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Quantitative uniqueness for Schrödinger operator

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Abstract

We give an upper bound on the vanishing order of solutions to Schrödinger equation on a compact smooth manifold. Our method is based on Carleman type inequalities, and gives a generalisation to a result of H. Donnelly and C. Fefferman [DF88] on eigenfunctions. It also sharpens previous results of I. Kukavica [Kuk98].

1 Introduction

Let (M, g) be a smooth, compact and connected, n -dimensional Riemannian manifold. It is well known that, if u is a non trivial solution of second order linear elliptic equation on M , then all zeros of u are of finite order ([Aro57, HS89]). The aim of this paper is to obtain quantitative estimate on the vanishing order of (non trivial) solutions to

$$\Delta u = Wu, \tag{1.1}$$

when $W \in C^1(M)$. In the particular case of eigenfunctions of the Laplacian (*i.e.* $W = \lambda$ is a constant), it has been shown by H. Donnelly and C. Fefferman [DF88] that the vanishing order is bounded by $C\sqrt{\lambda}$. In [Kuk98], I. Kukavica established some quantitative results for solution to (1.1). When W is a bounded function he obtained that the vanishing order of solutions to (1.1) is everywhere less than $C(1 + \sqrt{\|W\|_\infty} + (\text{osc}(W))^2)$, where $\text{osc}(W) = \sup W - \inf W$ and C a constant depending only on M . If W is C^1 he got the upper bound $C(1 + \|W\|_{C^1})$, with $\|W\|_{C^1} = \|W\|_\infty + \|\nabla W\|_\infty$. Our main result is the following

Theorem 1.1. *The vanishing order of solutions to (1.1) is everywhere less than*

$$C_1 \sqrt{\|W\|_{C^1}} + C_2,$$

where C_1 and C_2 are positive constants depending only on M .

More precisely theorem 1.1 is a direct consequence of the following doubling inequality on solutions (theorem 3.2) :

$$\|u\|_{L^2(B_{2r}(x_0))} \leq e^{C_1\sqrt{\|W\|_{C^1}}+C_2}\|u\|_{L^2(B_r(x_0))}. \quad (1.2)$$

The exponent $1/2$ on $\|W\|_{C^1}$ in this result is sharp and agrees with the result of H. Donnelly and C. Fefferman [DF88] when W is constant. Indeed, consider the homogeneous polynomials $f_k(x_1, x_2, \dots, x_{n+1}) = \Re(x_1 + ix_2)^k$ defined in \mathbb{R}^{n+1} . Set Y_k the restriction of f_k to \mathbb{S}^n . $(Y_k)_k$ is a sequence of spherical harmonics and $-\Delta_{\mathbb{S}^n} Y_k = k(k+n-1)Y_k = \lambda_k Y_k$. The vanishing order at the north pole $N = (0, \dots, 0, 1)$ of Y_k is $k \geq C\sqrt{\lambda_k}$.

Let us now discuss briefly the methods usually used to deal with quantitative uniqueness for linear partial differential equations. They are two principal methods : the first one is based on Carleman-type estimates [Aro57, DF88, DF90, Hör07, JK85, JL99] and the second one relies on the frequency function of solutions [Don92, GL86, Kuk98, Lin91]. The goal of both methods is to control the local behaviour of solutions. In the original works of Donnelly and Fefferman [DF88], the authors wrote down a Carleman estimate on the operator $\Delta + \lambda$. Later, several authors (F.-H. Lin [Lin91], D. Jerison & G. Lebeau [JL99], I. Kukavica [Kuk95],...) obtained some generalizations and simplifications in the proof. In particular, if u is an eigenfunction of the Laplace operator on M , with eigenvalue λ , then the function \tilde{u} defined on $M \times [-T, T]$ by

$$\tilde{u}(x, t) = \cosh(\sqrt{\lambda}t)u(x)$$

satisfies $(\Delta + \frac{\partial^2}{\partial t^2})\tilde{u} = 0$. The problem is then simplified since one has only to deal with the 0 eigenvalue of the operator $\Delta + \frac{\partial^2}{\partial t^2}$ on $M \times [-T, T]$. For example, in [JL99], D. Jerison and G. Lebeau established a Carleman estimate on $\Delta + \frac{\partial^2}{\partial t^2}$. However, it was pointed out by I. Kukavica [Kuk98] that, the method of [JL99] doesn't seem to extend easily when studying the more general equation (1.1). Despite this, the point of our paper is that one can successfully establish a Carleman estimate directly on the operator $\Delta + W$. Furthermore, when W is C^1 , it leads to a better upper bound on the vanishing order of solutions to (1.1), in terms of its dependance on $\|W\|_{C^1}$. The paper is organised as follows. In section 2 we establish Carleman estimate for the operator $\Delta + W$. Our method will differ slightly from [DF88], we will discuss it briefly in the remarks following theorem 2.1. In section 3 we deduce, in a standard manner, three balls theorem for solutions to (1.1), then using compactness we derive doubling inequality which gives immediately theorem 1.1. In a forthcoming paper we study the vanishing order of solutions when W is only a bounded function.

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2 Carleman estimates

Fix x_0 in M , and let $r = r(x) = d(x, x_0)$ the Riemannian distance from x_0 . We denote by $B_r(x_0)$ the geodesic ball centered at x_0 of radius r . We will denote by $\|\cdot\|$ the L^2 norm. Recall that Carleman estimates are weighted integral inequalities with a weight function $e^{\tau\phi}$, where the function ϕ satisfies some convexity properties. Let us now define the weight function we will use.

