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► To cite this version:

Pierre Cordesse, Marc Massot. Supplementary conservative law for non-linear systems of PDEs with non-conservative terms: application to the modelling and analysis of complex fluid flows using computer algebra. Communications in Mathematical Sciences, In press. hal-01978949v1

HAL Id: hal-01978949

<https://hal.science/hal-01978949v1>

Submitted on 12 Jan 2019 (v1), last revised 5 Nov 2019 (v2)

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Supplementary conservative law for non-linear systems of PDEs with non-conservative terms: application to the modelling and analysis of complex fluid flows using computer algebra

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In the present contribution, we investigate first-order nonlinear systems of partial differential equations which are constituted of two parts: a system of conservation laws and non-conservative first order terms. Whereas the theory of first-order systems of conservation laws is well established and the conditions for the existence of a supplementary conservation law for smooth solutions, well known, there exists so far no general extension when non-conservative terms are present. We propose a framework in order to extend the existing theory and show that the presence of non-conservative terms somewhat complexifies the problem since numerous combinations of the conservative and non-conservative terms can lead to a supplementary conservation law. We then identify a restricted framework in order to design and analyze physical models of complex fluid flows by means of computer algebra and thus obtain the entire ensemble of possible combination of conservative and non-conservative terms to obtain a supplementary conservation law. The theory as well as developed computer algebra tool are then applied to a Baer-Nunziato two-phase flow model and to a multicomponent plasma fluid model. The first one is a first-order fluid model, with non-conservative terms impacting on the linearly degenerate field and requires a closure since there is no way to derive interfacial quantities from averaging principles and we need guidance in order to close the pressure and velocity of the interface and the thermodynamics of the mixture. The second one involves first order terms for the heavy species coupled to second order terms for the electrons, the non-conservative terms impact the genuinely nonlinear fields and the model can be rigorously derived from kinetic theory. We show how the theory allows to recover the whole spectrum of closures obtained so far in the literature for the two-phase flow system as well as conditions when one aims at extending the thermodynamics and also applies to the plasma case, where we recover the usual entropy supplementary equation, thus assessing the effectiveness and scope of the proposed theory.

Keywords: Nonlinear PDEs with non-conservative terms, supplementary conservation law, entropy, computer algebra, two-phase flow, Baer-Nunziato model, multicomponent plasma fluid model

AMS Subject Classification: 35L60, 68W30, 76N15, 76T10, 82D10

1. Introduction

First-order nonlinear systems of partial differential equations and more specifically systems of conservation laws have been the subject of a vast literature since the

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second half of the twentieth century because they are ubiquitous in mathematical modeling of fluid flows and are used extensively for numerical simulation in applications and industrial context [1, 2]. Such systems of equation can either be rigorously derived from kinetic theory of gases through various expansion techniques [3, 4], or can be derived using rational thermodynamics and fluid mechanics [5, 6]. As far as Euler or Navier-Stokes equations are concerned for a gaseous flow field, the outcome of both approaches are similar and the mathematical properties of these systems have been thoroughly investigated for the past decades. In particular, these systems admit a supplementary conservative equation for smooth solution [7, 8], called also entropy equation, they are hyperbolic at any point where a locally convex entropy function exists [9], and when they are equipped with a strictly convex entropy, they can be symmetrized [8] [10] and thus are hyperbolic. These properties have been at the heart of the mathematical theory of existence and uniqueness of smooth solutions [11] [12], but they are also a corner stone for the study of weak solutions for which the work of [13] proves the well-posedness of Cauchy problem for one-dimensional systems.

