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Critical sets of eigenfunctions of the Laplacian

Bakri Laurent

Abstract

We give an upper bound for the $(n-1)$ -dimensional Hausdorff measure of the critical set of eigenfunctions of the Laplacian on compact analytic n -dimensional Riemannian manifolds. This is the analog of a result on nodal set of eigenfunctions by H. Donnelly and C. Fefferman.

1 Introduction and statement of the results

Let (M, g) be a smooth, compact and connected, n -dimensional Riemannian manifold ($n \geq 2$). For $u \in \mathcal{C}^1(M)$, we set

$$\mathcal{N}_u = \{x \in M : u(x) = 0\}$$

and

$$\mathcal{C}_u = \{x \in M : \nabla u(x) = 0\},$$

the nodal set of u and the critical set respectively. Our interest in this paper is to deal with the critical set \mathcal{C}_u of the eigenfunctions of the Laplacian :

$$-\Delta u = \lambda u, \tag{1.1}$$

Our main result is the following :

Theorem 1.1. *Let M be a n -dimensional, real analytic, compact, connected manifold with analytic metric. There exists $C > 0$ depending only on M such that for any non-constant solution u to (1.1) one has*

$$\mathcal{H}^{n-1}(\mathcal{C}_u) \leq C\sqrt{\lambda},$$

where \mathcal{H}^{n-1} denotes the $(n-1)$ -dimensional Hausdorff measure.

In some way theorem 1.1 gives an analog upper bound, on the measure of the critical set \mathcal{C}_u , of what was already known on the nodal set \mathcal{N}_u . To enlight this fact let us recall briefly some facts about the zero sets \mathcal{N}_u , \mathcal{C}_u and $\mathcal{N}_u \cap \mathcal{C}_u$.

First it is well known that if u is a non trivial solution of second order linear elliptic equation then, in general, all zeros of u are of finite order ([1, 15]), and then one can prove that the Hausdorff dimension of the nodal set \mathcal{N}_u is at most $n - 1$ (for example, see [3] or [11] for more precise results). Concerning the nodal set of eigenfunctions of the Laplacian operator, S. T. Yau, [25] has conjectured, that

Conjecture 1.2.

$$C_1\sqrt{\lambda} \leq \mathcal{H}^{n-1}(\mathcal{N}_u) \leq C_2\sqrt{\lambda}. \quad (1.2)$$

Here C_1, C_2 are positive constants depending only upon M . In case that both the manifold and the metric are real analytic, the problem was solved by H. Donnelly and C. Fefferman [5, 6]. Their method was based on Carleman-type inequalities which have proved to be a powerful tool in strong unique continuation theory ([1, 15, 20, 21, ...]). For smooth metric, the only known upper bound result ($n \geq 3$) is due to R. Hardt and L. Simon [11]. They proved that

$$\mathcal{H}^{n-1}(\mathcal{N}_u) \leq (c\sqrt{\lambda})^{c\sqrt{\lambda}}.$$

However this result is far from conjecture 1.2 and doesn't seem to be optimal. Recently, several authors [4, 12, 19, 22] obtained some lower bounds with polynomial decrease in λ .

Another interesting and related question is the description of the singular set $\mathcal{S}_u = \mathcal{N}_u \cap \mathcal{C}_u$ (some time called critical set or critical zero set) of solutions to partial differential equations. We refer to [2, 8, 9, 10, 18] and the references therein for further investigations. We only mentioned that \mathcal{S} has, in sufficiently favorable cases [8], Hausdorff dimension not greater than $(n - 2)$ and that there exists an analog conjecture to (1.2) which can be stated as follows in the particular case of equation (1.1) (see also [13, 18]) :

Conjecture 1.3.

$$\mathcal{H}^{n-2}(\mathcal{S}_u) \leq C\lambda. \quad (1.3)$$

On the opposite, not much is known about the critical set \mathcal{C}_u . We would like to recall some principal facts. Generically eigenfunctions are Morse functions [23] and therefore the critical set consists in isolated points. Moreover, D. Jakobson and N. Nadirashvili [16] have shown that there exists in dimension two a sequence of eigenfunctions for which the number of critical points is uniformly bounded. An open question is whether the number of critical points goes to infinity with the eigenvalue for generic metrics [24]. The only known result in this direction (to the knowledge of the author) is [26] where it is shown that there exists at least one non-trivial critical point. On the other hand, there exists simple examples for which the critical set has Hausdorff dimension $n - 1$:

Example 1.4. Let (N, g) be a $(n-1)$ -dimensional manifold and define $M = \mathbb{T}^1 \times N$ where \mathbb{T}^1 is the 1-dimensional Torus with standard metric, and M is equipped with the product metric. The function $f_k(x, y) = \sin(2\pi kx)$ is an eigenfunction of Δ_M with eigenvalue $\lambda := 4\pi^2 k^2$. The critical set, \mathcal{C}_{f_k} , of f_k is therefore a set of dimension $n - 1$.

It is also easy to find some surfaces of revolution with critical set of dimension $(n - 1)$, see [27] p 35. In the light of the above, it seems interesting to obtain an upper bound on the $(n - 1)$ -dimensional Hausdorff measure of the critical set, which is achieved with theorem 1.1. One should also notice that in example 1.4 one has $\mathcal{H}^{n-1}(\mathcal{C}_{f_k}) \geq C\sqrt{\lambda}$, where C depends only on M . Therefore the upper bound in theorem 1.1 is sharp.

The framework of this paper is the following. Recall that gradient of eigenfunctions satisfies a system of linear partial differential equations of the following type

$$\Delta V + \lambda V + A \cdot V + B \cdot \nabla V = 0,$$

where $\Delta + \lambda$ acts diagonally on vector valued functions and A (resp. B) is a $(1, 1)$ (resp. $(2, 1)$) tensor which only depend on (M, g) but not on λ . Then, since \mathcal{C}_u is the nodal set of $|\nabla u|$, we will adapt the method of strong uniqueness theory based on Carleman estimates to this system. Indeed from a Carleman estimate on $\Delta + \lambda$ we will be able to obtain the main ingredient in the proof of theorem 1.1 : a doubling inequality on gradient of eigenfunctions,

$$\|\nabla u\|_{B_{2r}} \leq e^{C(\sqrt{\lambda}+1)} \|\nabla u\|_{B_r}. \quad (1.4)$$

In the case M is analytic this doubling estimate will lead to theorem 1.1 by using an argument of Donnelly and Fefferman, see [5] theorem 4.1. When M is only assume to be smooth, we will also be able to give a local upper bound of the same kind on $\mathcal{H}^{n-1}(\mathcal{C}_u)$. To do this we will use a result of C. Bär (lemma 3.1 of [2]), which gives, for smooth functions, a local control on the measure of the nodal set by the vanishing order.

The paper is organised as follows. In section 2, we established a Carleman estimate for the operator $\Delta + \lambda$. In section 3 we establish our doubling inequality (1.4). This relies on compactness of M and standard arguments of quantitative uniqueness theory. Section 4 is devoted to derive estimates on the measure of the critical set.

From now on and until the end we will assume that solutions of (1.1) are non-constants or equivalently that the eigenvalue λ satisfies $\lambda > 0$.

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2 Carleman estimates

Recall that Carleman estimates are weighted integral inequalities with a weight function $e^{\tau\phi}$, where the function ϕ satisfy some convexity properties. Before proceeding to the proof of our Carleman estimate on the scalar operator $\Delta + \lambda$, let us define some standard notations. For a fixed point x_0 in M , we let : $r = r(x) = d(x, x_0)$ the Riemannian distance from x_0 . We denote by $B_r(x_0)$ the geodesic ball centered at x_0 of radius r . We will denote by $\|\cdot\|$ the L^2 norm, on eventually vector-valued functions, with obvious interpretation. Let us first define the weight function we will use.