For a fixed number ε such that $0 < \varepsilon < 1$ and $T_0 < 0$, we define the function f on $] -\infty, T_0[$ by $f(t) = t - e^{\varepsilon t}$. One can check easily that, for $|T_0|$ great enough, the function f verifies the following properties:

$$\begin{aligned} 1 - \varepsilon e^{\varepsilon T_0} &\leq f'(t) \leq 1 \quad \forall t \in] -\infty, T_0[, \\ \lim_{t \rightarrow -\infty} -e^{-t} f''(t) &= +\infty. \end{aligned} \quad (2.1)$$

Finally we define $\phi(x) = -f(\ln r(x))$. Now we can state the main result of this section:

Theorem 2.1. *There exist positive constants R_0, C, C_1, C_2 , which depend only on M and ε , such that, for any $W \in C^1(M)$, $x_0 \in M$, $u \in C_0^\infty(B_{R_0}(x_0) \setminus \{x_0\})$ and $\tau \geq C_1 \sqrt{\|W\|_{C^1}} + C_2$, one has*

$$C \left\| r^2 e^{\tau\phi} (\Delta u + Wu) \right\| \geq \tau^{\frac{3}{2}} \left\| r^{\frac{\varepsilon}{2}} e^{\tau\phi} u \right\| + \tau^{\frac{1}{2}} \left\| r^{1+\frac{\varepsilon}{2}} e^{\tau\phi} \nabla u \right\|. \quad (2.2)$$

Moreover, if

$$\text{supp}(u) \subset \{x \in M; r(x) \geq \delta > 0\},$$

then

$$\begin{aligned} C \left\| r^2 e^{\tau\phi} (\Delta u + Wu) \right\| &\geq \tau^{\frac{3}{2}} \left\| r^{\frac{\varepsilon}{2}} e^{\tau\phi} u \right\| \\ &+ \tau\delta \left\| r^{-1} e^{\tau\phi} u \right\| + \tau^{\frac{1}{2}} \left\| r^{1+\frac{\varepsilon}{2}} e^{\tau\phi} \nabla u \right\|. \end{aligned} \quad (2.3)$$

Remark 2.2. *This inequality can be seen as a generalization of previous Carleman type estimates in the case that W is a constant (see [DF88]). Indeed when $W = \lambda$ one has $\sqrt{\|W\|_{C^1}} = \sqrt{\lambda}$. The point is that since W is C^1 we will be allowed to integrate by parts, but then we have to take care of the derivatives of W .*

Remark 2.3. *In [DF88] the authors used a local conformal change, due to aronszajn [Aro57], $\tilde{g}_{ij} = e^{-2\nu r^2} g_{ij}$ to obtain the positiveness of a certain tensor (see [DF88] lemma 2.3 p.167 and the computation of J_4 p.168). Instead of doing this, we introduce a small function (f'') to handle some terms involving the derivatives of the metric, see by example the computation of J_2 below (2.13).*

Remark 2.4. The constants C , C_1 , C_2 go to infinity as ε goes to zero. Therefore the case $\varepsilon = 0$ is not include in our theorem.

Remark 2.5. In the inequalities (2.2) and (2.3) the gradient terms are not necessary to the purpose of this paper. We choose to include them for a more general statement.

Proof. Hereafter C , C_1 , C_2 and c denote positive constants depending only upon M , though their values may change from one line to another. Without loss of generality, we may suppose that all functions are real. We now introduce the polar geodesic coordinates (r, θ) near x_0 . Using Einstein notation, the Laplace operator takes the form :

$$r^2 \Delta u = r^2 \partial_r^2 u + r^2 \left(\partial_r \ln(\sqrt{\gamma}) + \frac{n-1}{r} \right) \partial_r u + \frac{1}{\sqrt{\gamma}} \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j u),$$

where $\partial_i = \frac{\partial}{\partial \theta_i}$ and for each fixed r , $\gamma_{ij}(r, \theta)$ is a metric on \mathbb{S}^{n-1} and $\gamma = \det(\gamma_{ij})$.

Since (M, g) is smooth, we have for r small enough :

$$\begin{aligned} \partial_r(\gamma^{ij}) &\leq C(\gamma^{ij}) \quad (\text{in the sense of tensors}); \\ |\partial_r(\gamma)| &\leq C; \\ C^{-1} \leq \gamma &\leq C. \end{aligned} \tag{2.4}$$

Set $r = e^t$, we have $\frac{\partial}{\partial r} = e^{-t} \frac{\partial}{\partial t}$. Then we will consider that the function u has support in $] -\infty, T_0[\times \mathbb{S}^{n-1}$, where $|T_0|$ will be chosen large enough. In this new variables, we can write :

$$e^{2t} \Delta u = \partial_t^2 u + (n-2 + \partial_t \ln \sqrt{\gamma}) \partial_t u + \frac{1}{\sqrt{\gamma}} \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j u).$$

The conditions (2.4) become

$$\begin{aligned} \partial_t(\gamma^{ij}) &\leq C e^t (\gamma^{ij}) \quad (\text{in the sense of tensors}); \\ |\partial_t(\gamma)| &\leq C e^t; \\ C^{-1} \leq \gamma &\leq C. \end{aligned} \tag{2.5}$$

Now we introduce the conjugate operator :

$$\begin{aligned} L_\tau(u) &= e^{2t} e^{\tau \phi} \Delta(e^{-\tau \phi} u) + e^{2t} W u \\ &= \partial_t^2 u + (2\tau f' + n-2 + \partial_t \ln \sqrt{\gamma}) \partial_t u \\ &\quad + \left(\tau^2 f'^2 + \tau f'' + (n-2)\tau f' + \tau \partial_t \ln \sqrt{\gamma} f' \right) u \\ &\quad + \Delta_\theta u + e^{2t} W u, \end{aligned} \tag{2.6}$$

with

$$\Delta_\theta u = \frac{1}{\sqrt{\gamma}} \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j u).$$

