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IRREDUCIBLE UNIRATIONAL AND UNIRULED COMPONENTS OF MODULI SPACES OF POLARIZED ENRIQUES SURFACES

CIRO CILIBERTO, THOMAS DEDIEU, CONCETTINA GALATI,
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ABSTRACT. We prove that, under suitable conditions, some moduli spaces of polarised Enriques surfaces are irreducible, unirational (resp. uniruled).

1. INTRODUCTION

Let \mathcal{E} denote the smooth, irreducible 10-dimensional moduli space parameterizing smooth Enriques surfaces over \mathbb{C} , which is known to be rational (cf. [9]), and $\mathcal{E}_{g,\phi}$ (respectively, $\overline{\mathcal{E}}_{g,\phi}$) denote the moduli space of *polarized* (resp., *numerically polarized*) *Enriques surfaces*, that is, pairs (S, H) (resp., $(S, [H])$) such that $[S] \in \mathcal{E}$ and $H \in \text{Pic}(S)$ (resp., $[H] \in \text{Num}(S)$) is ample with $H^2 = 2g - 2 \geq 2$ and $\phi = \phi(H)$ where

$$(1) \quad \phi(H) := \min \{ E \cdot H \mid E^2 = 0, E > 0 \},$$

cf., e.g., [6]. Thus, g is the arithmetic genus of all curves in the linear system $|H|$. There is an étale double cover $\rho : \mathcal{E}_{g,\phi} \rightarrow \overline{\mathcal{E}}_{g,\phi}$ mapping (S, H) and $(S, H + K_S)$ to $(S, [H])$ by [6, Prop. 4.1].

It is an interesting open problem to determine the Kodaira dimension of the various irreducible components of $\mathcal{E}_{g,\phi}$ and $\overline{\mathcal{E}}_{g,\phi}$ (cf. [5, §4]), as well as identifying its various irreducible components, as these spaces are reducible in many cases. It is also in general an open problem to determine on which irreducible components of $\overline{\mathcal{E}}_{g,\phi}$ the inverse image via ρ is irreducible or not. It had been conjectured that the moduli spaces of polarized Enriques surfaces are all unirational (or at least, of negative Kodaira dimension), but a recent paper of Gritsenko and Hulek [6] disproves this (cf. [5, §4]). On the other hand, it is known that $\mathcal{E}_{3,2}$ is irreducible and rational (cf. [2]), that $\mathcal{E}_{4,2}$ is irreducible and rational (this is the classical case of *Enriques sextics*, cf. [5, §3]) and that $\mathcal{E}_{6,3}$ is irreducible and unirational (cf. [11]). Moreover, $\overline{\mathcal{E}}_{g,\phi}$ has negative Kodaira dimension for all $g \leq 17$ (cf. [6]).

In this paper we improve the above results, by describing in many cases (for unbounded g and ϕ) the different irreducible components of $\mathcal{E}_{g,\phi}$ and $\overline{\mathcal{E}}_{g,\phi}$ and proving their unirationality or uniruledness. To explain our results, we need to introduce some notions.

Recall that by [8, Lemma 2.12], any effective line bundle H such that with $H^2 \geq 0$ on an Enriques surface may be written as

$$(2) \quad H \equiv a_1 E_1 + \cdots + a_n E_n$$

(here \equiv denotes numerical equivalence), where:

- all E_i are effective, non-zero, *isotropic*, i.e., $E_i^2 = 0$, and *primitive*, i.e., indivisible in $\text{Num}(S)$;

- all a_i are positive integers;
 - $n \leq 10$,
- and moreover

$$(3) \quad \begin{cases} \text{either } E_i \cdot E_j = 1 \text{ for all } i \neq j, \\ \text{or } E_1 \cdot E_2 = 2 \text{ and } E_i \cdot E_j = 1 \text{ for all other indices } i \neq j, \\ \text{or } E_1 \cdot E_2 = E_1 \cdot E_3 = 2 \text{ and } E_i \cdot E_j = 1 \text{ for all other indices } i \neq j, \end{cases}$$

up to reordering indices. We call this a *simple isotropic decomposition*.

We say that two polarized (respectively, numerically polarized) Enriques surfaces (S, H) and (S', H') in $\mathcal{E}_{g,\phi}$ (resp., $(S, [H])$ and $(S, [H'])$) in $\overline{\mathcal{E}}_{g,\phi}$ admit the same simple decomposition type if one can write

$$(4) \quad H \sim a_1 E_1 + \cdots + a_n E_n + \varepsilon K_S \quad \text{and} \quad H' \sim a_1 E'_1 + \cdots + a_n E'_n + \varepsilon K_{S'}, \quad \text{with } \varepsilon = 0 \text{ or } 1$$

$$\text{(resp. } H \equiv a_1 E_1 + \cdots + a_n E_n \text{ and } H' \equiv a_1 E'_1 + \cdots + a_n E'_n \text{)}$$

where \sim denotes linear equivalence, with

- all E_i and E'_i effective, non-zero, primitive, isotropic, such that

$$E_i \cdot E_j = E'_i \cdot E'_j \quad \text{for all } i \neq j;$$

- all a_i are positive integers;
- $n \leq 10$;
- (3) is satisfied for both H and H' , possibly after reordering indices.

We call n the *length* of the decomposition type.

If, possibly after reordering indices, there exists $r \leq n$ such that $a_1 = \cdots = a_r$ and $E_i \cdot E_j = 1$ for $1 \leq i \neq j \leq r$, then we say that (S, H) and (S', H') admit the same *simple r -symmetric decomposition type*.

We note that $\varepsilon = 1$ is only needed in (4) when all a_i s are even, otherwise one may substitute any E_i having odd coefficient with $E_i + K_S$.

The various irreducible components of $\overline{\mathcal{E}}_{g,\phi}$ are precisely the loci of pairs admitting the same simple decomposition type, cf. Proposition 3.4. We do not know if the same holds for components of $\mathcal{E}_{g,\phi}$, cf. Question 3.5, although it does in many cases by our results below. One of the advantages of writing polarizations in terms of such simple decompositions rather than in terms of a basis of $\text{Num}(S) \simeq U \oplus E_8(-1)$, is that it gives an efficient way to find all irreducible components of $\overline{\mathcal{E}}_{g,\phi}$ for fixed g and ϕ . Moreover, the value $\phi(H)$ can easily be read off from a simple decomposition, cf. Remark 3.9.

The main results of this paper prove that in many cases the irreducible components of $\mathcal{E}_{g,\phi}$ precisely parameterize pairs (S, H) , with H admitting a given simple decomposition type, and moreover they are unirational or uniruled:

Theorem 1.1. *The locus of pairs $(S, H) \in \mathcal{E}_{g,\phi}$ admitting the same simple decomposition type of length $n \leq 4$ is an irreducible, unirational component of $\mathcal{E}_{g,\phi}$.*

The locus of pairs $(S, H) \in \mathcal{E}_{g,\phi}$ admitting the same simple decomposition type of length 5 is an irreducible component of $\mathcal{E}_{g,\phi}$, which is unirational if all $E_i \cdot E_j = 1$ for all $i \neq j$ and uniruled otherwise.

Theorem 1.2. *The locus of pairs $(S, H) \in \mathcal{E}_{g,\phi}$ admitting the same simple 7-symmetric (respectively, 6-symmetric) decomposition type is an irreducible, unirational (resp., uniruled) component of $\mathcal{E}_{g,\phi}$.*

The theorems are immediate consequences of Propositions 4.5 and 4.6 proved in §4.

We remark that these theorems apply for unbounded g and ϕ , in the sense that line bundles admitting the simple decomposition types as in the statements occur for unbounded g and ϕ . For bounded g and ϕ we can deduce the following corollaries, proved in §4, which improve [6, Cor. 5.6]:

Corollary 1.3. *When $\phi \leq 4$ the different irreducible components of $\mathcal{E}_{g,\phi}$ are precisely the loci parameterizing pairs (S, H) admitting the same simple decomposition type and they are all unirational.*

Corollary 1.4. *When $g \leq 20$ the different irreducible components of $\mathcal{E}_{g,\phi}$ are precisely the loci parameterizing pairs (S, H) admitting the same simple decomposition type. Moreover, they are all unirational, except possibly for $\mathcal{E}_{16,5}$ and $\mathcal{E}_{17,5}$, which are irreducible and uniruled.*

In the cases of the corollaries one may write down all irreducible components of $\mathcal{E}_{g,\phi}$, cf. Lemma 2.5 below and the appendix. For instance, we have

- $\mathcal{E}_{g,1}$ is irreducible for all $g \geq 2$;
- $\mathcal{E}_{g,2}$ is irreducible for even g and for $g = 3$, has two irreducible components for $g \equiv 3 \pmod{4}$ ($g > 3$) and three irreducible components for $g \equiv 1 \pmod{4}$;
- $\mathcal{E}_{g,3}$ is irreducible for $g \leq 8$ and for $g \equiv 2 \pmod{3}$ and has two irreducible components for $g \not\equiv 2 \pmod{3}$ and $g \geq 9$.

We list all irreducible components of $\mathcal{E}_{g,\phi}$ and $\overline{\mathcal{E}}_{g,\phi}$ for $g \leq 30$ in the appendix, and there are only a few cases in which we cannot determine irreducibility, unirationality or uniruledness. However, we will not make use of this list in the present paper.

At the other extreme, our results can also be used to describe the irreducible components of $\mathcal{E}_{g,\phi}$ for the highest values of ϕ with respect to g . Indeed, one has $\phi^2 \leq 2(g-1)$ (cf. [4, Cor. 2.7.1]) and there are no cases with $\phi^2 < 2(g-1) < \phi^2 + \phi - 2$ (cf. [8, Prop. 1.4]). In the borderline cases, we obtain:

Corollary 1.5. *For each even ϕ , the space $\mathcal{E}_{\frac{\phi^2}{2}+1,\phi}$ is irreducible and unirational if $\phi \equiv 2 \pmod{4}$ and has two irreducible components, both unirational, if $\phi \equiv 0 \pmod{4}$.*

For each $\phi \geq 1$, the space $\mathcal{E}_{\frac{\phi(\phi+1)}{2},\phi}$ is irreducible and unirational when $\phi \neq 6$, and consists of three irreducible unirational components when $\phi = 6$.

The cases of the latter corollary are of particular interest from a Brill-Noether theoretical point of view, since they are precisely the cases where the gonality of a general curve in the complete linear system $|H|$ is less than both 2ϕ and $\lfloor \frac{g+3}{2} \rfloor$, the first being the lowest degree of the restriction of an elliptic pencil on the surface, the latter being the gonality of a general curve of genus g , cf. [8, Cor. 1.5].

The last application of our results concerns the map $\rho : \mathcal{E}_{g,\phi} \rightarrow \overline{\mathcal{E}}_{g,\phi}$ and precisely the question of the irreducibility of the preimage via ρ of a component of $\overline{\mathcal{E}}_{g,\phi}$. We give the following answer to [6, Question 4.2] in the cases described by Theorems 1.1 and 1.2:

Corollary 1.6. *Let $\mathcal{C} \subset \mathcal{E}_{g,\phi}$ be an irreducible component parameterizing classes admitting the same simple decomposition type of length ≤ 5 or being 6-symmetric. Then $\rho^{-1}(\rho(\mathcal{C}))$ is irreducible if and only if \mathcal{C} parameterizes classes that are not 2-divisible in $\text{Num}(S)$ or are as in Example 4.9.*

We stress that our results are completely independent from [6], except for the already mentioned Proposition 3.4, which is however logically independent of the rest of the article.

Our proofs of Theorems 1.1 and 1.2 are based on the fact that a general Enriques surface has a model in \mathbb{P}^3 as an Enriques sextic, i.e., a sextic surface singular along the six edges of a tetrahedron; such a model corresponds to the datum of an isotropic sequence (E_1, E_2, E_3) with $E_i \cdot E_j = 1$ for $i \neq j$, the E_i s corresponding to three edges of some face of the tetrahedron. The idea is then to exhibit various irreducible and rational (resp. uniruled) families \mathcal{F} of elliptic curves in \mathbb{P}^3 with prescribed intersection numbers with the edges of some fixed tetrahedron, such that a general Enriques sextic singular along this particular tetrahedron contains a member of \mathcal{F} . One thus gets incidence varieties that are irreducible and rational (resp. uniruled), and dominate the corresponding components of the moduli space of numerically polarized Enriques surfaces. Section 3 contains technical results which ensure that one can indeed apply this strategy to the situations of Theorems 1.1 and 1.2.

The paper is organised as follows. In §2 we collect some general facts we need about linear systems on Enriques surfaces. In particular, Lemma 2.5 contains the classification of all simple decomposition types of line bundles with $\phi \leq 5$. Section 3 is essentially devoted to proving the useful technical result Proposition 3.2 about sets of isotropic divisors, which implies that any simple isotropic decomposition can be written in terms of particular sets of isotropic divisors, cf. Corollary 3.3. This extends previously known results on isotropic divisors from [4] and we believe it is of independent interest. In §4 we prove our main results and corollaries stated in this introduction. We finish with the aforementioned appendix.

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2. GENERALITIES ON LINE BUNDLES ON ENRIQUES SURFACES

Any irreducible curve C on an Enriques surface S satisfies $C^2 \geq -2$, with equality if and only if C is smooth and rational. An Enriques surface containing such a curve is called *nodal*, otherwise it is called *unnodal*. On an unnodal Enriques surface, all divisors are nef and all divisors with positive self-intersection are ample. It is well-known that the general Enriques surface is unnodal, cf. references in [3, p. 577].

Recall that a divisor E is said to be *isotropic* if $E^2 = 0$ and $E \neq 0$. By Riemann-Roch, either E or $-E$ is effective. It is said to be *primitive* if it is non-divisible in $\text{Num}(S)$. On an unnodal surface, any effective primitive isotropic divisor E is represented by an irreducible curve of arithmetic genus one.

Let H be a line bundle with $H^2 > 0$ and $\phi(H)$ as in (1). By [4, Cor. 2.7.1] one has

$$(5) \quad \phi(H)^2 \leq H^2.$$

We recall from [4, p. 122] that an *isotropic r -sequence* on an Enriques surface S (called *exceptional sequence* in [3]) is a sequence of isotropic effective divisors $\{E_1, \dots, E_r\}$ such that $E_i \cdot E_j = 1$ for $i \neq j$. It is well-known that any Enriques surface contains such sequences for every $r \leq 10$; moreover, by [4, Cor. 2.5.6], we have

Proposition 2.1. *Any isotropic r -sequence with $r \neq 9$ can be extended to a 10-sequence.*

We will also make use of the following result:

Lemma 2.2. (a) Let $\{E_1, \dots, E_{10}\}$ be an isotropic 10-sequence. Then there exists a divisor D on S such that $D^2 = 10$, $\phi(D) = 3$ and $3D \sim E_1 + \dots + E_{10}$. Furthermore, for any $i \neq j$, we have

$$(6) \quad D \sim E_i + E_j + E_{i,j}, \quad \text{with } E_{i,j}^2 = 0, \quad E_{i,j} > 0 \quad \text{and } E_i \cdot E_{i,j} = E_j \cdot E_{i,j} = 2,$$

and $E_k \cdot E_{i,j} = 1$ for $k \neq i, j$.