For a fixed number ε such that $0 < \varepsilon < 1$ and $T_0 < 0$, we define the function f on $] - \infty, T_0[$ by $f(t) = t - e^{\varepsilon t}$. One can check easily that, for $|T_0|$ great enough, the function f verifies the following properties:

$$\begin{aligned} 1 - \varepsilon e^{\varepsilon T_0} &\leq f'(t) \leq 1 \quad \forall t \in] - \infty, T_0[, \\ \lim_{t \rightarrow -\infty} -e^{-t} f''(t) &= +\infty. \end{aligned} \quad (2.1)$$

Finally we define $\phi(x) = -f(\ln r(x))$. Now we can state the main result of this section:

Theorem 2.1. *There exist positive constants R_0, C, C_1, C_2 which depend only on M and ε , such that, for any $\lambda \geq 0$, any $x_0 \in M$, any $\delta \in (0, R_0)$, any $u \in C_0^\infty(B_{R_0}(x_0) \setminus B_\delta(x_0))$ and any $\tau \geq C_1\sqrt{\lambda} + C_2$, one has*

$$\begin{aligned} C \left\| r^2 e^{\tau\phi} (\Delta u + \lambda u) \right\| &\geq \tau^{\frac{3}{2}} \left\| r^{\frac{\varepsilon}{2}} e^{\tau\phi} u \right\| \\ &+ \tau\delta \left\| r^{-1} e^{\tau\phi} u \right\| + \tau^{\frac{1}{2}} \left\| r^{1+\frac{\varepsilon}{2}} e^{\tau\phi} \nabla u \right\|. \end{aligned} \quad (2.2)$$

Remark 2.2. This Carleman estimate is quite standard. However in this paper, an estimation including the gradient term in the right hand side is useful. Though it can be obtained from previous Carleman estimates ([1], [5] or [17] by example), we choose for the shake of completeness to give a proof.

The most important part to establish Theorem 2.1 is the following :

Lemma 2.3. *There exist positive constants R_0, C, C_1, C_2 which depend only on M and ε , such that, for any $\lambda > 0$, $x_0 \in M$, $u \in C_0^\infty(B_{R_0}(x_0) \setminus \{x_0\})$ and $\tau \geq C_1\sqrt{\lambda} + C_2$, one has*

$$C \left\| r^2 e^{\tau\phi} (\Delta u + \lambda u) \right\| \geq \tau^{\frac{3}{2}} \left\| r^{\frac{\varepsilon}{2}} e^{\tau\phi} u \right\| + \tau^{\frac{1}{2}} \left\| r^{1+\frac{\varepsilon}{2}} e^{\tau\phi} \nabla u \right\|. \quad (2.3)$$

proof of Lemma 2.3. Hereafter C, C_1 , and c denote positive constants depending only upon M , though their values may change from one line to another. Without loss of generality, we may suppose that all functions are

real. We now introduce the polar geodesic coordinates (r, θ) near x_0 . Using Einstein notation, the Laplace operator takes the form :

$$r^2 \Delta u = r^2 \partial_r^2 u + r^2 \left(\partial_r \ln(\sqrt{\gamma}) + \frac{n-1}{r} \right) \partial_r u + \frac{1}{\sqrt{\gamma}} \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j u),$$

where $\partial_i = \frac{\partial}{\partial \theta_i}$ and for each fixed r , $\gamma_{ij}(r, \theta)$ is a metric on \mathbb{S}^{n-1} and $\gamma = \det(\gamma_{ij})$.

Since (M, g) is smooth, we have for r small enough :

$$\begin{aligned} \partial_r(\gamma^{ij}) &\leq C(\gamma^{ij}) \quad (\text{in the sense of tensors}); \\ |\partial_r(\gamma)| &\leq C; \\ C^{-1} &\leq \gamma \leq C. \end{aligned} \tag{2.4}$$

Set $r = e^t$, we have $\frac{\partial}{\partial r} = e^{-t} \frac{\partial}{\partial t}$. Then the function u has supported in $] -\infty, T_0[\times \mathbb{S}^{n-1}$, where $|T_0|$ will be chosen large enough. In this new variables, we can write :

$$e^{2t} \Delta u = \partial_t^2 u + (n-2 + \partial_t \ln \sqrt{\gamma}) \partial_t u + \frac{1}{\sqrt{\gamma}} \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j u).$$

The inequalities (2.4) become

$$\begin{aligned} \partial_t(\gamma^{ij}) &\leq C e^t (\gamma^{ij}) \quad (\text{in the sense of tensors}); \\ |\partial_t(\gamma)| &\leq C e^t; \\ C^{-1} &\leq \gamma \leq C. \end{aligned} \tag{2.5}$$

Now we introduce the conjugate operator :

$$\begin{aligned} L_\tau(u) &= e^{2t} e^{\tau \phi} \Delta(e^{-\tau \phi} u) + e^{2t} \lambda u \\ &= \partial_t^2 u + (2\tau f' + n-2 + \partial_t \ln \sqrt{\gamma}) \partial_t u \\ &\quad + \left(\tau^2 f'^2 + \tau f'' + (n-2)\tau f' + \tau \partial_t \ln \sqrt{\gamma} f' \right) u \\ &\quad + \Delta_\theta u + e^{2t} \lambda u, \end{aligned} \tag{2.6}$$

with

$$\Delta_\theta u = \frac{1}{\sqrt{\gamma}} \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j u).$$

It will be useful for us to introduce the following L^2 norm on $] -\infty, T_0[\times \mathbb{S}^{n-1}$:

$$\|V\|_f^2 = \int_{]-\infty, T_0[\times \mathbb{S}^{n-1}} V^2 \sqrt{\gamma} f'^{-3} dt d\theta,$$

where $d\theta$ is the usual measure on \mathbb{S}^{n-1} . The corresponding inner product is denoted by $\langle \cdot, \cdot \rangle_f$, *i.e*

$$\langle u, v \rangle_f = \int uv \sqrt{\gamma} f'^{-3} dt d\theta.$$

We will estimate from below $\|L_\tau u\|_f^2$ by using elementary algebra and integrations by parts. We are concerned, in the computation, by the power of τ and exponential decay when t goes to $-\infty$. First by triangular inequality one has

$$\|L_\tau(u)\|_f^2 \geq \frac{1}{2}I - II, \quad (2.7)$$

with

$$\begin{aligned} I &= \left\| \partial_t^2 u + 2\tau f' \partial_t u + \tau^2 f'^2 u + \lambda e^{2t} u + \Delta_\theta u \right\|_f^2, \\ II &= \left\| \tau f'' u + (n-2)\tau f' u + \tau \partial_t \ln \sqrt{\gamma} f' u \right. \\ &\quad \left. + (n-2)\partial_t u + \partial_t \ln \sqrt{\gamma} \partial_t u \right\|_f^2. \end{aligned} \quad (2.8)$$

We will be able to absorb II later. Then we compute I :

$$I = I_1 + I_2 + I_3,$$

with

$$\begin{aligned} I_1 &= \|\partial_t^2 u + (\tau^2 f'^2 + \lambda e^{2t})u + \Delta_\theta u\|_f^2 \\ I_2 &= \|2\tau f' \partial_t u\|_f^2 \\ I_3 &= 2 \left\langle 2\tau f' \partial_t u, \partial_t^2 u + \tau^2 f'^2 u + \lambda e^{2t} u + \Delta_\theta u \right\rangle_f \end{aligned} \quad (2.9)$$

In order to compute I_3 we write it in a convenient way:

$$I_3 = J_1 + J_2 + J_3, \quad (2.10)$$

where the integrals J_i are defined by :

$$\begin{aligned} J_1 &= 2\tau \int f' \partial_t (|\partial_t u|^2) f'^{-3} \sqrt{\gamma} dt d\theta \\ J_2 &= 4\tau \int f' \partial_t u \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j u) f'^{-3} dt d\theta \\ J_3 &= \int (2\tau^3 (f')^3 + 2\lambda \tau f' e^{2t}) 2u \partial_t u f'^{-3} \sqrt{\gamma} dt d\theta. \end{aligned} \quad (2.11)$$

