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# NUMERICAL CHARACTERISATION OF QUADRICS 

THOMAS DEDIEU AND ANDREAS HÖRING


#### Abstract

Let $X$ be a Fano manifold such that $-K_{X} \cdot C \geq \operatorname{dim} X$ for every rational curve $C \subset X$. We prove that $X$ is a projective space or a quadric.


## 1. Introduction

Let $X$ be a Fano manifold, i.e. a complex projective manifold with ample anticanonical divisor $-K_{X}$. If the Picard number of $X$ is at least two, Mori theory shows the existence of at least two non-trivial morphisms $\varphi_{i}: X \rightarrow Y_{i}$ which encode some interesting information on the geometry of $X$. On the contrary, when the Picard number equals one Mori theory does not yield any information, and one is thus led to studying $X$ in terms of the positivity of the anticanonical bundle. A well-known example of such a characterisation is the following theorem of Kobayashi-Ochiai.
1.1. Theorem [KO73]. Let $X$ be a projective manifold of dimension $n$. Suppose that $-K_{X} \sim d H$ with $H$ an ample divisor on $X$.
a) Then one has $d \leq n+1$ and equality holds if and only if $X \simeq \mathbb{P}^{n}$.
b) If $d=n$, then $X \simeq \mathbb{Q}^{n}$.

The divisibility of $-K_{X}$ in the Picard group is a rather restrictive condition, so it is natural to ask for similar characterisations under (a priori) weaker assumptions. Based on Kebekus' study of singular rational curves [Keb02b], Cho, Miyaoka and Shepherd-Barron proved a generalisation of the first part of Theorem 1.1:
1.2. Theorem [CMSB02, Keb02a]. Let $X$ be a Fano manifold of dimension $n$. Suppose that

$$
-K_{X} \cdot C \geq n+1 \quad \text { for all rational curves } C \subset X
$$

Then $X \simeq \mathbb{P}^{n}$.

The aim of this paper is to prove the following, which is a similar generalisation for the second part of Theorem 1.1:
1.3. Theorem. Let $X$ be a Fano manifold of dimension $n$. Suppose that

$$
-K_{X} \cdot C \geq n \quad \text { for all rational curves } C \subset X \text {. }
$$

Then $X \simeq \mathbb{P}^{n}$ or $X \simeq \mathbb{Q}^{n}$.

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This statement already appeared in a paper of Miyaoka [Miy04, Thm.0.1], but the proof there is incomplete (cf. Remark 5.2 for instance). In this paper we borrow some ideas and tools from Miyaoka's, yet give a proof based on a completely different strategy. Note also that Hwang gave a proof under the additional assumption that the general VMRT (see below) is smooth [Hwa13, Thm.1.11], a property that does not hold for every Fano manifold [CD15, Thm.1.10].

In the proof of Theorem 1.3, we have to assume $n \geq 4$; for $n \leq 3$ the statement follows directly from classification results.

The assumption that $X$ is Fano assures that $\rho(X)=1$ because of the IonescuWiśniewski inequality [Ion86, Thm.0.4], [Wiś91, Thm.1.1] (see §4.1). It is possible to remove this assumption: the Ionescu-Wiśniewski inequality together with [HN13, Thm.1.3] enable one to deal with the case $\rho(X)>1$, and one gets the following.
1.4. Corollary. Let $X$ be a projective manifold of dimension $n$ containing a rational curve. If

$$
-K_{X} \cdot C \geq n \quad \text { for all rational curves } C \subset X
$$

then $X$ is a projective space, a hyperquadric, or a projective bundle over a curve.
(Note that under the assumptions of Corollary 1.4, if $\rho(X)=1$ then $X$ is Fano.)

Outline of the proof. In the situation of Theorem 1.3 let $\mathcal{K}$ be a family of minimal rational curves on $X$. By Mori's bend-and-break lemma a minimal curve $[l] \in \mathcal{K}$ satisfies $-K_{X} \cdot l \leq n+1$ and if equality holds then $X \simeq \mathbb{P}^{n}$ by [CMSB02]. By our assumption we are thus left to deal with the case $-K_{X} \cdot l=n$. Then, for a general point $x \in X$ the normalisation $\mathcal{K}_{x}$ of the space parametrising curves in $\mathcal{K}$ passing through $x$ has dimension $n-2$, and by [Keb02b, Thm.3.4] there exists a morphism

$$
\tau_{x}: \mathcal{K}_{x} \rightarrow \mathbb{P}\left(\Omega_{X, x}\right)
$$

which maps a general curve $[l] \in \mathcal{K}_{x}$ to its tangent direction $T_{l, x}^{\perp}$ at the point $x$. By [HM04, Thm.1] this map is birational onto its image $\mathcal{V}_{x}$, the variety of minimal rational tangents (VMRT) at $x$. We denote by $\mathcal{V} \subset \mathbb{P}\left(\Omega_{X}\right)$ the total VMRT, i.e. the closure of the locus covered by the VMRTs $\mathcal{V}_{x}$ for $x \in X$ general. To prove Theorem 1.3, we compute the cohomology class of the total VMRT $\mathcal{V} \subset \mathbb{P}\left(\Omega_{X}\right)$ in terms of the tautological class $\zeta$ and $\pi^{*} K_{X}$, where $\pi: \mathbb{P}\left(\Omega_{X}\right) \rightarrow X$ is the projection map. This computation is based on the construction, on the manifold $X$, of a family $\mathcal{W}^{\circ}$ of smooth rational curves such that for every $[C] \in \mathcal{W}^{\circ}$ one has

$$
\left.T_{X}\right|_{C} \simeq \mathcal{O}_{\mathbb{P}^{1}}(2)^{\oplus n}
$$

it lifts to a family of curves on $\mathbb{P}\left(\Omega_{X}\right)$ by associating to a curve $C \subset X$ the image $\tilde{C}$ of the morphism $C \rightarrow \mathbb{P}\left(\Omega_{X}\right)$ defined by the invertible quotient

$$
\left.\Omega_{X}\right|_{C} \rightarrow \Omega_{C}
$$

The main technical statement of this paper is:
1.5. Proposition. Let $X \nsim \mathbb{P}^{n}$ be a Fano manifold of dimension $n \geq 4$, and suppose that

$$
-K_{X} \cdot C \geq n \quad \text { for all rational curves } C \subset X
$$

Then, in the above notation, one has $\mathcal{V} \cdot \tilde{C}=0$ for all $[C] \in \mathcal{W}^{\circ}$.

Once we have shown this statement a similar intersection computation involving a general minimal rational curve $l$ yields that the VMRT $\mathcal{V}_{x} \subset \mathbb{P}\left(\Omega_{X, x}\right)$ is a hypersurface of degree at most two. We then conclude with some earlier results of Araujo, Hwang, and Mok [Ara06, Hwa07, Mok08].

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## 2. Notation and conventions

We work over the field $\mathbb{C}$ of complex numbers. Throughout the paper, $\mathbb{Q}^{n}$ designates a smooth quadric hypersurface in $\mathbb{P}^{n+1}$ for any positive integer $n$. Topological notions refer to the Zariski topology.

We use the modern notation for projective spaces, as introduced by Grothendieck: if $\mathcal{E}$ is a locally free sheaf on a scheme $X$, we let $\mathbb{P}(\mathcal{E})$ be $\operatorname{Proj}(\operatorname{Sym} \mathcal{E})$. If $L$ is a line in a vector space $V, L^{\perp}$ designates the corresponding point in $\mathbb{P}\left(V^{\vee}\right)$. The symbols $\equiv$ and $\sim_{\mathbb{Q}}$ refer to numerical and $\mathbb{Q}$-linear equivalence respectively.

A variety is an integral scheme of finite type over $\mathbb{C}$, a manifold is a smooth variety. A fibration is a proper surjective morphism with connected fibres $\varphi: X \rightarrow$ $Y$ such that $X$ and $Y$ are normal and $\operatorname{dim} X>\operatorname{dim} Y>0$.

We use the standard terminology and results on rational curves, as explained in [Kol96, Ch.II], [Deb01, Ch.2,3,4], and [Hwa01]. Let $X$ be a projective variety. We remind the reader that following [Kol96, II, Def.2.11], the notation RatCurves ${ }^{\text {n }} X$ refers to the union of the normalisations of those locally closed subsets of the Chow variety of $X$ parametrising irreducible rational curves (the superscript ${ }^{\mathrm{n}}$ is a reminder that we normalised, and has nothing to do with the dimension).

For technical reasons, we have to consider families of rational curves on $X$ as living alternately in RatCurves ${ }^{\mathrm{n}} X$ and in $\operatorname{Hom}\left(\mathbb{P}^{1}, X\right)$. Our general policy is to call $\operatorname{Hom}_{\mathcal{R}} \subset \operatorname{Hom}\left(\mathbb{P}^{1}, X\right)$ the family corresponding to a normal variety $\mathcal{R} \subset$ RatCurves $^{\mathrm{n}} X$.

