}}+\|V\|_\infty^2)$, one has*

$$\begin{aligned} C \left\| r^2 e^{\tau\phi} \left(\Delta u + Wu + g(V, \nabla u)\right) r^{-n/2} \right\|^2 \\ \geq \tau^3 \left\| \sqrt{|f''(\ln r)|} e^{\tau\phi} u r^{-n/2} \right\|^2 + \tau^2 \delta \left\| r^{-\frac{1}{2}} e^{\tau\phi} u r^{-n/2} \right\|^2 \\ + \tau \left\| r \sqrt{|f''(\ln r)|} e^{\tau\phi} |\nabla u|^2 r^{-n/2} \right\|^2. \end{aligned} \tag{2.3}$$

Remark 2.3. Like in [5], the important statement in theorem 2.2 is the assumption on the parameter τ which is related to the vanishing order:

$$\tau \geq C(1 + \|W\|_\infty^{\frac{2}{3}} + \|V\|_\infty^2).$$

The following lemma, which deals solely with Laplace operator, contains the crucial part of theorem 2.2:

Lemma 2.4. *There exist positive constants R_0, C, C_1, which depend only on M and f, such that, for any $x_0 \in M$, any $u \in C_0^\infty\left(B_{R_0}(x_0) \setminus B_\delta(x_0)\right)$ and any $\tau \geq C_1$, one has*

$$\begin{aligned} C \left\| r^2 e^{\tau\phi} \Delta u r^{-n/2} \right\|^2 \geq \tau^3 \left\| \sqrt{|f''(\ln r)|} e^{\tau\phi} u r^{-n/2} \right\|^2 \\ + \tau^2 \delta \left\| r^{-\frac{1}{2}} e^{\tau\phi} u r^{-n/2} \right\|^2 + \tau \left\| r \sqrt{|f''(\ln r)|} e^{\tau\phi} |\nabla u|^2 r^{-n/2} \right\|^2. \end{aligned} \tag{2.4}$$

The main interest of this Carleman estimates is the powers of τ in the right hand side. Such type of estimates, with the corresponding powers on τ, are standard and for this reason we choose to omit a complete proof. We refer to the following : [4], [9] formula (17.2.11), [14], [18]. Moreover one

should note that lemma 2.4 can be derived from Theorem 1.1 of [2]. Indeed letting $W = 0$, Theorem 1.1 of [2] is just a special case of Lemma 2.4 with the weight functions of example 2.1. But one can check that the proof is still valid for these weight functions since it only require the properties (2.1).

We can now derive from Lemma 2.4, the Carleman estimate on the Schrödinger operator $P = \Delta + W + g(V, \nabla\cdot)$.

Proof of Theorem 2.5. If V and W are bounded, one has from triangular inequality :

$$C \left\| r^2 e^{\tau\phi} \left(\Delta u + Wu + g(V, \nabla u)\right) r^{-\frac{n}{2}} \right\|^2 \geq \frac{C}{4} \left\| r^2 e^{\tau\phi} \Delta u r^{-\frac{n}{2}} \right\|^2 - \frac{C}{2} \|W\|_\infty^2 \cdot \left\| r^2 e^{\tau\phi} u r^{-\frac{n}{2}} \right\|^2 - \|V\|_\infty^2 \cdot \left\| r^2 e^{\tau\phi} \nabla u r^{-\frac{n}{2}} \right\|^2$$

Since for R_0 small enough one has $r \leq \sqrt{|f''(\ln r)|}$ from (2.1), assuming additionally that $\tau^3 \geq C_1 \|W\|_\infty^2$ and $\tau \geq C_1 \|V\|_\infty^2$, we can use (2.4) to absorb the negative terms and conclude the proof.

□

2.1 Special choice of weight functions

In this paragraph we derive a special case of theorem 2.2 which will be useful for the remaining of this paper. Let ε be a real number such that $0 < \varepsilon < 1$. As in previous example we consider on $]-\infty, T_0]$ the function defined by

$$f(t) = t - e^{\varepsilon t},$$

which satisfies the properties (2.1). Therefore we can apply theorem 2.2 with the weight function $\phi(x) = -f(\ln r) = -\ln r + r^\varepsilon$. Now we notice that

$$e^{\tau\phi} = e^{-\tau \ln r} e^{\tau r^\varepsilon}.$$

The point is that we can now obtain a Carleman estimate with the usual L^2 norm. Indeed one has

$$e^{\tau\phi} r^{-\frac{n}{2}} = e^{(\tau + \frac{n}{2})\phi} e^{-\frac{n}{2} r^\varepsilon}$$

and therefore for r small enough

$$\frac{1}{2} e^{(\tau + \frac{n}{2})\phi} \leq e^{\tau\phi} r^{-\frac{n}{2}} \leq e^{(\tau + \frac{n}{2})\phi}.$$

Then we have

Corollary 2.5. *There exist positive constants R_0, C, C_1, C_2, which depend only on M and ε, such that, for any $W \in L^\infty(M)$, $x_0 \in M$, any $u \in \mathcal{C}_0^\infty\left(B_{R_0}(x_0) \setminus B_\delta(x_0)\right)$ and $\tau \geq C_1(\|W\|_\infty^{\frac{2}{3}} + \|V\|^2 + 1)$, one has*

$$\begin{aligned} C\left\|r^2 e^{\tau\phi}\left(\Delta u + Wu + V\cdot\nabla u\right)\right\|^2 &\geq \tau^3\left\|r^{\frac{\varepsilon}{2}} e^{\tau\phi} u\right\|^2 \\ + \tau^2\delta\left\|r^{-\frac{1}{2}} e^{\tau\phi} u\right\|^2 &+ \tau\left\|r^{1+\frac{\varepsilon}{2}} e^{\tau\phi}\nabla u\right\|^2. \end{aligned} \tag{2.5}$$

We emphasis that ε is fixed and its value will not have any influence on ours statements as long as $0 < \varepsilon < 1$. Therefore we may and will omit the dependency of our constants on ε.

3 Vanishing order

We want to derive from our Carleman estimate, a control on the local behaviour of solutions. We will first give and proove an Hadamard three circles type theorem.

3.1 Three balls inequality

It is well-known that one can deduce three balls inequality from carleman estimate, one can see by example the following references [12, 13, 20] To obtain such kind of result, the basic idea is to apply Carleman estimate to χu where χ is an appropriate cut off functions and u a solution of (1.1).

Proposition 3.1 (Three balls inequality)**.** *There exist positive constants R_1, C_1 and $0 < \alpha < 1$ which depend only on (M, g) such that, if u is a solution to (1.1) with W of class L^∞, then for any $R < R_1$, and any $x_0 \in M$, one has*

$$\|u\|_{B_R(x_0)} \leq e^{C_1(1+\|W\|_\infty^{2/3}+\|V\|_\infty^2)}\|u\|^\alpha_{B_{\frac{R}{2}}(x_0)}\|u\|^{1-\alpha}_{B_{2R}(x_0)}. \tag{3.1}$$

Proof. Let x_0 a point in M. Let u be a solution to (1.1) and R such that $0 < R < \frac{R_0}{2}$ with R_0 as in corollary 2.5. We will denote by $\|v\|_{R_1,R_2}$ the L^2 norm of v on the set $A_{R_1,R_2} := \{x \in M;\, R_1 \leq r(x) \leq R_2\}$. Let $\psi \in \mathcal{C}_0^\infty(B_{2R})$, $0 \leq \psi \leq 1$, a function with the following properties:

- $\psi(x) = 0$ if $r(x) < \frac{R}{4}$ or $r(x) > \frac{5R}{3}$,
- $\psi(x) = 1$ if $\frac{R}{3} < r(x) < \frac{3R}{2}$,
- $|\nabla\psi(x)| \leq \frac{C}{R}$,
- $|\nabla^2\psi(x)| \leq \frac{C}{R^2}$.