Now we will use integration by parts to estimate each terms of (2.11). Note that f is radial and that $2\partial_t u \partial_t^2 u = \partial_t (|\partial_t u|^2)$. We find :

$$\begin{aligned} J_1 &= \int (4\tau f'') |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\ &\quad - \int 2\tau f' \partial_t \ln \sqrt{\gamma} |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \end{aligned}$$

The conditions (2.5) imply that $|\partial_t \ln \sqrt{\gamma}| \leq C e^t$. Then properties (2.1) on f gives, for large $|T_0|$ that $|\partial_t \ln \sqrt{\gamma}|$ is small compared to $|f''|$. Then one has

$$J_1 \geq -c\tau \int |f''| \cdot |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \quad (2.12)$$

Now in order to estimate J_2 we first integrate by parts with respect to ∂_i :

$$J_2 = -2 \int 2\tau f' \partial_t \partial_i u \gamma^{ij} \partial_j u f'^{-3} \sqrt{\gamma} dt d\theta.$$

Then we integrate by parts with respect to ∂_t . We get :

$$\begin{aligned} J_2 &= -4\tau \int f'' \gamma^{ij} \partial_i u \partial_j u f'^{-3} \sqrt{\gamma} dt d\theta \\ &\quad + \int 2\tau f' \partial_t \ln \sqrt{\gamma} \gamma^{ij} \partial_i u \partial_j u f'^{-3} \sqrt{\gamma} dt d\theta \\ &\quad + \int 2\tau f' \partial_t (\gamma^{ij}) \partial_i u \partial_j u f'^{-3} \sqrt{\gamma} dt d\theta. \end{aligned}$$

We denote $|D_\theta u|^2 = \partial_i u \gamma^{ij} \partial_j u$. Now using that $-f''$ is non-negative and τ is large, the conditions (2.1) and (2.5) gives for $|T_0|$ large enough:

$$J_2 \geq 3\tau \int |f''| \cdot |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \quad (2.13)$$

Similarly computation of J_3 gives :

$$\begin{aligned} J_3 &= -2\tau^3 \int \partial_t \ln(\sqrt{\gamma}) u^2 \sqrt{\gamma} dt d\theta \\ &\quad - \lambda \tau \int (4f' - 4f'' + 2f' \partial_t \ln \sqrt{\gamma}) e^{2t} u^2 f'^{-3} \sqrt{\gamma} dt d\theta. \end{aligned} \quad (2.14)$$

Now we assume that

$$\tau \geq C_1 \sqrt{\lambda} + C_2. \quad (2.15)$$

From (2.1) and (2.5) one can see that if C_1, C_2 and $|T_0|$ are large enough, then

$$J_3 \geq -c\tau^3 \int e^t |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \quad (2.16)$$

Thus far, using (2.12),(2.13) and (2.16), we have :

$$\begin{aligned} I_3 &\geq 3\tau \int |f''| |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta - c\tau^3 \int e^t |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\ &\quad - c\tau \int |f''| |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \end{aligned} \quad (2.17)$$

Now we consider I_1 :

$$I_1 = \left\| \partial_t^2 u + \left(\tau^2 f'^2 + \lambda e^{2t} \right) u + \Delta_\theta u \right\|_f^2.$$

Let $\rho > 0$ a small number to be chosen later. Since $|f''| \leq 1$ and $\tau \geq 1$, we have :

$$I_1 \geq \frac{\rho}{\tau} I'_1, \quad (2.18)$$

where I'_1 is defined by :

$$I'_1 = \left\| \sqrt{|f''|} \left[\partial_t^2 u + \left(\tau^2 f'^2 + \lambda e^{2t} \right) u + \Delta_\theta u \right] \right\|_f^2 \quad (2.19)$$

and one has

$$I'_1 = K_1 + K_2 + K_3, \quad (2.20)$$

with

$$\begin{aligned}
K_1 &= \left\| \sqrt{|f''|} (\partial_t^2 u + \Delta_\theta u) \right\|_f^2, \\
K_2 &= \left\| \sqrt{|f''|} (\tau^2 f'^2 + \lambda e^{2t}) u \right\|_f^2, \\
K_3 &= 2 \left\langle (\partial_t^2 u + \Delta_\theta u) |f''|, (\tau^2 f'^2 + \lambda e^{2t}) u \right\rangle_f.
\end{aligned} \tag{2.21}$$

Integrating by parts gives :

$$\begin{aligned}
K_3 &= 2 \int f'' (\tau^2 f'^2 + \lambda e^{2t}) |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\
&\quad + 2 \int \partial_t \left[f'' (\tau^2 f'^2 + \lambda e^{2t}) \right] \partial_t u u \sqrt{\gamma} f'^{-3} dt d\theta \\
&\quad - 6 \int \left(f''^2 f'^{-1} (\tau^2 f'^2 + \lambda e^{2t}) \right) \partial_t u u \sqrt{\gamma} f'^{-3} dt d\theta \\
&\quad + 2 \int f'' (\tau^2 f'^2 + \lambda e^{2t}) \partial_t \ln \sqrt{\gamma} \partial_t u u f'^{-3} \sqrt{\gamma} dt d\theta \\
&\quad + 2 \int f'' (\tau^2 f'^2 + \lambda e^{2t}) |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta.
\end{aligned} \tag{2.22}$$

Now since $2\partial_t u u \leq u^2 + |\partial_t u|^2$ and $\tau \geq C_1 \sqrt{\lambda} + C_2$, we can use conditions (2.1) and (2.5) to get

$$K_3 \geq -c\tau^2 \int |f''| (|\partial_t u|^2 + |D_\theta u|^2 + |u|^2) f'^{-3} \sqrt{\gamma} dt d\theta \tag{2.23}$$

We also have

$$K_2 \geq c\tau^4 \int |f''| \cdot |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \tag{2.24}$$

and since $K_1 \geq 0$,

$$\begin{aligned}
I_1 &\geq -\rho c\tau \int |f''| (|\partial_t u|^2 + |D_\theta u|^2) f'^{-3} \sqrt{\gamma} dt d\theta \\
&\quad + C\tau^3 \rho \int |f''| |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta.
\end{aligned} \tag{2.25}$$

Then using (2.17) and (2.25), we get :

$$\begin{aligned}
I &\geq 4\tau^2 \|f' \partial_t u\|_f^2 + 3\tau \int |f''| |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\
&\quad + C\tau^3 \rho \int |f''| |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta - c\tau^3 \int e^t |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\
&\quad - \rho c\tau \int |f''| (|u|^2 + |\partial_t u|^2 + |D_\theta u|^2) f'^{-3} \sqrt{\gamma} dt d\theta \\
&\quad - c\tau \int |f''| |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta.
\end{aligned} \tag{2.26}$$

Now one needs to check that every non-positive terms in the right hand side of (2.26) can be absorbed in the first three terms.

First fix ρ small enough such that

$$\rho c\tau \int |f''| \cdot |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \leq 2\tau \int |f''| \cdot |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta$$

where c is the constant appearing in (2.26). The other terms in the last integral of (2.26) can then be absorbed by comparing powers of τ (for C_2 large enough). Finally since conditions (2.1) imply that e^t is small compared to $|f''|$, we can absorb $-c\tau^3 e^t |u|^2$ in $C\tau^3 \rho |f''| |u|^2$.