(b) Any divisor D on S such that $D^2 = 10$ and $\phi(D) = 3$ satisfies $3D \sim E_1 + \dots + E_{10}$, for an isotropic 10-sequence $\{E_1, \dots, E_{10}\}$ consisting precisely of all isotropic divisors computing $\phi(D)$ up to numerical equivalence. Moreover, if F is a divisor satisfying $F^2 = 0$ and $F \cdot D = 4$, then $F \equiv E_{i,j}$ for some $i \neq j$, where $E_{i,j}$ is defined by (6).

Proof. (a) The existence of D is [3, Lemma 1.6.2(i)] or [4, Cor. 2.5.5]. Its properties are easily checked and $E_{i,j} := D - E_i - E_j$, cf. also [3, Lemma 1.6.2(ii)].

(b) The first statement follows from [4, Cor. 2.5.5] and its proof. For the last statement, note that $F \cdot E_i > 0$ for $i = 1, \dots, 10$ by [7, Lemma 2.1], whence, after permuting indices if necessary, one must have $F \cdot (E_1 + E_2) = 4$ and $F \cdot E_i = 1$ for $i = 3, \dots, 10$. Then $F \cdot E_{1,2} = 0$ and $E_3 \cdot F = E_3 \cdot E_{1,2} = 1$, so that $F \equiv E_{1,2}$ by [7, Lemma 2.1] again. \square

Notation 2.3. When writing a simple isotropic decomposition (2) verifying (3) (up to permutation of indices), we will usually adopt the convention that $E_i, E_j, E_{i,j}$ are primitive isotropic satisfying $E_i \cdot E_j = 1$ for $i \neq j$, $E_{i,j} \cdot E_i = E_{i,j} \cdot E_j = 2$ and $E_{i,j} \cdot E_k = 1$ for $k \neq i, j$. This notation has already been used in Lemma 2.2. (By Corollary 3.3 below, there is no ambiguity in this notation.)

Recall that there are no cases satisfying $\phi(H)^2 < H^2 < \phi(H)^2 + \phi(H) - 2$ by [8, Prop. 1.4]. Moreover [8, Prop. 1.4] also classifies the borderline cases as follows:

Proposition 2.4. Let H be an effective line bundle on an Enriques surface satisfying $\phi(H)^2 \leq H^2 \leq \phi(H)^2 + \phi(H) - 2$. Then one of the following occurs:

- (i) $H^2 = \phi(H)^2$, in which case $H \equiv \frac{\phi(H)}{2}(E_1 + E_{1,2})$,
- (ii) $H^2 = \phi(H)^2 + \phi(H) - 2$, in which case,
 - $H \sim \frac{\phi(H)-1}{2}(E_1 + E_{1,2}) + E_2$ if $\phi(H)$ is odd, and
 - $H \sim \frac{\phi(H)-2}{2}E_1 + \frac{\phi(H)}{2}E_{1,2} + E_2$ or $H \equiv 2(E_1 + E_2 + E_{1,2})$ if $\phi(H)$ is even.

The following lemma classifies all possible simple decomposition types with $\phi \leq 5$. Note that all decomposition types do exist on any Enriques surface, by Lemma 2.2(a) and the existence of isotropic 10-sequences.

Lemma 2.5. Assume H is an effective line bundle on an Enriques surface S such that $H^2 = 2(g-1) > 0$. If $1 \leq \phi(H) \leq 5$, the line bundle H has one and only one of the following simple isotropic decompositions:

- (i) If $\phi(H) = 1$, then $H \sim (g-1)E_1 + E_2$.
- (ii) If $\phi(H) = 2$, then
 - $H \sim \frac{g-2}{2}E_1 + E_2 + E_3$ if g is even,
 - $H \sim \frac{g-1}{2}E_1 + E_{1,2}$ or $H \equiv \frac{g-1}{2}E_1 + 2E_2$ (with $g \geq 5$), if g is odd.
- (iii) If $\phi(H) = 3$, then

- $H \sim \frac{g-3}{3}E_1 + E_2 + E_{1,2}$ or $H \sim \frac{g-3}{3}E_1 + 2E_2 + E_3$ (with $g \geq 9$) if $g \equiv 0 \pmod{3}$,
- $H \sim \frac{g-4}{3}E_1 + E_2 + E_3 + E_4$ or $H \sim \frac{g-1}{3}E_1 + 3E_2$ (with $g \geq 10$) if $g \equiv 1 \pmod{3}$,
- $H \sim \frac{g-2}{3}E_1 + E_2 + E_{1,3}$ if $g \equiv 2 \pmod{3}$.

(iv) If $\phi(H) = 4$, then

- $H \sim \frac{g-4}{4}E_1 + 3E_2 + E_3$, $g \geq 16$, or
 $H \sim \frac{g-4}{4}E_1 + E_2 + E_3 + E_{1,4}$, $g \geq 12$,
if $g \equiv 0 \pmod{4}$,
- $H \equiv \frac{g-1}{4}E_1 + 4E_2$, $g \geq 17$, or
 $H \equiv \frac{g-1}{4}E_1 + 2E_{1,2}$, $g \geq 9$, or
 $H \equiv \frac{g-5}{4}E_1 + 2E_2 + 2E_3$, $g \geq 13$, or
 $H \sim \frac{g-5}{4}E_1 + 2E_2 + E_{1,2}$, $g \geq 13$,
if $g \equiv 1 \pmod{4}$,
- $H \sim \frac{g-6}{4}E_1 + 2E_2 + E_3 + E_4$, $g \geq 14$, or
 $H \sim \frac{g-2}{4}E_{1,2} + E_1 + E_2$, $g \geq 10$,
if $g \equiv 2 \pmod{4}$,
- $H \sim \frac{g-3}{4}E_1 + 2E_2 + E_{1,3}$, $g \geq 15$ or
 $H \sim \frac{g-7}{4}E_1 + E_2 + E_3 + E_4 + E_5$, $g \geq 11$,
if $g \equiv 3 \pmod{4}$.

(v) If $\phi(H) = 5$, then

- $H \sim \frac{g-5}{5}E_1 + E_2 + 2E_{1,2}$, $g \geq 15$, or
 $H \sim \frac{g-10}{5}E_1 + 2E_2 + E_3 + E_4 + E_5$, $g \geq 20$, or
 $H \sim \frac{g-5}{5}E_1 + 4E_2 + E_3$, $g \geq 25$
if $g \equiv 0 \pmod{5}$,
- $H \sim \frac{g-11}{5}E_1 + E_2 + E_3 + E_4 + E_5 + E_6$, $g \geq 16$, or
 $H \sim \frac{g-6}{5}E_1 + 2E_2 + E_3 + E_{1,4}$, $g \geq 21$, or
 $H \sim \frac{g-1}{5}E_1 + 5E_2$, $g \geq 26$
if $g \equiv 1 \pmod{5}$,
- $H \sim \frac{g-7}{5}E_1 + E_2 + E_3 + E_4 + E_{1,5}$, $g \geq 17$, or
 $H \sim \frac{g-7}{5}E_1 + 3E_2 + E_{1,2}$, $g \geq 22$, or
 $H \sim \frac{g-7}{5}E_1 + 3E_2 + 2E_3$, $g \geq 22$
if $g \equiv 2 \pmod{5}$,
- $H \sim \frac{g-3}{5}E_1 + 2E_{1,3} + E_2$, $g \geq 18$, or
 $H \sim \frac{g-8}{5}E_1 + 2E_2 + E_3 + E_{1,2}$, $g \geq 18$, or
 $H \sim \frac{g-8}{5}E_1 + 3E_2 + E_3 + E_4$, $g \geq 23$
if $g \equiv 3 \pmod{5}$,
- $H \sim \frac{g-9}{5}E_1 + 2E_2 + 2E_3 + E_4$, $g \geq 19$, or
 $H \sim \frac{g-4}{5}E_{1,2} + E_1 + E_2 + E_3$, $g \geq 19$, or
 $H \sim \frac{g-4}{5}E_1 + 3E_2 + E_{1,3}$, $g \geq 24$
if $g \equiv 4 \pmod{5}$,

Proof. The proof is tedious but straightforward and similar to [8, pf. of Prop. 1.4 in §2.2], and we therefore will leave most of it to the reader. The idea is to pick an effective, isotropic E such that $E \cdot H = \phi(H)$, find a suitable integer k so that $\phi(H - kE) < \phi(H)$ (in which case we use the classification for lower ϕ), or so that $\phi(H - kE) = \phi(H)$ and

$H - kE$ is as in Proposition 2.4(i) or (ii). As a sample, we show how this works in the case $\phi(H) = 5$ and $g \equiv 3 \pmod{5}$.

We pick an effective, isotropic E such that $E \cdot H = \phi(H) = 5$ and set $k := \frac{g-13}{5}$. Then $(H - kE)^2 = 24$, so that $\phi(H - kE) \leq 4$ by (5).

Assume $\phi(H - kE) = 4$ and note that $E \cdot (H - kE) = E \cdot H = 5$. By the classification in the case $\phi = 4$, we have the three possibilities, where we use Notation 2.3:

- (a) $H - kE \sim 3F_1 + 2F_{1,2}$,
- (b) $H - kE \sim 2(F_1 + F_2 + F_3)$,
- (c) $H - kE \sim 2F_1 + 2F_2 + F_{1,2}$.

Case (b) is impossible, as $5 = E \cdot (H - kE)$.

In case (a) we have $F_1 \cdot (H - kE) = 4$ and $F_{1,2} \cdot H = 6$, hence $E \not\equiv F_1, F_{1,2}$. Thus, $E \cdot F_1 = E \cdot F_{1,2} = 1$. Let $F := F_1 + F_{1,2} - E$. Then $F^2 = 0$, $E \cdot F = 2$ and $F_1 \cdot F = 1$, so that F is effective, non-zero and we have

$$H \sim kE + 3F_1 + 2F_{1,2} \sim (k+2)E + F_1 + 2F.$$

Using Notation 2.3, we set $E_1 := E$, $E_2 := F_1$ and $E_{1,3} := F$ and, recalling that $k+2 = \frac{g-3}{5}$, we obtain the desired form

$$(7) \quad H \sim \frac{g-3}{5}E_1 + E_2 + 2E_{1,3}.$$

As $5 = \phi(H) \leq E_2 \cdot H = \frac{g-3}{5} + 2$, we have $g \geq 18$.

In case (c) we have $F_1 \cdot (H - kE) = F_2 \cdot (H - kE) = 4$ and $F_{1,2} \cdot (H - kE) = 8$, hence $E \not\equiv F_1, F_2, F_{1,2}$. Thus, $E \cdot F_1 = E \cdot F_2 = E \cdot F_{1,2} = 1$. Let $F := F_2 + F_{1,2} - E$. Then $F^2 = 0$, $E \cdot F = F_1 \cdot F = 2$ and $F_2 \cdot F = 1$ and we have

$$H \sim kE + 2F_1 + 2F_2 + F_{1,2} \sim (k+1)E + 2F_1 + F_2 + F.$$

Using Notation 2.3, we set $E_1 := E$, $E_2 := F_1$, $E_3 := F_2$ and $E_{1,2} := F$ and, recalling that $k+1 = \frac{g-8}{5}$, we obtain the desired form

$$(8) \quad H \sim \frac{g-8}{5}E_1 + 2E_2 + E_3 + E_{1,2}.$$

As $5 = \phi(H) \leq E_2 \cdot H = \frac{g-8}{5} + 3$, we have $g \geq 18$.

We claim that H cannot simultaneously have a simple isotropic decomposition as in (7) and (8). Indeed, there are two (respectively, three) isotropic, effective classes $F \in \text{Num}(S)$ such that $F \cdot H = \frac{g+7}{5}$ in case (8) if $g > 18$ (resp., $g = 18$), namely $F \equiv E_2, E_3$ (resp., $F \equiv E_1, E_2, E_3$), whereas there is only one (resp., two) such classes in case (7), namely $F \equiv E_2$ (resp., $F \equiv E_1, E_2$), as $E_{1,3} \cdot H = \frac{2g-1}{5} > \frac{g+7}{5}$ and $F \cdot H \geq \frac{g-3}{5} + 1 + 2 = \frac{g+12}{5}$ for $F \not\equiv E_1, E_2, E_{1,3}$ by [7, Lemma 2.1].

Assume $\phi(H - kE) = 3$. By the classification in the case $\phi = 3$, we have the two possibilities:

- (d) $H - kE \sim 3F_1 + F_2 + F_3 + F_4$,
- (e) $H - kE \sim 4F_1 + 3F_2$

In case (d) we have $F_1 \cdot (H - kE) = 3$, hence $E \not\equiv F_1$. Thus, we must have $E \cdot F_1 = 1$ and, possibly after rearranging indices, $E \cdot F_2 = E \cdot F_3 = 1$ and $E \equiv F_4$. Thus, using

again Notation 2.3, we set $E_1 := E$, $E_2 := F_1$, $E_3 := F_2$ and $E_4 := F_3$ and, recalling that $k + 1 = \frac{g-8}{5}$, we obtain the desired form

$$(9) \quad H \sim \frac{g-8}{5}E_1 + 3E_2 + E_3 + E_4,$$

possibly after substituting E_4 with $E_4 + K_S$. Since $5 = \phi(H) \leq E_2 \cdot H = \frac{g-8}{5} + 2$, we obtain $g \geq 23$.

Because of the different values of $\phi(H - kE)$, it is again not possible that H can be written both as in (9) and as in (7) or (8).

In case (e) we have $F_1 \cdot (H - kE) = 3$ and $F_2 \cdot (H - kE) = 4$, whence $E \neq F_1, F_2$. It follows that $E \cdot F_1 > 0$ and $E \cdot F_2 > 0$, so that $5 = E \cdot (H - kE) \geq 7$, a contradiction.

Assume $\phi(H - kE) = 2$. By the classification in the case $\phi = 2$, we have the two possibilities:

- (f) $H - kE \sim 6F_1 + F_{1,2}$,
- (g) $H - kE \equiv 6F_1 + 2F_2$.

In both cases, since $F_1 \cdot (H - kE) = 2$, we have $E \neq F_1$, whence the contradiction $5 = E \cdot (H - kE) \geq 6E \cdot F_1 \geq 6$.

Assume finally $\phi(H - kE) = 1$. By the classification in the case $\phi = 1$, we have $H - kE \sim 12F_1 + F_2$. As $F_1 \cdot (H - kE) = 1$, we have $E \neq F_1$, whence the contradiction $5 = E \cdot (H - kE) \geq 12E \cdot F_1 \geq 12$. \square

Remark 2.6. We will later use the observation immediately deduced from parts (i)-(ii) of Lemma 2.5 that for $\phi(H) \leq 2$ there are at most three numerical, effective, isotropic classes E such that $E \cdot H \leq 2$.

3. MORE ON SIMPLE, ISOTROPIC DECOMPOSITIONS

The main aim of this section is to prove that the isotropic divisors occurring in a simple isotropic decomposition can always be extended to an isotropic 10-sequence plus one of the divisors $E_{i,j}$ occurring in Lemma 2.2. This will be needed in the proof of our main results, see the comment right after Propositions 4.5 and 4.6.

Definition 3.1. A set $\{E_1, \dots, E_n\}$ of primitive isotropic divisors on an Enriques surface is called a simple isotropic set if it satisfies (3), possibly after permuting indices.

It is called a maximal simple isotropic set if it is of the form $\{E_1, \dots, E_{10}, E_{i,j}\}$, where $\{E_1, \dots, E_{10}\}$ is an isotropic 10-sequence and $E_{i,j}$ is defined up to numerical equivalence as in (6) for some $i \neq j$, that is, $E_{i,j} \equiv \frac{1}{3}(E_1 + \dots + E_{10}) - E_i - E_j$.