It will be useful for us to introduce the following L^2 norm on $] -\infty, T_0[\times \mathbb{S}^{n-1}$:

$$\|V\|_f^2 = \int_{]-\infty, T_0[\times \mathbb{S}^{n-1}} V^2 \sqrt{\gamma} f'^{-3} dt d\theta,$$

where $d\theta$ is the usual measure on \mathbb{S}^{n-1} . The corresponding inner product is denoted by $\langle \cdot, \cdot \rangle_f$, *i.e*

$$\langle u, v \rangle_f = \int uv \sqrt{\gamma} f'^{-3} dt d\theta.$$

We will estimate from below $\|L_\tau u\|_f^2$ by using elementary algebra and integrations by parts. We are concerned, in the computation, by the power of τ and exponential decay when t goes to $-\infty$. First by triangular inequality one has

$$\|L_\tau(u)\|_f \geq I - II, \quad (2.7)$$

with

$$\begin{aligned} I &= \left\| \partial_t^2 u + 2\tau f' \partial_t u + \tau^2 f'^2 u + e^{2t} W u + \Delta_\theta u \right\|_f, \\ II &= \left\| \tau f'' u + (n-2)\tau f' u + \tau \partial_t \ln \sqrt{\gamma} f' u \right\|_f \\ &+ \left\| (n-2)\partial_t u + \partial_t \ln \sqrt{\gamma} \partial_t u \right\|_f. \end{aligned} \quad (2.8)$$

We will be able to absorb II later. Then we compute I^2 :

$$I^2 = I_1 + I_2 + I_3,$$

with

$$\begin{aligned} I_1 &= \left\| \partial_t^2 u + (\tau^2 f'^2 + e^{2t} W) u + \Delta_\theta u \right\|_f^2 \\ I_2 &= \left\| 2\tau f' \partial_t u \right\|_f^2 \\ I_3 &= 2 \left\langle 2\tau f' \partial_t u, \partial_t^2 u + \tau^2 f'^2 u + e^{2t} W u + \Delta_\theta u \right\rangle_f \end{aligned} \quad (2.9)$$

In order to compute I_3 we write it in a convenient way:

$$I_3 = J_1 + J_2 + J_3, \quad (2.10)$$

where the integrals J_i are defined by :

$$\begin{aligned} J_1 &= 2\tau \int f' \partial_t (|\partial_t u|^2) f'^{-3} \sqrt{\gamma} dt d\theta \\ J_2 &= 4\tau \int f' \partial_t u \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j u) f'^{-3} dt d\theta \\ J_3 &= \int (2\tau^3 (f')^3 + 2\tau f' e^{2t} W) 2u \partial_t u f'^{-3} \sqrt{\gamma} dt d\theta. \end{aligned} \quad (2.11)$$

Now we will use integration by parts to estimate each terms of (2.11). Note that f is radial and that $2\partial_t u \partial_t^2 u = \partial_t(|\partial_t u|^2)$. We find that :

$$\begin{aligned} J_1 &= \int (4\tau f'') |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\ &\quad - \int 2\tau f' \partial_t \ln \sqrt{\gamma} |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \end{aligned}$$

The conditions (2.5) imply that $|\partial_t \ln \sqrt{\gamma}| \leq C e^t$. Then properties (2.1) on f gives, for large $|T_0|$ that $|\partial_t \ln \sqrt{\gamma}|$ is small compared to $|f''|$. Then one has

$$J_1 \geq -c\tau \int |f''| \cdot |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \quad (2.12)$$

In order to estimate J_2 we first integrate by parts with respect to ∂_i :

$$J_2 = -2 \int 2\tau f' \partial_t \partial_i u \gamma^{ij} \partial_j u f'^{-3} \sqrt{\gamma} dt d\theta.$$

Then we integrate by parts with respect to ∂_t . We get :

$$\begin{aligned} J_2 &= -4\tau \int f'' \gamma^{ij} \partial_i u \partial_j u f'^{-3} \sqrt{\gamma} dt d\theta \\ &\quad + \int 2\tau f' \partial_t \ln \sqrt{\gamma} \gamma^{ij} \partial_i u \partial_j u f'^{-3} \sqrt{\gamma} dt d\theta \\ &\quad + \int 2\tau f' \partial_t (\gamma^{ij}) \partial_i u \partial_j u f'^{-3} \sqrt{\gamma} dt d\theta. \end{aligned}$$

We denote $|D_\theta u|^2 = \partial_i u \gamma^{ij} \partial_j u$. Now using that $-f''$ is non-negative and τ is large, the conditions (2.1) and (2.5) gives for $|T_0|$ large enough:

$$J_2 \geq 3\tau \int |f''| \cdot |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \quad (2.13)$$

Similarly computation of J_3 gives :

$$\begin{aligned} J_3 &= -2 \int \tau^3 \partial_t \ln(\sqrt{\gamma}) u^2 \sqrt{\gamma} dt d\theta \\ &\quad - \int (4f' - 4f'' + 2f' \partial_t \ln \sqrt{\gamma}) \tau e^{2t} W u^2 f'^{-3} \sqrt{\gamma} dt d\theta \\ &\quad - \int 2\tau f' e^{2t} \partial_t W |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \end{aligned} \quad (2.14)$$

Now we assume that

$$\tau \geq C_1 \sqrt{\|W\|_{C^1}} + C_2. \quad (2.15)$$

From (2.1) and (2.5) one can see that if C_1 , C_2 and $|T_0|$ are large enough, then

$$J_3 \geq -c\tau^3 \int e^t |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \quad (2.16)$$

Thus far, using (2.12), (2.13) and (2.16), we have :

$$\begin{aligned} I_3 &\geq 3\tau \int |f''| |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta - c\tau^3 \int e^t |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\ &\quad - c\tau \int |f''| |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \end{aligned} \quad (2.17)$$

Now we consider I_1 :

$$I_1 = \left\| \partial_t^2 u + \left(\tau^2 f'^2 + e^{2t} W \right) u + \Delta_\theta u \right\|_f^2.$$