First since the function ψu is supported in the annulus $A_{\frac{R}{3},\frac{5R}{3}}$, we can apply estimate (2.5) of theorem 2.5. In particular we have, since the quotient between $\frac{R}{3}$ and $\frac{5R}{3}$ does not depend on R :

$$C\left\|r^2e^{\tau\phi}\left(\Delta\psi u+2\nabla u\cdot\nabla\psi+V\cdot u\nabla\psi\right)\right\|\geq\tau\left\|e^{\tau\phi}\psi u\right\|. \tag{3.2}$$

First note that for a solution u to (1.1) one can apply Carleman estimate (2.5) and, keeping appropriate term in the right hand side, one gets

$$C\left\|r^2e^{\tau\phi}\left(\Delta\psi u+2\nabla u\cdot\nabla\psi+V\cdot u\nabla\psi\right)\right\|\geq\tau\left\|e^{\tau\phi}\psi u\right\|. \tag{3.3}$$

Noticing that $\tau\geq C(1+\|W\|_\infty^{\frac{2}{3}}+\|V\|_\infty^2)\geq 1$, the estimate (3.3) leads to

$$\left\|e^{\tau\phi}\psi u\right\|\leq\left\|r^2e^{\tau\phi}\left(\Delta\psi u+2\nabla u\cdot\nabla\psi\right)\right\|+\|r^2e^{\tau\phi}u\nabla\psi\| \tag{3.4}$$

Now recall $|D^\alpha\psi|\leq\frac{c}{R^{|\alpha|}}$ and $r^2\leq r$. Then from properties of ψ and triangular inequality one gets :

$$\begin{aligned}\|e^{\tau\phi}u\|_{\frac{R}{3},\frac{3R}{2}} &\leq C\left(\|e^{\tau\phi}u\|_{\frac{R}{4},\frac{R}{3}}+\|e^{\tau\phi}u\|_{\frac{3R}{2},\frac{5R}{3}}\right)\\ &+ C\left(R\|e^{\tau\phi}\nabla u\|_{\frac{R}{4},\frac{R}{3}}+R\|e^{\tau\phi}\nabla u\|_{\frac{3R}{2},\frac{5R}{3}}\right).\end{aligned}$$

Recall that $\phi(x)=-\ln r(x)+r(x)^\varepsilon$. In particular ϕ is radial and decreasing (for small r). Then one has,

$$\begin{aligned}\|e^{\tau\phi}u\|_{\frac{R}{3},\frac{3R}{2}} &\leq C\left(e^{\tau\phi(\frac{R}{4})}\|u\|_{\frac{R}{4},\frac{R}{3}}+e^{\tau\phi(\frac{3R}{2})}\|u\|_{\frac{3R}{2},\frac{5R}{3}}\right)\\ &+ C\left(Re^{\tau\phi(\frac{R}{4})}\|\nabla u\|_{\frac{R}{4},\frac{R}{3}}+Re^{\tau\phi(\frac{3R}{2})}\|\nabla u\|_{\frac{3R}{2},\frac{5R}{3}}\right).\end{aligned} \tag{3.5}$$

Now we state a standard elliptic estimate to absorb the terms involving gradients (Caccioppoli estimates) : for solutions to (1.1) one has

$$\|\nabla u\|_{aR}\leq C\left(\frac{1}{(1-a)R}+\|W\|_\infty^{1/2}+\|V\|_\infty\right)\|u\|_R,\quad\text{for }\ 0<a<1, \tag{3.6}$$

Moreover since $A_{R_1,R_2}\subset B_{R_2}$, using formula (3.6) and properties of ϕ gives

$$e^{\tau\phi(\frac{R}{4})}\|\nabla u\|_{\frac{R}{4},\frac{R}{3}}\leq C\left(\frac{1}{R}+\|W\|_\infty^{1/2}+\|V\|_\infty\right)e^{\tau\phi(\frac{R}{4})}\|u\|_{\frac{R}{2}},$$

and

$$e^{\tau\phi(\frac{3R}{2})}\|\nabla u\|_{\frac{3R}{2},\frac{5R}{3}}\leq C\left(\frac{1}{R}+\|W\|_\infty^{1/2}+\|V\|_\infty\right)e^{\tau\phi(\frac{3R}{2})}\|u\|_{2R}.$$

Using (3.6) one has :

$$\|u\|_{\frac{R}{3},R} \leq C(1+\|W\|_\infty^{1/2}+\|V\|_\infty)\left(e^{\tau(\phi(\frac{R}{4})-\phi(R))}\|u\|_{\frac{R}{2}}+e^{\tau(\phi(\frac{3R}{2})-\phi(R))}\|u\|_{2R}\right).$$

Let $A_R = \phi(\frac{R}{4}) - \phi(R)$ and $B_R = -(\phi(\frac{3R}{2}) - \phi(R))$. From the definition of ϕ, we have $0 < A^{-1} \leq A_R \leq A$ and $0 < B \leq B_R \leq B^{-1}$ where A and B do not depend on R. We may assume that $C(1+\|W\|_\infty^{1/2}+\|V\|_\infty) \geq 2$. Then we can add $\|u\|_{\frac{R}{3}}$ to each side and bound it in the right hand side by $C(1+\|W\|_\infty^{1/2}+\|V\|_\infty)e^{\tau A}\|u\|_{\frac{R}{2}}$. We get :

$$\|u\|_R \leq C(1+\|W\|_\infty^{1/2}+\|V\|_\infty)\left(e^{\tau A}\|u\|_{\frac{R}{2}}+e^{-\tau B}\|u\|_{2R}\right). \tag{3.7}$$

Now we want to find τ such that

$$C(1+\|W\|_\infty^{1/2}+\|V\|_\infty)e^{-\tau B}\|u\|_{2R} \leq \frac{1}{2}\|u\|_R$$

which is true for $\tau \geq -\frac{1}{B}\ln\left(\frac{1}{2C(1+\|W\|_\infty^{1/2}+\|V\|_\infty)}\frac{\|u\|_R}{\|u\|_{2R}}\right)$. Since τ must also satisfy

$$\tau \geq C_1(1+\|W\|_\infty^{2/3}+\|V\|^2)$$

we choose

$$\tau = -\frac{1}{B}\ln\left(\frac{1}{2C(1+\|W\|_\infty^{1/2}+\|V\|_\infty)}\frac{\|u\|_R}{\|u\|_{2R}}\right)+C_1(1+\|W\|_\infty^{2/3}+\|V\|_\infty^2). \tag{3.8}$$

Up to a change of constant we may assume that $C(1+\|W\|_\infty^{1/2}+\|V\|_\infty) \leq C_1(1+\|W\|_\infty^{2/3}+\|V\|_\infty^2)$ then we can deduce from (3.7) that :

$$\|u\|_R^{\frac{B+A}{B}} \leq e^{C_1(1+\|W\|_\infty^{2/3}+\|V\|_\infty^2)}\|u\|_{2R}^{\frac{A}{B}}\|u\|_{\frac{R}{2}}, \tag{3.9}$$

Finally define $\alpha = \frac{A}{A+B}$ and taking exponent $\frac{B}{A+B}$ of (3.9) gives the result. □

3.2 Doubling estimates

Now we intend to show that the vanishing order of solutions to (1.1) is everywhere bound by $C(1+\|W\|_\infty^{\frac{2}{3}}+\|V\|_\infty^2)$. This is an immediate consequence of the following :

Theorem 3.2 (doubling estimate)**.** *There exists a positive constant C, depending only on M such that : if u is a solution to* (1.1) *on M with V, W bounded, then for any x_0 in M and any $r > 0$, one has*

$$\|u\|_{B_{2r}(x_0)} \leq e^{C(1+\|W\|_\infty^{2/3}+\|V\|_\infty^2)}\|u\|_{B_r(x_0)}. \tag{3.10}$$