Note that since any simple isotropic set of n elements contains members of an isotropic $(n-1)$ -sequence, any simple isotropic set contains at most 11 elements (cf. [4, p. 179]). Also note that by [3, Rem. p. 584] any maximal simple isotropic set form a basis of $\text{Num}(S)$.

We will prove the following result, which can be viewed as a generalization of Proposition 2.1, and which we hope is of independent interest.

Proposition 3.2. Any simple isotropic set can be extended to a maximal simple isotropic set.¹

¹This includes the case of simple isotropic sets of 11 elements, which means that such are automatically maximal.

Before giving the proof, we discuss some consequences.

Corollary 3.3. *Let H be any effective divisor on an Enriques surface such that $H^2 > 0$. Then there is an isotropic 10-sequence $\{E_1, \dots, E_{10}\}$ (depending on H) such that*

$$(10) \quad H \equiv a_0 E_{1,2} + a_1 E_1 + \dots + a_{10} E_{10},$$

where $E_{1,2} \equiv \frac{1}{3}(E_1 + \dots + E_{10}) - E_1 - E_2$ (cf. (6)) and a_0, a_1, \dots, a_{10} are nonnegative integers, at least one being 0.

More precisely, any simple isotropic decomposition of H occurs in this way.

Proof. By Proposition 3.2, any isotropic divisors occurring in any simple isotropic decomposition of the form (2) can be extended to a maximal simple isotropic set. \square

Proposition 3.4. *Two numerically polarized Enriques surfaces $(S, [H])$ and $(S', [H'])$ lie in the same irreducible component of $\overline{\mathcal{E}}_{g,\phi}$ if and only if they admit the same simple decomposition type.*

Proof. Since the torsion free part of $H^2(S, \mathbb{Z})$, which is equal to $\text{Num}(S) \simeq U \oplus E_8(-1)$ (see [1, Lemma VIII.15.1]), is constant among all $S \in \mathcal{E}$, the only if part is immediate.

Conversely, it is proved in [6] that the irreducible components of $\overline{\mathcal{E}}_{g,\phi}$ correspond precisely to the different orbits of the action of the orthogonal group on $U \oplus E_8(-1)$. Since this group acts transitively on the set of isotropic 10-sequences by [4, Lemma 2.5.2], and $E_{1,2} \equiv \frac{1}{3}(E_1 + \dots + E_{10}) - E_1 - E_2$, we see that any two numerical polarizations admitting the same simple decomposition type lie in the same irreducible component of $\overline{\mathcal{E}}_{g,\phi}$, as claimed. \square

Question 3.5. *Does Proposition 3.4 also hold for polarized Enriques surfaces? In other words, is it true that (S, H) and (S, H') lie in the same irreducible component of $\mathcal{E}_{g,\phi}$ if and only if H and H' admit the same simple decomposition type? (The “only if” part follows as in the first lines of the proof of 3.4, as $\text{Pic}(S) \simeq U \oplus E_8(-1) \oplus \mathbb{Z}/2\mathbb{Z}$ is also constant among all $S \in \mathcal{E}$.)*

Theorems 1.1 and 1.2 give a positive answer in the case of simple decomposition types of length ≤ 5 or 6-symmetric.

Proposition 3.2 is a consequence of Lemmas 3.6, 3.7 and 3.8 below, together with Proposition 2.1 and Lemma 2.2.

Lemma 3.6. *Let $\{E_1, \dots, E_r\}$ be an isotropic r -sequence with $2 \leq r \leq 9$, and F an isotropic divisor such that $F \cdot E_1 = F \cdot E_2 = 2$ and $F \cdot E_i = 1$ for all $i \in \{3, \dots, r\}$. Then there is an isotropic 10-sequence $\{E_1, \dots, E_r, E_{r+1}, \dots, E_{10}\}$ such that $F \cdot E_i = 1$ for all $i \in \{r+1, \dots, 10\}$.*

Proof. The divisor $D := E_1 + E_2 + F$ satisfies $D^2 = 10$ and $\phi(D) = 3 = E_i \cdot D$ for all $i \in \{1, \dots, r\}$. Thus, $3D \sim E_1 + \dots + E_{10}$ for an isotropic 10-sequence $\{E_1, \dots, E_{10}\}$ by Lemma 2.2(b). Since $F \cdot D = 4$, we have $F \not\equiv E_i$ for any i , hence $F \cdot E_i > 0$ for all i by [7, Lemma 2.1]. As $12 = 3F \cdot D = F \cdot (3D) = 4 + F \cdot (E_3 + \dots + E_{10})$, we must have $F \cdot E_i = 1$ for all i . \square

Lemma 3.7. *Let $\{E_1, \dots, E_8, F\}$ be an isotropic 9-sequence. Then, for any extension of $\{E_1, \dots, E_8\}$ to an isotropic 10-sequence $\{E_1, \dots, E_{10}\}$, we have either*

- (i) $F \equiv E_i$, for $i = 9$ or 10 , or
- (ii) $F \cdot E_9 = F \cdot E_{10} = 2$.

Proof. If $F \cdot E_i = 0$ for $i = 9$ or 10 , then $F \equiv E_i$ by [7, Lemma 2.1] and we are done. Otherwise, as $E_1 + \dots + E_{10}$ is 3-divisible by Lemma 2.2, we must have

$$F \cdot (E_9 + E_{10}) \equiv 1 \pmod{3} \quad \text{and} \quad F \cdot E_i > 0 \quad \text{for} \quad i = 9, 10.$$

We are therefore done if we show that

$$(11) \quad F \cdot E_i \leq 2, \quad \text{for} \quad i \in \{9, 10\}.$$

To prove this, assume by contradiction that $n := F \cdot E_9 \geq 3$, say. Set $k = \lfloor \frac{n-1}{2} \rfloor \geq 1$ and $B := F + E_9 - kE_1$. Then $B^2 \in \{2, 4\}$ and $E_i \cdot B = 2 - k \leq 1$ for all $i \in \{2, \dots, 8\}$, contradicting Remark 2.6. This proves (11), whence the lemma. \square

Lemma 3.8. *Let F_1 and F_2 be isotropic divisors such that $F_1 \cdot F_2 = 2$ and $\{E_1, \dots, E_r\}$ be an isotropic r -sequence, with $0 \leq r \leq 8$, such that $F_i \cdot E_j = 1$ for all $i \in \{1, 2\}$, $j \in \{1, \dots, r\}$.*

Then, for $k = 1$ or 2 , there is an isotropic 10-sequence $\{F_k, E_1, \dots, E_r, E_{r+1}, \dots, E_9\}$ such that, for $j \neq k$, $F_j \cdot E_i = 1$ for $i \in \{r+1, \dots, 8\}$ and $F_j \cdot E_9 = 2$.

Proof. Assume first that $r \leq 7$. By Proposition 2.1, the set \mathcal{A} of $A \in \text{Pic}(S)$ such that

$$A^2 = 0, \quad A \cdot F_1 = A \cdot E_1 = \dots = A \cdot E_r = 1, \quad A \not\equiv E_1 + E_2 + E_3 - F_2 \quad \text{if} \quad r = 3,$$

is nonempty. Pick $A \in \mathcal{A}$ such that $A \cdot F_2$ is minimal.

Claim. $A \cdot F_2 \leq 2$.

Assume, to get a contradiction, that $n := A \cdot F_2 \geq 3$. Let $k = \lfloor \frac{n-1}{3} \rfloor$ and set $B := A + F_2 - kF_1$. Then $2 \leq B^2 \leq 6$ and B has a simple isotropic decomposition containing at least two summands. None of these may be F_2 , since $B - F_2 = A - kF_1$ is not effective, unless $k = 0$, in which case $B = F_2 + A$ is not a simple isotropic decomposition.

Since $F_2 \cdot B = n - 2k$, the intersection of F_2 with each of the summands in the simple isotropic decomposition of B is smaller than n . Since $F_1 \cdot B = 3$, there is at least one of these summands, say E' , such that $F_1 \cdot E' = 1$. If $r = 0$, since $F_2 \cdot E' < n$, the curve E' contradicts the minimality of A and finishes the proof in this case.

If $r > 0$, then, as $E_i \cdot B = 2 - k$ for any $i \in \{1, \dots, r\}$, we must have $k \leq 1$.

Case $k = 0$. Then $n = 3$, $B \sim A + F_2$, $B^2 = 6$ and $\phi(B) = E_i \cdot B = 2$. Thus, by Lemma 2.5, B can be written as a sum of three isotropic divisors, containing all E_i for $i \in \{1, \dots, r\}$. This implies $r \leq 3$. Since $F_i \cdot B = 3$, for $i = 1, 2$, each summand has intersection one with F_i , for $i = 1, 2$. This implies $r = 3$. Indeed, if $r < 3$, then at least one of the summands of B , say E' , is different from the E_i s, and has $E' \cdot E_i = 1$ for $i = 1, \dots, r$. Hence $E' \in \mathcal{A}$ and $E' \cdot F_2 = 1$, contradicting the minimality of A . Since $r = 3$, we have $B \equiv E_1 + E_2 + E_3$. But then $A \equiv E_1 + E_2 + E_3 - F_2$, thus $A \notin \mathcal{A}$, a contradiction.

Case $k = 1$. One has $B \sim A + F_2 - F_1$ and $\phi(B) = E_1 \cdot B = 1$. Moreover $B^2 = 2n - 6$, hence $(n, B^2) \in \{(4, 2), (5, 4), (6, 6)\}$.

Subcase $(n, B^2) = (4, 2)$. As $E_i \cdot B = 1$, for $i \in \{2, \dots, r\}$, by Lemma 2.5(i) we have $r \leq 2$ and, if $r = 2$, we have $B \equiv E_1 + E_2$. But $3 = F_1 \cdot B = F_1 \cdot (E_1 + E_2) = 2$, a contradiction. Hence we have $r = 1$ and $B \sim E_1 + E'_2$ with $E'_2{}^2 = 0$ and $E_1 \cdot E'_2 = 1$.

We have $F_1 \cdot B = 3$, and since $F_1 \cdot E_1 = 1$, we have $F_1 \cdot E'_2 = 2$. Since $F_2 \cdot B = 2$ and $F_2 \cdot E_1 = 1$, we have $F_2 \cdot E'_2 = 1$. Set $G := F_1 + F_2 + E'_2$. Then $G^2 = 10$,

$F_1 \cdot G = 4$ and $\phi(G) = E_1 \cdot G = E'_2 \cdot G = F_2 \cdot G = 3$. By Lemma 2.2(b), we have $3G \sim E_1 + E'_2 + F_2 + F'_1 + \cdots + F'_7$ for an isotropic 10-sequence $\{E_1, E'_2, F_2, F'_1, \dots, F'_7\}$. As $F_1 \cdot (3G) = 12$, and $F_1 \cdot (E_1 + E'_2 + F_2) = 5$, it follows that $F_1 \cdot (F'_1 + \cdots + F'_7) = 7$, whence $F_1 \cdot F'_i = 1$ for all $i \in \{1, \dots, 7\}$. Since $F_2 \cdot F'_i = 1$ for all $i \in \{1, \dots, 7\}$, we find a contradiction to the minimality of A .

Subcase $(n, B^2) = (5, 4)$. As $E_i \cdot B = 1$, for $i \in \{2, \dots, r\}$, by Lemma 2.5(i) we have $r = 1$ and $B \sim 2E_1 + E'_2$ with $E'_2{}^2 = 0$ and $E_1 \cdot E'_2 = 1$. As $F_1 \cdot B = F_2 \cdot B = 3$, it follows that $F_1 \cdot E'_2 = F_2 \cdot E'_2 = 1$, contradicting the minimality of A .

Subcase $(n, B^2) = (6, 6)$. As $E_1 \cdot B = 1$ and $F_1 \cdot B = 3$, we must have $B \equiv 3E_1 + F_1$. But then we get the contradiction

$$4 = F_2 \cdot (A + F_2 - F_1) = F_2 \cdot B = 3E_1 \cdot F_2 + F_1 \cdot F_2 = 5.$$

Therefore, we have proved the claim that $A \cdot F_2 \leq 2$.

Assume now that $A \cdot F_2 = 2$. By Lemma 3.6, the isotropic sequence $\{F_1, A, E_1, \dots, E_r\}$ can be extended to an isotropic 10-sequence such that $F_2 \cdot F_1 = F_2 \cdot A$ and F_2 has intersection one with the remaining divisors in the sequence. Hence, we are done.

Assume next that $A \cdot F_2 = 1$. We then repeat the process starting with the isotropic $(r+1)$ -sequence $\{E_1, \dots, E_r, E_{r+1} := A\}$, unless $r+1 = 8$. We thus reduce to proving the lemma when $r = 8$.

For the rest of the proof we therefore let $r = 8$. Then we can by Proposition 2.1 extend $\{E_1, \dots, E_8\}$ to an isotropic 10-sequence $\{E_1, \dots, E_{10}\}$. We claim that

$$(12) \quad \text{there is an } i \in \{1, 2\} \text{ and a } j \in \{9, 10\} \text{ such that } F_i \equiv E_j.$$

Indeed, if not, by Lemma 3.7 we must have all $F_i \cdot E_j = 2$ for $i \in \{1, 2\}, j \in \{9, 10\}$. Set $B := F_1 + F_2 + E_9 + E_{10} - 2E_1$. Then $B^2 = 6$ and $E_j \cdot B = 2$ for all $j \in \{2, \dots, 8\}$, which is impossible by Remark 2.6. This proves (12).

By (12) we have, say, $F_1 \equiv E_{10}$. Then $E_9 \not\equiv F_2$, so $F_2 \cdot E_9 = 2$ by Lemma 3.7. Hence, $\{F_1, E_1, \dots, E_8, E_9\}$ is the desired isotropic 10-sequence. \square

We can finally give the:

Proof of Proposition 3.2. Consider the simple isotropic set $\{E_1, \dots, E_r\}$ satisfying (3). If $E_i \cdot E_j = 1$ for all $i \neq j$, and if $r \neq 9$, we apply Proposition 2.1. If instead $r = 9$, we apply Lemmas 3.7 and 2.2(b). If $E_1 \cdot E_2 = 2$ and otherwise $E_i \cdot E_j = 1$ for $i \neq j$, we apply Lemmas 3.8 and 2.2(b). Finally, if $E_1 \cdot E_2 = E_1 \cdot E_3 = 2$ and otherwise $E_i \cdot E_j = 1$ for $i \neq j$, we apply Lemmas 3.6 and 2.2(b). \square

Remark 3.9. Writing a simple isotropic decomposition of H as in (10) has the advantage that $\phi(H)$ is calculated by one among $E_{1,2}, E_1, \dots, E_{10}$. More precisely, setting $a := \sum_{i=0}^{10} a_i$, one has

$$(13) \quad \phi(H) = a - \max\{a_1 - a_0, a_2 - a_0, a_3, \dots, a_{10}, a_0 - a_1 - a_2\}.$$

Indeed, for any nontrivial isotropic effective $E \not\equiv E_{1,2}, E_1, \dots, E_{10}$, one has $E \cdot H \geq a \geq a - a_i = E_i \cdot H$, for any $i \geq 3$. Then (13) follows since $E_i \cdot H = a + a_0 - a_i$ for $i = 1, 2$ and $E_{1,2} \cdot H = a + a_1 + a_2 - a_0$. By symmetry and Lemma 3.7, one can furthermore make sure that

$$(14) \quad a_1 > 0, \quad a_1 \geq a_2, \quad a_3 \geq \cdots \geq a_{10} \quad \text{and either } a_0 > 0 \text{ or } a_2 \geq a_3,$$

in which case

$$(15) \quad \phi(H) = \min\{E_1 \cdot H, E_3 \cdot H, E_{1,2} \cdot H\} = a - \max\{a_1 - a_0, a_3, a_0 - a_1 - a_2\}.$$

Remark 3.10. Even imposing the conditions (14) does not make the decomposition type unique, and the properties such as the length or being r -symmetric may also vary with the different ways of writing the decompositions. Consider for instance the decomposition type $H \equiv 2E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_{1,7}$ (with $g = 30$ and $\phi(H) = 7$). This has length 7 and is 5-symmetric, but not 6-symmetric. By Proposition 3.2 we may extend $\{E_1, \dots, E_6\}$ to an isotropic 10-sequence so that $E_{1,7}$ is defined as in (6). Let also $E_{7,8}$ be as defined by (6). It follows that $E_1 + E_{1,7} \sim E_8 + E_{7,8}$. Thus, we may also write $H \equiv E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_8 + E_{7,8}$, which has length 8 and is 6-symmetric.