Let $\rho > 0$ a small number to be chosen later. Since $|f''| \leq 1$ and $\tau \geq 1$, we have :

$$I_1 \geq \frac{\rho}{\tau} I'_1, \quad (2.18)$$

where I'_1 is defined by :

$$I'_1 = \left\| \sqrt{|f''|} \left[\partial_t^2 u + \left(\tau^2 f'^2 + e^{2t} W \right) u + \Delta_\theta u \right] \right\|_f^2 \quad (2.19)$$

and one has

$$I'_1 = K_1 + K_2 + K_3, \quad (2.20)$$

with

$$\begin{aligned} K_1 &= \left\| \sqrt{|f''|} (\partial_t^2 u + \Delta_\theta u) \right\|_f^2, \\ K_2 &= \left\| \sqrt{|f''|} \left(\tau^2 f'^2 + e^{2t} W \right) u \right\|_f^2, \\ K_3 &= 2 \left\langle (\partial_t^2 u + \Delta_\theta u) |f''|, \left(\tau^2 f'^2 + e^{2t} W \right) u \right\rangle_f. \end{aligned} \quad (2.21)$$

Integrating by parts gives :

$$\begin{aligned} K_3 &= 2 \int f'' \left(\tau^2 f'^2 + e^{2t} W \right) |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\ &+ 2 \int \partial_t \left[f'' \left(\tau^2 f'^2 + e^{2t} W \right) \right] \partial_t u u \sqrt{\gamma} f'^{-3} dt d\theta \\ &- 6 \int \left(f''^2 f'^{-1} \left(\tau^2 f'^2 + e^{2t} W \right) \right) \partial_t u u \sqrt{\gamma} f'^{-3} dt d\theta \\ &+ 2 \int f'' \left(\tau^2 f'^2 + e^{2t} W \right) \partial_t \ln \sqrt{\gamma} \partial_t u u f'^{-3} \sqrt{\gamma} dt d\theta \\ &+ 2 \int f'' \left(\tau^2 f'^2 + e^{2t} W \right) |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\ &+ 2 \int f'' e^{2t} \partial_i W \cdot \gamma^{ij} \partial_j u u f'^{-3} \sqrt{\gamma} dt d\theta. \end{aligned} \quad (2.22)$$

The condition $\tau \geq C_1 \sqrt{\|W\|_{C^1}} + C_2$ implies,

$$\int |f''| e^{2t} |\partial_i W \gamma^{ij} \partial_j u u| \sqrt{\gamma} dt d\theta \leq c \tau^2 \int |f''| (|D_\theta u|^2 + |u|^2) f'^{-3} \sqrt{\gamma} dt d\theta.$$

Now since $2\partial_t u u \leq u^2 + |\partial_t u|^2$, we can use conditions (2.1) and (2.5) to get

$$K_3 \geq -c \tau^2 \int |f''| (|\partial_t u|^2 + |D_\theta u|^2 + |u|^2) f'^{-3} \sqrt{\gamma} dt d\theta \quad (2.23)$$

We also have

$$K_2 \geq c\tau^4 \int |f''| |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \quad (2.24)$$

and since $K_1 \geq 0$,

$$\begin{aligned} I_1 &\geq -\rho c\tau \int |f''| (|\partial_t u|^2 + |D_\theta u|^2) f'^{-3} \sqrt{\gamma} dt d\theta \\ &\quad + C\tau^3 \rho \int |f''| |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \end{aligned} \quad (2.25)$$

Then using (2.17) and (2.25)

$$\begin{aligned} I^2 &\geq 4\tau^2 \|f' \partial_t u\|_f^2 + 3\tau \int |f''| |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\ &\quad + C\tau^3 \rho \int |f''| |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta - c\tau^3 \int e^t |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\ &\quad - \rho c\tau \int |f''| (|u|^2 + |\partial_t u|^2 + |D_\theta u|^2) f'^{-3} \sqrt{\gamma} dt d\theta. \\ &\quad - c\tau \int |f''| |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \end{aligned} \quad (2.26)$$

Now one needs to check that every non-positive term in the right hand side of (2.26) can be absorbed in the first three terms.

First fix ρ small enough such that

$$\rho c\tau \int |f''| \cdot |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \leq 2\tau \int |f''| \cdot |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta$$

where c is the constant appearing in (2.26). The other terms in the last integral of (2.26) can then be absorbed by comparing powers of τ (for C_2 large enough). Finally since conditions (2.1) imply that e^t is small compared to $|f''|$, we can absorb $-c\tau^3 e^t |u|^2$ in $C\tau^3 \rho |f''| |u|^2$.

Thus we obtain :

$$\begin{aligned} I^2 &\geq C\tau^2 \int |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta + C\tau \int |f''| |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\ &\quad + C\tau^3 \int |f''| |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \end{aligned} \quad (2.27)$$

As before, we can check that II can be absorbed in I for $|T_0|$ and τ large enough. Then we obtain

$$\|L_\tau u\|_f^2 \geq C\tau^3 \|\sqrt{|f''|} u\|_f^2 + C\tau^2 \|\partial_t u\|_f^2 + C\tau \|\sqrt{|f''|} D_\theta u\|_f^2. \quad (2.28)$$

Note that, since τ is large and $\sqrt{|f''|} \leq 1$, one has

$$\|L_\tau u\|_f^2 \geq C\tau^3 \|\sqrt{|f''|} u\|_f^2 + c\tau \|\sqrt{|f''|} \partial_t u\|_f^2 + C\tau \|\sqrt{|f''|} D_\theta u\|_f^2, \quad (2.29)$$

and the constant c can be choosen arbitrary smaller than C . If we set $v = e^{-\tau\phi} u$, then we have

$$\begin{aligned} \|e^{2t} e^{\tau\phi} (\Delta v + Wv)\|_f^2 &\geq C\tau^3 \|\sqrt{|f''|} e^{\tau\phi} v\|_f^2 - c\tau^3 \|\sqrt{|f''|} f' e^{\tau\phi} v\|_f^2 \\ &\quad + \frac{c}{2}\tau \|\sqrt{|f''|} e^{\tau\phi} \partial_t v\|_f^2 + C\tau \|\sqrt{|f''|} e^{\tau\phi} D_\theta v\|_f^2. \end{aligned}$$