Remark 3.3. Using standard elliptic theory to bound the L^∞ norm of $|u|$ by a multiple of its L^2 norm on a greater ball (see by example [8] Theorem 8.17 and problem 8.3), show that the doubling estimate is still true with the L^∞ norm :

$$\|u\|_{L^\infty(B_{2r}(x_0))} \leq e^{C(1+\|W\|_\infty^{2/3}+\|V\|_\infty^2)}\|u\|_{L^\infty(B_r(x_0))}. \tag{3.11}$$

To prove theorem 3.2 we need to use the standard overlapping chains of balls argument ([5, 12, 17]) to show :

Proposition 3.4. *For any $R > 0$ there exists $C_R > 0$ such that for any $x_0 \in M$, any $W \in \mathcal{C}^1(M)$ and any solutions u to* (1.1) *:*

$$\|u\|_{B_R(x_0)} \geq e^{-C_R(1+\|W\|_\infty^{2/3}+\|V\|_\infty^2)}\|u\|_{L^2(M)}.$$

Proof. We may assume without loss of generality that $R < R_0$, with R_0 as in the three balls inequality (proposition 3.1). Up to multiplication by a constant, we can assume that $\|u\|_{L^2(M)} = 1$. We denote by $\bar{x}$ a point in M such that $\|u\|_{B_R(\bar{x})} = \sup_{x\in M}\|u\|_{B_R(x)}$. This implies that one has $\|u\|_{B_{R(\bar{x})}} \geq D_R$, where D_R depend only on M and R. One has from proposition (3.1) at an arbitrary point x of M :

$$\|u\|_{B_{R/2}(x)} \geq e^{-c(1+\|W\|_\infty^{2/3}+\|V\|_\infty^2)}\|u\|_{B_R(x)}^{\frac{1}{\alpha}}. \tag{3.12}$$

Let γ be a geodesic curve between x_0 and $\bar{x}$ and define $x_1, \cdots, x_m = \bar{x}$ such that $x_i \in \gamma$ and $B_{\frac{R}{2}}(x_{i+1}) \subset B_R(x_i)$, for any i from 0 to $m-1$. The number m depends only on $\mathrm{diam}(M)$ and R. Then the properties of $(x_i)_{1\leq i\leq m}$ and inequality (3.12) give for all i, $1 \leq i \leq m$:

$$\|u\|_{B_{R/2}(x_i)} \geq e^{-c(1+\|W\|_\infty^{2/3}+\ |V\|_\infty^2)}\|u\|_{B_{R/2}(x_{i+1})}^{\frac{1}{\alpha}}. \tag{3.13}$$

The result follows by iteration and the fact that $\|u\|_{B_R(\bar{x})} \geq D_R$. □

Similarly one also has,

Corollary 3.5. *For all $R > 0$, there exists a positive constant C_R depending only on M and R such that at any point x_0 in M one has*

$$\|u\|_{R,2R} \geq e^{-C_R(1+\|W\|_\infty^{2/3}+\|V\|_\infty^2)}\|u\|_{L^2(M)}.$$

Proof. Recall that $\|u\|_{R,2R} = \|u\|_{L^2(A_{R,2R})}$ with $A_{R,2R} := \{x; R \leq d(x,x_0) \leq 2R)\}$. Let $R < R_0$ where R_0 is from proposition 3.5, note that $R_0 \leq \mathrm{diam}(M)$. Since M is geodesically complete, there exists a point x_1 in $A_{R,2R}$ such that $B_{x_1}(\frac{R}{4}) \subset A_{R,2R}$. From proposition 3.4 one has

$$\|u\|_{B_{\frac{R}{4}}(x_1)} \geq e^{-C_R(1+\|W\|_\infty^{2/3}+\|V\|_\infty^2)}\|u\|_{L^2(M)}$$

wich gives the result. □

Proof of theorem 3.2. We proceed as in the proof of three balls inequality (proposition 3.5) except for the fact that now the radius of the first ball become arbitrary small in front of the others. Let $R = \frac{R_0}{4}$ with R_0 as in the three balls inequality, let δ such that $0 < 3\delta < \frac{R}{8}$, and define a smooth function ψ, with $0 \leq \psi \leq 1$ as follows:

- $\psi(x) = 0$ if $r(x) < \delta$ or if $r(x) > R$,
- $\psi(x) = 1$ if $r(x) \in [\frac{5\delta}{4}, \frac{R}{2}]$,
- $|\nabla\psi(x)| \leq \frac{C}{\delta}$ and $|\nabla^2\psi(x)| \leq \frac{C}{\delta^2}$ if $r(x) \in [\delta, \frac{5\delta}{4}]$,
- $|\nabla\psi(x)| \leq C$ and $|\nabla^2\psi(x)| \leq C$ if $r(x) \in [\frac{R}{2}, R]$.

Keeping appropriates terms in (2.5) applied to ψu gives :

$$\begin{aligned}\tau^{\frac{3}{2}}\|r^{\frac{\varepsilon}{2}}e^{\tau\phi}\psi u\| + \tau^{\frac{1}{2}}\delta^{\frac{1}{2}}\|r^{-\frac{1}{2}}e^{\tau\phi}\psi u\| \\ \leq C\left(\|r^2e^{\tau\phi}\nabla u\cdot\nabla\psi\| + \|r^2e^{\tau\phi}\Delta\psi u\| + \|r^2e^{\tau\phi}Vu\nabla\psi\|\right). \quad (3.14)\end{aligned}$$

Using properties of ψ and since $\tau \geq \|V\|_\infty$, one finds

$$\begin{aligned}\tau^{\frac{3}{2}}\|r^{\frac{\varepsilon}{2}}e^{\tau\phi}u\|_{\frac{R}{8},\frac{R}{4}} + \tau^{\frac{1}{2}}\|e^{\tau\phi}u\|_{\frac{5\delta}{4},3\delta} \leq C(\delta\|e^{\tau\phi}\nabla u\|_{\delta,\frac{5\delta}{4}} + \|e^{\tau\phi}\nabla u\|_{\frac{R}{2},R}) \\ + C(\|e^{\tau\phi}u\|_{\delta,\frac{5\delta}{4}} + \|e^{\tau\phi}u\|_{\frac{R}{2},R}) \\ + C\frac{\tau}{\delta}\|r^2e^{\tau\phi}u\|_{\delta,\frac{5\delta}{4}} + C\tau\|r^2e^{\tau\phi}u\|_{\frac{R}{2},R}.\end{aligned}$$

Now, we bound from above the two last terms of the previous inequality by $C\tau\left(\|e^{\tau\phi}u\|_{\delta,\frac{5\delta}{4}}+\|e^{\tau\phi}u\|_{\frac{R}{2},R}\right)$. Then we divide both sides of (3.15) by $\tau^{\frac{1}{2}}$. Noticing that $\tau\geq 1$, this yields to

$$\begin{aligned}\|r^{\frac{\varepsilon}{2}}e^{\tau\phi}u\|_{\frac{R}{8},\frac{R}{4}}+\|e^{\tau\phi}u\|_{\frac{5\delta}{4},3\delta}\leq C\tau^{\frac{1}{2}}&\left(\delta\|e^{\tau\phi}\nabla u\|_{\delta,\frac{5\delta}{4}}+\|e^{\tau\phi}\nabla u\|_{\frac{R}{2},R}\right)\\&+C\tau^{\frac{1}{2}}\left(\|e^{\tau\phi}u\|_{\delta,\frac{5\delta}{4}}+\|e^{\tau\phi}u\|_{\frac{R}{2},R}\right).\end{aligned}\tag{3.15}$$