Thus we obtain :

$$I \geq C\tau^2 \int |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta + C\tau \int |f''| |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta + C\tau^3 \int |f''| |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \quad (2.27)$$

As before, we can check that II can be absorbed in I for $|T_0|$ and τ large enough. Then we obtain

$$\|L_\tau u\|_f^2 \geq C\tau^3 \|\sqrt{|f''|} u\|_f^2 + C\tau^2 \|\partial_t u\|_f^2 + C\tau \|\sqrt{|f''|} D_\theta u\|_f^2. \quad (2.28)$$

Note that, since τ is large and $\sqrt{|f''|} \leq 1$, one has

$$\|L_\tau u\|_f^2 \geq C\tau^3 \|\sqrt{|f''|} u\|_f^2 + c\tau \|\sqrt{|f''|} \partial_t u\|_f^2 + C\tau \|\sqrt{|f''|} D_\theta u\|_f^2, \quad (2.29)$$

and the constant c can be chosen arbitrary smaller than C . If we set $v = e^{-\tau\phi} u$, then we have

$$\begin{aligned} \|e^{2t} e^{\tau\phi} (\Delta v + \lambda v)\|_f^2 &\geq C\tau^3 \|\sqrt{|f''|} e^{\tau\phi} v\|_f^2 - c\tau^3 \|\sqrt{|f''|} f' e^{\tau\phi} v\|_f^2 \\ &+ \frac{c}{2}\tau \|\sqrt{|f''|} e^{\tau\phi} \partial_t v\|_f^2 + C\tau \|\sqrt{|f''|} e^{\tau\phi} D_\theta v\|_f^2. \end{aligned} \quad (2.30)$$

Finally since f' is close to 1, one can absorb the negative term to obtain

$$\begin{aligned} \|e^{2t} e^{\tau\phi} (\Delta v + \lambda v)\|_f^2 &\geq C\tau^3 \|\sqrt{|f''|} e^{\tau\phi} v\|_f^2 \\ &+ C\tau \|\sqrt{|f''|} e^{\tau\phi} \partial_t v\|_f^2 + C\tau \|\sqrt{|f''|} e^{\tau\phi} D_\theta v\|_f^2. \end{aligned} \quad (2.31)$$

It remains to get back to the usual L^2 norm. First note that since f' is close to 1 (2.1), we can get the same estimate without the term $(f')^{-3}$ in the integrals. Recall that in polar coordinates (r, θ) the volume element is $r^{n-1} \sqrt{\gamma} dr d\theta$, we can deduce from (2.27) by substitution that :

$$\begin{aligned} \|r^2 e^{\tau\phi} (\Delta v + \lambda v) r^{-\frac{n}{2}}\|^2 &\geq C\tau^3 \|r^{\frac{\varepsilon}{2}} e^{\tau\phi} v r^{-\frac{n}{2}}\|^2 \\ &+ C\tau \|r^{1+\frac{\varepsilon}{2}} e^{\tau\phi} \nabla v r^{-\frac{n}{2}}\|^2. \end{aligned} \quad (2.32)$$

Finally one can get rid of the term $r^{-\frac{n}{2}}$ by replacing τ with $\tau + \frac{n}{2}$. Indeed from $e^{\tau\phi} r^{-\frac{n}{2}} = e^{(\tau+\frac{n}{2})\phi} e^{-\frac{n}{2}\phi} r^{-\frac{n}{2}}$ one can check easily that, for r small enough

$$\frac{1}{2} e^{(\tau+\frac{n}{2})\phi} \leq e^{\tau\phi} r^{-\frac{n}{2}} \leq e^{(\tau+\frac{n}{2})\phi}.$$

This achieves the proof of the first part of lemma 2.3. \square

Now we can use the additional information about how far from the origin $\text{supp}(u)$ is, to prove theorem 2.1. This extra term will be useful in section 3.

proof of Theorem 2.1. Suppose that $\text{supp}(u) \subset B_{R_0}(x_0) \setminus B_\delta(x_0)$ and define $T_1 = \ln \delta$.

Cauchy-Schwarz inequality apply to

$$\int \partial_t(u^2)e^{-t}\sqrt{\gamma}dtd\theta = 2 \int u\partial_t u e^{-t}\sqrt{\gamma}dtd\theta,$$

gives

$$\int \partial_t(u^2)e^{-t}\sqrt{\gamma}dtd\theta \leq 2 \left(\int (\partial_t u)^2 e^{-t}\sqrt{\gamma}dtd\theta \right)^{\frac{1}{2}} \left(\int u^2 e^{-t}\sqrt{\gamma}dtd\theta \right)^{\frac{1}{2}}. \quad (2.33)$$

On the other hand, integrating by parts gives

$$\int \partial_t(u^2)e^{-t}\sqrt{\gamma}dtd\theta = \int u^2 e^{-t}\sqrt{\gamma}dtd\theta - \int u^2 e^{-t}\partial_t(\ln(\sqrt{\gamma}))\sqrt{\gamma}dtd\theta. \quad (2.34)$$

Now since $|\partial_t \ln \sqrt{\gamma}| \leq C e^t$ for $|T_0|$ large enough we can deduce :

$$\int \partial_t(u^2)e^{-t}\sqrt{\gamma}dtd\theta \geq c \int u^2 e^{-t}\sqrt{\gamma}dtd\theta. \quad (2.35)$$

Combining (2.33) and (2.35) gives

$$\begin{aligned} c^2 \int u^2 e^{-t}\sqrt{\gamma}dtd\theta &\leq 4 \int (\partial_t u)^2 e^{-t}\sqrt{\gamma}dtd\theta \\ &\leq 4e^{-T_1} \int (\partial_t u)^2 \sqrt{\gamma}dtd\theta. \end{aligned}$$

Finally, dropping all terms except $\tau^2 \int |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dtd\theta$ in (2.27) gives :

$$C'I \geq \tau^2 \delta^2 \|e^{-t}u\|_f^2.$$

Inequality (2.27) can then be replaced by :

$$\begin{aligned} I &\geq C\tau^2 \int |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dtd\theta + C\tau \int |f''| \cdot |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dtd\theta \\ &\quad + C\tau^3 \int |f''| \cdot |u|^2 f'^{-3} \sqrt{\gamma} dtd\theta + C\tau^2 \delta^2 \int |u|^2 f'^{-3} \sqrt{\gamma} dtd\theta. \end{aligned} \quad (2.36)$$

The rest of the proof follows in a similar way than the proof of Lemma 2.3. \square

Now we will state a Carleman estimate for the operator $\Delta + \lambda$ acting on vector functions, which will be useful in the next section. For $U \in \mathcal{C}_0^\infty(B_{R_0}(x_0) \setminus \{x_0\}, \mathbb{R}^m)$, applying (2.3) to each components U^i of U and summing gives :

Corollary 2.4. *There exists non-negative constants R_0, C, C_1, C_2 , which depend only on M and ε , such that :*

For any $x_0 \in M$, any $\delta \in (0, R_0)$, any $U \in C_0^\infty(B_{R_0}(x_0) \setminus (B_\delta(x_0), \mathbb{R}^m))$, and any $\tau \geq C_1\sqrt{\lambda} + C_2$, one has :

$$\begin{aligned} C \left\| r^2 e^{-\tau\phi} (\Delta U + \lambda U) \right\| &\geq \tau^{\frac{3}{2}} \left\| r^{\frac{\varepsilon}{2}} e^{-\tau\phi} U \right\| \\ &+ \tau\delta \left\| r^{-1} e^{\tau\phi} U \right\| + \tau^{\frac{1}{2}} \left\| r^{1+\frac{\varepsilon}{2}} e^{\tau\phi} \nabla U \right\|. \end{aligned} \quad (2.37)$$

3 Doubling inequality

In this section we intend to prove a doubling property for gradient of eigenfunctions. First we establish a three balls theorem :

Proposition 3.1 (Three balls theorem). *There exist non-negative constants R_0, c and $0 < \alpha < 1$ which depend only on M such that, if u is a solution to (1.1) one has :*

for any R such that $0 < R < 2R < R_0$, and any $x_0 \in M$,

$$\|\nabla u\|_{B_R(x_0)} \leq e^{c(\sqrt{\lambda}+1)} \|\nabla u\|_{B_{\frac{R}{2}}(x_0)}^\alpha \|\nabla u\|_{B_{2R}(x_0)}^{1-\alpha} \quad (3.1)$$