Nonetheless, for a number of applications, where reduced-order fluid models have to be used for tractable mathematical modelling and numerical simulations, be it in the industry or in other disciplines, micro-macro kinetic-theory-like approaches as well as rational thermodynamics approaches often lead to system of conservation laws with the addition of non-conservative terms. Among the large spectrum of applications, we focus on two types of models, which exemplify the two approaches: 1- two phase flows models which rely on an hierarchy of diffuse interface models among which stands the Baer-Nunziato [14] model used when full disequilibrium of the phases must be taken into account. Since this model is derived through rational thermodynamics, the macroscopic set of equations can not be derived from physics at small scale of interface dynamics and thus require closure of interfacial pressure and velocity, 2- multicomponent fluid modelling of plasmas flows out of thermal equilibrium, where the equations can be derived rigorously from kinetic theory using a multi-scale Chapman-Enskog expansion mixing a hyperbolic scaling for the heavy species and a parabolic scaling for the electrons [15]. Concerning the thermodynamics, whereas for the first model it has to be postulated and requires assumptions, it can be obtained from kinetic theory in the second model. In both case, the models involve non-conservative terms, but these terms do not act on the same fields; linearly degenerate field is impacted for the two-phase flow model, whereas it acts on the genuinely nonlinear fields in the second [16]. Whereas hyperbolicity depends on the closure and is not guaranteed for the first class of models [17], the second is naturally hyperbolic [15] and also involves second-order terms and eventually source terms [18].

Thus, the presence of nonconservative terms encompasses several situations and requires a general theoretical framework. Nevertheless a unifying theory extending the standard approach for systems of conservations laws (supplementary entropy

conservation law, entropic symmetrization, Godunov-Mock theorem, hyperbolicity) is still missing for such systems even if some key advances exist. The system has been shown to be symmetrizable by [19] far from the resonance condition for which hyperbolicity degenerates. In [20], the model is proved to be partially symmetrizable in the sense of Godunov-Mock.

The present paper first proposes an extension of the theory for system of conservation laws to first-order nonlinear systems of partial differential equations which are constituted of two parts: a system of conservation laws and non-conservative first order terms. We emphasize how the presence of non-conservative terms somewhat complexifies the problem since numerous combinations of the conservative and non-conservative terms can lead to a supplementary conservation law. We then identify a restricted framework in order to design and analyze physical models of complex fluid flows by means of computer algebra and thus obtain the entire ensemble of possible combination of conservative and non-conservative terms to obtain a supplementary conservation law. The proposed theoretical approach is then applied to the two systems identified so far for their diversity of behaviour. For the two-phase flow model, assuming a thermodynamics of non-miscible phases, we derive conditions to obtain a supplementary conservative equation together with a compatible thermodynamics and closures for the non-conservative terms. Interestingly enough, all the closures proposed so far in the literature are recovered [14, 21, 22, 23, 24]. The strength of the formalism lays also in the capacity to derive such conditions for some level of mixing of the phases. By introducing a mixing term in the definition of the entropy to allow mixing of the phases, the new theory brings out constraints on the form of the added term. We recover not only the closure proposed to account for a configuration energy as in the context of deflagration-to-detonation [14] or in [25], but we also rigorously find new closures leading to a conservative system of equations^a. We also prove that the theory encompasses the plasma case, where we recover the usual entropy supplementary equation assessing the effectiveness and scope of the proposed theory.

The paper is organized as follows. The extension of the theory for system of conservation laws to first-order nonlinear systems of partial differential equations including non-conservative terms, as well as the framework to apply the theory by means of computer algebra are introduced in Section 2. These results are then applied first to the Baer-Nunziato model in Section 3 and then to the plasma model in Section 4.

Notations: Let $\mathbf{a} \in \mathbb{R}^p$, $\mathbf{b} \in \mathbb{R}^p$, $\mathcal{B} \in \mathbb{R}^{p \times p}$, $\mathcal{C} \in \mathbb{R}^{p \times p}$, $\mathcal{D} \in \mathbb{R}^{p \times p \times p}$ be a p -component line first-order tensor, a p -component column first-order tensor, two p -square second-order tensor and a third-order tensor respectively. We introduce the following notations:

^aSuch closure is similar to the one used in [26, 27] which led to a controversy [22, 28, 29]

- $\mathbf{a}\mathcal{B}$ is a line first-order tensor in \mathbb{R}^p whose i component are defined by

$$(\mathbf{a}\mathcal{B})_i = \sum_{j=1,p} \mathbf{a}_j \mathcal{B}_{j,i}, \quad (1.1)$$