From the elliptic estimate (3.6) and the decreasing of ϕ, we get

$$\begin{aligned}&e^{\tau\phi(\frac{R}{4})}\|u\|_{\frac{R}{8},\frac{R}{4}}+e^{\tau\phi(3\delta)}\|u\|_{\frac{5\delta}{4},3\delta}\\&\qquad\leq C\tau^{\frac{1}{2}}(1+\|W\|_\infty^{1/2}+\|V\|_\infty)\left(e^{\tau\phi(\delta)}\|u\|_{\frac{3\delta}{2}}+e^{\tau\phi(\frac{R}{3})}\|u\|_{\frac{5R}{3}}\right).\end{aligned}$$

Adding $e^{\tau\phi(3\delta)}\|u\|_{\frac{5\delta}{4}}$ to each side and noting that we can bound it from above by $C\tau^{\frac{1}{2}}(1+\|W\|_\infty^{\frac{1}{2}}+\|V\|_\infty)e^{\tau\phi(\delta)}\|u\|_{\frac{3\delta}{2}}$, we find that

$$\begin{aligned}&e^{\tau\phi(\frac{R}{4})}\|u\|_{\frac{R}{8},\frac{R}{4}}+e^{\tau\phi(3\delta)}\|u\|_{3\delta}\\&\qquad\leq C\tau^{\frac{1}{2}}(1+\|W\|_\infty^{1/2}+\|V\|_\infty)\left(e^{\tau\phi(\delta)}\|u\|_{\frac{3\delta}{2}}+e^{\tau\phi(\frac{R}{3})}\|u\|_{\frac{5R}{3}}\right).\end{aligned}$$

Now we want to choose τ such that

$$C\tau^{\frac{1}{2}}(1+\|W\|_\infty^{1/2}+\|V\|_\infty)e^{\tau\phi(\frac{R}{3})}\|u\|_{\frac{5R}{3}}\leq\frac{1}{2}e^{\tau\phi(\frac{R}{4})}\|u\|_{\frac{R}{8},\frac{R}{4}}.$$

For the same reasons as before we choose

$$\begin{aligned}\tau=\frac{2}{\phi(\frac{R}{3})-\phi(\frac{R}{4})}\ln\left(\frac{1}{2C(1+\|W\|_\infty^{1/2}+\|V\|_\infty)}\frac{\|u\|_{\frac{R}{8},\frac{R}{4}}}{\|u\|_{\frac{5R}{3}}}\right)&\\+C(1+\|W\|_\infty^{\frac{2}{3}}+\|V\|_\infty^2).&\end{aligned}$$

Define $D_R=-\left(\phi(\frac{R}{3})-\phi(\frac{R}{4})\right)^{-1}$; like before, one has $0<E^{-1}\leq D_R\leq E$, with E a fixed real number. Dropping the first term in the left hand side and noting that $0<\phi(\delta)-\phi(3\delta)\leq C$, one has

$$\|u\|_{3\delta}\leq e^{C(1+\|W\|_\infty^{\frac{2}{3}}+\|V\|_\infty^2)}\left(\frac{\|u\|_{\frac{R}{8},\frac{R}{4}}}{\|u\|_{\frac{5R}{3}}}\right)^{-E}\|u\|_{\frac{3\delta}{2}}$$

Finally, from Corollary 3.5, we define $r=\frac{3\delta}{2}$ to have :

$$\|u\|_{2r}\leq e^{C(1+\|W\|_\infty^{\frac{2}{3}}+\|V\|_\infty^2)}\|u\|_r.$$

Thus, the theorem is proved for all $r \leq \frac{R_1}{16}$. Using Proposition 3.4 we have for $r \geq \frac{R_1}{16}$:

$$\begin{aligned}\|u\|_{B_{x_0}(r)} \geq \|u\|_{B_{x_0}(\frac{R_0}{16})} &\geq e^{-C_0(1+\|W\|_\infty^{\frac{2}{3}}+\|V\|_\infty^2)}\|u\|_{L^2(M)} \\ &\geq e^{-C_1(1+\|W\|_\infty^{\frac{2}{3}}+\|V\|_\infty^2)}\|u\|_{B_{x_0}(2r)}.\end{aligned}$$

□

Finally theorem 1.1 is an easy and direct consequence of this doubling estimates.

4 Possible vanishing order for solutions

The aim of this is section is to prove theorem 1.2 which states that our exponent $\frac{2}{3}$ on $\|W\|_\infty$ in theorem 1.1 is sharp. More precisely, we will construct a sequence (u_k, W_k) verifying $\Delta u_k + W_k u_k = 0$ and such that :

- At the north pole N of $\mathbb{S}^2$, the function u_k vanishes at an order at least $C_1\|W_k\|_\infty^{\frac{2}{3}}+C_2$, where C_1 and C_2 are fixed numbers which don't depend on k.
- $\lim_{k\to+\infty}\|W_k\|_\infty = +\infty$.
- The potential W has (arbitrary small) compact support in $\mathbb{S}^2\backslash(\{N, S\})$, with S the south pole of $\mathbb{S}^2$.

Our construction will be inspired by the previous work of Meshkov. In [21], he has shown that one can find non trivial, complex valued, solutions of (1.1) in $\mathbb{R}^n$ $(n \geq 2)$, with the following exponential decay at infinity : $|u(x)| \leq e^{-C|x|^{\frac{4}{3}}}$. This gives an negative answer to a question of Landis, [19]. He has also shown that $\frac{4}{3}$ is the greatest exponent for which one can hope find non trivial solutions to (1.1) in $\mathbb{R}^n$.
Our main point is to construct on $\mathbb{R}^2$ a solution u to $\Delta u + Wu = 0$ with appropriate decay at infinity with respect to the L^∞ norm of W. Then, we use stereographic projection to obtain theorem 1.2. This suggests in particular that we not only need the function W to be bounded, since we have to pull it back to the sphere, but we need the stronger condition

$$|W(x,y)| \leq \frac{C}{1+|x|^2+|y|^2}.$$

It actually appears that we can obtain W with compact support. We recall that our construction relies crucially on the fact that W and u are complex valued functions.

From now on, we will denote by (r, φ) polar coordinates of a point in $\mathbb{R}^2$ and by $[a, b]$ the annulus $[a, b] := \{(r, \varphi) \in \mathbb{R}^+ \times [0, 2\pi[, a \leq r \leq b\}$. We have the following on $\mathbb{R}^2$:

Proposition 4.1. *Let $\rho > 0$ a real number and N an integer. For ρ and N great enough their exits u and W verifying $\Delta u + Wu = 0$ on $\mathbb{R}^2$ such that :*

- $|u(x)| = C|x|^{-N}$ *for* $|x| \geq \rho + 6$
- $\|W\|_{L^\infty} \leq CN^{\frac{3}{2}}$
- *W has support in* $]\frac{\rho}{4}, \rho + 6]$

Proof.
Let $\rho \geq 1$, without loss of generality we suppose that N is the square of an integer. Let define $\delta = \frac{1}{\sqrt{N}}$. For j an integer from 10 to $\sqrt{N}$, we note $n_j = j^2$ and $k_j = 2j + 1$ and $\rho_j = \rho + 6(j - 1)\delta$. On each annulus $A_j = [\rho_j, \rho_{j+1}]$ we construct a solution of $\Delta u + Wu = 0$ such that $|u| = a_j r^{-n_j}$ on the sphere $\{\rho_j\}$ and $|u| = a_{j+1} r^{-n_j - k_j} = a_{j+1} r^{-n_{j+1}}$ on the sphere $\{\rho_{j+1}\}$. We set $a_{10} = 1$, the numbers a_j, for $j > 10$, still have to be defined. When working in the annulus A_j, we will write a, n, k, ρ instead of a_j, n_j, k_j, ρ_j. Before proceeding to the proof it is useful to notice that

$$\begin{gathered} k \leq C_1\sqrt{n} \leq C_1\sqrt{N} \\ \tfrac{n}{k} \leq C_2\sqrt{N} \\ \tfrac{1}{\delta} = \sqrt{N} \\ k\delta \leq 3 \end{gathered} \tag{4.1}$$

To build (u, W) in A_j, it is convenient to divide the construction into four steps corresponding to the four annulus $[\rho, \rho + 2\delta], [\rho + 2\delta, \rho + 3\delta], [\rho + 3\delta, \rho + 4\delta], [\rho + 4\delta, \rho + 6\delta]$.