## 3. Preliminaries on conic bundles

In this section, we establish some basic facts about conic bundles over a curve and compute some intersection numbers which will turn out to be crucial for the proof of Proposition 1.5. All these statements appear in one form or another in [Miy04, §2], but we recall them and their proofs for the clarity of exposition.
3.1. Definition. $A$ conic bundle is an equidimensional projective fibration $\varphi$ : $X \rightarrow Y$ such that there exists a rank three vector bundle $V \rightarrow Y$ and an embedding $X \hookrightarrow \mathbb{P}(V)$ that maps every $\varphi$-fibre $\varphi^{-1}(y)$ onto a conic (i.e. the zero scheme of a degree 2 form) in $\mathbb{P}\left(V_{y}\right)$. The set

$$
\Delta:=\left\{y \in Y \mid \varphi^{-1}(y) \text { is not smooth }\right\}
$$

is called the discriminant locus of the conic bundle.

[^0]3.2. Lemma. Let $S$ be a smooth surface admitting a projective fibration $\varphi: S \rightarrow T$ onto a smooth curve such that the general fibre is $\mathbb{P}^{1}$, and such that $-K_{S}$ is $\varphi$-nef. Let $F$ be a reducible $\varphi$-fibre and suppose that
$$
F=C_{1}+C_{2}+F^{\prime}
$$
where the $C_{i}$ are $(-1)$-curves and $C_{i} \not \subset S u p p\left(F^{\prime}\right)$. Then $F^{\prime}=\sum E_{j}$ is a reduced chain of $(-2)$-curves and the dual graph of $F$ is as depicted in Figure 1.

## Figure 1



Proof. Write $F^{\prime}=\sum_{j=1}^{k} a_{j} E_{j}, a_{j} \in \mathbb{N}$, where $E_{1}, \ldots, E_{k}$ are the irreducible components of $F^{\prime}$. First note that since $-K_{S} \cdot F=2$ and $-K_{S} \cdot C_{i}=1$, the fact that $-K_{S}$ is $\varphi$-nef implies $-K_{S} \cdot E_{j}=0$ for all $j$. Since $E_{j}$ is an irreducible component of a reducible fibre, we have $E_{j}^{2}<0$. Thus we see that each $E_{j}$ is a $(-2)$-curve.

We will now proceed by induction on the number of irreducible components of $F^{\prime}$, the case $F^{\prime}=0$ being trivial. Let $\mu: S \rightarrow S^{\prime}$ be the blow-down of the $(-1)$ curve $C_{2}$; then by the rigidity lemma [Deb01, Lemma 1.15], there is a morphism $\varphi^{\prime}: S^{\prime} \rightarrow T$ such that $\varphi=\varphi^{\prime} \circ \mu$. Note that $S^{\prime}$ is smooth and $-K_{S^{\prime}}$ is $\varphi^{\prime}$-nef. We also have

$$
0=C_{2} \cdot F=-1+C_{2} \cdot\left(C_{1}+\sum_{i=1}^{k} a_{i} E_{i}\right)
$$

so $C_{2}$ meets $C_{1}+\sum_{i=1}^{k} a_{i} E_{i}$ transversally in exactly one point. If $C_{2} \cdot C_{1}>0$, then $\mu_{*}\left(C_{1}\right)$ has self-intersection 0 , yet it is also an irreducible component of the reducible fibre $\mu_{*}\left(C_{1}+\sum_{i=1}^{k} a_{i} E_{i}\right)$, a contradiction. Thus (up to renumbering) we can suppose that $C_{2} \cdot E_{1}=1$ and $a_{1}=1$. In particular $\mu_{*}\left(E_{1}\right)$ is a $(-1)$-curve, so

$$
\mu_{*}\left(C_{1}+\sum_{i=1}^{k} a_{i} E_{i}\right)=\mu_{*}\left(C_{1}\right)+\mu_{*}\left(E_{1}\right)+\mu_{*}\left(\sum_{i=2}^{k} a_{i} E_{i}\right)
$$

satisfies the induction hypothesis.

In the following we use that for every normal surface one can define an intersection theory using the Mumford pull-back to the minimal resolution, cf. [Sak84].
3.3. Lemma. Let $S$ be a normal surface admitting a projective fibration $\varphi: S \rightarrow T$ onto a smooth curve such that the general fibre is $\mathbb{P}^{1}$ and such that every fibre is reduced and has at most two irreducible components. Then
a) $\varphi$ is a conic bundle;
b) $S$ has at most $A_{k}$-singularities; and
c) if $s \in S_{\text {sing }}$, then $s=F_{\varphi(s), 1} \cap F_{\varphi(s), 2}$ where $F_{\varphi(s)}=F_{\varphi(s), 1}+F_{\varphi(s), 2}$ is the decomposition of the fibre over $\varphi(s)$ in its irreducible components. In particular $F_{\varphi(s)}$ is a reducible conic.

Proof. If a fibre $\varphi^{-1}(t)$ is irreducible, then $\varphi$ is a $\mathbb{P}^{1}$-bundle over a neighbourhood of $t$ [Kol96, II, Thm.2.8]. Thus we only have to consider points $t \in T$ such that $S_{t}:=\varphi^{-1}(t)$ is reducible. Since $p_{a}\left(S_{t}\right)=0$ and $S_{t}=C_{1}+C_{2}$ is reduced, we see that $S_{t}$ is a union of two $\mathbb{P}^{1}$ 's meeting transversally in a point. Since $S_{t}=\varphi^{*} t$ is a Cartier divisor, this already implies c).

Let $\varepsilon: \hat{S} \rightarrow S$ be the canonical modification [Kol13, Thm.1.31] of the singular points lying on $S_{t}$. Then we have

$$
K_{\hat{S}} \equiv \varepsilon^{*} K_{S}-E
$$

with $E$ an effective $\varepsilon$-exceptional $\mathbb{Q}$-divisor whose support is equal to the $\varepsilon$ exceptional locus. Denote by $\hat{C}_{i}$ the proper transform of $C_{i}$. If $K_{\hat{S}} \cdot \hat{C}_{i}<-1$, then $\hat{C}_{i}$ deforms in $\hat{S}$ [Kol96, II, Thm.1.15]. Yet $\hat{C}_{i}$ is an irreducible component of a reducible $\varphi \circ \varepsilon$-fibre, so this is impossible. So we have

$$
K_{S} \cdot C_{i} \geq K_{\hat{S}} \cdot \hat{C}_{i} \geq-1
$$

for $i=1,2$. Since $K_{S} \cdot\left(C_{1}+C_{2}\right)=-2$, this implies that $K_{S} \cdot C_{i}=-1$ and $E=0$. Thus $S$ has canonical singularities. Since canonical surface singularities are Gorenstein we see that $-K_{S}$ is Cartier, $\varphi$-ample and defines an embedding

$$
S \subset \mathbb{P}\left(V:=\varphi_{*}\left(\mathcal{O}_{S}\left(-K_{S}\right)\right)\right)
$$

into a $\mathbb{P}^{2}$-bundle mapping each fibre onto a conic. This proves a).
Let now $\tilde{\varepsilon}: \tilde{S} \rightarrow S$ be the minimal resolution. It is crepant, so the divisor $-K_{\tilde{S}}$ is $\varphi \circ \tilde{\varepsilon}$-nef. Moreover the proper transforms $\tilde{C}_{i}$ of the curves $C_{i}$ are $(-1)$-curves in $\tilde{S}$. By Lemma 3.2 this proves b).

The following fundamental lemma should be seen as an analogue of the basic fact that a projective bundle over a curve contains at most one curve with negative self-intersection.
3.4. Lemma [Miy04, Prop.2.4]. Let $S$ be a normal projective surface that is a conic bundle $\varphi: S \rightarrow T$ over a smooth curve $T$, and denote by $\Delta$ the discriminant locus. Suppose that $\varphi$ has two disjoint sections $\sigma_{1}$ and $\sigma_{2}$, both contained in the smooth locus of $S$. Suppose moreover that for every $t \in \Delta$, the fibre $F_{t}$ has a decomposition $F_{t}=F_{t, 1}+F_{t, 2}$ such that

$$
\begin{equation*}
\sigma_{i} \cdot F_{t, j}=\delta_{i, j} \tag{C1}
\end{equation*}
$$

(Kronecker's delta). Assume also that we have

$$
\begin{equation*}
\sigma_{1}^{2}<0 \text { and } \sigma_{2}^{2}<0 \tag{C2}
\end{equation*}
$$

Let $\varepsilon: \hat{S} \rightarrow S$ be the minimal resolution. Let $\sigma$ be a $\varphi$-section, and $\hat{\sigma} \subset \hat{S}$ its proper transform. Then the following holds:
a) If $(\hat{\sigma})^{2}<0$, then $\sigma=\sigma_{1}$ or $\sigma=\sigma_{2}$.
b) If $(\hat{\sigma})^{2}=0$ then $\sigma$ is disjoint from $\sigma_{1} \cup \sigma_{2}$.
3.5. Remarks. 1. In the situation above all the fibres are reduced, since there exists a section that is contained in the smooth locus.
2. The two inequalities ( C 2 ) are satisfied if there exists a birational morphism $S \rightarrow S^{\prime}$ onto a projective surface $S^{\prime}$ that contracts $\sigma_{1}$ and $\sigma_{2}$. More generally, the

Hodge index theorem implies that (C2) holds if there exists a nef and big divisor $H$ on $S$ such that $H \cdot \sigma_{1}=H \cdot \sigma_{2}=0$.