Proof. Let x_0 a point in M and (x_1, x_2, \dots, x_n) local coordinates around x_0 . Let u be a solution to (1.1) and define $V = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n})$. Let $R_0 > 0$ as in theorem (2.3) and R such that $0 < R < 2R < R_0$. We still denote $r(x)$ the riemannian distance between x and x_0 . We also denote by B_r the geodesic ball centered at x_0 of radius r . If v is a function defined in a neighborhood of x_0 , we denote by $\|v\|_R$ the L^2 norm of v on B_R and by $\|v\|_{R_1, R_2}$ the L^2 norm of v on the set $A_{R_1, R_2} := \{x \in M; R_1 \leq r(x) \leq R_2\}$. Let $\psi \in C_0^\infty(B_{2R})$, $0 \leq \psi \leq 1$, a radial function with the following properties :

- $\psi(x) = 0$ if $r(x) < \frac{R}{4}$ or if $r(x) > \frac{5R}{3}$,
- $\psi(x) = 1$ if $\frac{R}{3} < r(x) < \frac{3R}{2}$,
- $|\nabla\psi(x)| \leq \frac{C}{R}$, $|\nabla^2\psi(x)| \leq \frac{C}{R^2}$.

Now, applying ∂_k to each side of (1.1) gives

$$-\Delta\partial_k u + [\Delta, \partial_k]u = \lambda\partial_k u$$

where $[\Delta, \partial_k]$ is the commutator of Δ and ∂_k . It is a second order operator with no zero order term and with coefficients depending only of M . The function $V = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n})$ is therefore a solution of the system :

$$\Delta V + \lambda V + AV + B \cdot \nabla V = 0 \quad (3.2)$$

where A and B depend only on the metric g of M and its derivatives. Now we apply the Carleman estimate (2.37) to the function ψV , we get :

$$C \left\| r^2 e^{\tau\phi} (\Delta(\psi V) + \lambda\psi V) \right\| \geq \tau^{\frac{3}{2}} \left\| r^{\frac{\varepsilon}{2}} e^{\tau\phi} \psi V \right\| \\ + \tau R \left\| r^{-1} e^{\tau\phi} \psi V \right\| + \tau^{\frac{1}{2}} \left\| r^{1+\frac{\varepsilon}{2}} e^{\tau\phi} \nabla(\psi V) \right\|.$$

Using that V is a solution of (3.2), we have :

$$C \left\| r^2 e^{\tau\phi} (\psi AV + \psi B \cdot \nabla V + 2\nabla V \cdot \nabla \psi + \Delta\psi V) \right\| \geq \tau^{\frac{3}{2}} \left\| r^{\frac{\varepsilon}{2}} e^{\tau\phi} \psi V \right\| \\ + \tau R \left\| r^{-1} e^{\tau\phi} \psi V \right\| + \tau^{\frac{1}{2}} \left\| r^{1+\frac{\varepsilon}{2}} e^{\tau\phi} \nabla(\psi V) \right\|$$

Now from triangular inequality we get

$$C \left\| r^2 e^{\tau\phi} (\Delta\psi V + 2\nabla V \cdot \nabla \psi) \right\| \geq \tau^{\frac{3}{2}} \left\| r^{\frac{\varepsilon}{2}} e^{\tau\phi} \psi V \right\| - C \left\| r^2 e^{\tau\phi} \psi AV \right\| \\ + \tau R \left\| r^{-1} e^{\tau\phi} \psi V \right\| + \tau^{\frac{1}{2}} \left\| r^{1+\frac{\varepsilon}{2}} e^{-\tau\phi} \nabla(\psi V) \right\| - C \left\| r^2 e^{\tau\phi} \psi B \cdot \nabla V \right\|$$

and

$$\tau^{\frac{1}{2}} \left\| r^{1+\frac{\varepsilon}{2}} e^{\tau\phi} \nabla(\psi V) \right\| \geq \tau^{\frac{1}{2}} \left\| r^{1+\frac{\varepsilon}{2}} e^{\tau\phi} \psi \nabla V \right\| - \tau^{\frac{1}{2}} \left\| r^{1+\frac{\varepsilon}{2}} e^{\tau\phi} \nabla \psi V \right\| \\ \geq \tau^{\frac{1}{2}} \left\| r^{1+\frac{\varepsilon}{2}} e^{\tau\phi} \psi \nabla V \right\| - \tau^{\frac{1}{2}} \left\| r^{\frac{\varepsilon}{2}} e^{\tau\phi} V \right\|$$

Then for τ great enough and for sufficient small R_0 ,

$$C \left\| r^2 e^{\tau\phi} (\Delta\psi V + 2\nabla V \cdot \nabla \psi) \right\| \geq \tau^{\frac{3}{2}} \left\| r^{\frac{\varepsilon}{2}} e^{-\tau\phi} \psi V \right\| \\ + \tau R \left\| r^{-1} e^{-\tau\phi} \psi V \right\| + \tau^{\frac{1}{2}} \left\| r^{1+\frac{\varepsilon}{2}} e^{-\tau\phi} \psi \nabla V \right\|.$$

In particular we have :

$$C \left\| r^2 e^{\tau\phi} (\Delta\psi V + 2\nabla V \cdot \nabla \psi) \right\| \geq \tau \left\| e^{\tau\phi} \psi V \right\|$$

Assume that $\tau \geq 1$ and use properties of ψ gives :

$$\|e^{\tau\phi} V\|_{\frac{R}{3}, \frac{3R}{2}} \leq C \left(\|e^{\tau\phi} V\|_{\frac{R}{4}, \frac{R}{3}} + \|e^{\tau\phi} V\|_{\frac{3R}{2}, \frac{5R}{3}} \right) \\ + C \left(R \|e^{\tau\phi} \nabla V\|_{\frac{R}{4}, \frac{R}{3}} + R \|e^{\tau\phi} \nabla V\|_{\frac{3R}{2}, \frac{5R}{3}} \right). \quad (3.3)$$

Furthermore, since ϕ is radial and decreasing, one has

$$\|e^{\tau\phi} V\|_{\frac{R}{3}, \frac{3R}{2}} \leq C \left(e^{\tau\phi(\frac{R}{4})} \|V\|_{\frac{R}{4}, \frac{R}{3}} + e^{\tau\phi(\frac{3R}{2})} \|V\|_{\frac{3R}{2}, \frac{5R}{3}} \right) \\ + C \left(R e^{\tau\phi(\frac{R}{4})} \|\nabla V\|_{\frac{R}{4}, \frac{R}{3}} + R e^{\tau\phi(\frac{3R}{2})} \|\nabla V\|_{\frac{3R}{2}, \frac{5R}{3}} \right). \quad (3.4)$$

Now we recall the following elliptic estimates : since V satisfies (3.2) then it is not hard to see that :

$$\|\nabla V\|_{(1-a)R} \leq C \left(\frac{1}{(1-a)R} + \sqrt{\lambda} \right) \|V\|_R, \quad \text{for } 0 < a < 1 \quad (3.5)$$

Since $\|e^{\tau\phi}\nabla V\|_{\frac{R}{4}, \frac{R}{3}}$ is bounded by $\|e^{\tau\phi}\nabla V\|_{\frac{R}{3}}$, using the formula (3.5) gives:

$$e^{\tau\phi(\frac{R}{4})}\|\nabla V\|_{\frac{R}{4}, \frac{R}{3}} \leq C \left(\frac{1}{R} + \sqrt{\lambda} \right) e^{\tau\phi(\frac{R}{4})}\|V\|_{\frac{R}{2}}. \quad (3.6)$$

Similarly, we also have,

$$e^{\tau\phi(\frac{3R}{2})}\|\nabla V\|_{\frac{3R}{2}, \frac{5R}{3}} \leq C \left(\frac{1}{R} + \sqrt{\lambda} \right) e^{\tau\phi(\frac{3R}{2})}\|V\|_{2R}. \quad (3.7)$$