Finally since f' is close to 1 one can absorb the negative term to obtain

$$\begin{aligned} \|e^{2t}e^{\tau\phi}(\Delta v + Wv)\|_f^2 &\geq C\tau^3\|\sqrt{|f''|}e^{\tau\phi}v\|_f^2 \\ &+ C\tau\|\sqrt{|f''|}e^{\tau\phi}\partial_t v\|_f^2 + C\tau\|\sqrt{|f''|}e^{\tau\phi}D_\theta v\|_f^2. \end{aligned} \quad (2.30)$$

It remains to get back to the usual L^2 norm. First note that since f' is close to 1 (2.1), we can get the same estimate without the term $(f')^{-3}$ in the integrals. Recall that in polar coordinates (r, θ) the volume element is $r^{n-1}\sqrt{\gamma}drd\theta$, we can deduce from (2.27) by substitution that :

$$\begin{aligned} \|r^2e^{\tau\phi}(\Delta v + Wv)r^{-\frac{n}{2}}\|^2 &\geq C\tau^3\|r^{\frac{\varepsilon}{2}}e^{\tau\phi}vr^{-\frac{n}{2}}\|^2 \\ &+ C\tau\|r^{1+\frac{\varepsilon}{2}}e^{\tau\phi}\nabla v r^{-\frac{n}{2}}\|^2. \end{aligned} \quad (2.31)$$

Finally one can get rid of the term $r^{-\frac{n}{2}}$ by replacing τ with $\tau + \frac{n}{2}$. Indeed from $e^{\tau\phi}r^{-\frac{n}{2}} = e^{(\tau+\frac{n}{2})\phi}e^{-\frac{n}{2}r^\varepsilon}$ one can check easily that, for r small enough

$$\frac{1}{2}e^{(\tau+\frac{n}{2})\phi} \leq e^{\tau\phi}r^{-\frac{n}{2}} \leq e^{(\tau+\frac{n}{2})\phi}.$$

This achieves the proof of the first part of theorem 2.1.

Now suppose that $\text{supp}(u) \subset \{x \in M; r(x) \geq \delta > 0\}$ and define $T_1 = \ln \delta$.

Cauchy-Schwarz inequality apply to

$$\int \partial_t(u^2)e^{-t}\sqrt{\gamma}dtd\theta = 2 \int u\partial_t u e^{-t}\sqrt{\gamma}dtd\theta$$

gives

$$\int \partial_t(u^2)e^{-t}\sqrt{\gamma}dtd\theta \leq 2 \left(\int (\partial_t u)^2 e^{-t}\sqrt{\gamma}dtd\theta \right)^{\frac{1}{2}} \left(\int u^2 e^{-t}\sqrt{\gamma}dtd\theta \right)^{\frac{1}{2}}. \quad (2.32)$$

On the other hand, integrating by parts gives

$$\int \partial_t(u^2)e^{-t}\sqrt{\gamma}dtd\theta = \int u^2 e^{-t}\sqrt{\gamma}dtd\theta - \int u^2 e^{-t}\partial_t(\ln(\sqrt{\gamma}))\sqrt{\gamma}dtd\theta. \quad (2.33)$$

Now since $|\partial_t \ln \sqrt{\gamma}| \leq Ce^t$ for $|T_0|$ large enough we can deduce :

$$\int \partial_t(u^2)e^{-t}\sqrt{\gamma}dtd\theta \geq c \int u^2 e^{-t}\sqrt{\gamma}dtd\theta. \quad (2.34)$$

Combining (2.32) and (2.34) gives

$$\begin{aligned} c^2 \int u^2 e^{-t}\sqrt{\gamma}dtd\theta &\leq 4 \int (\partial_t u)^2 e^{-t}\sqrt{\gamma}dtd\theta \\ &\leq 4e^{-T_1} \int (\partial_t u)^2 \sqrt{\gamma}dtd\theta. \end{aligned}$$

Finally, dropping all terms except $\tau^2 \int |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta$ in (2.27) gives :

$$C' I^2 \geq \tau^2 \delta^2 \|e^{-t} u\|_f^2.$$

Inequality (2.27) can then be replaced by :

$$\begin{aligned} I^2 &\geq C\tau^2 \int |\partial_t u|^2 f'^{-3} \sqrt{\gamma} dt d\theta + C\tau \int |f''| \cdot |D_\theta u|^2 f'^{-3} \sqrt{\gamma} dt d\theta \\ &+ C\tau^3 \int |f''| \cdot |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta + C\tau^2 \delta^2 \int |u|^2 f'^{-3} \sqrt{\gamma} dt d\theta. \end{aligned} \quad (2.35)$$

The rest of the proof follows in a similar way than the first part. \square

3 Doubling inequality

In this section we prove a doubling inequality for solutions of (1.1). First we deduce from Carleman estimate a three balls theorem for solutions. The standard way to do so is to apply such estimate, to ψu where ψ is an appropriate cut off function, u a solution, and then make a appropriate choice of the parameter τ (see [JL99]). We give a proof, following the method of Donnely and Fefferman [DF88], adapted to our particular choice of weight functions in the Carleman estimate.