Proof. Preparation: contraction to a smooth ruled surface. Lemma 3.3 applies to the surface $S$. It follows that $S$ has an $A_{k_{t}}$-singularity ( $k_{t} \geq 0$ ) in $F_{t, 1} \cap F_{t, 2}$ for every $t \in \Delta$, and no further singularity. In particular, the dual graph of $(\varphi \circ \varepsilon)^{-1}(t)$ is as described in Figure 1 for every $t \in \Delta$.

We consider the birational morphism

$$
\hat{\mu}: \hat{S} \rightarrow S^{b}
$$

defined as the composition, for every $t \in \Delta$, of the blow-down of the proper transform $\hat{F}_{t, 1}$ of $F_{t, 1}$ and of all the $k_{t}(-2)$-curves contained in $(\varphi \circ \varepsilon)^{-1}(t)$. Since $\hat{\mu}$ is a composition of blow-down of $(-1)$-curves, the surface $S^{b}$ is smooth. By the rigidity lemma [Deb01, Lemma 1.15], there is a morphism $\varphi^{b}: S^{b} \rightarrow T$. All its fibres are irreducible rational curves, so it is a $\mathbb{P}^{1}$-bundle by [Kol96, II, Thm.2.8]. Again by the rigidity lemma, $\hat{\mu}$ factors through $\varepsilon$, i.e. there is a birational morphism $\mu: S \rightarrow S^{b}$ such that $\hat{\mu}=\mu \circ \varepsilon$; it is the contraction of all the curves $F_{t, 1}, t \in \Delta$.

Since $\sigma_{1}$ meets $F_{t, 1}$ in a smooth point of $S$, the proper transforms $\hat{\sigma}_{1}$ and $\hat{F}_{t, 1}$ meet in the same point. Thus (the successive images of) $\hat{\sigma}_{1}$ meets the exceptional divisor of all the blow-downs of $(-1)$-curves composing $\hat{\mu}$, and since the section $\sigma_{1}^{b}:=\hat{\mu}\left(\hat{\sigma}_{1}\right)$ is smooth, all the intersections are transversal. Vice versa we can say that $\hat{S}$ is obtained from $S^{b}$ by blowing up points on (the successive proper transforms of) $\sigma_{1}^{b}$.

By the symmetry condition ( C 1 ) the curve $\sigma_{2}$ is disjoint from the $\mu$-exceptional locus, so if we set $\sigma_{2}^{b}:=\mu\left(\sigma_{2}\right)$, then we have $\left(\sigma_{2}^{b}\right)^{2}=\left(\sigma_{2}\right)^{2}<0$. Thus, in the notation of [Har77, V,Ch.2], $\varphi^{b}: S^{b} \rightarrow T$ is a ruled surface with invariant $-e:=$ $\left(\sigma_{2}^{b}\right)^{2}>0$. In particular the Mori cone $\overline{\mathrm{NE}}\left(S^{b}\right)$ is generated by a general $\varphi^{b}$-fibre $F$ and $\sigma_{2}^{b}$. Since $\sigma_{1}^{b} \cdot \sigma_{2}^{b}=0$ and $\sigma_{1}^{b} \cdot F=1$, we have

$$
\begin{equation*}
\sigma_{1}^{b} \equiv \sigma_{2}^{b}+e F \tag{3.5.1}
\end{equation*}
$$

Conclusion. Let now $\sigma \subset S$ be a section that is distinct from both $\sigma_{1}$ and $\sigma_{2}$. Then $\sigma^{b}:=\mu(\sigma)$ is distinct from both $\sigma_{1}^{b}$ and $\sigma_{2}^{b}$. Since $\sigma^{b} \neq \sigma_{2}^{b}$ we have

$$
\begin{equation*}
\sigma^{b} \equiv \sigma_{2}^{b}+c F \tag{3.5.2}
\end{equation*}
$$

for some $c \geq e\left[\operatorname{Har} 77\right.$, V, Prop.2.20]. Since $\sigma^{b} \neq \sigma_{1}^{b}$ we have

$$
\begin{equation*}
\sigma^{b} \cdot \sigma_{1}^{b} \geq \sum_{t \in \Delta} \tau_{t} \tag{3.5.3}
\end{equation*}
$$

where $\tau_{t}$ is the intersection multiplicity of $\sigma^{b}$ and $\sigma_{1}^{b}$ at the point $F_{t} \cap \sigma_{1}^{b}$. Denote by $\hat{\sigma} \subset \hat{S}$ the proper transform of $\sigma \subset S$, which is also the proper transform of $\sigma^{b} \subset S^{b}$. By our description of $\hat{\mu}$ as a sequence of blow-ups in $\sigma_{1}^{b}$ we obtain

$$
(\hat{\sigma})^{2}=\left(\sigma^{b}\right)^{2}-\sum_{t \in \Delta} \min \left(\tau_{t}, k_{t}+1\right) \geq\left(\sigma^{b}\right)^{2}-\sum_{t \in \Delta} \tau_{t}
$$

By (3.5.3) this implies

$$
(\hat{\sigma})^{2} \geq\left(\sigma^{b}\right)^{2}-\sigma^{b} \cdot \sigma_{1}^{b}=\sigma^{b} \cdot\left(\sigma^{b}-\sigma_{1}^{b}\right)
$$

Plugging in (3.5.1) and (3.5.2) we obtain

$$
\begin{equation*}
(\hat{\sigma})^{2} \geq c-e \geq 0 \tag{3.5.4}
\end{equation*}
$$

This shows statement a).
Suppose now that $(\hat{\sigma})^{2}=0$. Then by (3.5.4) we have $c=e$, hence $\sigma^{b} \cdot \sigma_{2}^{b}=0$. Being distinct, the two curves $\sigma^{b}$ and $\sigma_{2}^{b}$ are therefore disjoint, and so are their proper transforms $\hat{\sigma}$ and $\hat{\sigma}_{2}$. Note now that $\varepsilon$ is an isomorphism in a neighbourhood of $\hat{\sigma}_{2}$, so $\sigma=\varepsilon(\hat{\sigma})$ is disjoint from $\sigma_{2}=\varepsilon\left(\hat{\sigma}_{2}\right)$. In order to see that $\sigma$ and $\sigma_{1}$ are disjoint, we repeat the same argument but contract those fibre components which meet $\sigma_{2}$. This proves statement b).

## 4. The main construction

4.1. Set-up. For the whole section, we let $X \not 千 \mathbb{P}^{n}$ be a Fano manifold of dimension $n \geq 4$, and suppose that

$$
\begin{equation*}
-K_{X} \cdot C \geq n \quad \text { for all rational curves } C \subset X \tag{4.1.1}
\end{equation*}
$$

this is the situation of Proposition 1.5. It then follows from the Ionescu-Wiśniewski inequality that the Picard number $\rho(X)$ equals 1, see [Miy04, Lemma 4.1].

Recall that a family of minimal rational curves is an irreducible component $\mathcal{K}$ of RatCurves ${ }^{\mathrm{n}}(X)$ such that the curves in $\mathcal{K}$ dominate $X$, and for $x \in X$ general the algebraic set $\mathcal{K}_{x}^{b} \subset \mathcal{K}$ parametrising curves passing through $x$ is proper. We will use the following simple observation:
4.2. Lemma. In the situation of Proposition 1.5, let $l \subset X$ be a rational curve such that $-K_{X} \cdot l=n$. Then any irreducible component $\mathcal{K}$ of RatCurves $^{n} X$ containing [l] is a family of minimal rational curves.

Proof. Condition (4.1.1) implies the properness of $\mathcal{K}[K o l 96$, II, (2.14)]. On the other hand, we know by [Kol96, IV, Cor.2.6.2] that the curves parametrised by $\mathcal{K}$ dominate $X$.
4.3. Minimal rational curves and VMRTs. Since $X$ is Fano, it contains a rational curve $l\left[\operatorname{Mor} 79\right.$, Thm.6]. Since $X \nsimeq \mathbb{P}^{n}$, there exists a rational curve with $-K_{X} \cdot l=n$ by [CMSB02], and by Lemma 4.2 there exists a family of minimal rational curves containing the point $[l] \in \operatorname{RatCurves}^{\mathrm{n}}(X)$. We fix once and for all such a family, which we call $\mathcal{K}$.

For $x \in X$ general, denote by $\mathcal{K}_{x}$ the normalisation of the algebraic set $\mathcal{K}_{x}^{b} \subset \mathcal{K}$ parametrising curves passing through $x$. Every member of $\mathcal{K}_{x}^{b}$ is a free curve (this follows from the argument of [Kol96, II, proof of Thm.3.11]), so $\mathcal{K}_{x}$ is smooth and has dimension $n-2 \geq 2$ [Kol96, II, (1.7) and (2.16)].