On the other hand, using properties of ϕ leads to :

$$\|e^{\tau\phi}V\|_{\frac{R}{3}, \frac{3R}{2}} \geq \|e^{\tau\phi}V\|_{\frac{R}{3}, R} \geq e^{\tau\phi(R)}\|V\|_{\frac{R}{3}, R}. \quad (3.8)$$

Now using (3.6),(3.7) and (3.8) in (3.3) we get :

$$\|V\|_{\frac{R}{3}, R} \leq C\sqrt{\lambda} \left(e^{\tau(\phi(\frac{R}{4})-\phi(R))}\|V\|_{\frac{R}{2}} + e^{\tau(\phi(\frac{3R}{2})-\phi(R))}\|V\|_{2R} \right)$$

Let $A_R = \phi(\frac{R}{4})-\phi(R)$ and $B_R = -(\phi(\frac{3R}{2})-\phi(R))$. Because of the properties of ϕ , we have $0 < C_1 \leq A_R \leq C_2$ and $0 < C_1 \leq B_R \leq C_2$ where C_1 and C_2 don't depend on R . We may assume that $C\sqrt{\lambda} \geq 2$. We can add $\|V\|_{\frac{R}{3}}$ to each member and bound it in the right hand side by $C\sqrt{\lambda}e^{\tau A}\|V\|_{\frac{R}{2}}$, for τ large enough. Then replacing C by $2C$ gives :

$$\|V\|_R \leq C\sqrt{\lambda} \left(e^{\tau A}\|V\|_{\frac{R}{2}} + e^{-\tau B}\|V\|_{2R} \right). \quad (3.9)$$

Now we want to find τ such that

$$C\sqrt{\lambda}e^{-\tau B}\|V\|_{2R} \leq \frac{1}{2}\|V\|_R$$

Since τ must satisfy $\tau \geq C_1\sqrt{\lambda} + C_2$, we choose

$$\tau = -\frac{1}{B} \ln \left(\frac{1}{2C\sqrt{\lambda}} \frac{\|V\|_R}{\|V\|_{2R}} \right) + C_1\sqrt{\lambda} + C_2. \quad (3.10)$$

Inequality (3.9) becomes

$$\|V\|_R \leq C\sqrt{\lambda}e^{C_1\sqrt{\lambda}+C_2}e^{\frac{-A}{B} \ln \left(\frac{1}{2C\sqrt{\lambda}} \frac{\|V\|_R}{\|V\|_{2R}} \right)}\|V\|_{\frac{R}{2}},$$

wich gives easily

$$\|V\|_R \leq e^{(C_1\sqrt{\lambda}+C_2)\frac{B}{A+B}} \|V\|_{2R}^{\frac{A}{A+B}} \|V\|_{\frac{R}{2}}^{\frac{B}{B+A}}.$$

Finally define $\alpha = \frac{A}{A+B}$ gives easily

$$\|V\|_R \leq e^{C_5\sqrt{\lambda}+C_6} \|V\|_{2R}^\alpha \|V\|_{\frac{R}{2}}^{1-\alpha}.$$

□

From now on we assume that M is compact. Thus we can derive from three balls theorem above uniform doubling estimates on gradient of eigenfunctions.

Theorem 3.2 (doubling estimates). *There exists non-negative constants R_0, C_1, C_2 depending only on M such that : if u is a solution to (1.1) on M , then for any $x_0 \in M$ and any $r > 0$,*

$$\|\nabla u\|_{B_{2r}(x_0)} \leq e^{C_1\sqrt{\lambda}+C_2} \|\nabla u\|_{B_r(x_0)}. \quad (3.11)$$

Remark 3.3. Using standard elliptic theory (see [7]) to bound the L^∞ norm of $|V|$ by a multiple of its L^2 norm and rescaling arguments gives for $\delta > 0$:

$$\|V\|_{L^\infty(B_\delta(x_0))} \leq (C_1\lambda + C_2)^{\frac{n}{2}} \delta^{-n/2} \|u\|_{L^2(B_{2\delta}(x_0))}$$

Then one can see that the doubling estimate is still true with the L^∞ norm

$$\|V\|_{L^\infty(B_{2r}(x_0))} \leq e^{C(\sqrt{\lambda}+1)} \|V\|_{L^\infty(B_r(x_0))} \quad (3.12)$$

To prove the theorem 3.2 we need the following

Proposition 3.4. *For any $R > 0$, there exists $C_R > 0$, such that, for any $x_0 \in M$ one has :*

$$\|\nabla u\|_{B_R(x_0)} \geq e^{-C_R(\sqrt{\lambda}+1)} \|\nabla u\|_{L^2(M)}.$$

Proof. Let $R > 0$ and assume without loss of generality that $R < R_0$, with R_0 from the three balls theorem (theorem 3.1). Up to multiplication by a constant, we can assume that $\|\nabla u\|_{L^2(M)} = 1$. We denote by \bar{x} a point in M such that $\|\nabla u\|_{B_R(\bar{x})} = \sup_{x \in M} \|\nabla u\|_{B_R(x)}$. This implies that one has $\|\nabla u\|_{B_R(\bar{x})} \geq D_R$, where D_R depend only on M and R . One has from proposition (3.1) at an arbitrary point x of M :

$$\|\nabla u\|_{B_{R/2}(x)} \geq e^{-c(\sqrt{\lambda}+1)} \|\nabla u\|_{B_R(x)}^{\frac{1}{\alpha}} \quad (3.13)$$

Let γ be a geodesic curve between x and \bar{x} and define $x_0 = x, x_1, \dots, x_m = \bar{x}$ such that $x_i \in \gamma$ and $B_{\frac{R}{2}}(x_{i+1}) \subset B_R(x_i), \forall i = 1, \dots, m$. The constant m depends only on $\text{diam}(M)$ and R . Then the properties of $(x_i)_{1 \leq i \leq m}$ and inequality (3.13) give for all $i, 1 \leq i \leq m$:

$$\|\nabla u\|_{B_{R/2}(x_i)} \geq e^{-c_i(\sqrt{\lambda}+1)} \|\nabla u\|_{B_{R/2}(x_{i+1})}^{\frac{1}{\alpha}}. \quad (3.14)$$

The result follows by induction and the fact that $\|\nabla u\|_{B_R(\bar{x})} \geq D_R$. \square

Corollary 3.5. *For any $R > 0$, there exists a positive constant C_R depending only on M and R such that at any point x_0 in M one has*

$$\|\nabla u\|_{\frac{R}{4}, \frac{R}{8}} \geq e^{-C_R(\sqrt{\lambda}+1)} \|\nabla u\|_{L^2(M)}.$$

Proof. Let $R < R_0$ where R_0 is from the three balls theorem, note that $R_0 \leq \text{diam}(M)$. Recall that we defined locally near a point x_0 the set A_{r_1, r_2} by $\{x \in M; r_1 \leq d(x, x_0) \leq r_2\}$. As M is geodesically complete, there exists a point x_1 in $A_{\frac{R}{8}, \frac{R}{4}}$ such that $B_{x_1}(\frac{R}{16}) \subset A_{\frac{R}{8}, \frac{R}{4}}$. From proposition 3.4 one has $\|\nabla u\|_{B_{\frac{R}{16}}(x_1)} \geq e^{-C_R(\sqrt{\lambda}+1)} \|\nabla u\|_{L^2(M)}$ which gives the result. \square

Proof of theorem 3.2. We proceed in a similar way than the proof of three balls theorem except for the fact that we want now one ball (*i.e.* δ below) to be arbitrary small in front of the others. Let $R = \frac{R_0}{4}$ where R_0 is from the three spheres theorems, let δ such that $0 < \delta < 2\delta < 3\delta < \frac{R}{8} < \frac{R}{2} < R$, and define a smooth radial function ψ , with $0 \leq \psi \leq 1$ as follows:

- $\psi(x) = 0$ if $r(x) < \delta$ or $r(x) > R$,
- $\psi(x) = 1$ if $\frac{5\delta}{4} < r(x) < \frac{R}{2}$,
- $|\nabla\psi(x)| \leq \frac{C}{\delta}$ if $r(x) \in [\delta, \frac{5\delta}{4}]$ and $|\nabla\psi(x)| \leq C$ if $r(x) \in [\frac{R}{2}, R]$,
- $|\nabla^2\psi(x)| \leq \frac{C}{\delta^2}$ if $r(x) \in [\delta, \frac{5\delta}{4}]$ and $|\nabla^2\psi(x)| \leq C$ if $r(x) \in [\frac{R}{2}, R]$.