4. IRREDUCIBILITY, UNIRATIONALITY AND UNIRULEDNESS OF MODULI SPACES

We have the natural forgetful map

$$(16) \quad \mathcal{E}_{g,\phi} \longrightarrow \mathcal{E},$$

whose differential at a point (S, H) is the linear map

$$(17) \quad H^1(S, \mathcal{E}_H) \longrightarrow H^1(S, \mathcal{T}_S)$$

coming from the *Atiyah extension of H*

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{E}_H \longrightarrow \mathcal{T}_S \longrightarrow 0,$$

by [10, Prop. 3.3.12]. Since $h^1(\mathcal{O}_S) = h^2(\mathcal{O}_S) = 0$, the map (17) is an isomorphism, hence $\mathcal{E}_{g,\phi}$ is smooth and the map (16) is an étale cover.

We note that $\mathcal{E}_{g,\phi}$ need not be irreducible. Moreover, as already mentioned in Question 3.5, all members in an irreducible component of $\mathcal{E}_{g,\phi}$ must admit the same simple decomposition type. Theorems 1.1 and 1.2 prove the converse of this in many cases.

We now extend a construction from [11]. First we recall some basic facts about classical Enriques sextic surfaces in \mathbb{P}^3 (see [4]).

Fix homogeneous coordinates $(x_0 : x_1 : x_2 : x_3)$ on \mathbb{P}^3 and let

$$T = Z(x_0x_1x_2x_3)$$

be the *coordinate tetrahedron*. Consider the linear system \mathcal{S} of surfaces of degree 6 which are singular along the edges of T . They are called *Enriques sextic surfaces*. The surfaces in \mathcal{S} have equations of the form

$$(18) \quad c_3(x_0x_1x_2)^2 + c_2(x_0x_1x_3)^2 + c_1(x_0x_2x_3)^2 + c_0(x_1x_2x_3)^2 + Qx_0x_1x_2x_3 = 0,$$

where $Q = \sum_{i < j} q_{ij}x_ix_j$. This shows that $\dim(\mathcal{S}) = 13$ and we may identify \mathcal{S} with the \mathbb{P}^{13} with homogeneous coordinates

$$q = (c_0 : c_1 : c_2 : c_3 : q_{00} : q_{01} : q_{02} : q_{03} : q_{11} : q_{12} : q_{13} : q_{22} : q_{23} : q_{33}).$$

If $\Sigma \in \mathcal{S}$ is a general surface, its normalization $\varphi : S \rightarrow \Sigma$ is an Enriques surface and $H = \varphi^*(\mathcal{O}_\Sigma(1))$ is an ample divisor class with $H^2 = 6$ and $\phi(H) = 2$. More precisely, $H \sim E_1 + E_2 + E_3$, with the usual Notation 2.3, and the edges of T are the images of the curves E_i and $E'_i \sim E_i + K_S$, with $i = 1, 2, 3$. (Recall that for a primitive, isotropic E , the complete linear system $|E + K_S|$ has a unique element.) We will write ℓ_i (resp. ℓ'_i) for the line image of E_i (resp. of E'_i). This means that we have marked one face of T

and chosen an ordering of its three edges. We denote by v the vertex of T not contained in the marked face.

We thus have a natural rational map

$$p : \mathcal{S} \dashrightarrow \mathcal{E}_{4,2},$$

assigning to a general surface $\Sigma \in \mathcal{S}$ the pair (S, H) , where $\varphi : S \rightarrow \Sigma$ is the normalization and $H = \varphi^*(\mathcal{O}_\Sigma(1))$. Composing with the forgetful map $\mathcal{E}_{4,2} \rightarrow \mathcal{E}$, we have a rational map $\mathcal{S} \dashrightarrow \mathcal{E}$, which is dominant. Indeed, given a general, whence unnodal, Enriques surface S , we can find a 3-isotropic sequence $\{E_1, E_2, E_3\}$. If we set $H = E_1 + E_2 + E_3$, then $(S, H) \in \mathcal{E}_{4,2}$ and the linear system $|H|$ determines a morphism $\varphi_H : S \rightarrow \mathbb{P}^3$, cf., e.g., [4, Thm. 4.6.3 and 4.7.2], and, up to a change of coordinates, $\Sigma = \varphi_H(S)$ is an Enriques sextic surface. Accordingly, the map p is dominant. If (S, H) is a point of $\mathcal{E}_{4,2}$, the fibre $p^{-1}(S, H)$ consists of the orbit of $\Sigma = \varphi_H(S)$ via the 3-dimensional group of projective transformations which fix T .

Next we denote by \mathcal{F}_i , $i = 0, 1, 2$, the family of smooth cubic (resp., quartic, quintic) elliptic curves $F \subset \mathbb{P}^3$ such that $v \notin F$ and F meets

- all edges of T exactly once, if $i = 0$;
- the edges ℓ_1 and ℓ'_1 of T exactly twice, and the remaining edges exactly once, if $i = 1$;
- the edges ℓ_3 and ℓ'_3 of T exactly once, and the remaining edges exactly twice, if $i = 2$.

Lemma 4.1. (a) *The family \mathcal{F}_2 is irreducible, 10-dimensional and rational, and each $F \in \mathcal{F}_2$ is contained in a 3-dimensional linear system of Enriques sextics.*

(b) *The family \mathcal{F}_1 is irreducible, 8-dimensional and rational, and each $F \in \mathcal{F}_1$ is contained in a 5-dimensional linear system of Enriques sextics.*

(c) *The family \mathcal{F}_0 is irreducible, 6-dimensional, and rational, and each $F \in \mathcal{F}_0$ is contained in a 7-dimensional linear system of Enriques sextics.*

Proof. We first prove (b) (resp. (c)). Let $F \in \mathcal{F}_1$ (resp. $F \in \mathcal{F}_0$). The linear system \mathcal{S} cuts out on F a linear system of divisors with base locus (containing) $T \cap F$ and a moving part \mathfrak{g} of degree (at most) 8 (resp., 6). Note that \mathcal{S} contains the 9-dimensional linear system formed by surfaces of the form $T + Q$, where Q is a general quadric in \mathbb{P}^3 : looking at equation (18), these are the surfaces obtained by setting $c_i = 0$, for $i = 1, \dots, 4$. Since quadrics cut out on F a complete linear system, we see that \mathfrak{g} is complete, of dimension 7 (resp. 5). This proves that the linear system of Enriques sextics containing F has dimension 5 (resp., 7).

The family of quartic elliptic curves in \mathbb{P}^3 is 16-dimensional. Let Z be a configuration of eight points outside the vertices of T and distributed as two points on each of ℓ_1 and ℓ'_1 and one point on each of the remaining edges of T . Then there is a unique quartic elliptic curve through Z . Indeed, otherwise there would be a net \mathcal{Q} of quadrics through these 8 points. Fix the attention on a face Π of T containing four of these points (on three edges). By imposing to the quadrics in \mathcal{Q} to contain two general points of Π , the plane Π splits off the quadrics of \mathcal{Q} . Consequently, the remaining four of the eight points should be coplanar, a contradiction. Hence, the set of quartics with the given incidences with the edges of T is irreducible, 8-dimensional, and birational to $\text{Sym}^2(\ell_1) \times \text{Sym}^2(\ell'_1) \times \ell_2 \times \ell'_2 \times \ell_3 \times \ell'_3 \simeq \mathbb{P}^8$. This proves (b).

If $F \in \mathcal{F}_0$, then F spans a plane $\Pi_F \subset \mathbb{P}^3$, which intersects the set of edges of T in six points. The set of plane cubics through these six points is a linear system of dimension 3. Thus, \mathcal{F}_0 is a \mathbb{P}^3 -bundle over $|\mathcal{O}_{\mathbb{P}^3}(1)| \simeq \mathbb{P}^3$, and is therefore irreducible, rational and 6-dimensional. This proves (c).

As for item (a), the fact that \mathcal{F}_2 is irreducible, 10-dimensional and rational is proved in [11, Prop. 1.1 and §2]. The rest of the assertion is proved exactly in the same way we did it for cases (b) and (c) above. \square

We next define \mathcal{F}_{00} to be the family of ordered pairs (F, F') of smooth cubic elliptic curves $F, F' \subset \mathbb{P}^3$ such that $F, F' \in \mathcal{F}_0$ and F and F' intersect exactly in one point not on T , with distinct tangent lines.

Lemma 4.2. *The family \mathcal{F}_{00} is irreducible, 11-dimensional and rational and each pair $(F, F') \in \mathcal{F}_{00}$ is contained in a 2-dimensional linear system of Enriques sextics.*

Proof. The family \mathcal{F}_{00} can be constructed in the following way: fix a pair of general planes Π and Π' in \mathbb{P}^3 intersecting along a line ℓ , and fix a point $p \in \ell$. Consider in both Π and Π' the family of cubic curves passing through p and the six intersection points of Π and Π' , respectively, with the edges of T ; each of these is a two-dimensional linear system. Varying Π , Π' and p and taking the two families of cubic curves, we obtain all elements of \mathcal{F}_{00} . This description shows the rationality and the dimension.

Now fix $(F, F') \in \mathcal{F}_{00}$ and let $\mathcal{S}_{F+F'}$ be the linear system of Enriques sextics containing $F \cup F'$. First we prove that $\dim(\mathcal{S}_{F+F'}) \geq 2$. Indeed, the linear system \mathcal{S}_F of Enriques sextics containing F is 7-dimensional by Lemma 4.1(c). It cuts on F' a linear system of divisors with base locus (containing) $T \cap F$ and $p = F \cap F'$ and a moving part of degree (at most) 5, hence of dimension at most 4. Therefore, containing F' imposes at most 5 conditions on \mathcal{S}_F .

Next we prove that $\dim(\mathcal{S}_{F+F'}) \leq 2$, which will finish our proof. Consider the pair $F \subset \Pi$ and $F' \subset \Pi'$ in \mathcal{F}_{00} , with the planes they span. Set $\ell = \Pi \cap \Pi'$ and $F \cap \ell = \{a, b, p\}$ and $F' \cap \ell = \{a', b', p\}$. Let $\Sigma \in \mathcal{S}_{F+F'}$ be general. Then ℓ intersects Σ in six points, among these are $\{a, b, a', b', p\}$, call p' the sixth point. The surface Σ intersects Π (resp., Π') in a cubic G off F (resp., G' off F'), passing through a', b' and p' (resp., a, b and p'), in addition to the six intersection points of Π (resp., Π') with the edges of T . Thus, G and G' are uniquely determined. Consequently, $\Sigma \cap (\Pi \cup \Pi')$ may at most vary with the point $p' \in \ell$. Thus the restriction $\mathcal{S}_{\Pi \cup \Pi'}$ of $\mathcal{S}_{F+F'}$ to $\Pi \cup \Pi'$ is at most one-dimensional. Consider the restriction map $\mathcal{S}_{F+F'} \dashrightarrow \mathcal{S}_{\Pi \cup \Pi'}$, which is linear, rational and surjective by assumption. Its indeterminacy locus is the unique surface $T \cup \Pi \cup \Pi'$. Since $\dim(\mathcal{S}_{\Pi \cup \Pi'}) \leq 1$, we deduce that $\dim(\mathcal{S}_{F+F'}) \leq 2$, as desired. \square

We next define \mathcal{F}_{0i} , for $i = 1, 2$, to be the family of ordered pairs of smooth elliptic curves (F, F') in \mathbb{P}^3 such that $F \in \mathcal{F}_0$, $F' \in \mathcal{F}_1$ and F and F' intersect exactly in i points not on T , with distinct tangent lines.

Lemma 4.3. *The family \mathcal{F}_{0i} is irreducible, uniruled and $(14 - i)$ -dimensional and each pair $(F, F') \in \mathcal{F}_{0i}$ is contained in a linear system $\mathcal{S}_{F+F'}$ of Enriques sextics of dimension at least $i - 1$. If $(F, F') \in \mathcal{F}_{0i}$ is contained in an Enriques sextic Σ whose normalization S is an Enriques surface, then $\mathcal{S}_{F+F'}$ has dimension exactly $i - 1$, unless $F + F'$ is contained in only nodal Enriques sextics (that is, Enriques sextics whose normalizations contain smooth rational curves).*

Proof. We have a natural dominant map $q : \mathcal{F}_{0i} \rightarrow \mathcal{F}_1 \times (\mathbb{P}^3)^\vee$ sending the pair (F, F') to $F' \in \mathcal{F}_1$ and the plane Π_F spanned by F in $(\mathbb{P}^3)^\vee$.

For $i = 1$, the fiber of q over (F', Π) consists of the union of four 2-dimensional linear systems of cubics in Π through the six intersection points of Π with the edges of T and

one of the four intersection points of Π with F' . This proves the irreducibility because the monodromy action of the four intersection points is the symmetric group, and shows also the uniruledness. The dimension also follows easily.

For $i = 2$, the fiber of q over (F', Π) consists of the union of six 1-dimensional linear systems of cubics in Π through the six intersection points of Π with the edges of T and two of the four intersection points of Π with F' . As above, this proves irreducibility, uniruledness and the dimension.

The dimension of the linear system of Enriques sextics $\mathcal{S}_{F'}$ containing a fixed $F' \in \mathcal{F}_1$ is 5 by Lemma 4.1(b). Containing an additional cubic $F \in \mathcal{F}_0$ intersecting F' in i points, imposes at most $6 - i$ conditions, arguing as in the proof of Lemma 4.2. Therefore, the linear system of Enriques sextics $\mathcal{S}_{F+F'}$ containing a pair $(F, F') \in \mathcal{F}_{0i}$ has dimension at least $5 - (6 - i) = i - 1$.

Let Σ be an Enriques sextic containing $F + F'$ such that its normalization $\varphi : S \rightarrow \Sigma$ is an unnodal Enriques surface. The linear system \mathcal{S} cuts on Σ a linear system whose pull-back on S via φ is the sublinear system of $|6(E_1 + E_2 + E_3)|$ with base locus twice the sum of the pullback of the edges of the tetrahedron, which is

$$2\left(E_1 + E_2 + E_3 + (E_1 + K_S) + (E_2 + K_S) + (E_3 + K_S)\right) \sim 4(E_1 + E_2 + E_3).$$

Hence, the free part is $|2(E_1 + E_2 + E_3)|$. So we have a linear, rational restriction map

$$\mathcal{S}_{F+F'} \dashrightarrow |B|, \quad \text{with } B := 2(E_1 + E_2 + E_3) - (F + F')$$

whose indeterminacy locus is just the surface Σ .