Proposition 3.1 (Three balls inequality). *There exist positive constants R_1 , C_1 , C_2 and $0 < \alpha < 1$ wich depend only on M such that, if u is a solution to (1.1) with W of class \mathcal{C}^1 , then for any $R < R_1$, and any $x_0 \in M$, one has*

$$\|u\|_{B_R(x_0)} \leq e^{C_1 \sqrt{\|W\|_{\mathcal{C}^1}} + C_2} \|u\|_{B_{\frac{R}{2}}(x_0)}^\alpha \|u\|_{B_{2R}(x_0)}^{1-\alpha}. \quad (3.1)$$

Proof. Let x_0 a point in M . Let u be a solution to (1.1) and R such that $0 < R < \frac{R_0}{2}$ with R_0 as in theorem 2.1. Recall that $r(x)$ is the riemannian distance between x and x_0 and B_r the geodesic ball centered at x_0 of radius r . If v is a function defined in a neighborhood of x_0 , we denote by $\|v\|_R$ the L^2 norm of v on B_R and by $\|v\|_{R_1, R_2}$ the L^2 norm of v on the set $A_{R_1, R_2} := \{x \in M; R_1 \leq r(x) \leq R_2\}$. Let $\psi \in \mathcal{C}_0^\infty(B_{2R})$, $0 \leq \psi \leq 1$, a function with the following properties:

- $\psi(x) = 0$ if $r(x) < \frac{R}{4}$ or $r(x) > \frac{5R}{3}$,
- $\psi(x) = 1$ if $\frac{R}{3} < r(x) < \frac{3R}{2}$,
- $|\nabla \psi(x)| \leq \frac{C}{R}$,
- $|\nabla^2 \psi(x)| \leq \frac{C}{R^2}$.

First since the function ψu is supported in the annulus $A_{\frac{R}{3}, \frac{5R}{3}}$ we can apply estimate (2.3) of theorem 2.1. In particular we have :

$$C \left\| r^2 e^{\tau\phi} (\Delta\psi u + 2\nabla u \cdot \nabla\psi) \right\| \geq \tau \left\| e^{\tau\phi} \psi u \right\|. \quad (3.2)$$

Assume that $\tau \geq 1$, and use properties of ψ to get :

$$\begin{aligned} \|e^{\tau\phi} u\|_{\frac{R}{3}, \frac{3R}{2}} &\leq C \left(\|e^{\tau\phi} u\|_{\frac{R}{4}, \frac{R}{3}} + \|e^{\tau\phi} u\|_{\frac{3R}{2}, \frac{5R}{3}} \right) \\ &+ C \left(R \|e^{\tau\phi} \nabla u\|_{\frac{R}{4}, \frac{R}{3}} + R \|e^{\tau\phi} \nabla u\|_{\frac{3R}{2}, \frac{5R}{3}} \right). \end{aligned} \quad (3.3)$$

Recall that $\phi(x) = -\ln r(x) + r(x)^\varepsilon$. In particular ϕ is radial and decreasing (for small r). Then one has,

$$\begin{aligned} \|e^{\tau\phi} u\|_{\frac{R}{3}, \frac{3R}{2}} &\leq C \left(e^{\tau\phi(\frac{R}{4})} \|u\|_{\frac{R}{4}, \frac{R}{3}} + e^{\tau\phi(\frac{3R}{2})} \|u\|_{\frac{3R}{2}, \frac{5R}{3}} \right) \\ &+ C \left(R e^{\tau\phi(\frac{R}{4})} \|\nabla u\|_{\frac{R}{4}, \frac{R}{3}} + R e^{\tau\phi(\frac{3R}{2})} \|\nabla u\|_{\frac{3R}{2}, \frac{5R}{3}} \right). \end{aligned}$$

Now we recall the following elliptic estimates : since u satisfies (1.1) then it is not hard to see that :

$$\|\nabla u\|_{(1-a)r} \leq C \left(\frac{1}{(1-a)R} + \|W\|_\infty^{1/2} \right) \|u\|_{BR}, \quad \text{for } 0 < a < 1. \quad (3.4)$$

Moreover since $A_{R_1, R_2} \subset B_{R_2}$, using formula (3.4) and properties of ϕ gives

$$e^{\tau\phi(\frac{3R}{2})} \|\nabla u\|_{\frac{3R}{2}, \frac{5R}{3}} \leq C \left(\frac{1}{R} + \|W\|_\infty^{1/2} \right) e^{\tau\phi(\frac{3R}{2})} \|u\|_{2R}.$$

Using (3.3) one has :

$$\|u\|_{\frac{R}{3}, R} \leq C(\|W\|_\infty^{1/2} + 1) \left(e^{\tau(\phi(\frac{R}{4}) - \phi(R))} \|u\|_{\frac{R}{2}} + e^{\tau(\phi(\frac{3R}{2}) - \phi(R))} \|u\|_{2R} \right).$$

Let $A_R = \phi(\frac{R}{4}) - \phi(R)$ and $B_R = -(\phi(\frac{3R}{2}) - \phi(R))$. From the properties of ϕ , we have $0 < A^{-1} \leq A_R \leq A$ and $0 < B \leq B_R \leq B^{-1}$ where A and B don't depend on R . We may assume that $C(\|W\|_\infty^{1/2} + 1) \geq 2$. Then we can add $\|u\|_{\frac{R}{3}}$ to each member and bound it in the right hand side by $C(\|W\|_\infty^{1/2} + 1)e^{\tau A} \|u\|_{\frac{R}{2}}$. We get :

$$\|u\|_R \leq C(\|W\|_\infty^{1/2} + 1) \left(e^{\tau A} \|u\|_{\frac{R}{2}} + e^{-\tau B} \|u\|_{2R} \right). \quad (3.5)$$