Step I. Construction on the annulus $[\rho, \rho + 2\delta]$

Define

$$u_1 = ar^{-n}e^{-in\varphi},$$

and

$$u_2 = -br^{-n+2k}e^{iF(\varphi)},$$

with $b = a(\rho+\delta)^{-2k}$ such that $\left|\frac{u_2(\rho+\delta)}{u_1(\rho+\delta)}\right| = 1$, the function $F(\varphi)$ will be made explicit later. For m from 0 to $2n+2k-1$, we set $\varphi_m = \frac{2\pi m}{2n+2k}$ and $T = \frac{\pi}{n+k}$.

We will need the following

Lemma 4.2 ([21] p. 350)**.** *Their exists a real function h of class $\mathcal{C}^2$, periodic of period T, with the following properties :*

- $|h(\varphi)| \leq 5kT, \quad \forall \varphi \in \mathbb{R},$ (4.2)
- $|h^{'}(\varphi)| \leq 5k, \quad \forall \varphi \in \mathbb{R},$ (4.3)
- $|h^{''}(\varphi)| \leq Ckn, \quad \forall \varphi \in \mathbb{R},$ (4.4)
- $h(\varphi) = -4k(\varphi - \varphi_m), \quad \textit{for} \;\; \varphi_m - \dfrac{T}{5} \leq \varphi \leq \varphi_m + \dfrac{T}{5}.$ (4.5)

Let now choose F as follows

$$F(\varphi) = (n+2k)\varphi + h(\varphi). \tag{4.6}$$

We also consider two smooths radial functions ψ_1, ψ_2 with the following properties :

- $0 \leq \psi_i \leq 1,$ (4.7)
- $|\psi_i^{(p)}| \leq C\delta^{-p},$ (4.8)
- $\psi_1 = 1$ on $[\rho, \rho + \dfrac{5}{3}\delta]$ and $\psi_1 = 0$ on $[\rho + (2 - \dfrac{1}{10})\delta, \rho + 2\delta],$ (4.9)
- $\psi_2 = 0$ on $[\rho, \rho + \dfrac{1}{10}\delta]$ and $\psi_2 = 1$ on $[\rho + \dfrac{1}{3}\delta, \rho + 2\delta].$ (4.10)

Finally we set $u = \psi_1 u_1 + \psi_2 u_2$. The estimate on $[\rho, \rho+2\delta]$ is divided among four cases.

Step I.a. The set $[\rho, \rho + \frac{\delta}{3}]$

We have $u = u_1 + \psi_2 u_2$, and $|\frac{u_2}{u_1}| = \left(\frac{r}{\rho+\delta}\right)^{2k}$. Then we introduce α as an upper bound of $|\frac{u_2}{u_1}|$:

$$\left|\frac{u_2}{u_1}\right| \leq \left(\frac{\rho + \frac{\delta}{3}}{\rho+\delta}\right)^{2k} := \alpha < 1.$$

We notice that

$$|u| \geq |u_1| - |\psi_2 u_2| \geq |u_1| - |u_2| \geq \frac{1-\alpha}{\alpha}|u_2|,$$

Since $\alpha = e^{2k\ln(1-\frac{2\delta}{3(\rho+\delta)})}$, a computation of $\frac{\alpha}{1-\alpha}$ leads to the following inequality:

$$|u_2| \leq \frac{3(\rho+\delta)}{4k\delta}|u|. \tag{4.11}$$

Now we compute Δu :

$$\Delta u = \psi_2 \Delta u_2 + 2\frac{d\psi_2}{dr}\frac{\partial u_2}{\partial r} + \left(\frac{1}{r}\frac{d\psi_2}{dr} + \frac{d^2\psi_2}{dr^2}\right) u_2. \tag{4.12}$$

First we estimate Δu_2. One has

$$\Delta u_2 \quad = \quad \frac{1}{r^2}\left((-n+2k)^2 + iF'' - (F')^2\right) u_2.$$

From the definition of F (4.6) this gives :

$$\Delta u_2 = \frac{1}{r^2}\left[-8kn + ih'' - 2(n+2k)h' - (h')^2\right] u_2. \tag{4.13}$$

Using properties of h , we obtain

$$|\Delta u_2| \quad \leq \quad \frac{1}{r^2}\left(8kn + Ckn + C_2k^2\right)|u_2| \tag{4.14}$$

Then from inequality (4.11) we have ,

$$|\Delta u_2| \leq \frac{1}{r^2}Ckn\frac{3(\rho+\delta)}{k\delta}|u|,$$

Now (4.1) and $\frac{\rho+\delta}{r^2} \leq 1$ gives:

$$|\Delta u_2| \leq C\frac{n}{\delta}|u| \leq N^{3/2}|u|$$

Now we estimate the other terms in the right and side of (4.12). The conditions on ψ (4.8) and estimate of u_2 (4.11) lead to

$$\left|\frac{\partial\psi_2}{\partial r}\frac{\partial u_2}{\partial r}\right| \leq \frac{Cn}{k\delta^2}|u|. \tag{4.15}$$

Similarly, we have

$$\left|\left(\frac{d\psi_2}{dr} + \frac{d^2\psi_2}{dr^2}\right)\frac{1}{r}u_2\right| \quad \leq \quad C\frac{1}{\delta^3}|u| \tag{4.16}$$

Finally,

$$|\Delta u| \leq C(\frac{n}{\delta} + \frac{n}{k\delta^2} + \frac{1}{\delta^3})|u|.$$

Then we have shown, using relations (4.1) between k, n, δ and N that :

$$|\Delta u| \leq CN^{\frac{3}{2}}|u|. \tag{4.17}$$

Step I.b. $[\rho + \frac{5}{3}\delta, \rho + 2\delta]$

We want to estimate $\frac{|\Delta u|}{|u|}$, with $u = u_2 + \psi_1 u_1$. First since,

$$\frac{|u_2|}{|u_1|} = \frac{b}{a} r^{2k} \geq \left(\frac{\rho + \frac{5}{3}\delta}{\rho + \delta}\right)^{2k} = \beta > 1,$$

we have

$$|u| \geq |u_2| - |\psi_1 u_1| \geq |u_2| - |u_1| \geq \left(\frac{\beta - 1}{\beta}\right)|u_2| > 0. \tag{4.18}$$

One also has

$$\Delta u = \Delta u_2 + 2\frac{d\psi_1}{dr}\frac{\partial u_1}{\partial r} + \left(\frac{1}{r}\frac{d\psi_1}{dr} + \frac{d^2\psi_1}{dr^2}\right) u_1. \tag{4.19}$$

First since $r \geq \rho \geq 1$ one has from (4.18)

$$|\Delta u_2| \leq \frac{Ckn}{r^2}|u_2| \leq Ckn\frac{\beta}{\beta - 1}|u|. \tag{4.20}$$

Then we estimate $\frac{\beta}{\beta-1}$. On the real set $[0, \frac{4}{3}]$, we have :

$$\begin{array}{rcccl} (3/4)\ln(4x/3) & \leq & \ln(1+x) & \leq & x \\ 1 + x & \leq & e^x & \leq & 1 + 3/4(e^{4/3} - 1)x \end{array}$$