We have $B^2 = 2(i - 2)$. If $i = 1$ and S is unnodal, then $|B| = \emptyset$, which shows that $\mathcal{S}_{F+F'} = \{\Sigma\}$ has dimension 0, as wanted. If $i = 2$, then $B^2 = 0$ and $E_1 \cdot B = 1$, hence $h^0(B) = 1$ by Riemann–Roch. This implies that $\dim(\mathcal{S}_{F+F'}) \leq 1$, proving the assertion. \square

Consider now the incidence varieties

$$\mathcal{G}_i := \{(F, \Sigma) \in \mathcal{F}_i \times \mathcal{S} \mid F \subset \Sigma\},$$

for $i = 0, 1, 2$, and

$$\mathcal{G}_{00} := \{(F, F', \Sigma) \in \mathcal{F}_{00} \times \mathcal{S} \mid F + F' \subset \Sigma\},$$

which are irreducible, rational and 13-dimensional, by Lemmas 4.1 and 4.2. Similarly, for $i = 1, 2$, let

$$\mathcal{G}_{0i} := \{(F, F', \Sigma) \in \mathcal{F}_{0i} \times \mathcal{S} \mid \Sigma \text{ is unnodal, } F + F' \subset \Sigma\},$$

which are irreducible, uniruled and 13-dimensional, by Lemma 4.3.

Proposition 4.4. *If \mathcal{G} is any of the incidence varieties \mathcal{G}_i , for $i = 0, 1, 2$, \mathcal{G}_{00} , \mathcal{G}_{0i} , for $i = 1, 2$, the obvious projection $\pi : \mathcal{G} \rightarrow \mathcal{S}$ is dominant, hence generically finite. Accordingly, if $\xi \in \mathcal{G}$ is a general point, then $\Sigma = \pi(\xi)$ is a general element of \mathcal{S} and its normalization S is a general Enriques surface.*

Proof. We prove the assertion for $\mathcal{G} = \mathcal{G}_{00}$, the proof in the other cases being similar.

Let S be a general Enriques surface. There is an isotropic 5-sequence $\{E_1, \dots, E_5\}$ on S . Set $H = E_1 + E_2 + E_3$. Then $\varphi_H : S \rightarrow \Sigma \subset \mathbb{P}^3$ maps S , up to a projective transformation, to a general surface in \mathcal{S} . Moreover E_4, E_5 are mapped to two elliptic cubic curves F, F' meeting at a point. This proves the assertion. \square

We now define various maps from these incidence varieties to some $\mathcal{E}_{g,\phi}$ s, for various g and ϕ , which we eventually prove to be dominant, establishing irreducibility and unirationality or uniruledness.

Consider a general element (F, Σ) of \mathcal{G}_i , for $i = 0, 1, 2$. Then the normalization S of Σ is an Enriques surface and on S we have the three curves E_1, E_2, E_3 , plus the strict transform of F which, by abuse of notation, we still denote by F . Similar convention we introduce for \mathcal{G}_{0i} , for $i = 0, 1, 2$.

Fix four nonnegative integers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, at least two nonzero. Then, for each $i = 0, 1, 2$, we have a rational map

$$f_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}^i : \mathcal{G}_i \dashrightarrow \mathcal{E}_{g, \phi}$$

sending the general point $(F, \Sigma) \in \mathcal{G}_i$ to $(S, \mathcal{O}_S(\alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 F))$, where $g = p_a(\alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 F)$ and $\phi = \phi(\alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 F)$. Similarly, for each $i = 0, 1, 2$, we have a rational map

$$f_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}^{i'} : \mathcal{G}_i \dashrightarrow \mathcal{E}_{g, \phi}$$

sending the general point $(F, \Sigma) \in \mathcal{G}_i$ to $(S, \mathcal{O}_S(\alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 F + K_S))$.

Next, fix five positive integers $\alpha_1, \dots, \alpha_5$, at least two nonzero. For each $i = 0, 1, 2$, we have a rational map

$$f_{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5}^{0i} : \mathcal{G}_{0i} \dashrightarrow \mathcal{E}_{g, \phi}$$

sending a general $(F, F', \Sigma) \in \mathcal{G}_{0i}$ to $(S, \mathcal{O}_S(\alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 F + \alpha_5 F'))$, where $g = p_a(\alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 F + \alpha_5 F')$ and $\phi := \phi(\alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 F + \alpha_5 F')$. Similarly we have a map

$$f_{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5}^{0i'} : \mathcal{G}_{0i} \dashrightarrow \mathcal{E}_{g, \phi}$$

sending a general $(F, F', \Sigma) \in \mathcal{G}_{0i}$ to $(S, \mathcal{O}_S(\alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 F + \alpha_5 F' + K_S))$.

Let now $(F, \Sigma) \in \mathcal{G}_2$ be general and consider the curves E_1, E_2, E_3, F on S . Then $E_1 + E_2 + F$ satisfies the conditions of Lemma 2.2(b). Since $E_i \cdot (E_1 + E_2 + F) = 3$, for $i = 1, 2, 3$, we obtain an isotropic 10-sequence $\{E_1, E_2, E_3, E_4, \dots, E_{10}\}$ such that

$$3(E_1 + E_2 + F) \sim E_1 + \dots + E_{10}.$$

Note that each E_i for $i \geq 4$ is uniquely determined up to numerical equivalence class and permutation of indices; in particular, $E_4 + \dots + E_{10} \sim 2E_1 + 2E_2 + 3F - E_3$ is a well-defined element of $\text{Pic}(S)$. For any five nonnegative integers $\alpha_0, \dots, \alpha_4$ such that at least one among $\alpha_0, \dots, \alpha_3$ is zero, we can consider the rational map

$$h_{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4} : \mathcal{G}_2 \dashrightarrow \mathcal{E}_{g, \phi}$$

sending $(F, \Sigma) \in \mathcal{G}_2$ to $(S, \mathcal{O}_S(\alpha_0 F + \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4(E_4 + \dots + E_{10})))$, where $g := p_a(\alpha_0 F + \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4(E_4 + \dots + E_{10}))$ and $\phi := \phi(\alpha_0 F + \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4(E_4 + \dots + E_{10}))$. Similarly we have a map

$$h'_{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4} : \mathcal{G}_2 \dashrightarrow \mathcal{E}_{g, \phi}$$

sending $(F, \Sigma) \in \mathcal{G}_2$ to $(S, \mathcal{O}_S(\alpha_0 F + \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4(E_4 + \dots + E_{10})) + K_S)$.

Finally, let $(F, F', \Sigma) \in \mathcal{G}_{02}$ be a general point and consider E_1, E_2, E_3, F, F' curves in S . Then $F + F' + E_1$ satisfies the conditions of Lemma 2.2(b). Since $F \cdot (E_1 + F + F') = E_i \cdot (E_1 + F + F') = 3$, for $i = 1, 2, 3$, we obtain an isotropic 10-sequence $\{E_1, E_2, E_3, E_4 := F, E_5, \dots, E_{10}\}$ such that

$$3(E_1 + F + F') \sim E_1 + \dots + E_{10}.$$

Note that each E_i for $i \geq 5$ is uniquely determined up to numerical equivalence class and permutation of indices; in particular, $E_5 + \cdots + E_{10} \sim 2E_1 + 2F + 3F' - E_2 - E_3$ is a well-defined element of $\text{Pic}(S)$. For any six nonnegative integers $\alpha_0, \dots, \alpha_5$ such that at least one among $\alpha_0, \dots, \alpha_4$ is zero, we have a map

$$h_{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5}^0 : \mathcal{G}_{02} \dashrightarrow \mathcal{E}_{g, \phi}$$

sending (F, F', Σ) to $(S, \mathcal{O}_S(\alpha_0 F' + \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 F + \alpha_5(E_5 + \cdots + E_{10})))$, where $g := p_a(\alpha_0 F' + \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 E_4 + \alpha_5(E_5 + \cdots + E_{10}))$ and $\phi := \phi(\alpha_0 F' + \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 E_4 + \alpha_5(E_5 + \cdots + E_{10}))$. Similarly we have a map

$$h_{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5}^{0'} : \mathcal{G}_{02} \dashrightarrow \mathcal{E}_{g, \phi}$$

sending (F, F', Σ) to $(S, \mathcal{O}_S(\alpha_0 F' + \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 F + \alpha_5(E_5 + \cdots + E_{10})) + K_S)$.

Our main results, Theorem 1.1 and Theorem 1.2, are, respectively, immediate consequences of the following two propositions.

Proposition 4.5. *Let $i \in \{0, 1, 2\}$ and $\alpha_1, \dots, \alpha_4 \in \mathbb{N}$, at least two nonzero. The map $f_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}^i$ (respectively, $f_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}^{i'}$) is dominant onto the locus of pairs $(S, H) \in \mathcal{E}_{g, \phi}$ admitting the same simple decomposition type as $\alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 F$ (resp., $\alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 F + K_S$).*

Let $i \in \{0, 1, 2\}$ and $\alpha_1, \dots, \alpha_5 \in \mathbb{N}$, at least two nonzero. The map $f_{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5}^{0i}$ (respectively, $f_{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5}^{0i'}$) is dominant onto the locus of pairs $(S, H) \in \mathcal{E}_{g, \phi}$ admitting the same simple decomposition type as $\alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 F + \alpha_5 F'$ (resp., $\alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 F + \alpha_5 F' + K_S$).

Proposition 4.6. *Let $\alpha_0, \dots, \alpha_4 \in \mathbb{N}$, with at least one among $\alpha_0, \dots, \alpha_3$ nonzero. The map $h_{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4}$ (respectively, $h'_{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4}$) is dominant onto the locus of pairs $(S, H) \in \mathcal{E}_{g, \phi}$ admitting the same simple decomposition type as $\alpha_0 F + \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4(E_4 + \cdots + E_{10})$ (resp., $\alpha_0 F + \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4(E_4 + \cdots + E_{10}) + K_S$).*

Let $\alpha_0, \dots, \alpha_5 \in \mathbb{N}$, with at least one among $\alpha_0, \dots, \alpha_4$ nonzero. The map $h_{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5}^0$ (respectively, $h_{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5}^{0'}$) is dominant onto the locus of pairs $(S, H) \in \mathcal{E}_{g, \phi}$ admitting the same simple decomposition type as $\alpha_0 F' + \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 F + \alpha_5(E_5 + \cdots + E_{10})$ (resp., $\alpha_0 F' + \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 F + \alpha_5(E_5 + \cdots + E_{10}) + K_S$).

The proofs of Propositions 4.5 and 4.6 require the results of Section 3 to make sure we have enough isotropic divisors in the decompositions of H to map S to an Enriques sextic in the appropriate way. For instance, if $H \equiv \alpha_1 E_1 + \alpha_2 E_{12}$, one writes $H \equiv \alpha_1 E_1 + \alpha_2 E_{12} + 0E_2 + 0E_3$ so that $E_1 + E_2 + E_3$ defines a mapping of S to an Enriques sextic (following Notation 2.3 everywhere).

We use the following definition in the proofs of Propositions 4.5 and 4.6.

Definition 4.7. *Given an isotropic 3-sequence $\mathfrak{J} = \{E_1, E_2, E_3\}$ on the Enriques surface S , the set $\tilde{\mathcal{F}}_i(\mathfrak{J})$, for $i = 0, 1, 2$, will denote the set of all primitive, isotropic divisors F on S satisfying*

$$(F \cdot E_1, F \cdot E_2, F \cdot E_3) = \begin{cases} (1, 1, 1) & \text{if } F \in \tilde{\mathcal{F}}_0(\mathfrak{J}), \\ (2, 1, 1) & \text{if } F \in \tilde{\mathcal{F}}_1(\mathfrak{J}), \\ (2, 2, 1) & \text{if } F \in \tilde{\mathcal{F}}_2(\mathfrak{J}) \end{cases}$$

and the set $\tilde{\mathcal{F}}_{0i}(\mathcal{J})$, for $i = 0, 1, 2$, will denote the set of all pairs (F, F') of primitive, isotropic divisors F, F' on S such that $F \in \tilde{\mathcal{F}}_0(\mathcal{J})$ and

- $F' \in \tilde{\mathcal{F}}_0(\mathcal{J})$ and $F \cdot F' = 1$, if $i = 0$,
- $F' \in \tilde{\mathcal{F}}_1(\mathcal{J})$ and $F \cdot F' = 1$, if $i = 1$,
- $F' \in \tilde{\mathcal{F}}_1(\mathcal{J})$ and $F \cdot F' = 2$, if $i = 2$.

Proof of Proposition 4.5. Let (S, H) be as in either of the statements of the proposition. In particular, H admits a simple decomposition type of length n , with $2 \leq n \leq 5$. By Proposition 3.2, if $n \leq 4$, we may write $H \equiv \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 F$ with $\mathcal{J} = \{E_1, E_2, E_3\}$ an isotropic 3-sequence and $F \in \tilde{\mathcal{F}}_i(\mathcal{J})$, possibly allowing some of the α_i s to be 0. If $n = 5$, we may write $H \equiv \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 F + \alpha_5 F'$ with $(F, F') \in \tilde{\mathcal{F}}_{0i}(\mathcal{J})$. We may assume (S, H) to be general, in particular, S is unnodal. Then by [4, Thm. 4.6.3 and 4.7.2] the complete linear system $|E_1 + E_2 + E_3|$ maps S birationally onto an Enriques sextic in \mathbb{P}^3 , with double lines along the edges of the tetrahedron T defined by the images of all E_i and $E'_i := E_i + K_S$. Under this map, F (respectively, (F, F')) is mapped to an element of \mathcal{F}_i (resp., \mathcal{F}_{0i}), finishing the proof. \square

Proof of Proposition 4.6. To prove the surjectivity of $h_{\alpha_0, \dots, \alpha_4}$, assume (S, H) admits the given simple decomposition type as in the statement. We may assume that $\alpha_4 > 0$, otherwise the result follows from Proposition 4.5. By Corollary 3.3, we may always write $H \sim \alpha_0 E_{1,2} + \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 (E_4 + \dots + E_{10})$, possibly allowing more than one among $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ to be zero. Since $E_{1,2} \in \tilde{\mathcal{F}}_2(E_1, E_2, E_3)$, the result follows as in the proof of Proposition 4.5. The proof of the surjectivity of $h'_{\alpha_0, \dots, \alpha_4}$ is identical.

To prove the surjectivity of $h_{\alpha_0, \dots, \alpha_5}^0$, assume (S, H) admits the given simple decomposition type as in the statement. We may again assume that $\alpha_5 > 0$. By Corollary 3.3, we may always write $H \sim \alpha_0 E_{1,4} + \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 E_4 + \alpha_5 (E_5 + \dots + E_{10})$, possibly allowing more than one among $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ to be zero. Then $(E_4, E_{1,4}) \in \tilde{\mathcal{F}}_{02}(E_1, E_2, E_3)$ and the result follows as in the proof of Proposition 4.5. The proof of the surjectivity of $h_{\alpha_0, \dots, \alpha_5}^{0'}$ is identical. \square

Next we prove the:

Lemma 4.8. *Let \mathcal{C} be an irreducible component of $\mathcal{E}_{g,\phi}$, such that (S, H) and (S', H') both lie in \mathcal{C} if and only if they admit the same simple decomposition type as in (4). If some of the a_i s are odd, then in (4) one may always assume $\varepsilon = 0$, so that both pairs (S, H) and $(S, H + K_S)$ lie in \mathcal{C} .*

If all the a_i s are even, then there is a different irreducible component \mathcal{C}' of $\mathcal{E}_{g,\phi}$ such that (S, H) lies in \mathcal{C}' if and only if it admits the simple decomposition type $H \sim a_1 E_1 + \dots + a_n E_n + (1 - \varepsilon) K_S$.