Now we want to find τ such that

$$C(\|W\|_\infty^{1/2} + 1)e^{-\tau B} \|u\|_{2R} \leq \frac{1}{2} \|u\|_R$$

wich is true for $\tau \geq -\frac{1}{B} \ln \left(\frac{1}{2C(\|W\|_\infty^{1/2} + 1)} \frac{\|u\|_R}{\|u\|_{2R}} \right)$. Since τ must also satisfy

$$\tau \geq C_1 \sqrt{\|W\|_{C^1}} + C_2,$$

we choose

$$\tau = -\frac{1}{B} \ln \left(\frac{1}{2C(\|W\|_\infty^{1/2} + 1)} \frac{\|u\|_R}{\|u\|_{2R}} \right) + C_1 \sqrt{\|W\|_{C^1}} + C_2. \quad (3.6)$$

Since $\|W\|_\infty \leq \|W\|_{C^1}$ we can deduce from (3.5) that :

$$\|u\|_R^{\frac{B+A}{B}} \leq e^{C_1 \sqrt{\|W\|_{C^1}} + C_2} \|u\|_{2R}^{\frac{A}{B}} \|u\|_{\frac{R}{2}}^{\frac{B}{2}}, \quad (3.7)$$

Finally define $\alpha = \frac{A}{A+B}$ and taking exponent $\frac{B}{A+B}$ of (3.7):

$$\|u\|_R \leq e^{C_1 \sqrt{\|W\|_{C^1}} + C_2} \|u\|_{2R}^\alpha \|u\|_{\frac{R}{2}}^{1-\alpha}.$$

□

From now on we assume that M is compact. Thus we can derive from three balls theorem above uniform doubling estimate on solutions.

Theorem 3.2 (doubling estimate). *There exist two positive constants C_1 and C_2 , depending only on M such that : if u is a solution to (1.1) on M with W of class C^1 then for any x_0 in M and any $r > 0$, one has*

$$\|u\|_{B_{2r}(x_0)} \leq e^{C_1 \sqrt{\|W\|_{C^1}} + C_2} \|u\|_{B_r(x_0)}. \quad (3.8)$$

Remark 3.3. *Using standard elliptic theory to bound the L^∞ norm of $|u|$ by a multiple of its L^2 norm (see [GT01] theorem 8.17, or [CFG86]) and rescaling arguments gives for $\delta > 0$:*

$$\|u\|_{L^\infty(B_\delta(x_0))} \leq (C_1 \|W\|_\infty + C_2)^{\frac{n}{2}} \delta^{-n/2} \|u\|_{L^2(B_{2\delta}(x_0))}.$$

Then one can see that the doubling estimate is still true with the L^∞ norm:

$$\|u\|_{L^\infty(B_{2r}(x_0))} \leq e^{C_1 \sqrt{\|W\|_{C^1}} + C_2} \|u\|_{L^\infty(B_r(x_0))}. \quad (3.9)$$

Remark 3.4. *We recall also that, it is necessary to assume that M is compact to obtain an uniform upper bound on the vanishing order, and therefore a doubling estimate, on solutions. Indeed, consider the harmonic function, $f_k = \Re(x_1 + ix_2)^k$ defined in \mathbb{R}^2 , so f_k satisfies (1.1) with $W = 0$. The functions f_k can vanish at arbitrary high order at 0.*

To prove the theorem 3.2 we need to use the standard overlapping chains of balls argument ([DF88, JL99, Kuk98]) to show :

Proposition 3.5. *For any $R > 0$ there exists $C_R > 0$ such that for any $x_0 \in M$, any $W \in C^1(M)$ and any solutions u to (1.1) :*

$$\|u\|_{B_R(x_0)} \geq e^{-C_R(1 + \sqrt{\|W\|_{C^1}})} \|u\|_{L^2(M)}.$$

Proof. We may assume without loss of generality that $R < R_0$, with R_0 as in the three balls inequality (proposition 3.1). Up to multiplication by a constant, we can assume that $\|u\|_{L^2(M)} = 1$. We denote by \bar{x} a point in M such that $\|u\|_{B_R(\bar{x})} = \sup_{x \in M} \|u\|_{B_R(x)}$. This implies that one has $\|u\|_{B_R(\bar{x})} \geq D_R$, where D_R depend only on M and R . One has from proposition (3.1) at an arbitrary point x of M :

$$\|u\|_{B_{R/2}(x)} \geq e^{-c(1+\sqrt{\|W\|_{C^1}})} \|u\|_{B_R(x)}^{\frac{1}{\alpha}}. \quad (3.10)$$

Let γ be a geodesic curve between x_0 and \bar{x} and define $x_1, \dots, x_m = \bar{x}$ such that $x_i \in \gamma$ and $B_{\frac{R}{2}}(x_{i+1}) \subset B_R(x_i)$, for any i from 0 to $m-1$. The number m depends only on $\text{diam}(M)$ and R . Then the properties of $(x_i)_{1 \leq i \leq m}$ and inequality (3.10) give for all i , $1 \leq i \leq m$:

$$\|u\|_{B_{R/2}(x_i)} \geq e^{-c(1+\sqrt{\|W\|_{C^1}})} \|u\|_{B_{R/2}(x_{i+1})}^{\frac{1}{\alpha}}. \quad (3.11)$$

The result follows by iteration and the fact that $\|u\|_{B_R(\bar{x})} \geq D_R$. \square

Corollary 3.6. *For all $R > 0$, there exists a positive constant C_R depending only on M and R such that at any point x_0 in M one has*

$$\|u\|_{B_{2R}(x_0)} \geq e^{-C_R(1+\sqrt{\|W\|_{C^1}})} \|u\|_{L^2(M)}.$$

Proof. Recall that $\|u\|_{B_{2R}(x_0)} = \|u\|_{L^2(A_{R,2R})}$ with $A_{R,2R} := \{x \in M; R \leq d(x, x_0) \leq 2R\}$. Let $R < R_0$ where R_0 is from proposition 3.3, note that $R_0 \leq \text{diam}(M)$. Since M is geodesically complete, there exists a point x_1 in $A_{R,2R}$ such that $B_{x_1}(\frac{R}{4}) \subset A_{R,2R}$. From proposition 3.5 one has $\|u\|_{B_{\frac{R}{4}}(x_1)} \geq e^{-C_R(1+\sqrt{\|W\|_{C^1}})} \|u\|_{L^2(M)}$ which gives the result. \square