Keeping appropriate terms in (3.3) gives :

$$\begin{aligned} \|r^{\frac{5}{2}} e^{\tau\phi} \psi V\| + \tau\delta \|r^{-1} e^{\tau\phi} \psi V\| &\leq C (\|r^2 e^{\tau\phi} \nabla V \cdot \nabla\psi\| + \|r^2 e^{\tau\phi} \Delta\psi V\|) \\ &\leq \frac{C}{\delta} \|r^2 e^{\tau\phi} \nabla V\|_{\delta, \frac{5\delta}{4}} + C \|e^{\tau\phi} \nabla V\|_{\frac{R}{2}, R} \\ &\quad + \frac{C}{\delta^2} \|r^2 e^{\tau\phi} V\|_{\delta, \frac{5\delta}{4}} + C \|e^{\tau\phi} V\|_{\frac{R}{2}, R}. \end{aligned}$$

Using properties of ψ we obtain,

$$\begin{aligned}
& \|r^{\frac{\varepsilon}{2}} e^{\tau\phi} V\|_{\frac{5\delta}{4}, 3\delta} + \|r^{\frac{\varepsilon}{2}} e^{\tau\phi} V\|_{\frac{R}{8}, \frac{R}{4}} \\
& \quad + \tau\delta \|r^{-1} e^{\tau\phi} V\|_{\frac{5\delta}{4}, 3\delta} + \tau\delta \|r^{-1} e^{\tau\phi} V\|_{\frac{R}{8}, \frac{R}{4}} \\
& \leq \frac{C}{\delta} \|r^2 e^{\tau\phi} \nabla V\|_{\delta, \frac{5\delta}{4}} + C \|e^{\tau\phi} \nabla V\|_{\frac{R}{2}, R} \\
& \quad + \frac{C}{\delta^2} \|r^2 e^{\tau\phi} V\|_{\delta, \frac{5\delta}{4}} + C \|e^{\tau\phi} V\|_{\frac{R}{2}, R}.
\end{aligned}$$

Now we drop the first and last terms of the left hand side to get :

$$\begin{aligned}
& \|r^{\frac{\varepsilon}{2}} e^{\tau\phi} V\|_{\frac{R}{8}, \frac{R}{4}} + \|e^{\tau\phi} V\|_{\frac{5\delta}{4}, 3\delta} \leq C \left(\delta \|e^{\tau\phi} \nabla V\|_{\delta, \frac{5\delta}{4}} + \|e^{\tau\phi} \nabla V\|_{\frac{R}{2}, R} \right) \\
& \quad + C \left(\|e^{\tau\phi} V\|_{\delta, \frac{5\delta}{4}} + \|e^{\tau\phi} V\|_{\frac{R}{2}, R} \right).
\end{aligned}$$

Since R and ε are both fixed we can replace, up to a change of constant, $\|r^{\frac{\varepsilon}{2}} e^{\tau\phi} V\|_{\frac{R}{8}, \frac{R}{4}}$ by $\|e^{\tau\phi} V\|_{\frac{R}{8}, \frac{R}{4}}$. Then using elliptic estimates (3.5) and properties of ϕ gives,

$$\begin{aligned}
& \|e^{\tau\phi} V\|_{\frac{R}{8}, \frac{R}{4}} + \|e^{\tau\phi} V\|_{\frac{5\delta}{4}, 3\delta} \leq C\sqrt{\lambda} \left(e^{\tau\phi(\delta)} \|V\|_{\frac{2\delta}{3}, \frac{3\delta}{2}} + e^{\tau\phi(\frac{R}{3})} \|V\|_{\frac{R}{5}, \frac{5R}{3}} \right) \\
& \quad + C\sqrt{\lambda} \left(e^{\tau\phi(\delta)} \|V\|_{\delta, \frac{5\delta}{4}} + e^{\tau\phi(\frac{R}{3})} \|V\|_{\frac{R}{2}, R} \right).
\end{aligned}$$

This leads easily to

$$e^{\tau\phi(\frac{R}{4})} \|V\|_{\frac{R}{8}, \frac{R}{4}} + e^{\tau\phi(3\delta)} \|V\|_{\frac{5\delta}{4}, 3\delta} \leq C\sqrt{\lambda} \left(e^{\tau\phi(\delta)} \|V\|_{\frac{3\delta}{2}} + e^{\tau\phi(\frac{R}{3})} \|V\|_{\frac{5R}{3}} \right).$$

Adding $e^{\tau\phi(3\delta)} \|V\|_{\frac{5\delta}{4}}$ and absorbing it in the right hand side by $e^{\tau\phi(\delta)} \|V\|_{\frac{3\delta}{2}}$ gives

$$e^{\tau\phi(\frac{R}{4})} \|V\|_{\frac{R}{8}, \frac{R}{4}} + e^{\tau\phi(3\delta)} \|V\|_{3\delta} \leq C\sqrt{\lambda} \left(e^{\tau\phi(\delta)} \|V\|_{\frac{3\delta}{2}} + e^{\tau\phi(\frac{R}{3})} \|V\|_{\frac{5R}{3}} \right).$$

Now we want to choose τ such that

$$C\sqrt{\lambda} e^{\tau\phi(\frac{R}{3})} \|V\|_{\frac{5R}{3}} \leq \frac{1}{2} e^{\tau\phi(\frac{R}{4})} \|V\|_{\frac{R}{8}, \frac{R}{4}}.$$

For the same reasons than before we choose

$$\tau = \frac{1}{\phi(\frac{R}{3}) - \phi(\frac{R}{4})} \ln \left(\frac{1}{2C\sqrt{\lambda}} \frac{\|u\|_{\frac{R}{8}, \frac{R}{4}}}{\|u\|_{\frac{5R}{3}}} \right) + C_1\sqrt{\lambda} + C_2.$$

Define $A = (\phi(\frac{R}{3}) - \phi(\frac{R}{4}))^{-1}$; like before one can assume without loss that $-A$ is positive and independent of R . Then,

$$e^{\tau\phi(\frac{R}{4})} \|V\|_{\frac{R}{8}, \frac{R}{4}} + e^{\tau\phi(3\delta)} \|V\|_{3\delta} \leq C\sqrt{\lambda} e^{\tau\phi(\delta)} \|V\|_{\frac{3\delta}{2}}.$$

One can then ignore the first term of the right hand side to get :

$$e^{\tau\phi(3\delta)}\|V\|_{3\delta} \leq C\sqrt{\lambda} e^{A\ln\left(\frac{1}{2C\sqrt{\lambda}}\frac{\|V\|_{\frac{R}{8},\frac{R}{4}}}{\|V\|_{\frac{5R}{3}}}\right)+C_1\sqrt{\lambda}}\|V\|_{\frac{3\delta}{2}}$$

$$\|V\|_{3\delta} \leq e^{C\sqrt{\lambda}}\left(\frac{\|V\|_{\frac{R}{8},\frac{R}{4}}}{\|V\|_{\frac{5R}{3}}}\right)^A\|V\|_{\frac{3\delta}{2}}.$$