- $\mathcal{B}\mathbf{b}$ is a column first-order tensor in \mathbb{R}^p whose i component is defined by

$$(\mathcal{B}\mathbf{b})_i = \sum_{j=1,p} \mathcal{B}_{i,j} \mathbf{b}_j, \quad (1.2)$$

- $\mathcal{B} \times \mathcal{C}$ is p -square second-order tensor whose (i, j) component is defined by

$$(\mathcal{B} \times \mathcal{C})_{i,j} = \sum_{k=1,p} \mathcal{B}_{(i,k)} \mathcal{C}_{(k,j)}, \quad (1.3)$$

- $\mathbf{a} \otimes \mathbb{D}$ is a p -square second-order tensor whose (i, j) component is defined by

$$(\mathbf{a} \otimes \mathbb{D})_{(i,j)} = \sum_k \mathbf{a}_k \times \mathbb{D}_{k,i,j}. \quad (1.4)$$

Hereafter, we will name zero- first- and second-order tensors by scalar, vector and matrix respectively and for convenience we will use vector and matrix representations of functions. Moreover, given a scalar function S , the partial differentiation of S by a column vector \mathbf{a} , $\partial_{\mathbf{a}}S$ is a line vector in \mathbb{R}^p . Finally, \cdot denotes the Euclidean scalar product in \mathbb{R}^p .

2. Supplementary conservation law

First we recall the theory of the existence of a supplementary conservative equation for first-order nonlinear systems of conservation laws. Second, this notion is extended to systems containing first order non-conservative terms. Third, we introduce a framework to apply this new theory to design and analyze physical models using computer algebra.

2.1. First-order nonlinear conservative systems

The homogeneous form of a first-order nonlinear system of p conservative laws writes

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = \mathbf{0}, \quad (2.1)$$

where $\mathbf{u} \in \Omega \subset \mathbb{R}^p$ denotes the conservative variables with Ω an open convex of \mathbb{R}^p and $\mathbf{f} : \mathbf{u} \in \Omega \mapsto \mathbb{R}^p$ the conservative fluxes. Focusing on smooth solution of the system (2.1), its quasi-linear form is given by

$$\partial_t \mathbf{u} + \partial_{\mathbf{u}} \mathbf{f}(\mathbf{u}) \partial_x \mathbf{u} = \mathbf{0}. \quad (2.2)$$

Theorem 2.1. Let $H : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ be a scalar function, not necessarily convex. The following statements are equivalent:

(C₁) System (2.1) admits a supplementary conservative equation

$$\partial_t H(\mathbf{u}) + \partial_x G(\mathbf{u}) = \mathbf{0}, \quad (2.3)$$

where $\mathbf{u} \in \mathbb{R}^p$ is a smooth solution of System (2.1) and $G : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ is a scalar function.

(C₂) There exists a scalar function $G : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ such that

$$\partial_{\mathbf{u}} H(\mathbf{u}) \partial_{\mathbf{u}} \mathbf{f}(\mathbf{u}) = \partial_{\mathbf{u}} G(\mathbf{u}). \quad (2.4)$$

(C₃) $\partial_{\mathbf{u}\mathbf{u}} H(\mathbf{u}) \times \partial_{\mathbf{u}} \mathbf{f}(\mathbf{u})$ is a p -square symmetric matrix.