Proof of theorem 3.2. We proceed as in the proof of three balls inequality (proposition 3.3) except for the fact that now we want the first ball to become arbitrary small in front of the others. Let $R = \frac{R_0}{4}$ with R_0 as in the three balls inequality, let δ such that $0 < 3\delta < \frac{R}{8}$, and define a smooth function ψ , with $0 \leq \psi \leq 1$ as follows:

- $\psi(x) = 0$ if $r(x) < \delta$ or if $r(x) > R$,
- $\psi(x) = 1$ if $r(x) \in [\frac{5\delta}{4}, \frac{R}{2}]$,
- $|\nabla\psi(x)| \leq \frac{C}{\delta}$ and $|\nabla^2\psi(x)| \leq \frac{C}{\delta^2}$ if $r(x) \in [\delta, \frac{5\delta}{4}]$,
- $|\nabla\psi(x)| \leq C$ and $|\nabla^2\psi(x)| \leq C$ if $r(x) \in [\frac{R}{2}, R]$.

Keeping appropriate terms in (2.3) applied to ψu gives :

$$\|r^{\frac{\varepsilon}{2}} e^{\tau\phi} \psi u\| + \tau\delta \|r^{-1} e^{\tau\phi} \psi u\| \leq C \left(\|r^2 e^{\tau\phi} \nabla u \cdot \nabla \psi\| + \|r^2 e^{\tau\phi} \Delta \psi u\| \right).$$

Using properties of ψ , one has

$$\begin{aligned} \|r^{\frac{\varepsilon}{2}} e^{\tau\phi} u\|_{\frac{R}{8}, \frac{R}{4}} + \|e^{\tau\phi} u\|_{\frac{5\delta}{4}, 3\delta} &\leq C \left(\delta \|e^{\tau\phi} \nabla u\|_{\delta, \frac{5\delta}{4}} + \|e^{\tau\phi} \nabla u\|_{\frac{R}{2}, R} \right) \\ &+ C \left(\|e^{\tau\phi} u\|_{\delta, \frac{5\delta}{4}} + \|e^{\tau\phi} u\|_{\frac{R}{2}, R} \right). \end{aligned}$$

Using (3.4) and properties of ϕ , we get

$$\begin{aligned} e^{\tau\phi(\frac{R}{4})} \|u\|_{\frac{R}{8}, \frac{R}{4}} + e^{\tau\phi(3\delta)} \|u\|_{\frac{5\delta}{4}, 3\delta} \\ \leq C(1 + \|W\|_{\infty}^{1/2}) \left(e^{\tau\phi(\delta)} \|u\|_{\frac{3\delta}{2}} + e^{\tau\phi(\frac{R}{3})} \|u\|_{\frac{5R}{3}} \right). \end{aligned}$$

Then adding $e^{\tau\phi(3\delta)} \|u\|_{\frac{5\delta}{4}}$ to each side leads to

$$\begin{aligned} e^{\tau\phi(\frac{R}{4})} \|u\|_{\frac{R}{8}, \frac{R}{4}} + e^{\tau\phi(3\delta)} \|u\|_{3\delta} \\ \leq C(1 + \|W\|_{\infty}^{1/2}) \left(e^{\tau\phi(\delta)} \|u\|_{\frac{3\delta}{2}} + e^{\tau\phi(\frac{R}{3})} \|u\|_{\frac{5R}{3}} \right). \end{aligned}$$

Now we want to choose τ such that

$$C(1 + \|W\|_{\infty}^{1/2}) e^{\tau\phi(\frac{R}{3})} \|u\|_{\frac{5R}{3}} \leq \frac{1}{2} e^{\tau\phi(\frac{R}{4})} \|u\|_{\frac{R}{8}, \frac{R}{4}}.$$

For the same reasons than before we choose

$$\tau = \frac{1}{\phi(\frac{R}{3}) - \phi(\frac{R}{4})} \ln \left(\frac{1}{2C(1 + \|W\|_{\infty}^{1/2})} \frac{\|u\|_{\frac{R}{8}, \frac{R}{4}}}{\|u\|_{\frac{5R}{3}}} \right) + C_1(1 + \sqrt{\|W\|_{c^1}}).$$

Define $D_R = (\phi(\frac{R}{3}) - \phi(\frac{R}{4}))^{-1}$; like before one has $0 < A \leq -D_R \leq A^{-1}$. Dropping the first term in the left hand side, one has

$$\|u\|_{3\delta} \leq e^{C(1 + \sqrt{\|W\|_{c^1}})} \left(\frac{\|u\|_{\frac{R}{8}, \frac{R}{4}}}{\|u\|_{\frac{5R}{3}}} \right)^{-A} \|u\|_{\frac{3\delta}{2}}$$

Finally from corollary 3.6, define $r = \frac{3\delta}{2}$ to have :

$$\|u\|_{2r} \leq e^{C(1 + \sqrt{\|W\|_{c^1}})} \|u\|_r.$$

Thus, the theorem is proved for all $r \leq \frac{R_0}{16}$. Using proposition 3.5 we have for $r \geq \frac{R_0}{16}$:

$$\begin{aligned} \|u\|_{B_{x_0}(r)} &\geq \|u\|_{B_{x_0}(\frac{R_0}{16})} \geq e^{-C_0(1 + \sqrt{\|W\|_{c^1}})} \|u\|_{L^2(M)} \\ &\geq e^{-C_1(1 + \sqrt{\|W\|_{c^1}})} \|u\|_{B_{x_0}(2r)}. \end{aligned}$$

□

As stated before, the upper bound on vanishing order of solutions (theorem 1.1) is a direct consequence of theorem 3.2 for non trivial solutions to (1.1).

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