Finally from corollary 3.5, define $r = \frac{3\delta}{2}$ to have :

$$\|V\|_{2r} \leq e^{C(\sqrt{\lambda}+1)}\|V\|_r.$$

Thus, the theorem is proved for all $r \leq \frac{R_0}{16}$. Using proposition 3.4 we have for $r \geq \frac{R_0}{16}$:

$$\begin{aligned} \|\nabla u\|_{B_{x_0}(r)} &\geq \|\nabla u\|_{B_{x_0}\left(\frac{R_0}{16}\right)} \\ &\geq e^{-C_0(\sqrt{\lambda}+1)}\|\nabla u\|_{L^2(M)} \geq e^{-C_1(\sqrt{\lambda}+1)}\|\nabla u\|_{B_{x_0}(2r)}. \end{aligned}$$

□

4 Measure of the Critical set

From here we will follow the method of Donnelly and Fefferman [5] to establish an upper bound for the $(n-1)$ -dimensional measure of the critical set of eigenfunctions. We will assume from now, unless stated differently, that M is analytic and λ is sufficiently large ($\lambda \geq 1$). Recall that $\mathcal{N}_u = \{x \in M : u(x) = 0\}$ and $\mathcal{C}_u = \{x \in M : \nabla u(x) = 0\}$. Define $B_{\mathbb{C}}(r)$ the complex ball :

$$B_{\mathbb{C}}(r) = \{z \in \mathbb{C}^n : |z| < r\}$$

and $B(r)$ the standard ball in \mathbb{R}^n centered at 0 of radius r . The main point to deduce from our doubling inequality (3.11) an estimate on the Hausdorff measure of the critical set is the following result of Donnelly and Fefferman:

Theorem 4.1 ([5] p. 180). *Let F be an, non-zero, holomorphic function on $B_{\mathbb{C}}(1)$ and suppose there exists $\alpha > 1$, such that*

$$\max_{B_{\mathbb{C}}(1)} |F| \leq e^{\alpha} \max_{B\left(\frac{1}{2}\right)} |F|,$$

then

$$\mathcal{H}^{n-1}\left(\mathcal{N}_F \cap B\left(\frac{1}{4}\right)\right) \leq C\alpha.$$

where \mathcal{N}_F is the zero set of F in \mathbb{R}^n and C a constant depending only on the dimension.

Let u be a solution to (1.1). Fix x_0 in M and consider (x_1, \dots, x_n) a chart around x_0 . We assume that the chart contains the euclidean ball B_2 . We define

$$F(x) = \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2,$$

In this chart the nodal set of F is the critical set of u . One has :

Proposition 4.2. *The function F can be extended to an analytic function on $B_{\mathbb{C}}(1)$ and :*

$$\|F\|_{L^\infty(B_{\mathbb{C}}(1))} \leq e^{C\sqrt{\lambda}} \|F\|_{L^\infty(B(\frac{1}{2}))}$$

where C is a constant depending only on M .

Lemma 4.3. *Let u be an eigenfunction of the laplace operator on $B(1)$, for all multi-index β , with $|\beta| \geq 1$ one has :*

$$|D^\beta u(0)| \leq \beta! C^{|\beta|} \sqrt{\lambda}^{|\beta|} \|\nabla u\|_{L^\infty(B(\frac{C_1}{\sqrt{\lambda}}))} \quad (4.1)$$

where C_1 is a constant small enough.

Proof of lemma 4.3. Like in [5], this result can be obtained by rescaling the equation and using the hypoellipticity proof ([14], p.178) for an elliptic operator whose coefficients have uniform bounded derivatives.

Indeed, note first that we may assume $\|\nabla u\|_{L^\infty(M)} = 1$. Now writing in our local chart $\Delta = \sum_{1 \leq |\alpha| \leq 2} a_\alpha D^\alpha$ and consider the function $u_\lambda(x) = u(\frac{C_1}{\sqrt{\lambda}}x)$, where C_1 will be fix below. One can see that u_λ is a solution to the elliptic equation

$$P_\lambda u_\lambda = u_\lambda$$

with $P_\lambda = \sum_{1 \leq |\alpha| \leq 2} b_\alpha D^\alpha$ and

$$b_\alpha(x) = \frac{\lambda^{-1 + \frac{|\alpha|}{2}}}{C_1^{|\alpha|}} a_\alpha \left(\frac{C_1 x}{\sqrt{\lambda}} \right).$$

A short computation of $D^\beta b_\alpha$, gives for C_1 small enough and any multi-index β :

$$\sup_{B_1} |D^\beta b_\alpha(x)| \leq C_2 |\beta|!, \quad \forall 1 \leq |\alpha| \leq 2$$

where C_2 is a constant depending only on M . Then one can use the hypoellipticity proof [14] with simple modifications to get for any multi-index β with $|\beta| > 1$:

$$|D^\beta u_\lambda(0)| \leq A^{|\beta|} \beta!.$$

□

Proof of proposition 4.2. Expanding $V = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n})$ in its Taylor series gives

$$V(z) = \sum_{|\alpha| \geq 0} \frac{z^\alpha}{\alpha!} D^\alpha V(0),$$

where for $\alpha = (\alpha_1, \dots, \alpha_n)$ in \mathbb{N}^n and $z = (z_1, \dots, z_n)$ in \mathbb{C}^n we have set $z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}$ and $\alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$. Now using (4.1) and summing a geometric series gives for a constant ρ small enough

$$\sup_{B_{\mathbb{C}}(0, \frac{\rho}{\sqrt{\lambda}})} |V(z)| \leq C \sup_{B(0, \frac{C_1}{\sqrt{\lambda}})} |V(x)|. \quad (4.2)$$

Then by translating, in the complex ball $B_{\mathbb{C}}(1)$, the equation and iterating the estimate (4.2) a multiple of $\sqrt{\lambda}$ times one has

$$\forall z \in B_{\mathbb{C}}(1), |V(z)| \leq C^{\sqrt{\lambda}} \sup_{B(2)} |V(x)|$$

This implies

$$\sup_{B_{\mathbb{C}}(1)} |F(z)| \leq e^{C\sqrt{\lambda}} \sup_{B(2)} |F(x)| \quad (4.3)$$

which gives proposition 4.2 by using doubling inequality (3.11). \square

Proof of theorem 1.1. Let u be a solution to (1.1), let $r_0 > 0$ a fixed number not larger than the injectivity radius of M and p a arbitrary point in M . Let consider (x_1, x_2, \dots, x_n) , a normal chart around p . By proposition 4.2 one has that $F = \sum_{i=1..n} \left| \frac{\partial u}{\partial x_i} \right|^2$ satisfy the hypothesis of theorem 4.1. Then since the nodal set of F is the critical set of u one has

$$\mathcal{H}^{n-1}(\mathcal{C}_u \cap B(p, r_0)) \leq C\sqrt{\lambda} \quad (4.4)$$

where C depends only on r_0 and M .

The Theorem 1.1 follows by a covering argument since M is compact. \square

In the case of a smooth manifold, we can obtain a local version of theorem 1.1 :

Theorem 4.4. *Let M be a n -dimensionnal, smooth, compact, connected manifold. There exists $C > 0$ depending only on M such that for any non-constant solution u to (1.1), any point x_0 in M , there exists r_0 such that*

$$\mathcal{H}^{n-1}(\mathcal{C}_u \cap B(x_0, r)) \leq C\sqrt{\lambda}, \quad \text{for } r \in (0, r_0)$$

where \mathcal{C}_u is the critical set of u .

Proof. It is well known that the doubling estimate (3.11) implies that the vanishing order of gradient of eigenfunctions is everywhere less than $C\sqrt{\lambda}$. Then theorem 4.4 follows from lemma 3 of [2]. However there is no control on the radius r_0 . \square

Remark 4.5. It seems natural to assume that the estimates $\mathcal{H}^{n-1}(\mathcal{C}_u) \leq c\sqrt{\lambda}$ is still true for the global measure of the critical set in the case that M is only smooth.

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