Now from the definition of β and since $k\delta \leq 3$ this estimates gives,

$$\frac{\beta}{\beta - 1} \leq \frac{C}{k\delta},$$

then from (4.20) we have :

$$|\Delta u_2| \leq C\frac{n}{\delta}|u|. \tag{4.21}$$

Moreover we also have

$$\left|\frac{\partial u_1}{\partial r}\frac{d\psi_1}{dr}\right| \leq \frac{C}{\delta}\left|\frac{\partial u_1}{\partial r}\right|,$$

with

$$\left|\frac{\partial u_1}{\partial r}\right| \leq Cn|u_1| \leq C\frac{n}{\beta-1}|u| \leq \frac{Cn}{k\delta}|u|,$$

thus

$$\left|\frac{\partial u_1}{\partial r}\frac{d\psi_1}{dr}\right| \leq C\frac{n}{k\delta^2}|u|.$$

This also gives

$$\left|\frac{\partial u_1}{\partial r}\frac{d\psi_1}{dr}\right| \leq CN^{\frac{3}{2}}|u|. \tag{4.22}$$

Similarly we have,

$$\begin{aligned}\left|\frac{d^2\psi_1}{dr^2}+\frac{1}{r}\frac{d\psi_1}{dr}\right||u_1| &\leq C\left(\frac{1}{\delta^2}+\frac{1}{\delta}\right)|u_1| \\ &\leq C\frac{1}{\delta^2}|u_1| \\ &\leq CN^{\frac{3}{2}}|u|.\end{aligned}$$

Here again we have shown

$$|\Delta u| \leq CN^{\frac{3}{2}}|u|.$$

Step I.c $\{(r,\varphi); r \in [\rho+\frac{1}{3}\delta, \rho+\frac{5}{3}\delta],\ \varphi_m+\frac{T}{5} \leq \varphi \leq \varphi_m+\frac{4T}{5}\}$

On this set we have $u = u_1 + u_2$ and

$$|u| = |u_2|\left|e^{i(F(\varphi)+n\varphi)} - \frac{a}{br^{2k}}\right|. \tag{4.23}$$

Let $S(\varphi) = F(\varphi) + n\varphi$, we have $S(\varphi_m) = 2\pi m$ and

$$S'(\varphi) = 2n + 2k + h'(\varphi)$$

Since $|h'(\varphi)| \leq 5k$ we have

$$S'(\varphi) \geq 2n - 3k$$

Now since $n = j^2$ and $k = 2j + 1$, the condition $j \geq 10$ force $S'(\varphi) > n$ for $\varphi \in [\varphi_m, \varphi_{m+1}]$. Then for

$$\varphi_m + \frac{T}{5} \leq \varphi \leq \varphi_{m+1} - \frac{T}{5}$$

we have

$$S(\varphi) \geq S(\varphi_m + \frac{T}{5}) \geq S(\varphi_m) + \frac{nT}{5}$$

Using the same argument to get an upper bound on $S(\varphi)$ we can state that

$$2\pi m + \frac{nT}{5} \leq S(\varphi) \leq 2\pi(m+1) - \frac{nT}{5}$$

This can be written

$$2\pi m + \frac{n\pi}{5(n+k)} \leq S(\varphi) \leq 2\pi(m+1) - \frac{n\pi}{5(n+k)}$$

Now since $j \geq 10$, we clearly have $\frac{n\pi}{5(n+k)} \geq \frac{\pi}{7}$. Then we can state that

$$\left|e^{iS(\varphi)} - \lambda\right| \geq \sin(\frac{\pi}{7}),$$

for any real number λ. This leads to

$$|u| \geq \sin(\frac{\pi}{7})|u_2|. \tag{4.24}$$

Finally using that $\Delta u = \Delta u_2$, (4.14) and (4.24) we have

$$\begin{aligned} |\Delta u| = |\Delta u_2| &\leq Ckn|u_2| \leq ckn|u| \\ |\Delta u| &\leq CN^{\frac{3}{2}}|u| \end{aligned} \tag{4.25}$$

Step I.d) $\{r \in [\rho + \frac{1}{3}\delta, \rho + \frac{5}{3}\delta], |\varphi - \varphi_m| < \frac{T}{5}\}$

Here we just need to notice that $u_2 = -br^{-n+2k}e^{i(n-2k)\varphi}e^{4k\varphi_m}$ is harmonic and that $\psi_1 = \psi_2 = 1$. Then u is harmonic and we simply set $W = 0$.

Step II. Construction on the annulus $[\rho + 2\delta, \rho + 3\delta]$.

Recall that $u_2 = -br^{-n+2k}e^{iF(\varphi)}$. We now define

$$u_3 = -br^{-n+2k}e^{i(n+2k)\varphi}.$$

To pass from u_2 to u_3, we consider a smooth, radial function ψ on $[\rho+2\delta, \rho+3\delta]$ with the following properties :

- $0 \leq \psi(r) \leq 1$ (4.26)
- $\psi(r) = 1$ if $r \leq \rho + \frac{7}{3}\delta$ (4.27)
- $\psi(r) = 0$ if $r \geq \rho + \frac{8}{3}\delta$ (4.28)
- $|\psi^{(p)}(r)| \leq \dfrac{C}{\delta^p}$ (4.29)

Then we set on $[\rho + 2\delta, \rho + 3\delta]$,

$$u = -br^{-n+2k} e^{i[\psi(r)h(\varphi)+(n+2k)\varphi]}$$

with h, the function of lemma 4.2.
Now we prepare the computation of Δu, one has:

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \left(\frac{-n+2k}{r} + i\psi^{'}(r)h(\varphi)\right) u \\
\frac{\partial^2 u}{\partial r^2} &= \frac{(-n+2k)(-n+2k-1)}{r^2} u \\
&+ 2i\frac{-n+2k}{r}\psi'(r)h(\varphi)u + i\psi^{''}(r)h(\varphi)u - \psi^{'}(r)^2 h(\varphi)^2 u \\
\frac{\partial^2 u}{\partial \varphi^2} &= i\psi(r)h^{''}(\varphi)u + \left(i\psi(r)h^{'}(\varphi) + (n+2k)\right)^2 u
\end{aligned}$$

$$\begin{aligned}
\Delta u &= \left(\frac{(-n+2k)(-n+2k-1)}{r^2} + 2i\frac{-n+2k}{r}\psi'(r)h(\varphi)\right) u \\
&+ \left(\frac{-n+2k}{r^2} + \frac{i\psi^{'}(r)h(\varphi)}{r} + i\psi^{''}(r)h(\varphi) - \psi^{'}(r)^2 h(\varphi)^2\right) u \\
&+ \left(\frac{i\psi(r)h^{''}(\varphi)}{r^2} + \frac{\left(i\psi(r)h^{'}(\varphi) + (n+2k)\right)^2}{r^2}\right) u
\end{aligned}$$

We get

$$\begin{aligned}
\Delta u &= \left[\left(\frac{-n+2k}{r} + i\psi^{'}(r)h(\varphi)\right)^2 + ih(r)\left(\frac{\psi^{'}(r)}{r} + \psi^{''}(r)\right)\right] u \\
&+ \left[\frac{i\psi(r)h^{''}(\varphi)}{r^2} - \frac{\left(n+2k+\psi(r)h^{'}(\phi)\right)^2}{r^2}\right] u
\end{aligned}$$

After simplification, the main point is that there is no term of order n^2 left, we obtain :

$$|\Delta u| \leq C(kn + k^2 + |\psi^{'}h|^2 + n|\psi^{'}h| + +|h\psi^{''}| + |\psi h^{''}| + n|\psi||h^{'}| + |\psi^2||h'|^2)u$$

Now using (4.1), the properties of ψ (4.26-4.29) and of h (4.2-4.5), we have

$$|\Delta u| \leq CN^{\frac{3}{2}}|u|.$$

Step III. Construction on the annulus $[\rho + 3\delta, \rho + 4\delta]$.