Proof. The proofs of the theorem can be found in the literature. We would like to recall how the last statement is obtained. Assuming (C₂), differentiating Equation (2.4) leads to

$$\partial_{\mathbf{u}\mathbf{u}} H(\mathbf{u}) \times \partial_{\mathbf{u}} \mathbf{f}(\mathbf{u}) + \partial_{\mathbf{u}} H(\mathbf{u}) \otimes \partial_{\mathbf{u}\mathbf{u}} \mathbf{f}(\mathbf{u}) = \partial_{\mathbf{u}\mathbf{u}} G(\mathbf{u}), \quad (2.5)$$

where $\partial_{\mathbf{u}} H(\mathbf{u}) \otimes \partial_{\mathbf{u}\mathbf{u}} \mathbf{f}(\mathbf{u})$ is a p -square matrix defined as $\sum_i \partial_{\mathbf{u}_i} H(\mathbf{u}) \partial_{\mathbf{u}\mathbf{u}} \mathbf{f}_i(\mathbf{u})$ which is a linear combination of Hessian matrices and hence symmetric. Moreover, the RHS of Equation (2.5) $\partial_{\mathbf{u}\mathbf{u}} G(\mathbf{u})$ is symmetric. Therefore $\partial_{\mathbf{u}\mathbf{u}} H(\mathbf{u}) \times \partial_{\mathbf{u}} \mathbf{f}(\mathbf{u})$ is symmetric. \square

Remark 2.1. Other formulations of Theorem 2.1 can be found in the literature [10, 30, 31].

We can then define the notions of *entropy* and *entropic variables* in the following two definitions.

Definition 2.1. $H : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ is said to be an *entropy* of the system (2.1) if $H(\mathbf{u})$ is a convex scalar function of the variables \mathbf{u} which fulfills Theorem 2.1. The supplementary conservative equation (2.3) is then named the *entropy equation* and $G : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ is the associated *entropy flux*.

Definition 2.2. Let $H : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ be a scalar function, not necessarily convex. Given a first-order nonlinear conservative system (2.7), let us define the *entropic variables* $\mathbf{v} : \mathbf{u} \in \Omega \mapsto \mathbb{R}^p$ such that

$$\mathbf{v}(\mathbf{u}) = (\partial_{\mathbf{u}} H(\mathbf{u}))^t. \quad (2.6)$$

The entropic variables have been studied in [12] in order to obtain symmetric and normal forms of the system of equation and used in the framework of gaseous mixtures, where the mathematical entropy H is usually defined as the opposite of a physical entropy density per unit volume of the system [12].

2.2. Extension to systems of non-conservative equations

Let us now consider the homogeneous form of a first-order nonlinear system of partial differential equations constituted of two parts: conservations laws and first-order non-conservative terms. Its quasi-linear form can be written as

$$\partial_t \mathbf{u} + [\partial_{\mathbf{u}} \mathbf{f}(\mathbf{u}) + \mathcal{N}(\mathbf{u})] \partial_x \mathbf{u} = \mathbf{0}, \quad (2.7)$$

where $\mathbf{u} \in \Omega \subset \mathbb{R}^p$ is a smooth solution with Ω an open convex of \mathbb{R}^p , $\mathbf{f} : \mathbf{u} \in \Omega \mapsto \mathbb{R}^p$ the conservative fluxes, $\mathcal{N} : \mathbf{u} \in \Omega \mapsto \mathbb{R}^{p \times p}$ the p -square matrix containing the first-order non-conservative terms.

In the following we extend the theory introduced in Section 2.1 to system (2.7). Given a scalar function $H : \mathbf{u} \in \Omega \mapsto \mathbb{R}$, multiplying system (2.7) by the line vector $\partial_{\mathbf{u}} H(\mathbf{u})$ yields

$$\partial_t H + \partial_{\mathbf{u}} H(\mathbf{u}) [\partial_{\mathbf{u}} \mathbf{f}(\mathbf{u}) + \mathcal{N}(\mathbf{u})] \partial_x \mathbf{u} = 0. \quad (2.8)$$

Compared to Equation (2.3), the presence of the non-conservative terms in Equation (2.8) complexifies the question of the existence of a supplementary conservative equation. Therefore we propose to decompose in a specific way the conservative and non-conservative terms in Definition 2.3.