By results of Kebekus, a general curve $[l] \in \mathcal{K}_{x}^{b}$ is smooth [Keb02b, Thm.3.3], and the tangent map

$$
\tau_{x}: \mathcal{K}_{x} \rightarrow \mathbb{P}\left(\Omega_{X, x}\right)
$$

which to a general curve $[l]$ associates its tangent direction $T_{l, x}^{\perp}$ at the point $x$ is a finite morphism [Keb02b, Thm.3.4]. Its image $\mathcal{V}_{x}$ is called the variety of minimal rational tangents (VMRT) at $x$. The map $\tau_{x}$ is birational by [HM04, Thm.1], so the normalisation of $\mathcal{V}_{x}$ is $\mathcal{K}_{x}$, which is smooth (this is [HM04, Cor.1]). Also, one can associate to a general point $v \in \mathcal{V}_{x}$ a unique minimal curve $[l] \in \mathcal{K}_{x}$. We denote
by $\mathcal{V} \subset \mathbb{P}\left(\Omega_{X}\right)$ the total $V M R T$, i.e. the closure of the locus covered by the VMRTs $\mathcal{V}_{x}$ for $x \in X$ general. Since $\mathcal{K}_{x}$ has dimension $n-2$, the total VMRT $\mathcal{V}$ is a divisor in $\mathbb{P}\left(\Omega_{X}\right)$.

For a general $[l] \in \mathcal{K}$, one has

$$
\begin{equation*}
\left.T_{X}\right|_{l} \simeq \mathcal{O}_{\mathbb{P}^{1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{1}}(1)^{\oplus n-2} \oplus \mathcal{O}_{\mathbb{P}^{1}} \tag{4.3.1}
\end{equation*}
$$

[Kol96, IV, Cor.2.9]. We call a minimal rational curve $[l] \in \mathcal{K}$ standard if $l$ is smooth and the bundle $\left.T_{X}\right|_{l}$ has the same splitting type as in (4.3.1).
4.4. Smoothing pairs of minimal curves. For a general point $x_{1} \in X$ the curves parametrised by $\mathcal{K}_{x_{1}}$ cover a divisor $D_{x_{1}} \subset X$ [Kol96, IV, Prop.2.5]. This divisor is ample because $\rho(X)=1$, so for $x_{2} \in X$ and $\left[l_{2}\right] \in \mathcal{K}_{x_{2}}$ the curve $l_{2}$ intersects $D_{x_{1}}$. Thus for a general point $x_{2} \in X$ we can find a chain of two standard minimal curves $l_{1} \cup l_{2}$ connecting the points $x_{1}$ and $x_{2}$. By [Kol96, II, Ex.7.6.4.1] the union $l_{1} \cup l_{2}$ is dominated by a transverse union $\mathbb{P}^{1} \cup \mathbb{P}^{1}$. Since both rational curves are free we can smooth the tree $\mathbb{P}^{1} \cup \mathbb{P}^{1}$ keeping the point $x_{1}$ fixed [Kol96, II, Thm.7.6.1]. Since $x_{1}$ is general in $X$ this defines a family of rational curves dominating $X$, and we denote by $\mathcal{W}$ the normalisation of the irreducible component of Chow $(X)$ containing these rational curves.
4.5. Since a general member $[C]$ of the family $\mathcal{W}$ is free and $-K_{X} \cdot C=2 n$, we have $\operatorname{dim} \mathcal{W}=3 n-3$. We pick an arbitrary irreducible component of the subset of $\mathcal{W}$ parametrising cycles containing $x_{1}$, and let $\mathcal{W}_{x_{1}}$ be its normalisation; then we have $\operatorname{dim} \mathcal{W}_{x_{1}}=2 n-2$. Let $\mathcal{U}_{x_{1}}$ be the normalisation of the universal family of cycles over $\mathcal{W}_{x_{1}}$. The evaluation map $\mathrm{ev}_{x_{1}}: \mathcal{U}_{x_{1}} \rightarrow X$ is surjective: its image is irreducible, and it contains both the divisor $D_{x_{1}}$ (because it is contained in the image of the restriction of $\mathrm{ev}_{x_{1}}$ to those members of $\mathcal{W}_{x_{1}}$ that contain a minimal curve through $x_{1}$ ) and the point $x_{2}$ which is general in $X$ (in particular $x_{2} \notin D_{x_{1}}$ ).

Next, we choose an arbitrary irreducible component of the subset of $\mathcal{W}$ parametrising cycles passing through $x_{1}$ and $x_{2}$, and let $\mathcal{W}_{x_{1}, x_{2}}$ be its normalisation, $\mathcal{U}_{x_{1}, x_{2}}$ the normalisation of the universal family over $\mathcal{W}_{x_{1}, x_{2}}$. We denote by

$$
q: \mathcal{U}_{x_{1}, x_{2}} \rightarrow \mathcal{W}_{x_{1}, x_{2}}, \quad \text { ev }: \mathcal{U}_{x_{1}, x_{2}} \rightarrow X
$$

the natural maps. It follows from the considerations above that $\mathcal{W}_{x_{1}, x_{2}}$ is non-empty of dimension $n-1$.

By construction, a general curve $[C] \in \mathcal{W}_{x_{1}, x_{2}}$ is smooth at $x_{i}, i \in\{1,2\}$, so the preimage $\mathrm{ev}^{-1}\left(x_{i}\right)$ contains a unique divisor $\sigma_{i}$ that surjects onto $\mathcal{W}_{x_{1}, x_{2}}$. Since ev is finite on the $q$-fibres and $\mathcal{W}_{x_{1}, x_{2}}$ is normal, we obtain that the degree one map $\sigma_{i} \rightarrow \mathcal{W}_{x_{1}, x_{2}}$ is an isomorphism. We call the divisors $\sigma_{i}$ the distinguished sections of $q$. We denote by $\Delta \subset \mathcal{W}_{x_{1}, x_{2}}$ the locus parametrising non-integral cycles.

Let $\operatorname{loc}_{x_{1}}^{1}$ be the locus covered by all the minimal rational curves of $X$ passing through $x_{1}$. It is itself a divisor, but may be bigger than $D_{x_{1}}$ since in general there are finitely many families of minimal curves. From now on we choose a general point $x_{2} \in X$ such that $x_{2} \notin \operatorname{loc}_{x_{1}}^{1}$ (which implies $x_{1} \notin \operatorname{loc}_{x_{2}}^{1}$ ).
4.6. Lemma. In the situation of Proposition 1.5 and using the notation introduced above, let $C$ be a non-integral cycle corresponding to a point $[C] \in \Delta$. Then $C=$ $l_{1}+l_{2}$, with the $l_{i}$ minimal rational curves such that $x_{i} \in l_{j}$ if and only if $i=j$.

Remark. Note that we do not claim that the curves $l_{i}$ belong to the family $\mathcal{K}$. However by construction of the family $\mathcal{W}$ as smoothings of pairs $l_{1} \cup l_{2}$ in $\mathcal{K}$ there exists an irreducible component $\Delta_{\mathcal{K}} \subset \Delta$ such that $l_{i} \in \mathcal{K}$ when $\left[l_{1}+l_{2}\right] \in \Delta_{\mathcal{K}}$.

Proof. We can write $C=\sum a_{i} l_{i}$ where the $a_{i}$ are positive integers and $l_{i}$ integral curves. By [Kol96, II, Prop.2.2] all the irreducible components $l_{i}$ are rational curves. We can suppose that up to renumbering one has $x_{1} \in l_{1}$. If $a_{1} \geq 2$, then $-K_{X} \cdot C=$ $2 n$ and $-K_{X} \cdot l_{1} \geq n$ implies that $C=2 l_{1}$ and $l_{1}$ is a minimal rational curve. Yet this contradicts the assumption $x_{2} \notin \operatorname{loc}_{x_{1}}^{1}$. Thus we have $a_{1}=1$ and since $C$ is not integral there exists a second irreducible component $l_{2}$. Again $-K_{X} \cdot C=2 n$ and $-K_{X} \cdot l_{i} \geq n$ implies $C=l_{1}+l_{2}$ and the $l_{i}$ are minimal rational curves by Lemma 4.2. The last property now follows by observing that $x_{2} \notin \operatorname{loc}_{x_{1}}^{1}$ implies that $x_{1} \notin \operatorname{loc}_{x_{2}}^{1}$.

By [Kol96, II, Thm.2.8], the fibration $q: \mathcal{U}_{x_{1}, x_{2}} \rightarrow \mathcal{W}_{x_{1}, x_{2}}$ is a $\mathbb{P}^{1}$-bundle over the open set $\mathcal{W}_{x_{1}, x_{2}} \backslash \Delta$. Although Lemma 4.6 essentially says that the singular fibres are reducible conics, it is a priori not clear that $q$ is a conic bundle (cf. Definition 3.1). This becomes true after we make a base change to a smooth curve.
4.7. Lemma. In the situation of Proposition 1.5 and using the notation introduced above, let $Z \subset \mathcal{W}_{x_{1}, x_{2}}$ be a curve such that a general point of $Z$ parametrises an irreducible curve. Then there exists a finite morphism $T \rightarrow Z$ such that the normalisation $S$ of the fibre product $\mathcal{U}_{x_{1}, x_{2}} \times \mathcal{W}_{x_{1}, x_{2}} T$ has a conic bundle structure $\varphi: S \rightarrow T$ that satisfies the conditions of Lemma 3.4.