Recall that

$$u_3 = -br^{-n+2k}e^{i(n+2k)\varphi}.$$

In this step, we want to pass from u_3 to the harmonic function:

$$u_4 = -b_1 r^{-n-2k}e^{i(n+2k)\varphi}.$$

Let $d = (\rho + 3\delta)^{4k}$, $b_1 = bd$ and

$$g(r) = dr^{-4k} = \left(\frac{\rho + 3\delta}{r}\right)^{4k}.$$

Then $|g^p(r)| \leq C^p k^p$, furthermore

$$g(r) \geq \left(\frac{\rho + 3\delta}{\rho + 4\delta}\right)^{4k} \geq e^{4k\ln(1-\frac{\delta}{\rho+4\delta})}$$

We have $0 \leq \frac{\delta}{\rho+4\delta} \leq \frac{1}{4}$ and $\ln(1-x) \geq 4\ln(\frac{3}{4})x$ for x in the real set $[0, \frac{1}{4}]$, then

$$g(r) \geq e^{-16k\ln(\frac{4}{3})\frac{k\delta}{\rho+4\delta}} \geq e^{-16\ln\frac{4}{3}} \geq e^{-5}$$

We know consider a smooth radial function ψ, with the following properties:

- $0 \leq \psi \leq 1$
- $|\psi^{(p)}| \leq \frac{C}{\delta^p}$
- $\psi(r) = 1$ if $r \in [\rho + 3\delta, \rho + (3 + \frac{1}{3})\delta]$
- $\psi(r) = 0$ if $r \in [\rho + (3 + \frac{2}{3})\delta]$

We let $f(r) = \psi(r) + (1 - \psi(r))g(r)$ so one has

$$e^{-5} \leq f(r) \leq 1.$$

A computation of $f'(r)$ and $f^{(2)}(r)$ leads to the following estimates:

$$\begin{aligned} |f^{'}(r)| &\leq C\left(|\psi^{'}(r)| + |\psi^{'}(r)g(r)| + |\psi g^{'}|\right), \\ |f^{'}(r)| &\leq \frac{C}{\delta} + Ck \leq \frac{C}{\delta}, \end{aligned}$$

and

$$|f^{''}(r)| \leq \frac{C}{\delta^2}.$$

We finally set $u = u_3 f$, which implies that $|u| > 0$ and $|u_3| \leq C|u|$. To compute Δu_3 we observe that

$$\begin{aligned} \frac{\partial u_3}{\partial r} &= \frac{-n+2k}{r} u_3, \\ \frac{\partial^2 u_3}{\partial r^2} &= \frac{(-n+2k)(-n+2k-1)}{r^2} u_3, \\ &\text{and} \\ \frac{\partial^2 u_3}{\partial \varphi^2} &= -(n+2k)^2 u_3. \end{aligned}$$

Then we have :

$$\Delta u_3 = \frac{1}{r^2}\left((-n+2k)(-n+2k-1) + (-n+2k) - (n+2k)^2\right) u_3$$

Thus we have the following estimates

$$|\Delta u_3| \leq Ckn|u_3| \leq Ckn|u| \tag{4.30}$$

From $\left|\frac{\partial u_3}{\partial r}\right| \leq Cnu_3$, and $\left|\frac{d^p f}{dr^p}\right| \leq \frac{C}{\delta^p}$ one can deduce that:

$$2\left|\frac{df}{dr}\frac{\partial u_3}{\partial r}\right| \leq C\frac{n}{\delta}u_3 \leq C\frac{n}{\delta}u,$$

and

$$\left|\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr}\right||u_3| \leq \frac{C}{\delta^2}|u_3| \leq \frac{C}{\delta^2}|u|.$$

We then have the following estimate on Δu :

$$\begin{aligned} |\Delta u| &\leq C\left(\frac{n}{\delta} + kn + \frac{1}{\delta^2}\right)|u| \\ |\Delta u| &\leq CN^{\frac{3}{2}}|u| \end{aligned}$$

Step IV. Construction on the annulus $[\rho+4\delta,\rho+6\delta]$
Recall that

$$u_4=-b_1r^{-(n+2k)}e^{i(n+2k)\varphi}$$

and let

$$u_5=cr^{-n-k}e^{-i(n+k)\varphi}. \tag{4.31}$$

We choose c such that

$$\left|\frac{u_5(\rho+5\delta)}{u_4(\rho+5\delta)}\right|=1 \tag{4.32}$$

then

$$\left|\frac{u_5}{u_4}\right|=\frac{r^k}{(\rho+5\delta)^k}.$$

Like step I we consider two smooth radial functions ψ_4 and ψ_5 with the following properties

- $\psi_4=1$ on $[\rho+4\delta,\rho+(4+\frac{5}{3})\delta]$ and $\psi_4=0$ on $[\rho+(5,9)\delta,\rho+6\delta]$
- $\psi_5=0$ on $[\rho+4\delta,\rho+(4,1)\delta]$ and $\psi_5=1$ on $[\rho+(4+\frac{1}{3})\delta,\rho+6\delta]$

We let

$$u=\psi_4u_4+\psi_5u_5.$$

Now we estimate Δu. Here again it is convenient to divide this into three steps:

Step IV.a $[\rho+(4+\frac{1}{3})\delta,\rho+(4+\frac{5}{3})\delta]$

We just have $\psi_4=\psi_5=1$ then $\Delta u=0$ and we set $W=0$.

Step IV.b $[\rho+(4+\frac{5}{3})\delta,\rho+6\delta]$

We have $u=\psi_4u_4+u_5$, then $|u|\geq|u_5|-|u_4|$.

$$\left|\frac{u_5}{u_4}\right|=\frac{r^k}{(\rho+5\delta)^k}\geq\left(\frac{\rho+(4+\frac{5}{3})\delta}{\rho+5\delta}\right)^k\geq e^{k\ln(1+\frac{2\delta}{3(\rho+5\delta)})}$$

Let C_1 such that $\ln(1+x)\geq C_1x$ on $[0,\frac{2}{15}]$.
Then we get

$$\left|\frac{u_5}{u_4}\right|\geq e^{C_1\frac{2k\delta}{3(\rho+5\delta)}}\geq 1+C_1\frac{2k\delta}{3(\rho+5\delta)},$$

since $|u|\geq|u_5|-|u_4|$ this gives

$$|u|\geq C_1\frac{2k\delta}{3(\rho+5\delta)}|u_4|$$

and then

$$|u_4| \leq \frac{C}{k\delta}|u|. \tag{4.33}$$

We also have

$$\Delta u \;=\; 2\frac{d\psi_1}{dr}\frac{\partial u_4}{\partial r} + \left(\frac{\partial^2\psi_4}{\partial r^2} + \frac{1}{r}\frac{\psi_4}{\partial r}\right)u_4.$$

Since $|\frac{\partial u_4}{\partial r}| \leq Cn|u_4|$, we get

$$|\Delta u| \;\leq\; C\frac{n}{\delta}|u_4| + \frac{C}{\delta^2}|u_4| + \frac{C}{\delta}|u_4|.$$

Here again we have obtain from (4.33)

$$|\Delta u| \leq CN^{\frac{3}{2}}|u|.$$

Step IV.c $[\rho + 4\delta, \rho + (4 + \frac{1}{3})\delta]$

This step is similar to step IV.b and therefore the estimation is omitted.