Definition 2.3. Given a scalar function $H : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ and a first-order nonlinear non-conservative system (2.7), let us define the four p -square matrices, $\mathcal{C}_1(\mathbf{u})$, $\mathcal{Z}_1(\mathbf{u})$, $\mathcal{C}_2(\mathbf{u})$ and $\mathcal{Z}_2(\mathbf{u})$ in $\mathbb{R}^{p \times p}$ such that

$$\partial_{\mathbf{u}} \mathbf{f}(\mathbf{u}) = \mathcal{C}_1(\mathbf{u}) + \mathcal{Z}_1(\mathbf{u}), \quad (2.9)$$

$$\mathcal{N}(\mathbf{u}) = \mathcal{C}_2(\mathbf{u}) + \mathcal{Z}_2(\mathbf{u}), \quad (2.10)$$

with the condition

$$\partial_{\mathbf{u}} H(\mathbf{u}) [\mathcal{Z}_1(\mathbf{u}) + \mathcal{Z}_2(\mathbf{u})] = \mathbf{0}. \quad (2.11)$$

In light of Definition 2.3, Theorem 2.1 can be extended as follows:

Theorem 2.2. *Let $H : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ be a scalar function, not necessarily convex. Given a first-order nonlinear system of non-conservation laws (2.7), if we introduce the decomposition as in Definition 2.3, then the following statements are equivalent:*

(C₁) *System (2.7) admits a supplementary conservative equation*

$$\partial_t H(\mathbf{u}) + \partial_x G(\mathbf{u}) = 0, \quad (2.12)$$

where $\mathbf{u} \in \mathbb{R}^p$ is a smooth solution of System (2.7) and $G : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ is a scalar function.

(C₂) *There exists a scalar function $G : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ such that*

$$\partial_{\mathbf{u}} H(\mathbf{u}) [\mathcal{C}_1(\mathbf{u}) + \mathcal{C}_2(\mathbf{u})] = \partial_{\mathbf{u}} G(\mathbf{u}). \quad (2.13)$$

(C₃) $\partial_{\mathbf{u}\mathbf{u}}\mathbf{H}(\mathbf{u}) \times [\mathbf{C}_1(\mathbf{u}) + \mathbf{C}_2(\mathbf{u})] + \partial_{\mathbf{u}}\mathbf{H}(\mathbf{u}) \otimes \partial_{\mathbf{u}}[\mathbf{C}_1(\mathbf{u}) + \mathbf{C}_2(\mathbf{u})]$ is a p -square symmetric matrix.

Proof. Rewriting Equation (2.8) using the decomposition of the conservative and non-conservative terms as

$$\partial_t \mathbf{H}(\mathbf{u}) + \partial_{\mathbf{u}}\mathbf{H}(\mathbf{u}) [\mathbf{C}_1(\mathbf{u}) + \mathbf{C}_2(\mathbf{u})] \partial_x \mathbf{u} = -\partial_{\mathbf{u}}\mathbf{H}(\mathbf{u}) [\mathbf{Z}_1(\mathbf{u}) + \mathbf{Z}_2(\mathbf{u})] \partial_x \mathbf{u} \quad (2.14)$$

outlines the result. \square

Remark 2.2. In Definition 2.3, the condition (2.11) implies that the conservative and non-conservative terms depend only on the variables \mathbf{u} , and not on their gradient. Some authors have allowed the matrices \mathbf{Z}_k to depend also on the gradients of the variables \mathbf{u} , then a more general condition for the decomposition can be written

$$\partial_{\mathbf{u}}\mathbf{H}(\mathbf{u}) [\mathbf{Z}_1(\mathbf{u}, \partial_x \mathbf{u}) + \mathbf{Z}_2(\mathbf{u}, \partial_x \mathbf{u})] \partial_x \mathbf{u} \leq 0. \quad (2.15)$$

In Section 3, we will see that such a condition has been chosen to close the Baer-Nunziato model [24]. However, since it changes the mathematical nature of the PDE under investigation, we will not include it in our study.

From a modelling perspective, System (2.7) under consideration is not necessarily closed. Therefore, the following corollary yields conditions on the model to obtain a supplementary conservative equation once we have postulated the thermodynamics.