Proof. Let $\nu: \tilde{Z} \rightarrow Z$ be the normalisation, and let $N$ be the normalisation of $\mathcal{U}_{x_{1}, x_{2}} \times \mathcal{W}_{x_{1}, x_{2}} \tilde{Z}, f_{N}: N \rightarrow X$ the morphism induced by ev : $\mathcal{U}_{x_{1}, x_{2}} \rightarrow X$. Since all the curves pass through $x_{1}$ and $x_{2}$ there exists a curve $Z_{1} \subset N\left(\right.$ resp. $\left.Z_{2} \subset N\right)$ that is contracted by $f_{N}$ onto the point $x_{1}$ (resp. $x_{2}$ ). Since ev is finite on the $q$-fibres, the curves $Z_{1}$ and $Z_{2}$ are multisections of $N \rightarrow \tilde{Z}$. If $\tilde{Z}_{i}$ is the normalisation of $Z_{i}$, then the fibration $\left(N \times_{\tilde{Z}} \tilde{Z}_{i}\right) \rightarrow \tilde{Z}_{i}$ has a section given by $c \mapsto(c, c)$. Thus there exists a finite base change $T \rightarrow \tilde{Z}$ such that the normalisation $\varphi: S \rightarrow T$ of the fibre product $\left(\mathcal{U}_{x_{1}, x_{2}} \times \mathcal{W}_{x_{1}, x_{2}} T\right) \rightarrow T$ has a natural morphism $f: S \rightarrow X$ induced by ev: $\mathcal{U}_{x_{1}, x_{2}} \rightarrow X$ and contracts two $\varphi$-sections $\sigma_{1}$ and $\sigma_{2}$ on $x_{1}$ and $x_{2}$ respectively.

Since $Z \not \subset \Delta$, the general $\varphi$-fibre is $\mathbb{P}^{1}$. Moreover by Lemma 4.6 all the $\varphi$-fibres are reduced and have at most two irreducible components. By Lemma 3.3 this implies that $\varphi$ is a conic bundle and if $s \in S_{\text {sing }}$, then $F_{\varphi(s)}$ is a reducible conic and the two irreducible components meet in $s$. Thus we have $\sigma_{i} \subset S_{s m}$, where $S_{s m}$ denotes the smooth locus, since otherwise both irreducible components would pass through $x_{i}$, thereby contradicting the property that $x_{2} \notin \operatorname{loc}_{x_{1}}^{1}$. For the same reason we can decompose any reducible $\varphi$-fibre $F_{t}$ by defining $F_{t, i}$ as the unique component meeting the section $\sigma_{i}$. Since $\sigma_{i} \cdot F=1$ for a general $\varphi$-fibre we see that (C1) holds. Condition (C2) holds with $H$ the pull-back of an ample divisor on $X$.

From this one deduces with Lemma 3.4 the following statement, in the spirit of the bend-and-break lemma [Deb01, Prop.3.2].
4.8. Lemma. The restriction of the evaluation map ev : $\mathcal{U}_{x_{1}, x_{2}} \rightarrow X$ to the complement of $\sigma_{1} \cup \sigma_{2}$ is quasi-finite. In particular ev is generically finite onto its image.

Proof. We argue by contradiction. Since ev is finite on the $q$-fibres there exists a curve $Z \subset \mathcal{W}_{x_{1}, x_{2}}$ such that the natural map from the surface $q^{-1}(Z)$ onto $\operatorname{ev}\left(q^{-1}(Z)\right)$ contracts three disjoint curves $\sigma_{1}, \sigma_{2}$ and $\sigma$ onto the points $x_{1}, x_{2}$ and $x:=\operatorname{ev}(\sigma)$.

If $Z \not \subset \Delta$, then by Lemma 4.7 we can suppose, possibly up to a finite base change, that $q^{-1}(Z) \rightarrow Z$ satisfies the conditions (C1) of Lemma 3.4. After a further base change we can assume that $\sigma$ is a section. Since $\sigma$ is contracted by ev we have $\sigma^{2}<0$. By Lemma 3.4, a), this implies $\sigma=\sigma_{1}$ or $\sigma=\sigma_{2}$, a contradiction.

If $Z \subset \Delta$, then all the fibres over $Z$ are unions of two minimal rational curves. Thus the normalisation of $q^{-1}(Z)$ is a union of two $\mathbb{P}^{1}$-bundles mapping onto $Z$ and by construction they contain three curves which are mapped onto points. However a ruled surface contains at most one contractible curve, a contradiction.
4.9. Since $\operatorname{dim} \mathcal{U}_{x_{1}, x_{2}}=\operatorname{dim} X$, one deduces from Lemma 4.8 above that the cycles $[C] \in \mathcal{W}$ passing through $x_{1}, x_{2}$ cover the manifold $X$. By [Deb01, 4.10] this implies that a general member $[C] \in \mathcal{W}_{x_{1}, x_{2}}$ is a 2-free rational curve [Deb01, Defn.4.5]. Since $-K_{X} \cdot C=2 n$, this forces

$$
\begin{equation*}
f^{*} T_{X} \simeq \mathcal{O}_{\mathbb{P}^{1}}(2)^{\oplus n} \tag{4.9.1}
\end{equation*}
$$

where $f: \mathbb{P}^{1} \rightarrow C \subset X$ is the normalisation of $C$. As a consequence, one sees from [Kol96, II, Thm.3.14.3] that a general member $[C] \in \mathcal{W}$ is a smooth rational curve in $X$.

Let $\operatorname{Hom}_{\mathcal{W}}^{\circ} \subset \operatorname{Hom}\left(\mathbb{P}^{1}, X\right)$ be the irreducible open set parametrising morphisms $f: \mathbb{P}^{1} \rightarrow X$ such that the image $C:=f\left(\mathbb{P}^{1}\right)$ is smooth, the associated cycle $[C] \in \operatorname{Chow}(X)$ is a point in $\mathcal{W}$, and $f^{*} T_{X}$ has the splitting type (4.9.1). By what precedes, the image of $\operatorname{Hom}_{\mathcal{W}}^{\circ}$ in $\mathcal{W}$ under the natural map $\operatorname{Hom}\left(\mathbb{P}^{1}, X\right) \rightarrow$ $\operatorname{Chow}(X)$ is a dense open set $\mathcal{W}^{\circ} \subset \mathcal{W}$.
4.10. Denote by $\pi: \mathbb{P}\left(\Omega_{X}\right) \rightarrow X$ the projection map. We define an injective map

$$
i: \operatorname{Hom}_{\mathcal{W}}^{\circ} \hookrightarrow \operatorname{Hom}\left(\mathbb{P}^{1}, \mathbb{P}\left(\Omega_{X}\right)\right)
$$

by mapping $f: \mathbb{P}^{1} \rightarrow X$ to the morphism $\tilde{f}: \mathbb{P}^{1} \rightarrow \mathbb{P}\left(\Omega_{X}\right)$ corresponding to the invertible quotient $f^{*} \Omega_{X} \rightarrow \Omega_{\mathbb{P}^{1}}$. Correspondingly, for $[C] \in \mathcal{W}^{\circ}$ with normalisation $f$, we call $[\tilde{C}]$ the member of $\operatorname{Chow}\left(\mathbb{P}\left(\Omega_{X}\right)\right)$ corresponding to the lifting $\tilde{f}$.

We let $\operatorname{Hom}_{\mathcal{W}}$ be the image of $i$. Note that it parametrises a family of rational curves that dominates $\mathbb{P}\left(\Omega_{X}\right)$, but it is not an irreducible component of $\operatorname{Hom}\left(\mathbb{P}^{1}, \mathbb{P}\left(\Omega_{X}\right)\right)$. Indeed, $\operatorname{Hom}_{\mathcal{W}}$ is contained in a (much bigger) irreducible component defined by morphisms corresponding to arbitrary quotients $f^{*} \Omega_{X} \rightarrow$ $\mathcal{O}_{\mathbb{P}^{1}}(-2)$.

The following property is well-known to experts. Since Hom $\tilde{\mathcal{W}}$ is not an open set of the space $\operatorname{Hom}\left(\mathbb{P}^{1}, \mathbb{P}\left(\Omega_{X}\right)\right)$, we have to adapt the proof of [Kol96, II,Prop.3.7].
4.11. Lemma. In the situation of Proposition 1.5, let $\mathcal{V}_{0} \subset \mathcal{V}$ be a dense, Zariski open set in the total VMRT $\mathcal{V}$, and let $\tilde{C}:=\tilde{f}\left(\mathbb{P}^{1}\right)$ be a rational curve parametrised by a general point of $\mathrm{Hom}_{\mathcal{W}}^{\sim}$. Then one has

$$
(\mathcal{V} \cap \tilde{C}) \subset\left(\mathcal{V}_{0} \cap \tilde{C}\right)
$$