Proof. The first assertion is trivial. So assume that all the a_i s are even. Assume in (4) one has $\varepsilon = 0$. This means that H is divisible by 2 in $\text{Pic}(S)$. The pairs $(S, H + K_S)$ do fill up an irreducible component \mathcal{C}' of $\mathcal{E}_{g,\phi}$, and we claim that it is different from \mathcal{C} . Indeed if $\mathcal{C} = \mathcal{C}'$, given a general Enriques surface S , we would have a relation of the form $H \sim H' + K_S$, with H and H' divisible by 2 in $\text{Pic}(S)$. This is impossible, since K_S is not divisible by 2 in $\text{Pic}(S)$. The proof is similar if $\varepsilon = 1$. \square

We finally give the proofs of the four corollaries in the introduction.

Proof of Corollary 1.3. By Lemma 2.5, all cases with $\phi \leq 4$ admit simple decomposition types of length $n \leq 4$, except for the decomposition type $\frac{g-7}{4}E_1 + E_2 + E_3 + E_4 + E_5$. The result thus follows from Theorem 1.1. \square

Proof of Corollary 1.4. Since $g \leq 20$, we have $H^2 \leq 38$, whence $\phi \leq 6$ by (5), with equality $\phi = 6$ possible only for $H^2 = 36$ by Proposition 2.4, in which case the simple decomposition type has length 2. Thus the result follows from Theorem 1.1 in this case.

We have left to treat the cases where $\phi \leq 5$. By Lemma 2.5, all cases with $\phi \leq 5$ and $g \leq 20$ have decomposition types of length $n \leq 5$, except for the type $E_1 + E_2 + E_3 + E_4 + E_5 + E_6$ for $(g, \phi) = (16, 5)$, which is the only type occurring for these values of g and ϕ . Hence $\mathcal{E}_{16,5}$ is irreducible and uniruled by Theorem 1.2. Again by Lemma 2.5, all remaining cases with $\phi \leq 5$ and $g \leq 20$ admit simple decomposition types of length $n \leq 4$ or of length 5 with all nonzero intersections occurring equal to one, except for the type $2E_1 + E_2 + E_3 + E_4 + E_{1,5}$ for $(g, \phi) = (17, 5)$, which is the only type occurring for these values of g and ϕ . Hence $\mathcal{E}_{17,5}$ is irreducible and uniruled and all irreducible components of the remaining $\mathcal{E}_{g,\phi}$ are unirational by Theorem 1.1. \square

Proof of Corollary 1.5. When $g = \frac{\phi^2}{2} + 1$, equivalently $H^2 = \phi^2$, then Proposition 2.4, Theorem 1.1 and Lemma 4.8 yield that, when $\frac{\phi}{2}$ is even, i.e., $\phi \equiv 0 \pmod{4}$ (respectively, when $\frac{\phi}{2}$ is odd, i.e., $\phi \equiv 2 \pmod{4}$) then $\mathcal{E}_{\frac{\phi^2}{2}+1,\phi}$ has two irreducible, unirational components (resp. only one irreducible, unirational component), corresponding to the simple decomposition types $\frac{\phi}{2}(E_1 + E_{1,2})$ and $\frac{\phi}{2}(E_1 + E_{1,2}) + K_S$ (resp. $\frac{\phi}{2}(E_1 + E_{1,2})$).

When $g = \frac{\phi(\phi+1)}{2}$, Proposition 2.4 yields that there is a unique simple decomposition type, of length 3, for each ϕ , except for $\phi = 6$, where there are three possible types

$$2E_1 + 3E_{1,2} + E_2, \quad 2(E_1 + E_2 + E_{1,2}), \quad 2(E_1 + E_2 + E_{1,2}) + K_S,$$

The result follows from Theorem 1.1 and Lemma 4.8. \square

Before proving Corollary 1.6, we need an example and an auxiliary result.

Example 4.9. Let S be an Enriques surface and H a line bundle with a simple isotropic decomposition type of the form

$$H \equiv a_0E_{1,2} + a_1E_1 + a_3E_3 + \cdots + a_{10}E_{10},$$

where a_1 is an even nonnegative integer, and a_0, a_3, \dots, a_{10} are odd positive integers. Then H is numerically 2-divisible, that is, its class in $\text{Num}(S)$ is 2-divisible. Indeed, the claim is equivalent to $B := E_{1,2} + E_3 + \cdots + E_{10}$ being numerically 2-divisible. As

$$\begin{aligned} B &\equiv 3(E_{1,2} + E_1 + E_2) + (E_1 + E_2 + E_3 + \cdots + E_{10}) - 2E_{1,2} - 4E_1 - 4E_2 \\ &\equiv 2(E_1 + \cdots + E_{10}) - 2E_{1,2} - 4E_1 - 4E_2, \end{aligned}$$

using Lemma 2.2(a), the claim follows.

Lemma 4.10. *A line bundle H on an Enriques surface is numerically 2-divisible if and only if either all coefficients in any simple isotropic decomposition in $\text{Num}(S)$ are even or H is as in Example 4.9.*

Proof. The if part is clear. To prove the converse, assume that H is numerically 2-divisible and

$$H \equiv a_0E_{1,2} + a_1E_1 + a_2E_2 + a_3E_3 + \cdots + a_{10}E_{10},$$

by Corollary 3.3, where the a_i s are nonnegative integers, at least one being zero. By symmetry, we may consider the three cases $a_0 = 0$, $a_2 = 0$ and $a_{10} = 0$. We let $E_{i,j}$ be defined as in (6).

Assume $a_0 = 0$. Since $(E_{i,j} - E_i) \cdot H = 2a_i + a_j$, for $i \neq j$, and H is numerically 2-divisible, we must have all a_j even, as desired.

Assume $a_{10} = 0$. For $i = 1, 2$, we have $(E_{i,10} - E_{10}) \cdot H = a_i$, hence a_1 and a_2 are even. For $i \geq 3$, we have $(E_i - E_{10}) \cdot H = -a_i$, hence also a_i for $i \geq 3$ must be even. Moreover $E_3 \cdot H = a_0 + a_1 + a_2 + a_4 + \cdots + a_9$, and since a_1, \dots, a_9 are all even, also a_0 is even, as desired.

Assume $a_2 = 0$. Since $(E_2 - E_1) \cdot H = a_1$, we have that a_1 is even. For $i \neq j$ and $i, j \geq 3$, we have $(E_j - E_i) \cdot H = a_i - a_j$. Hence $a_i + a_j$ is even for all $i, j \geq 3$. For $j \geq 3$, we have $(E_{1,2} - E_{2,j}) \cdot H = -a_0 - a_j + a_1$. Hence $a_0 + a_j$ is even for all $j \geq 3$. It follows that all a_0, a_3, \dots, a_{10} have the same parity. Thus, either all coefficients are even, or H is as in Example 4.9, as claimed. \square

Proof of Corollary 1.6. If $[H] \in \text{Num}(S)$ is not 2-divisible or as in Example 4.9, then some simple decomposition types of H and $H + K_S$ have not all even coefficients in front of the isotropic, primitive summands. Hence, by substituting one E_i with odd coefficient with $E_i + K_S$, we see that H and $H + K_S$ admit the same simple decomposition type, and thus belong to the same irreducible component of $\mathcal{E}_{g,\phi}$, by Theorems 1.1 or 1.2 and Lemma 4.8, by the assumption on the decomposition types. Hence $\rho^{-1}(\rho(\mathcal{C}))$ is irreducible.

Conversely, assume $[H] \in \text{Num}(S)$ is 2-divisible and not as in Example 4.9 and with a simple decomposition type of length at most 5. By Corollary 3.3 there is an isotropic 10-sequence $\{E_1, \dots, E_{10}\}$ so that

$$H \equiv a_0 E_{1,2} + a_1 E_1 + a_2 E_2 + a_3 E_3 + a_4 E_4 + a_5 E_5,$$

where at least one of a_0, \dots, a_5 is zero. There may be more such sequences, but by Lemma 4.10, for any such, all coefficients a_i are even. Thus, by Lemma 4.8, H and $H + K_S$ do not admit the same simple decomposition type and therefore do not lie in the same irreducible component of $\mathcal{E}_{g,\phi}$, whence $\rho^{-1}(\rho(\mathcal{C}))$ consists of two disjoint components.

The argument is the same if H admits a simple decomposition type that is 6-symmetric. \square

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APPENDIX: IRREDUCIBLE COMPONENTS OF $\overline{\mathcal{E}}_{g,\phi}$ AND $\mathcal{E}_{g,\phi}$ FOR $g \leq 30$

Using Proposition 3.4 (and Notation 2.3) we list all irreducible components of the moduli spaces $\overline{\mathcal{E}}_{g,\phi}$ for $g \leq 30$, marking them with roman numbers, and describe the properties of ρ^{-1} of these components obtained by Theorems 1.1 and 1.2 and Corollary 1.6. We thus obtain information about all irreducible components of the moduli spaces $\mathcal{E}_{g,\phi}$, with few exceptions. The various decomposition types can be obtained from Lemma 2.5 and Proposition 2.4, and an ad hoc treatment as in the proof of Lemma 2.5 for the cases $\phi = 6$ and 7. The fact that all decomposition types below are truly different can be checked by computing suitable intersections as in the proof of Lemma 2.5, and the fact that they all do exist on any Enriques surface follows from Lemma 2.2(a).

| g | ϕ | comp. | dec. type | ρ^{-1} |
|-----|--------|--|-------------------------------|--------------------------|
| 2 | 1 | $\overline{\mathcal{E}}_{2,1}$ | $E_1 + E_2$ | irred. unirat. |
| 3 | 1 | $\overline{\mathcal{E}}_{3,1}$ | $2E_1 + E_2$ | irred. unirat. |
| 3 | 2 | $\overline{\mathcal{E}}_{3,2}$ | $E_1 + E_{1,2}$ | irred. rational [2] |
| 4 | 1 | $\overline{\mathcal{E}}_{4,1}$ | $3E_1 + E_2$ | irred. unirat. |
| 4 | 2 | $\overline{\mathcal{E}}_{4,2}$ | $E_1 + E_2 + E_3$ | irred. rational, [5, §3] |
| 5 | 1 | $\overline{\mathcal{E}}_{5,1}$ | $4E_1 + E_2$ | irred. unirat. |
| 5 | 2 | $\overline{\mathcal{E}}_{5,2}^{(I)}$ | $2E_1 + E_{1,2}$ | irred. unirat. |
| 5 | 2 | $\overline{\mathcal{E}}_{5,2}^{(II)}$ | $2(E_1 + E_2)$ | two unirat. components |
| 6 | 1 | $\overline{\mathcal{E}}_{6,1}$ | $5E_1 + E_2$ | irred. unirat. |
| 6 | 2 | $\overline{\mathcal{E}}_{6,2}$ | $2E_1 + E_2 + E_3$ | irred. unirat. |
| 6 | 3 | $\overline{\mathcal{E}}_{6,3}$ | $E_1 + E_2 + E_{1,2}$ | irred. unirat. [11] |
| 7 | 1 | $\overline{\mathcal{E}}_{7,1}$ | $6E_1 + E_2$ | irred. unirat. |
| 7 | 2 | $\overline{\mathcal{E}}_{7,2}^{(I)}$ | $3E_1 + E_{1,2}$ | irred. unirat. |
| 7 | 2 | $\overline{\mathcal{E}}_{7,2}^{(II)}$ | $3E_1 + 2E_2$ | irred. unirat. |
| 7 | 3 | $\overline{\mathcal{E}}_{7,3}$ | $E_1 + E_2 + E_3 + E_4$ | irred. unirat. |
| 8 | 1 | $\overline{\mathcal{E}}_{8,1}$ | $7E_1 + E_2$ | irred. unirat. |
| 8 | 2 | $\overline{\mathcal{E}}_{8,2}$ | $3E_1 + E_2 + E_3$ | irred. unirat. |
| 8 | 3 | $\overline{\mathcal{E}}_{8,3}$ | $2E_1 + E_2 + E_{1,3}$ | irred. unirat. |
| 9 | 1 | $\overline{\mathcal{E}}_{9,1}$ | $8E_1 + E_2$ | irred. unirat. |
| 9 | 2 | $\overline{\mathcal{E}}_{9,2}^{(I)}$ | $4E_1 + E_{1,2}$ | irred. unirat. |
| 9 | 2 | $\overline{\mathcal{E}}_{9,2}^{(II)}$ | $2(2E_1 + E_2)$ | two unirat. components |
| 9 | 3 | $\overline{\mathcal{E}}_{9,3}^{(I)}$ | $2E_1 + E_2 + E_{1,2}$ | irred. unirat. |
| 9 | 3 | $\overline{\mathcal{E}}_{9,3}^{(II)}$ | $2E_1 + 2E_2 + E_3$ | irred. unirat. |
| 9 | 4 | $\overline{\mathcal{E}}_{9,4}$ | $2(E_1 + E_{1,2})$ | two unirat. components |
| 10 | 1 | $\overline{\mathcal{E}}_{10,1}$ | $9E_1 + E_2$ | irred. unirat. |
| 10 | 2 | $\overline{\mathcal{E}}_{10,2}$ | $4E_1 + E_2 + E_3$ | irred. unirat. |
| 10 | 3 | $\overline{\mathcal{E}}_{10,3}^{(I)}$ | $2E_1 + E_2 + E_3 + E_4$ | irred. unirat. |
| 10 | 3 | $\overline{\mathcal{E}}_{10,3}^{(II)}$ | $3(E_1 + E_2)$ | irred. unirat. |
| 10 | 4 | $\overline{\mathcal{E}}_{10,4}$ | $2E_{1,2} + E_1 + E_2$ | irred. unirat. |
| 11 | 1 | $\overline{\mathcal{E}}_{11,1}$ | $10E_1 + E_2$ | irred. unirat. |
| 11 | 2 | $\overline{\mathcal{E}}_{11,2}^{(I)}$ | $5E_1 + E_{1,2}$ | irred. unirat. |
| 11 | 2 | $\overline{\mathcal{E}}_{11,2}^{(II)}$ | $5E_1 + 2E_2$ | irred. unirat. |
| 11 | 3 | $\overline{\mathcal{E}}_{11,3}$ | $3E_1 + E_2 + E_{1,3}$ | irred. unirat. |
| 11 | 4 | $\overline{\mathcal{E}}_{11,4}$ | $E_1 + E_2 + E_3 + E_4 + E_5$ | irred. unirat. |