Corollary 2.1. *Let $\mathbf{H} : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ be a scalar function, not necessarily convex. Given a first-order nonlinear system of non-conservation laws (2.7) where $\mathbf{f} : \mathbf{u} \in \Omega \mapsto \mathbb{R}^p$ and $\mathcal{N} : \mathbf{u} \in \Omega \mapsto \mathbb{R}^{p \times p}$ are unknown functions to be modelled. If we introduce the decomposition as in Definition 2.3, then System (2.7) admits a supplementary conservative equation*

$$\partial_t \mathbf{H}(\mathbf{u}) + \partial_x \mathbf{G}(\mathbf{u}) = 0, \quad (2.16)$$

where $\mathbf{u} \in \Omega \subset \mathbb{R}^p$ is a smooth solution of System (2.7) and $\mathbf{G} : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ a scalar function, if and only if the following conditions hold

(C₁) $\partial_{\mathbf{u}\mathbf{u}}\mathbf{H}(\mathbf{u}) \times [\mathbf{C}_1(\mathbf{u}) + \mathbf{C}_2(\mathbf{u})] + \partial_{\mathbf{u}}\mathbf{H}(\mathbf{u}) \otimes \partial_{\mathbf{u}}[\mathbf{C}_1(\mathbf{u}) + \mathbf{C}_2(\mathbf{u})]$ is a p -square symmetric matrix.

(C₂) $\partial_{\mathbf{u}}\mathbf{H}(\mathbf{u}) [\mathbf{Z}_1(\mathbf{u}) + \mathbf{Z}_2(\mathbf{u})] = \mathbf{0}$.

2.3. Design or analysis of physical models using computer algebra

We would like to apply the theory on first-order nonlinear non-conservative systems introduced in Section 2.2 to physical models such as the Baer-Nunziato model and

the plasma model in order to design and analyze them. However, the difficulty is manifold:

- The combination of the non-conservative terms and conservative terms proposed in Definition 2.3 to build a supplementary conservative equations is not unique and thus many degrees of freedom exist in defining the matrices \mathcal{C}_k and \mathcal{Z}_k .
- When the model is derived through rational thermodynamics, terms in the system of equations might need closure and the thermodynamics has to be postulated. Therefore, the matrices \mathcal{C}_k and \mathcal{Z}_k can contain unknowns related to the system and the definition of H.
- The calculations needed to derive a supplementary conservative equation are heavy and choice-based. Any change of \mathcal{C}_k and \mathcal{Z}_k that respects Definition 2.3, or any new postulated thermodynamics would require to derive again all the equations, and eventually a very limited range of possibilities would be examined.

These difficulties to apply the theory and examine all the possibilities makes computer algebra very appealing since it allows symbolic operations to be implemented and thus can derive equations systematically and quasi-instantaneously for any combinations of conservative and non-conservative terms as well as model closure and H definition. Still, the generic level handled by computer algebra is not unlimited and therefore Definition 2.3 requires further assumptions to circumscribe the number of degrees of freedom that can be accounted for.

Definition 2.4. Given a scalar function $H : \mathbf{u} \in \Omega \mapsto \mathbb{R}$, a first-order nonlinear non-conservative system (2.7), and the four p -square matrices $\mathcal{C}_1(\mathbf{u})$, $\mathcal{Z}_1(\mathbf{u})$, $\mathcal{C}_2(\mathbf{u})$ and $\mathcal{Z}_2(\mathbf{u})$ in $\mathbb{R}^{p \times p}$ defined in Definition 2.3, we introduce the unknown line vector $\mathbf{t} : \mathbf{u} \in \Omega \mapsto \mathbb{R}^p$ such that

$$\partial_{\mathbf{u}}H(\mathbf{u}) [\mathcal{C}_1(\mathbf{u}) + \mathcal{C}_2(\mathbf{u})] = \partial_{\mathbf{u}}H(\mathbf{u}) \partial_{\mathbf{u}}\mathbf{f}(\mathbf{u}) + \mathbf{t}(\mathbf{u}), \quad (2.17)$$

$$\partial_{\mathbf{u}}H(\mathbf{u}) [\mathcal{Z}_1(\mathbf{u}) + \mathcal{Z}_2(\mathbf{u})] = \partial_{\mathbf{u}}H(\mathbf{u}) \mathcal{N}(\mathbf{u}) - \mathbf{t}(\mathbf{u}). \quad (2.18)$$