Proof. Set $Z:=\mathcal{V} \backslash \mathcal{V}_{0}$. A point $z \in \mathbb{P}\left(\Omega_{X}\right)$ is $z=\left(v_{z}^{\perp}, x\right)$, where $\mathbb{C} v_{z} \subset T_{X, x}$ is a tangent direction in $X$ at $x=\pi(z)$. So for all $p \in \mathbb{P}^{1}, z=\left(v_{z}^{\perp}, x\right) \in \mathbb{P}\left(\Omega_{X}\right)$, the morphisms $[\tilde{f}] \in \operatorname{Hom}_{\mathcal{W}} \tilde{}$ mapping $p$ to $z$ correspond to morphisms $f: \mathbb{P}^{1} \rightarrow X$ in $\operatorname{Hom}_{\mathcal{W}}{ }^{\circ}$ mapping $p$ to $x$ with tangent direction $\mathbb{C} v_{z}$. Since $f$ has the splitting type (4.9.1), the set of these morphisms has dimension exactly $n$. It follows that

$$
\operatorname{Hom}_{\mathcal{W}, Z}:=\left\{[\tilde{f}] \in \operatorname{Hom}_{\mathcal{W}} \mid \tilde{f}\left(\mathbb{P}^{1}\right) \cap Z \neq \emptyset\right\}=\bigcup_{z \in Z} \bigcup_{p \in \mathbb{P}^{1}}\left\{[\tilde{f}] \in \operatorname{Hom}_{\mathcal{W}} \mid \tilde{f}(p)=z\right\}
$$

has dimension at most $\operatorname{dim} Z+1+n$.
Now $\mathcal{V} \subset \mathbb{P}\left(\Omega_{X}\right)$ is a divisor, and $Z$ has codimension at least one in $\mathcal{V}$, so $Z$ has dimension at most $2 n-3$, and the set $\operatorname{Hom}_{\mathcal{W}, Z}$ above has dimension at most $3 n-2$. Since $\operatorname{Hom}_{\mathcal{W}}^{\circ}$ has dimension $3 n$ and $\operatorname{Hom}_{\mathcal{W}}^{\circ} \rightarrow \operatorname{Hom}_{\mathcal{W}}^{\sim}$ is injective, a general point $[\tilde{f}] \in \operatorname{Hom}_{\mathcal{W}}$ is not in $\operatorname{Hom}_{\mathcal{W}, Z}$.

We need one more technical statement:
4.12. Lemma. In the situation of Proposition 1.5 and using the notation introduced above, let $[f] \in \operatorname{Hom}_{\mathcal{W}}^{\circ}$ be a general point. Then for every $x \in f\left(\mathbb{P}^{1}\right)$ we have $f\left(\mathbb{P}^{1}\right) \not \subset \operatorname{loc}_{x}^{1}$.

Proof. Fix two general points $x_{1}, x_{2} \in X$. A general morphism $[f] \in \operatorname{Hom}_{\mathcal{W}}^{\circ}$ passing through $x_{1}$ and $x_{2}$ is 2-free and up to reparametrisation we have $f(0)=x_{1}, f(\infty)=$ $x_{2}$. Set $g:=\left.f\right|_{\{0, \infty\}}$, then $f$ is free over $g$ [Kol96, II, Defn.3.1]. Suppose now that such a curve has the property $f\left(\mathbb{P}^{1}\right) \subset \operatorname{loc}_{x_{0}}^{1}$ for some $x_{0} \in f\left(\mathbb{P}^{1}\right)$. Thus $x_{1}, x_{2} \in \operatorname{loc}_{x_{0}}^{1}$, hence by symmetry $x_{0} \in\left(\operatorname{loc}_{x_{1}}^{1} \cap \operatorname{loc}_{x_{2}}^{1}\right)$. Yet the intersection

$$
\operatorname{loc}_{x_{1}}^{1} \cap \operatorname{loc}_{x_{2}}^{1}
$$

has codimension two in $X$. By [Kol96, II, Prop.3.7] a general deformation of $f$ over $g$ is disjoint from this set.
4.13. Proof of Proposition 1.5. Arguing by contradiction, we suppose that $\mathcal{V} \cdot \tilde{C}>0\left(\tilde{C}\right.$ is not contained in $\mathcal{V}$ for the general $\left.[C] \in \mathcal{W}^{\circ}\right)$. Applying Lemma 4.11 with

$$
\mathcal{V}_{0}:=\left\{v^{\perp} \in \mathcal{V} \mid \mathbb{C} v=T_{l, \pi(v)} \text { where }[l] \in \mathcal{K} \text { is standard }\right\}
$$

we see that for a general point $[C] \in \mathcal{W}$ there exists a point $x_{1} \in C$ and a standard curve $[l] \in \mathcal{K}_{x_{1}}$ such that

$$
\begin{equation*}
T_{C, x_{1}}=T_{l, x_{1}} \tag{4.13.1}
\end{equation*}
$$

We shall now reformulate the property (4.13.1) in terms of the universal family $\mathcal{U}_{x_{1}, x_{2}}$, with $x_{2}$ a point chosen in $C \backslash \operatorname{loc}_{x_{1}}^{1}$ thanks to Lemma 4.12. Consider the blow-up $\varepsilon: \tilde{X} \rightarrow X$ at the point $x_{1}$, with exceptional divisor $E_{1}$. There is a rational map $\tilde{\mathrm{ev}}: \mathcal{U}_{x_{1}, x_{2}} \rightarrow \tilde{X}$ such that $\varepsilon \circ \tilde{\mathrm{ev}}=\mathrm{ev}$ (on the locus where $\tilde{\mathrm{ev}}$ is defined); since the general member of $\mathcal{W}_{x_{1}, x_{2}}$ is smooth at $x_{1}$, this map $\tilde{\text { ev }}$ is well-defined in a general point of $\sigma_{1}$, and restricts to a rational map $\sigma_{1} \rightarrow E_{1}$. The latter is dominant and therefore generically finite, because the general member of $\mathcal{W}_{x_{1}, x_{2}}$ is 2 -free. In particular we may assume it is finite in a neighbourhood of the point $C \cap \sigma_{1}$.

We then consider the proper transform $\tilde{l}$ of $l$ under $\varepsilon$, and let $\Gamma$ be an irreducible component of $\tilde{\mathrm{ev}}^{-1}(\tilde{l})$ passing through $C \cap \sigma_{1}$. It is a curve that is mapped to a
curve in $\mathcal{W}_{x_{1}, x_{2}}$ by $q$. Also, applying the same construction to the divisor $D_{x_{1}} \subset X$, one gets a prime divisor $G \subset \mathcal{U}_{x_{1}, x_{2}}$ mapping surjectively onto $D_{x_{1}}$ and $\mathcal{W}_{x_{1}, x_{2}}$ respectively.

In general the curve $\Gamma$ could be contained in the locus where $\left.q\right|_{G}$ or $\left.\mathrm{ev}\right|_{G}$ are not étale. However the standard rational curves $[l] \in \mathcal{K}$ such that a corresponding curve $\Gamma$ is not contained in these ramification loci form a non-empty Zariski open set in $\mathcal{K}$. Hence their tangent directions define a non-empty Zariski open set in $\mathcal{V}$. Applying Lemma 4.11 a second time we can thus replace $C$ by a general curve $C^{\prime}$ such that $\left[C^{\prime}\right] \in \mathcal{W}^{\circ} \cap \mathcal{W}_{x_{1}, x_{2}}$ and hence $l$ by a general $\left[l^{\prime}\right] \in \mathcal{K}_{x_{1}}$ such that there exists a curve $\Gamma^{\prime} \subset G$ such that $q\left(\Gamma^{\prime}\right)$ is a curve, $\operatorname{ev}\left(\Gamma^{\prime}\right)=l^{\prime}$, and both maps $\left.q\right|_{G}$ and $\left.\mathrm{ev}\right|_{G}$ are étale at the general point $x \in \Gamma^{\prime}$. By construction the point $C^{\prime} \cap \sigma_{1}$ lies on $\Gamma^{\prime}$. This is a contradiction to Proposition 4.14 below.
4.14. Proposition [Miy04, Lemma 3.9]. In the situation of Proposition 1.5, let $x_{1}, x_{2} \in X$ be general points, and $[l]$ a general member of $\mathcal{K}_{x_{1}}$. Consider an irreducible curve $\Gamma \subset \mathcal{U}_{x_{1}, x_{2}}$ such that $\mathrm{ev}(\Gamma)=l$ and $q(\Gamma)$ is a curve, and assume there exists a prime divisor $G \subset \mathcal{U}_{x_{1}, x_{2}}$ mapped onto $D_{x_{1}}$ by ev and containing $\Gamma$, such that both maps $\left.q\right|_{G}$ and $\left.\mathrm{ev}\right|_{G}$ are étale at a general point of $\Gamma$. Then $\Gamma \cap \sigma_{1}$ does not contain any point $C \cap \sigma_{1}$ with $[C] \in \mathcal{W}^{\circ} \cap \mathcal{W}_{x_{1}, x_{2}}$.

We give the proof for the sake of completeness.
Proof. Since [l] is general in $\mathcal{K}_{x_{1}}$, we have

$$
\left.T_{X}\right|_{l} \simeq \mathcal{O}_{\mathbb{P}^{1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{1}}(1)^{n-2} \oplus \mathcal{O}_{\mathbb{P}^{1}}
$$

and $\mathcal{K}_{x_{1}}$ is smooth with tangent space $H^{0}\left(l, N_{l / X}^{+} \otimes \mathcal{O}_{l}\left(-x_{1}\right)\right)$ at $[l]$, where $\mathcal{E}^{+}$ denotes the ample part of a vector bundle $\mathcal{E} \rightarrow \mathbb{P}^{1}$, i.e. its ample subbundle of maximal rank.