Then, defining $a_{j+1} = c$ with c from (4.31) and(4.32), recall that $n_{j+1} = n_j + k_j = j^2 + 2j + 1$, and define $u(r, \varphi) = u_j(r, \varphi)$ for $\rho_j \leq r \leq \rho_{j+1}$ we have construct on the set

$$I_N = \bigcup_{10\leq j\leq\sqrt{N}} A_j = \bigcup_{10\leq j\leq\sqrt{N}} [\rho + 6(j-1)\delta, \rho + 6j\delta] = [\rho + 54\delta, \rho + 6[,$$

a solution of $\Delta u + Wu = 0$ with $|W| \leq CN^{\frac{3}{2}}$. Now we extend our construction to the whole $\mathbb{R}^2$ in the following way: On $[\rho + 6\delta, +\infty[$ we just keep the harmonic function u obtained on the last annulus $A_{\sqrt{N}}$: $u = a_N r^{-N} e^{-iN\varphi}$. On $[0, \rho + 54\delta]$ we define $g(r)$ a smooth radial function such that

- $g(r) = r^{n_1}$ in $[0, \frac{\rho}{4}]$
- $g(r) = r^{-n_1}$ in $[\frac{3}{4}\rho, \rho + 54\delta]$
- $|g^{(p)}(r)| \leq C\rho^{-1}$, for all r in $[0, \rho + 54\delta]$

The constant C in the last point does not depend on N (the gap from n_1 to $-n_1$ is fixed since $n_1 = 10$). Now define $u = g(r)e^{-in_1\varphi}$. On the compact set $[0, \rho + 10\delta]$ on easily get $\left|\frac{\Delta u}{u}\right| \leq C$ with C independent on N. Finally we have constructed on $\mathbb{R}^2$ a solution of $\Delta u + Wu = 0$ with $\|W\|_{L^\infty} \leq CN^{\frac{3}{2}}$ and $|u(x)| = |x|^{-N}$ for $|x| \geq \rho + 6$. This end the proof of proposition 4.1. $\square$

Now we can proceed to the proof on theorem 1.2.

Proof of theorem 1.2. Then we consider the inverse of the stereographic projection :

$$\begin{array}{rcl} \pi : \mathbb{R}^2 & \to & \mathbb{S}^2 \setminus \mathrm{N} \\ (x,y) & \mapsto & \frac{1}{x^2+y^2+1}(2x, 2y, x^2+y^2-1) \end{array}$$

In the Chart $(\mathbb{S}^2 \setminus (0,0,1), \pi^{-1})$, the canonical metric is written

$$g_{\mathbb{S}^2} = \begin{pmatrix} \frac{4}{(x^2+y^2+1)^2} & 0 \\ 0 & \frac{4}{(x^2+y^2+1)^2} \end{pmatrix}$$

On $\mathbb{S}^2 \setminus \mathrm{N}$, we have

$$\Delta_{\mathbb{S}^2} = \frac{1}{4}(x^2+y^2+1)^2 \Delta_{\mathbb{R}^2}$$

Let $\bar{W}$ and $\bar{u}$ two real valued functions defined on $\mathbb{S}^2 \setminus (0,0,1)$ and C a positive constant. We consider $u = \bar{u} \circ \pi^{-1}$ and $W = \bar{W} \circ \pi^{-1}$, So we have

$$\left\{ \begin{array}{rcl} \Delta_{\mathbb{S}^2} \bar{u} & = & \bar{W}\bar{u} \\ |\bar{W}(x)| & \leq & C \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{rcl} \Delta_{\mathbb{R}^2} u & = & Wu \\ |W(x,y)| & \leq & \frac{C}{(1+x^2+y^2)^2} \end{array} \right.$$

Since the function W we have constructed is compactly supported with $|W(x)| \leq CN^{\frac{3}{2}}$, their exists C' such that

$$\frac{4}{(1+x^2+y^2)}|W(x,y)| \leq C' N^{\frac{3}{2}}, \ \forall (x,y) \in \mathbb{R}^2.$$

It follows that the function $\bar{u}$ is a solution of $\Delta_{\mathbb{S}^2}\bar{u} = \bar{W}\bar{u}$ on $\mathbb{S}^2$, with $\bar{u}$ vanishing at order N at the north pole and with $N \geq \|\bar{W}\|_\infty^{2/3}$.

Finally note that with our construction the potential W in $\mathbb{R}^2$ has support in $[\frac{\rho}{4}, \rho+6]$ where ρ can be chosen arbitrary large. The measure of the image of this set under the inverse of the stereographic projection clearly goes to zero as ρ goes to infinity. This conclude the proof of theorem 1.2 $\square$

References

[1] N. Aronszajn, A. Krzywicki, and J. Szarski. A unique continuation theorem for exterior differential forms on Riemannian manifolds. *Ark. Mat.*, 4:417–453 (1962), 1962.

[2] Laurent Bakri. Quantitative uniqueness for Schrödinger operator. *Indiana Univ. Math. J.* , to appear, available at `http://www.iumj.indiana.edu/IUMJ/Preprints/4713.pdf`.

[3] Jean Bourgain and Carlos E. Kenig. On localization in the continuous Anderson-Bernoulli model in higher dimension. *Invent. Math.*, 161(2):389–426, 2005.

[4] Ferruccio Colombini and Herbert Koch. Strong unique continuation for products of elliptic operators of second order. *Trans. Amer. Math. Soc.*, 362(1):345–355, 2010.

[5] Harold Donnelly and Charles Fefferman. Nodal sets of eigenfunctions on Riemannian manifolds. *Invent. Math.*, 93(1):161–183, 1988.

[6] Harold Donnelly and Charles Fefferman. Nodal sets of eigenfunctions: Riemannian manifolds with boundary. In *Analysis, et cetera*, pages 251–262. Academic Press, Boston, MA, 1990.

[7] Nicola Garofalo and Fang-Hua Lin. Monotonicity properties of variational integrals, A_p weights and unique continuation. *Indiana Univ. Math. J.*, 35(2):245–268, 1986.

[8] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[9] Lars Hörmander. *The analysis of linear partial differential operators. III*. Classics in Mathematics. Springer, Berlin, 2007. Pseudo-differential operators, Reprint of the 1994 edition.

[10] Victor Isakov. Carleman estimates and applications to inverse problems. *Milan J. Math.*, 72:249–271, 2004.

[11] David Jerison and Carlos E. Kenig. Unique continuation and absence of positive eigenvalues for Schrödinger operators. *Ann. of Math. (2)*, 121(3):463–494, 1985. With an appendix by E. M. Stein.

[12] David Jerison and Gilles Lebeau. Nodal sets of sums of eigenfunctions. In *Harmonic analysis and partial differential equations (Chicago, IL, 1996)*, Chicago Lectures in Math., pages 223–239. Univ. Chicago Press, Chicago, IL, 1999.

[13] Carlos E. Kenig. Some recent applications of unique continuation. In *Recent developments in nonlinear partial differential equations*, volume 439 of *Contemp. Math.*, pages 25–56. Amer. Math. Soc., Providence, RI, 2007.

[14] Herbert Koch and Daniel Tataru. Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients. *Comm. Pure Appl. Math.*, 54(3):339–360, 2001.

[15] Herbert Koch and Daniel Tataru. Carleman estimates and absence of embedded eigenvalues. *Comm. Math. Phys.*, 267(2):419–449, 2006.

[16] Herbert Koch and Daniel Tataru. Carleman estimates and unique continuation for second order parabolic equations with nonsmooth coefficients. *Comm. Partial Differential Equations*, 34(4-6):305–366, 2009.

[17] Igor Kukavica. Quantitative uniqueness for second-order elliptic operators. *Duke Math. J.*, 91(2):225–240, 1998.

[18] Jérôme Le Rousseau and Gilles Lebeau. On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations. *ESAIM - Control Optimisation and Calculus of Variations*, page 36pp, 2011.

[19] Fang-Hua Lin. Nodal sets of solutions of elliptic and parabolic equations. *Comm. Pure Appl. Math.*, 44(3):287–308, 1991.

[20] E. Malinnikova and S. Vessella. Quantitative uniqueness for elliptic equations with singular lower order terms. *Math. Ann.*, 353(4):1157–1181, 2012.

[21] V. Z. Meshkov. On the possible rate of decrease at infinity of the solutions of second-order partial differential equations. *Mat. Sb.*, 182(3):364–383, 1991.

[22] Rachid Regbaoui. Unique continuation for differential equations of Schrödinger's type. *Comm. Anal. Geom.*, 7(2):303–323, 1999.

[23] Christopher D. Sogge. Strong uniqueness theorems for second order elliptic differential equations. *Amer. J. Math.*, 112(6):943–984, 1990.