| g | ϕ | comp. | dec. type | ρ^{-1} |
|-----|--------|------------------------------------|-------------------------------------|------------------------|
| 12 | 1 | $\bar{\mathcal{E}}_{12,1}$ | $11E_1 + E_2$ | irred. unirat. |
| 12 | 2 | $\bar{\mathcal{E}}_{12,2}$ | $5E_1 + E_2 + E_3$ | irred. unirat. |
| 12 | 3 | $\bar{\mathcal{E}}_{12,3}^{(I)}$ | $3E_1 + 2E_2 + E_3$ | irred. unirat. |
| 12 | 3 | $\bar{\mathcal{E}}_{12,3}^{(II)}$ | $3E_1 + E_2 + E_{1,2}$ | irred. unirat. |
| 12 | 4 | $\bar{\mathcal{E}}_{12,4}$ | $2E_1 + E_2 + E_3 + E_{1,4}$ | irred. unirat. |
| 13 | 1 | $\bar{\mathcal{E}}_{13,1}$ | $12E_1 + E_2$ | irred. unirat. |
| 13 | 2 | $\bar{\mathcal{E}}_{13,2}^{(I)}$ | $6E_1 + E_{1,2}$ | irred. unirat. |
| 13 | 2 | $\bar{\mathcal{E}}_{13,2}^{(II)}$ | $2(3E_1 + E_2)$ | two unirat. components |
| 13 | 3 | $\bar{\mathcal{E}}_{13,3}^{(I)}$ | $3E_1 + E_2 + E_3 + E_4$ | irred. unirat. |
| 13 | 3 | $\bar{\mathcal{E}}_{13,3}^{(II)}$ | $4E_1 + 3E_2$ | irred. unirat. |
| 13 | 4 | $\bar{\mathcal{E}}_{13,4}^{(I)}$ | $2E_1 + 2E_2 + E_{1,2}$ | irred. unirat. |
| 13 | 4 | $\bar{\mathcal{E}}_{13,4}^{(II)}$ | $2(E_1 + E_2 + E_3)$ | two unirat. components |
| 13 | 4 | $\bar{\mathcal{E}}_{13,4}^{(III)}$ | $3E_1 + 2E_{1,2}$ | irred. unirat. |
| 14 | 1 | $\bar{\mathcal{E}}_{14,1}$ | $13E_1 + E_2$ | irred. unirat. |
| 14 | 2 | $\bar{\mathcal{E}}_{14,2}$ | $6E_1 + E_2 + E_3$ | irred. unirat. |
| 14 | 3 | $\bar{\mathcal{E}}_{14,3}$ | $4E_1 + E_2 + E_{1,3}$ | irred. unirat. |
| 14 | 4 | $\bar{\mathcal{E}}_{14,4}^{(I)}$ | $2E_1 + 2E_2 + E_3 + E_4$ | irred. unirat. |
| 14 | 4 | $\bar{\mathcal{E}}_{14,4}^{(II)}$ | $3E_{1,2} + E_1 + E_2$ | irred. unirat. |
| 15 | 1 | $\bar{\mathcal{E}}_{15,1}$ | $14E_1 + E_2$ | irred. unirat. |
| 15 | 2 | $\bar{\mathcal{E}}_{15,2}^{(I)}$ | $7E_1 + E_{1,2}$ | irred. unirat. |
| 15 | 2 | $\bar{\mathcal{E}}_{15,2}^{(II)}$ | $7E_1 + 2E_2$ | irred. unirat. |
| 15 | 3 | $\bar{\mathcal{E}}_{15,3}^{(I)}$ | $4E_1 + 2E_2 + E_3$ | irred. unirat. |
| 15 | 3 | $\bar{\mathcal{E}}_{15,3}^{(II)}$ | $4E_1 + E_2 + E_{1,2}$ | irred. unirat. |
| 15 | 4 | $\bar{\mathcal{E}}_{15,4}^{(I)}$ | $2E_1 + E_2 + E_3 + E_4 + E_5$ | irred. unirat. |
| 15 | 4 | $\bar{\mathcal{E}}_{15,4}^{(II)}$ | $3E_1 + 2E_2 + E_{1,3}$ | irred. unirat. |
| 15 | 5 | $\bar{\mathcal{E}}_{15,5}$ | $2E_1 + E_2 + 2E_{1,2}$ | irred. unirat. |
| 16 | 1 | $\bar{\mathcal{E}}_{16,1}$ | $15E_1 + E_2$ | irred. unirat. |
| 16 | 2 | $\bar{\mathcal{E}}_{16,2}$ | $7E_1 + E_2 + E_3$ | irred. unirat. |
| 16 | 3 | $\bar{\mathcal{E}}_{16,3}^{(I)}$ | $4E_1 + E_2 + E_3 + E_4$ | irred. unirat. |
| 16 | 3 | $\bar{\mathcal{E}}_{16,3}^{(II)}$ | $5E_1 + 3E_2$ | irred. unirat. |
| 16 | 4 | $\bar{\mathcal{E}}_{16,4}^{(I)}$ | $3E_1 + 3E_2 + E_3$ | irred. unirat. |
| 16 | 4 | $\bar{\mathcal{E}}_{16,4}^{(II)}$ | $3E_1 + E_2 + E_3 + E_{1,4}$ | irred. unirat. |
| 16 | 5 | $\bar{\mathcal{E}}_{16,5}$ | $E_1 + E_2 + E_3 + E_4 + E_5 + E_6$ | irred. uniruled |
| 17 | 1 | $\bar{\mathcal{E}}_{17,1}$ | $16E_1 + E_2$ | irred. unirat. |
| 17 | 2 | $\bar{\mathcal{E}}_{17,2}^{(I)}$ | $8E_1 + E_{1,2}$ | irred. unirat. |
| 17 | 2 | $\bar{\mathcal{E}}_{17,2}^{(II)}$ | $2(4E_1 + E_2)$ | two unirat. components |
| 17 | 3 | $\bar{\mathcal{E}}_{17,3}$ | $5E_1 + E_2 + E_{1,3}$ | irred. unirat. |
| 17 | 4 | $\bar{\mathcal{E}}_{17,4}^{(I)}$ | $3E_1 + 2E_2 + 2E_3$ | irred. unirat. |
| 17 | 4 | $\bar{\mathcal{E}}_{17,4}^{(II)}$ | $3E_1 + 2E_2 + E_{1,2}$ | irred. unirat. |
| 17 | 4 | $\bar{\mathcal{E}}_{17,4}^{(III)}$ | $2(2E_1 + E_{1,2})$ | two unirat. components |
| 17 | 4 | $\bar{\mathcal{E}}_{17,4}^{(IV)}$ | $4(E_1 + E_2)$ | two unirat. components |
| 17 | 5 | $\bar{\mathcal{E}}_{17,5}$ | $2E_1 + E_2 + E_3 + E_4 + E_{1,5}$ | irred. uniruled |

| g | ϕ | comp. | dec. type | ρ^{-1} |
|-----|--------|------------------------------------|---|------------------------|
| 18 | 1 | $\bar{\mathcal{E}}_{18,1}$ | $17E_1 + E_2$ | irred. unirat. |
| 18 | 2 | $\bar{\mathcal{E}}_{18,2}$ | $8E_1 + E_2 + E_3$ | irred. unirat. |
| 18 | 3 | $\bar{\mathcal{E}}_{18,3}^{(I)}$ | $5E_1 + 2E_2 + E_3$ | irred. unirat. |
| 18 | 3 | $\bar{\mathcal{E}}_{18,3}^{(II)}$ | $5E_1 + E_2 + E_{1,2}$ | irred. unirat. |
| 18 | 4 | $\bar{\mathcal{E}}_{18,4}^{(I)}$ | $3E_1 + 2E_2 + E_3 + E_4$ | irred. unirat. |
| 18 | 4 | $\bar{\mathcal{E}}_{18,4}^{(II)}$ | $4E_{1,2} + E_1 + E_2$ | irred. unirat. |
| 18 | 5 | $\bar{\mathcal{E}}_{18,5}^{(I)}$ | $3E_1 + E_2 + 2E_{1,3}$ | irred. unirat. |
| 18 | 5 | $\bar{\mathcal{E}}_{18,5}^{(II)}$ | $2E_1 + 2E_2 + E_3 + E_{1,2}$ | irred. unirat. |
| 19 | 1 | $\bar{\mathcal{E}}_{19,1}$ | $18E_1 + E_2$ | irred. unirat. |
| 19 | 2 | $\bar{\mathcal{E}}_{19,2}^{(I)}$ | $9E_1 + E_{1,2}$ | irred. unirat. |
| 19 | 2 | $\bar{\mathcal{E}}_{19,2}^{(II)}$ | $9E_1 + 2E_2$ | irred. unirat. |
| 19 | 3 | $\bar{\mathcal{E}}_{19,3}^{(I)}$ | $5E_1 + E_2 + E_3 + E_4$ | irred. unirat. |
| 19 | 3 | $\bar{\mathcal{E}}_{19,3}^{(II)}$ | $3(2E_1 + E_2)$ | irred. unirat. |
| 19 | 4 | $\bar{\mathcal{E}}_{19,4}^{(I)}$ | $3E_1 + E_2 + E_3 + E_4 + E_5$ | irred. unirat. |
| 19 | 4 | $\bar{\mathcal{E}}_{19,4}^{(II)}$ | $4E_1 + 2E_2 + E_{1,3}$ | irred. unirat. |
| 19 | 5 | $\bar{\mathcal{E}}_{19,5}^{(I)}$ | $2E_1 + 2E_2 + 2E_3 + E_4$ | irred. unirat. |
| 19 | 5 | $\bar{\mathcal{E}}_{19,5}^{(II)}$ | $3E_{1,2} + E_1 + E_2 + E_3$ | irred. unirat. |
| 19 | 6 | $\bar{\mathcal{E}}_{19,6}$ | $3(E_1 + E_{1,2})$ | irred. unirat. |
| 20 | 1 | $\bar{\mathcal{E}}_{20,1}$ | $19E_1 + E_2$ | irred. unirat. |
| 20 | 2 | $\bar{\mathcal{E}}_{20,2}$ | $9E_1 + E_2 + E_3$ | irred. unirat. |
| 20 | 3 | $\bar{\mathcal{E}}_{20,3}$ | $6E_1 + E_2 + E_{1,3}$ | irred. unirat. |
| 20 | 4 | $\bar{\mathcal{E}}_{20,4}^{(I)}$ | $4E_1 + 3E_2 + E_3$ | irred. unirat. |
| 20 | 4 | $\bar{\mathcal{E}}_{20,4}^{(II)}$ | $4E_1 + E_2 + E_3 + E_{1,4}$ | irred. unirat. |
| 20 | 5 | $\bar{\mathcal{E}}_{20,5}^{(I)}$ | $2E_1 + 2E_2 + E_3 + E_4 + E_5$ | irred. unirat. |
| 20 | 5 | $\bar{\mathcal{E}}_{20,5}^{(II)}$ | $3E_1 + E_2 + 2E_{1,2}$ | irred. unirat. |
| 21 | 1 | $\bar{\mathcal{E}}_{21,1}$ | $20E_1 + E_2$ | irred. unirat. |
| 21 | 2 | $\bar{\mathcal{E}}_{21,2}^{(I)}$ | $10E_1 + E_{1,2}$ | irred. unirat. |
| 21 | 2 | $\bar{\mathcal{E}}_{21,2}^{(II)}$ | $10E_1 + 2E_2$ | two unirat. components |
| 21 | 3 | $\bar{\mathcal{E}}_{21,3}^{(I)}$ | $6E_1 + E_2 + E_{1,2}$ | irred. unirat. |
| 21 | 3 | $\bar{\mathcal{E}}_{21,3}^{(II)}$ | $6E_1 + 2E_2 + E_3$ | irred. unirat. |
| 21 | 4 | $\bar{\mathcal{E}}_{21,4}^{(I)}$ | $5E_1 + 4E_2$ | irred. unirat. |
| 21 | 4 | $\bar{\mathcal{E}}_{21,4}^{(II)}$ | $5E_1 + 2E_{1,2}$ | irred. unirat. |
| 21 | 4 | $\bar{\mathcal{E}}_{21,4}^{(III)}$ | $4E_1 + 2E_2 + 2E_3$ | two unirat. components |
| 21 | 4 | $\bar{\mathcal{E}}_{21,4}^{(IV)}$ | $4E_1 + 2E_2 + E_{1,2}$ | irred. unirat. |
| 21 | 5 | $\bar{\mathcal{E}}_{21,5}^{(I)}$ | $2E_1 + E_2 + E_3 + E_4 + E_5 + E_6$ | ?? |
| 21 | 5 | $\bar{\mathcal{E}}_{21,5}^{(II)}$ | $3E_1 + 2E_2 + E_3 + E_{1,4}$ | irred. unirat. |
| 21 | 6 | $\bar{\mathcal{E}}_{21,6}$ | $2(E_1 + E_2 + E_{1,2})$ | two unirat. components |
| 22 | 1 | $\bar{\mathcal{E}}_{22,1}$ | $21E_1 + E_2$ | irred. unirat. |
| 22 | 2 | $\bar{\mathcal{E}}_{22,2}$ | $10E_1 + E_2 + E_3$ | irred. unirat. |
| 22 | 3 | $\bar{\mathcal{E}}_{22,3}^{(I)}$ | $6E_1 + E_2 + E_3 + E_4$ | irred. unirat. |
| 22 | 3 | $\bar{\mathcal{E}}_{22,3}^{(II)}$ | $7E_1 + 3E_2$ | irred. unirat. |
| 22 | 4 | $\bar{\mathcal{E}}_{22,4}^{(I)}$ | $4E_1 + 2E_2 + E_3 + E_4$ | irred. unirat. |
| 22 | 4 | $\bar{\mathcal{E}}_{22,4}^{(II)}$ | $5E_{1,2} + E_1 + E_2$ | irred. unirat. |
| 22 | 5 | $\bar{\mathcal{E}}_{22,5}^{(I)}$ | $3E_1 + E_2 + E_3 + E_4 + E_{1,5}$ | irred. uniruled |
| 22 | 5 | $\bar{\mathcal{E}}_{22,5}^{(II)}$ | $3E_1 + 3E_2 + E_{1,2}$ | irred. unirat. |
| 22 | 5 | $\bar{\mathcal{E}}_{22,5}^{(III)}$ | $3E_1 + 3E_2 + 2E_3$ | irred. unirat. |
| 22 | 6 | $\bar{\mathcal{E}}_{22,6}$ | $E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7$ | irred. unirat. |