Let $x \in \Gamma$ be a general point, and set $y=\mathrm{ev}(x) \in l$. For some analytic neighbourhood $V \subset \mathcal{K}_{x_{1}}$ of $[l]$, we have an evaluation map

$$
\mathbb{P}^{1} \times V \longrightarrow D_{x_{1}}
$$

which is étale at $(y,[l])$, and the tangent space to $D_{x_{1}}$ at $y$ is thus

$$
T_{D_{x_{1}}, y}=T_{l, y} \oplus\left(N_{l / X}^{+} \otimes \mathcal{O}_{l}\left(-x_{1}\right)\right)_{y}=\left.T_{X}\right|_{l, y} ^{+}
$$

Since ev $\left.\right|_{G}$ is étale in $x$, we obtain that the tangent map

$$
d_{x} \mathrm{ev}: T_{\mathcal{U}_{x_{1}, x_{2}}, x} \rightarrow \operatorname{ev}^{*}\left(T_{X, \operatorname{ev}(x)}\right)
$$

maps $T_{G, x}$ isomorphically into the ample part i.e. we have

$$
\begin{equation*}
d_{x} \mathrm{ev}\left(T_{G, x}\right) \simeq \mathrm{ev}^{*}\left(\left.T_{X}\right|_{l, \mathrm{ev}(x)} ^{+}\right) \tag{4.14.1}
\end{equation*}
$$

We argue by contradiction and suppose that there exists $[C] \in \mathcal{W}^{\circ} \cap \mathcal{W}_{x_{1}, x_{2}}$ such that $\left(C \cap \sigma_{1}\right) \in\left(\Gamma \cap \sigma_{1}\right)$. Since $\Gamma$ maps onto $l$ it is not contained in the divisor $\sigma_{1}$. Since the smooth rational curve $C$ is 2 -free, there exists by semicontinuity a neighbourhood $U$ of $[C] \in \mathcal{W}_{x_{1}, x_{2}}$ parametrising 2 -free smooth rational curves. For a 2 -free rational curve, the evaluation morphism ev is smooth in the complement of the distinguished divisors $\sigma_{i}$ [Kol96, II, Prop.3.5.1]. Thus if we denote by $R \subset$ $\mathcal{U}_{x_{1}, x_{2}}$ the ramification divisor of ev, $\sigma_{1}$ is the unique irreducible component of $R$
containing the point $C \cap \sigma_{1}$. Thus $\Gamma$ is not contained in the ramification divisor of ev.

Since $q(\Gamma)$ is a curve, there exists by Lemma 4.7 a finite base change $T \rightarrow$ $q(\Gamma)$ with $T$ a smooth curve, such that the normalisation $S$ of the fibre product $T \times \mathcal{W}_{x_{1}, x_{2}} \mathcal{U}_{x_{1}, x_{2}}$ is a surface with a conic bundle structure $\varphi: S \rightarrow T$ satisfying the conditions of Lemma 3.4. After a further base change we may suppose that there exists a $\varphi$-section $\Gamma_{1}$ that maps onto $\Gamma$. Note that since we obtained $S$ by a base change from $\mathcal{U}_{x_{1}, x_{2}}$, the ramification divisor of the map $\mu: S \rightarrow \mathcal{U}_{x_{1}, x_{2}}$ is contained in the $\varphi$-fibres, i.e. its image by $\varphi$ has dimension 0 . In particular $\Gamma_{1}$ is not contained in this ramification locus.

Since the rational curve $C$ is smooth and 2 -free, the universal family $\mathcal{U}_{x_{1}, x_{2}}$ is smooth in a neighbourhood of $C \cap \sigma_{1}$. Thus $\sigma_{1}$ is a Cartier divisor in a neighbourhood of $C \cap \sigma_{1}$, and we can use the projection formula to see that

$$
\Gamma_{1} \cdot \mu^{*} \sigma_{1}=\mu_{*}\left(\Gamma_{1}\right) \cdot \sigma_{1}>0
$$

In particular $\Gamma_{1}$ is not disjoint from the distinguished sections in the conic bundle $S \rightarrow T$. Let now $\varepsilon: \hat{S} \rightarrow S$ be the minimal resolution of singularities, and $\hat{\Gamma}_{1}$ the proper transform of $\Gamma_{1}$. Since the distinguished sections are in the smooth locus of $S$, the section $\hat{\Gamma}_{1}$ is not disjoint from the distinguished sections of $\hat{S} \rightarrow T$. We shall now show that

$$
\left(\hat{\Gamma}_{1}\right)^{2} \leq 0
$$

which is a contradiction to Lemma 3.4.
Denote by $f: \hat{\Gamma}_{1} \rightarrow l$ the restriction of ev $\circ \mu \circ \varepsilon: \hat{S} \rightarrow X$. Since $\hat{\Gamma}_{1}$ is not in the ramification locus of $\mu \circ \varepsilon$ and $\Gamma$ is not in the ramification divisor of ev, the tangent map

$$
\left.\left.T_{\hat{S}}\right|_{\hat{\Gamma}_{1}} \rightarrow f^{*} T_{X}\right|_{l}
$$

is generically injective. Since $\hat{\Gamma}_{1}$ is a $\varphi \circ \varepsilon$-section, we have an isomorphism

$$
\begin{equation*}
T_{\hat{S} / T} \mid \hat{\Gamma}_{1} \simeq N_{\hat{\Gamma}_{1} / \hat{S}} \tag{4.14.2}
\end{equation*}
$$

Since $l$ has the standard splitting type (4.3.1) we have a (unique) trivial quotient $\left.f^{*} T_{X}\right|_{l} \rightarrow \mathcal{O}_{\hat{\Gamma}_{1}}$, and thanks to (4.14.2) we are done if we prove that the natural map

$$
\left.\left.\left.T_{\hat{S} / T}\right|_{\hat{\Gamma}_{1}} \hookrightarrow T_{\hat{S}}\right|_{\hat{\Gamma}_{1}} \rightarrow f^{*} T_{X}\right|_{l} \rightarrow \mathcal{O}_{\hat{\Gamma}_{1}}
$$

is not zero. It is sufficient to check this property for a general point in $\hat{\Gamma}_{1}$, and since $\hat{\Gamma}_{1} \rightarrow \Gamma$ is generically étale, it is sufficient to check that for a general $x \in \Gamma$, the natural map

$$
T_{\mathcal{U}_{x_{1}, x_{2}} / \mathcal{W}_{x_{1}, x_{2}, x}} \rightarrow \operatorname{ev}^{*}\left(T_{X, \operatorname{ev}(x)}\right)
$$

does not have its image into the ample part $\operatorname{ev}^{*}\left(\left.T_{X}\right|_{l, \text { ev }(x)} ^{+}\right)$. Yet if $T_{\mathcal{U}_{x_{1}, x_{2}} / \mathcal{W}_{x_{1}, x_{2}, x}}$ maps into the ample part, the decomposition $T_{\mathcal{U}_{x_{1}, x_{2}, x}}=T_{\mathcal{U}_{x_{1}, x_{2}} / \mathcal{W}_{x_{1}, x_{2}, x}} \oplus T_{G, x}$ (given by the fact that $\left.q\right|_{G}$ is étale in $x$ ) combined with (4.14.1) implies that the tangent map

$$
d_{x} \mathrm{ev}: T_{\mathcal{U}_{x_{1}, x_{2}}, x} \rightarrow \operatorname{ev}^{*}\left(T_{X, \operatorname{ev}(x)}\right)
$$

cannot be surjective. Since $\Gamma$ is not contained in the ramification locus of ev this is impossible.

## 5. Proof of the main theorem

5.1. Proof of Theorem 1.3. If $X \simeq \mathbb{P}^{n}$ we are done, so suppose that this is not the case. Then consider the family of minimal rational curves $\mathcal{K}$ constructed in Section 4 and the associated total VMRT $\mathcal{V}$. Denote by $d \in \mathbb{N}$ the degree of a general VMRT $\mathcal{V}_{x} \subset \mathbb{P}\left(\Omega_{X, x}\right)$.
Step 1. Using the family $\mathcal{W}^{\circ}$. In this step we prove that

$$
\begin{equation*}
\mathcal{V} \sim_{\mathbb{Q}} d\left(\zeta-\frac{1}{n} \pi^{*} K_{X}\right) \tag{5.1.1}
\end{equation*}
$$

where $\zeta$ is the tautological divisor class on $\mathbb{P}\left(\Omega_{X}\right)$. Note that $\mathbb{P}\left(\Omega_{X}\right)$ has Picard number two, so we can always write

$$
\mathcal{V} \sim_{\mathbb{Q}} a \zeta+b \frac{-1}{n} \pi^{*} K_{X}
$$

with $a, b \in \mathbb{Q}$. Let now $\mathcal{W}^{\circ}$ be the family of rational curves constructed in Section 4 , and let $\tilde{C}$ be the lifting of a curve $C \in \mathcal{W}^{\circ}$. By Proposition 1.5 we have $\mathcal{V} \cdot \tilde{C}=0$. Since by the definition of $\tilde{C}$ one has $\zeta \cdot \tilde{C}=-2$ and $-\frac{1}{n} \pi^{*} K_{X} \cdot \tilde{C}=2$, it follows that $a=b$. Since $\mathcal{V}_{x}=\left.\left.\mathcal{V}\right|_{\mathbb{P}\left(\Omega_{X, x}\right)} \sim_{\mathbb{Q}} d \zeta\right|_{\mathbb{P}\left(\Omega_{X, x}\right)}$, we have $a=b=d$. This proves (5.1.1).