| g | ϕ | comp. | dec. type | ρ^{-1} |
|-----|--------|---|--|--------------------------|
| 23 | 1 | $\overline{\mathcal{E}}_{23,1}$ | $22E_1 + E_2$ | irred. unirated. |
| 23 | 2 | $\overline{\mathcal{E}}_{23,2}^{(I)}$ | $11E_1 + E_{1,2}$ | irred. unirated. |
| 23 | 2 | $\overline{\mathcal{E}}_{23,2}^{(II)}$ | $11E_1 + 2E_2$ | irred. unirated. |
| 23 | 3 | $\overline{\mathcal{E}}_{23,3}$ | $7E_1 + E_2 + E_{1,3}$ | irred. unirated. |
| 23 | 4 | $\overline{\mathcal{E}}_{23,4}^{(I)}$ | $5E_1 + 2E_2 + E_{1,3}$ | irred. unirated. |
| 23 | 4 | $\overline{\mathcal{E}}_{23,4}^{(II)}$ | $4E_1 + E_2 + E_3 + E_4 + E_5$ | irred. unirated. |
| 23 | 5 | $\overline{\mathcal{E}}_{23,5}^{(I)}$ | $4E_1 + E_2 + 2E_{1,3}$ | irred. unirated. |
| 23 | 5 | $\overline{\mathcal{E}}_{23,5}^{(II)}$ | $3E_1 + 2E_2 + E_3 + E_{1,2}$ | irred. unirated. |
| 23 | 5 | $\overline{\mathcal{E}}_{23,5}^{(III)}$ | $3E_1 + 3E_2 + E_3 + E_4$ | irred. unirated. |
| 23 | 6 | $\overline{\mathcal{E}}_{23,6}$ | $2E_1 + E_2 + E_3 + E_4 + E_5 + E_{1,6}$ | ?? |
| 24 | 1 | $\overline{\mathcal{E}}_{24,1}$ | $23E_1 + E_2$ | irred. unirated. |
| 24 | 2 | $\overline{\mathcal{E}}_{24,2}$ | $11E_1 + E_2 + E_3$ | irred. unirated. |
| 24 | 3 | $\overline{\mathcal{E}}_{24,3}^{(I)}$ | $7E_1 + E_2 + E_{1,2}$ | irred. unirated. |
| 24 | 3 | $\overline{\mathcal{E}}_{24,3}^{(II)}$ | $7E_1 + 2E_2 + E_3$ | irred. unirated. |
| 24 | 4 | $\overline{\mathcal{E}}_{24,4}^{(I)}$ | $5E_1 + 3E_2 + E_3$ | irred. unirated. |
| 24 | 4 | $\overline{\mathcal{E}}_{24,4}^{(II)}$ | $5E_1 + E_2 + E_3 + E_{1,4}$ | irred. unirated. |
| 24 | 5 | $\overline{\mathcal{E}}_{24,5}^{(I)}$ | $3E_1 + 2E_2 + 2E_3 + E_4$ | irred. uniruled |
| 24 | 5 | $\overline{\mathcal{E}}_{24,5}^{(II)}$ | $4E_{1,2} + E_1 + E_2 + E_3$ | irred. unirated. |
| 24 | 5 | $\overline{\mathcal{E}}_{24,5}^{(III)}$ | $4E_1 + 3E_2 + E_{1,3}$ | irred. unirated. |
| 24 | 6 | $\overline{\mathcal{E}}_{24,6}^{(I)}$ | $3E_1 + E_2 + E_3 + 2E_{1,4}$ | irred. unirated. |
| 24 | 6 | $\overline{\mathcal{E}}_{24,6}^{(II)}$ | $2E_1 + 2E_2 + E_3 + E_4 + E_{1,2}$ | irred. uniruled |
| 25 | 1 | $\overline{\mathcal{E}}_{25,1}$ | $24E_1 + E_2$ | irred. unirated. |
| 25 | 2 | $\overline{\mathcal{E}}_{25,2}^{(I)}$ | $12E_1 + E_{1,2}$ | irred. unirated. |
| 25 | 2 | $\overline{\mathcal{E}}_{25,2}^{(II)}$ | $2(6E_1 + E_2)$ | two unirated. components |
| 25 | 3 | $\overline{\mathcal{E}}_{25,3}^{(I)}$ | $7E_1 + E_2 + E_3 + E_4$ | irred. unirated. |
| 25 | 3 | $\overline{\mathcal{E}}_{25,3}^{(II)}$ | $8E_1 + 3E_2$ | irred. unirated. |
| 25 | 4 | $\overline{\mathcal{E}}_{25,4}^{(I)}$ | $2(3E_1 + 2E_2)$ | two unirated. components |
| 25 | 4 | $\overline{\mathcal{E}}_{25,4}^{(II)}$ | $2(3E_1 + E_{1,2})$ | two unirated. components |
| 25 | 4 | $\overline{\mathcal{E}}_{25,4}^{(III)}$ | $5E_1 + 2E_2 + 2E_3$ | irred. unirated. |
| 25 | 4 | $\overline{\mathcal{E}}_{25,4}^{(IV)}$ | $5E_1 + 2E_2 + E_{1,2}$ | irred. unirated. |
| 25 | 5 | $\overline{\mathcal{E}}_{25,5}^{(I)}$ | $4E_1 + E_2 + 2E_{1,2}$ | irred. unirated. |
| 25 | 5 | $\overline{\mathcal{E}}_{25,5}^{(II)}$ | $3E_1 + 2E_2 + E_3 + E_4 + E_5$ | irred. unirated. |
| 25 | 5 | $\overline{\mathcal{E}}_{25,5}^{(III)}$ | $4E_1 + 4E_2 + E_3$ | irred. unirated. |
| 25 | 6 | $\overline{\mathcal{E}}_{25,6}^{(I)}$ | $4E_1 + 3E_{1,2}$ | irred. unirated. |
| 25 | 6 | $\overline{\mathcal{E}}_{25,6}^{(II)}$ | $2(E_1 + E_2 + E_3 + E_4)$ | two unirated. components |
| 25 | 6 | $\overline{\mathcal{E}}_{25,6}^{(III)}$ | $3E_{1,2} + E_1 + E_2 + E_3 + E_4$ | irred. uniruled |

| g | ϕ | comp. | dec. type | ρ^{-1} |
|-----|--------|------------------------------------|--|-----------------|
| 26 | 1 | $\bar{\mathcal{E}}_{26,1}$ | $25E_1 + E_2$ | irred. unirat. |
| 26 | 2 | $\bar{\mathcal{E}}_{26,2}$ | $12E_1 + E_2 + E_3$ | irred. unirat. |
| 26 | 3 | $\bar{\mathcal{E}}_{26,3}$ | $8E_1 + E_2 + E_{1,3}$ | irred. unirat. |
| 26 | 4 | $\bar{\mathcal{E}}_{26,4}^{(I)}$ | $5E_1 + 2E_2 + E_3 + E_4$ | irred. unirat. |
| 26 | 4 | $\bar{\mathcal{E}}_{26,4}^{(II)}$ | $6E_{1,2} + E_1 + E_2$ | irred. unirat. |
| 26 | 5 | $\bar{\mathcal{E}}_{26,5}^{(I)}$ | $3E_1 + E_2 + E_3 + E_4 + E_5 + E_6$ | ?? |
| 26 | 5 | $\bar{\mathcal{E}}_{26,5}^{(II)}$ | $4E_1 + 2E_2 + E_3 + E_{1,4}$ | irred. unirat. |
| 26 | 5 | $\bar{\mathcal{E}}_{26,5}^{(III)}$ | $5(E_1 + E_2)$ | irred. unirat. |
| 26 | 6 | $\bar{\mathcal{E}}_{26,6}^{(I)}$ | $3E_1 + E_2 + E_3 + 2E_{1,2}$ | irred. unirat. |
| 26 | 6 | $\bar{\mathcal{E}}_{26,6}^{(II)}$ | $2E_1 + 2E_2 + 2E_3 + E_4 + E_5$ | irred. unirat. |
| 27 | 1 | $\bar{\mathcal{E}}_{27,1}$ | $26E_1 + E_2$ | irred. unirat. |
| 27 | 2 | $\bar{\mathcal{E}}_{27,2}^{(I)}$ | $13E_1 + E_{1,2}$ | irred. unirat. |
| 27 | 2 | $\bar{\mathcal{E}}_{27,2}^{(II)}$ | $13E_1 + 2E_2$ | irred. unirat. |
| 27 | 3 | $\bar{\mathcal{E}}_{27,3}^{(I)}$ | $8E_1 + E_2 + E_{1,2}$ | irred. unirat. |
| 27 | 3 | $\bar{\mathcal{E}}_{27,3}^{(II)}$ | $8E_1 + 2E_2 + E_3$ | irred. unirat. |
| 27 | 4 | $\bar{\mathcal{E}}_{27,4}^{(I)}$ | $6E_1 + 2E_2 + E_{1,3}$ | irred. unirat. |
| 27 | 4 | $\bar{\mathcal{E}}_{27,4}^{(II)}$ | $5E_1 + E_2 + E_3 + E_4 + E_5$ | irred. unirat. |
| 27 | 5 | $\bar{\mathcal{E}}_{27,5}^{(I)}$ | $4E_1 + E_2 + E_3 + E_4 + E_{1,5}$ | irred. uniruled |
| 27 | 5 | $\bar{\mathcal{E}}_{27,5}^{(II)}$ | $4E_1 + 3E_2 + E_{1,2}$ | irred. unirat. |
| 27 | 5 | $\bar{\mathcal{E}}_{27,5}^{(III)}$ | $4E_1 + 3E_2 + 2E_3$ | irred. unirat. |
| 27 | 6 | $\bar{\mathcal{E}}_{27,6}^{(I)}$ | $3E_1 + 2E_2 + 2E_{1,2}$ | irred. unirat. |
| 27 | 6 | $\bar{\mathcal{E}}_{27,6}^{(II)}$ | $3E_1 + 2E_2 + 2E_3 + E_{1,4}$ | irred. unirat. |
| 27 | 6 | $\bar{\mathcal{E}}_{27,6}^{(III)}$ | $2E_1 + 2E_2 + E_3 + E_4 + E_5 + E_6$ | ?? |
| 28 | 1 | $\bar{\mathcal{E}}_{28,1}$ | $27E_1 + E_2$ | irred. unirat. |
| 28 | 2 | $\bar{\mathcal{E}}_{28,2}$ | $13E_1 + E_2 + E_3$ | irred. unirat. |
| 28 | 3 | $\bar{\mathcal{E}}_{28,3}^{(I)}$ | $8E_1 + E_2 + E_3 + E_4$ | irred. unirat. |
| 28 | 3 | $\bar{\mathcal{E}}_{28,3}^{(II)}$ | $3(3E_1 + E_2)$ | irred. unirat. |
| 28 | 4 | $\bar{\mathcal{E}}_{28,4}^{(I)}$ | $6E_1 + 3E_2 + E_3$ | irred. unirat. |
| 28 | 4 | $\bar{\mathcal{E}}_{28,4}^{(II)}$ | $6E_1 + E_2 + E_3 + E_{1,4}$ | irred. unirat. |
| 28 | 5 | $\bar{\mathcal{E}}_{28,5}^{(I)}$ | $5E_1 + E_2 + 2E_{1,3}$ | irred. unirat. |
| 28 | 5 | $\bar{\mathcal{E}}_{28,5}^{(II)}$ | $4E_1 + 2E_2 + E_3 + E_{1,2}$ | irred. unirat. |
| 28 | 5 | $\bar{\mathcal{E}}_{28,5}^{(III)}$ | $4E_1 + 3E_2 + E_3 + E_4$ | irred. unirat. |
| 28 | 6 | $\bar{\mathcal{E}}_{28,6}^{(I)}$ | $2E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7$ | irred. uniruled |
| 28 | 6 | $\bar{\mathcal{E}}_{28,6}^{(II)}$ | $3(E_1 + E_2 + E_3)$ | irred. unirat. |
| 28 | 6 | $\bar{\mathcal{E}}_{28,6}^{(III)}$ | $3E_1 + 2E_2 + E_3 + E_4 + E_{1,5}$ | irred. uniruled |
| 28 | 7 | $\bar{\mathcal{E}}_{28,7}$ | $3E_1 + E_2 + 3E_{1,2}$ | irred. unirat. |

| g | ϕ | comp. | dec. type | ρ^{-1} |
|-----|--------|---|--|--------------------------------|
| 29 | 1 | $\overline{\mathcal{E}}_{29,1}$ | $28E_1 + E_2$ | irred. unirat. |
| 29 | 2 | $\overline{\mathcal{E}}_{29,2}^{(I)}$ | $14E_1 + E_{1,2}$ | irred. unirat. |
| 29 | 2 | $\overline{\mathcal{E}}_{29,2}^{(II)}$ | $2(7E_1 + E_2)$ | two unirat. components |
| 29 | 3 | $\overline{\mathcal{E}}_{29,3}$ | $9E_1 + E_2 + E_{1,3}$ | irred. unirat. |
| 29 | 4 | $\overline{\mathcal{E}}_{29,4}^{(I)}$ | $7E_1 + 4E_2$ | irred. unirat. |
| 29 | 4 | $\overline{\mathcal{E}}_{29,4}^{(II)}$ | $7E_1 + 2E_{1,2}$ | irred. unirat. |
| 29 | 4 | $\overline{\mathcal{E}}_{29,4}^{(III)}$ | $2(3E_1 + E_2 + E_3)$ | two unirat. components |
| 29 | 4 | $\overline{\mathcal{E}}_{29,4}^{(IV)}$ | $6E_1 + 2E_2 + E_{1,2}$ | two unirat. components |
| 29 | 5 | $\overline{\mathcal{E}}_{29,5}^{(I)}$ | $4E_1 + 2E_2 + 2E_3 + E_4$ | irred. unirat. |
| 29 | 5 | $\overline{\mathcal{E}}_{29,5}^{(II)}$ | $5E_{1,2} + E_1 + E_2 + E_3$ | irred. unirat. |
| 29 | 5 | $\overline{\mathcal{E}}_{29,5}^{(III)}$ | $5E_1 + 3E_2 + E_{1,3}$ | irred. unirat. |
| 29 | 6 | $\overline{\mathcal{E}}_{29,6}^{(I)}$ | $3E_1 + E_2 + E_3 + E_4 + E_5 + E_{1,6}$ | ?? |
| 29 | 6 | $\overline{\mathcal{E}}_{29,6}^{(II)}$ | $2(2E_1 + E_2 + E_{1,3})$ | two unirat. components |
| 29 | 6 | $\overline{\mathcal{E}}_{29,6}^{(III)}$ | $3E_1 + 3E_2 + E_3 + E_{1,2}$ | irred. unirat. |
| 30 | 1 | $\overline{\mathcal{E}}_{30,1}$ | $29E_1 + E_2$ | irred. unirat. |
| 30 | 2 | $\overline{\mathcal{E}}_{30,2}$ | $14E_1 + E_2 + E_3$ | irred. unirat. |
| 30 | 3 | $\overline{\mathcal{E}}_{30,3}^{(I)}$ | $9E_1 + 2E_2 + E_3$ | irred. unirat. |
| 30 | 3 | $\overline{\mathcal{E}}_{30,3}^{(II)}$ | $9E_1 + E_2 + E_{1,2}$ | irred. unirat. |
| 30 | 4 | $\overline{\mathcal{E}}_{30,4}^{(I)}$ | $6E_1 + 2E_2 + E_3 + E_4$ | irred. unirat. |
| 30 | 4 | $\overline{\mathcal{E}}_{30,4}^{(II)}$ | $7E_{1,2} + E_1 + E_2$ | irred. unirat. |
| 30 | 5 | $\overline{\mathcal{E}}_{30,5}^{(I)}$ | $5E_1 + E_2 + 2E_{1,2}$ | irred. unirat. |
| 30 | 5 | $\overline{\mathcal{E}}_{30,5}^{(II)}$ | $4E_1 + 2E_2 + E_3 + E_4 + E_5$ | irred. unirat. |
| 30 | 5 | $\overline{\mathcal{E}}_{30,5}^{(III)}$ | $5E_1 + 4E_2 + E_3$ | irred. unirat. |
| 30 | 6 | $\overline{\mathcal{E}}_{30,6}^{(I)}$ | $4E_1 + E_2 + E_3 + 2E_{1,4}$ | irred. unirat. |
| 30 | 6 | $\overline{\mathcal{E}}_{30,6}^{(II)}$ | $3E_1 + 2E_2 + E_3 + E_4 + E_{1,2}$ | irred. unirat. |
| 30 | 6 | $\overline{\mathcal{E}}_{30,6}^{(III)}$ | $3E_1 + 3E_2 + 2E_3 + E_4$ | irred. unirat. |
| 30 | 6 | $\overline{\mathcal{E}}_{30,6}^{(IV)}$ | $4E_{1,2} + E_1 + E_2 + 2E_3$ | irred. unirat. |
| 30 | 7 | $\overline{\mathcal{E}}_{30,7}^{(I)}$ | $2E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_{1,7}$ | irred. unirat. (cf. Rem. 3.10) |
| 30 | 7 | $\overline{\mathcal{E}}_{30,7}^{(II)}$ | $2E_1 + 4E_2 + E_3 + E_4 + E_5$ | irred. unirat. |

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