Step 2. Bounding the degree $d$. Denote by $\mathcal{K}^{\circ} \subset \mathcal{K}$ the open set parametrising smooth standard rational curves in $\mathcal{K}$. We define an injective map

$$
j: \mathcal{K}^{\circ} \hookrightarrow \operatorname{RatCurves}^{\mathrm{n}}\left(\mathbb{P}\left(\Omega_{X}\right)\right)
$$

by mapping a curve $l$ to the image $\tilde{l}$ of the morphism $s: l \rightarrow \mathbb{P}\left(\Omega_{X}\right)$ defined by the invertible quotient $\left.\Omega_{X}\right|_{l} \rightarrow \Omega_{l}$. We denote by $\tilde{\mathcal{K}}^{\circ}$ the image of $j$. Let us start by showing that $\tilde{\mathcal{K}}^{\circ}$ is dense in an irreducible component of RatCurves ${ }^{\mathrm{n}}\left(\mathbb{P}\left(\Omega_{X}\right)\right)$. Since $l$ is standard, the relative Euler sequence restricted to $\tilde{l}$ implies that $H^{0}\left(\tilde{l},\left.T_{\mathbb{P}\left(\Omega_{X}\right) / X}\right|_{\tilde{l}}\right)=0$. Then, using the exact sequence

$$
\left.\left.\left.\left.0 \rightarrow T_{\mathbb{P}\left(\Omega_{X}\right) / X}\right|_{\tilde{l}} \rightarrow T_{\mathbb{P}\left(\Omega_{X}\right)}\right|_{\tilde{l}} \rightarrow\left(\pi^{*} T_{X}\right)\right|_{\tilde{\imath}} \simeq T_{X}\right|_{l} \rightarrow 0
$$

we obtain that the Zariski tangent space of $\operatorname{Hom}\left(\mathbb{P}^{1}, \mathbb{P}\left(\Omega_{X}\right)\right)$ at a point corresponding to the rational curve $\tilde{l}$ has dimension at most $h^{0}\left(l,\left.T_{X}\right|_{l}\right)=2 n$. Thus we can use [Kol96, II, Thm.2.15] to see that RatCurves ${ }^{\mathrm{n}}\left(\mathbb{P}\left(\Omega_{X}\right)\right)$ has dimension at most $2 n-3$ at the point $[\tilde{l}]$, which is exactly the dimension of $\tilde{\mathcal{K}}^{\circ}$.

By construction the lifted curves $\tilde{l}$ are contained in $\mathcal{V}$. Thus the open set $\tilde{\mathcal{K}}_{0} \subset$ RatCurves ${ }^{\mathrm{n}}\left(\mathbb{P}\left(\Omega_{X}\right)\right)$ is actually an open set in RatCurves ${ }^{\mathrm{n}}(\mathcal{V})$. Since $\mathcal{V} \subset \mathbb{P}\left(\Omega_{X}\right)$ is a hypersurface, the algebraic set $\mathcal{V}$ has lci singularities. Thus we can apply [Kol96, II, Thm.1.3, Thm.2.15] and obtain

$$
2 n-3=\operatorname{dim} \tilde{\mathcal{K}}_{0} \geq\left.\operatorname{deg} \omega_{\mathcal{V}}^{-1}\right|_{\tilde{l}}+(2 n-2)-3
$$

We thus have $\left.\operatorname{deg} \omega_{\mathcal{V}}^{-1}\right|_{\tilde{l}} \leq 2$.
Now by construction we have $-\frac{1}{n} \pi^{*} K_{X} \cdot \tilde{l}=1$ and $\zeta \cdot \tilde{l}=-2$. Since $K_{\mathbb{P}\left(\Omega_{X}\right)}=$ $2 \pi^{*} K_{X}-n \zeta$, the adjunction formula and (5.1.1) yield

$$
2 \geq\left.\operatorname{deg} \omega_{\mathcal{V}}^{-1}\right|_{\tilde{l}}=-\left(K_{\mathbb{P}\left(\Omega_{X}\right)}+\mathcal{V}\right) \cdot \tilde{l}=d
$$

Step 3. Conclusion. If $d=1$ or $d=2$ but $\mathcal{V}_{x}$ is reducible, we obtain a contradiction to [Hwa07, Thm.1.5] (cf. also [Ara06, Thm.3.1]). If $d=2$ and $\mathcal{V}_{x}$ is irreducible, $\mathcal{V}_{x}$ is normal [Har77, II,Ex.6.5(a)], and therefore isomorphic to its normalisation $\mathcal{K}_{x}$
which is smooth (see $\S 4.3$ ). It is thus a smooth quadric and we conclude by [Mok08, Main Thm.].
5.2. Remark. Let us explain the difference of our proof with Miyaoka's approach: in the notation of Section 4 , he considers the family $\mathcal{W}_{x_{1}, x_{2}}$. As we have seen above the evaluation map ev : $\mathcal{U}_{x_{1}, x_{2}} \rightarrow X$ is generically finite and his goal is to prove that ev is birational. He therefore analyses the preimage $\mathrm{ev}^{-1}\left(l_{1} \cup l_{2}\right)$, where the $l_{i} \subset X$ are general minimal curves passing through $x_{i}$ respectively such that $\left[l_{1} \cup l_{2}\right] \in \mathcal{W}_{x_{1}, x_{2}}$. If $\Gamma \subset \mathrm{ev}^{-1}\left(l_{1} \cup l_{2}\right)$ is an irreducible curve mapping onto $l_{1}$ one can make a case distinction: if $q(\Gamma)$ is a curve that is not contained in the discriminant locus $\Delta \subset \mathcal{W}_{x_{1}, x_{2}}$ (Case $\mathbf{C}$ in [Miy04, p.227]) Miyaoka makes a very interesting observation which we stated as Proposition 4.14. However the analysis of the 'trivial' case (Case A in [Miy04, p.227]) where $q(\Gamma)$ is a point is not correct: it is not clear that $q(\Gamma)=\left[l_{1} \cup l_{2}\right]$, because there might be another curve in $\mathcal{W}_{x_{1}, x_{2}}$ which is of the form $l_{1} \cup l_{2}^{\prime}$ with $l_{2} \neq l_{2}^{\prime}$. This possibility is an obvious obstruction to the birationality of ev and invalidates [Miy04, Cor.3.11(2), Cor.3.13(1)]. The following example shows that this possibility does indeed occur in certain cases.
5.3. Example. Let $H \subset \mathbb{P}^{n}$ be a hyperplane and $A \subset H \subset \mathbb{P}^{n}$ a projective manifold $A$ of dimension $n-2$ and degree $3 \leq a \leq n$. Let $\mu: X \rightarrow \mathbb{P}^{n}$ be the blowup of $\mathbb{P}^{n}$ along $A$. Then $X$ is a Fano manifold [Miy04, Rem.4.2] and $-K_{X} \cdot C \geq n$ for every rational curve $C \subset X$ passing through a general point (the $\mu$-fibres are however rational curves with $-K_{X} \cdot C=1$ ). The general member of a family of minimal rational curves $\mathcal{K}$ is the proper transform of a line that intersects $A$. Consider the family $\mathcal{W}$ whose general member is the strict transform of a reduced, connected degree two curve $C$ such that $A \cap C$ is a finite scheme of length two. For general points $x_{1}, x_{2} \in X$ the (normalised) universal family $\mathcal{U}_{x_{1}, x_{2}} \rightarrow \mathcal{W}_{x_{1}, x_{2}}$ is a conic bundle and the evaluation map ev : $\mathcal{U}_{x_{1}, x_{2}} \rightarrow X$ is generically finite. We claim that ev is not birational.

Proof of the claim. For simplicity of notation we denote by $x_{1}, x_{2}$ also the corresponding points in $\mathbb{P}^{n}$. Let $l_{1} \subset \mathbb{P}^{n}$ be a general line through $x_{1}$ that intersects $A$. Since $x_{2} \in \mathbb{P}^{n}$ is general there exists a unique plane $\Pi$ containing $l_{1}$ and $x_{2}$. Moreover the intersection $\Pi \cap A$ consists of exactly $a$ points, one of them the point $A \cap l_{1}$. For every point $x \in \Pi \cap A$ other than $A \cap l_{1}$, there exists a unique line $l_{2, x}$ through $x$ and $x_{2}$. By Bezout's theorem $l_{1} \cup l_{2}$ is connected, so its proper transform belongs to $\mathcal{W}_{x_{1}, x_{2}}$. Yet this shows that $\mathrm{ev}^{-1}\left(l_{1}\right)$ contains $a-1>1$ copies of $l_{1}$, one for each point $x \in \Pi \cap A \backslash l_{1} \cap A$. This proves the claim.

Let us conclude this example by mentioning that the conic bundle $\mathcal{U}_{x_{1}, x_{2}} \rightarrow$ $\mathcal{W}_{x_{1}, x_{2}}$ does not satisfy the symmetry conditions of Lemma 3.4.

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