The condition of Equation (2.11) rewrites into

$$\partial_{\mathbf{u}}H(\mathbf{u}) \mathcal{N}(\mathbf{u}) - \mathbf{t}(\mathbf{u}) = \mathbf{0}. \quad (2.19)$$

Remark 2.3. Since Definition 2.4 is a projection of the matrix equations of Definition 2.3 on the vector $\partial_{\mathbf{u}}H(\mathbf{u})$, it may be interesting to introduce an unknown matrix $\mathcal{T}(\mathbf{u}) \in \mathbb{R}^{p \times p}$ associated to the unknown line vector $\mathbf{t}(\mathbf{u})$ such that

$$\mathbf{t}(\mathbf{u}) = \partial_{\mathbf{u}}H(\mathbf{u}) \mathcal{T}(\mathbf{u}). \quad (2.20)$$

Thus, Definition 2.4 can be formulated as follows

$$\mathcal{C}_1(\mathbf{u}) + \mathcal{C}_2(\mathbf{u}) = \partial_{\mathbf{u}}\mathbf{f}(\mathbf{u}) + \mathcal{T}(\mathbf{u}), \quad (2.21)$$

$$\mathcal{Z}_1(\mathbf{u}) + \mathcal{Z}_2(\mathbf{u}) = \mathcal{N}(\mathbf{u}) - \mathcal{T}(\mathbf{u}), \quad (2.22)$$

with the condition

$$\partial_{\mathbf{u}}\mathbf{H}(\mathbf{u}) [\mathcal{N}(\mathbf{u}) - \mathcal{T}(\mathbf{u})] = \mathbf{0}. \quad (2.23)$$

The unknown functional line vector $\mathbf{t}(\mathbf{u}) \in \mathbb{R}^7$ represents the transfer of non-conservative terms to the conservative terms. In the degenerate case where $\mathbf{t} = \mathbf{0}$, \mathcal{C}_k receives all the conservative terms and \mathcal{Z}_k all the non-conservative terms. Condition (2.19) forces all the non-conservative terms to vanish and System (2.7) is fully conservative, hence the theory of conservative system can be applied.

Definition 2.4 being more restrictive than Definition 2.3, computer algebra is now applicable to analyse the properties of a first-order nonlinear non-conservative systems leading to a reformulation of Theorem 2.2.

Theorem 2.3. *Let $\mathbf{H} : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ be a scalar function, not necessarily convex. Consider a first-order nonlinear system of non-conservation laws (2.7). If we introduce the decomposition as in Definition 2.4, then the following statements are equivalent:*

(C₁) *System (2.7) admits a supplementary conservative equation*

$$\partial_t \mathbf{H}(\mathbf{u}) + \partial_x \mathbf{G}(\mathbf{u}) = 0, \quad (2.24)$$

where $\mathbf{u} \in \mathbb{R}^p$ is a smooth solution of System (2.7) and $\mathbf{G} : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ is a scalar function.

(C₂) *There exists a scalar function $\mathbf{G} : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ such that*

$$\partial_{\mathbf{u}}\mathbf{H}(\mathbf{u}) \partial_{\mathbf{u}}\mathbf{f}(\mathbf{u}) + \mathbf{t}(\mathbf{u}) = \partial_{\mathbf{u}}\mathbf{G}(\mathbf{u}). \quad (2.25)$$

(C₃) *$\partial_{\mathbf{u}\mathbf{u}}\mathbf{H}(\mathbf{u}) \times \partial_{\mathbf{u}}\mathbf{f}(\mathbf{u}) + \partial_{\mathbf{u}}\mathbf{t}(\mathbf{u})$ is a p -square symmetric matrix.*

Proof. Injecting Definition 2.4 into Theorem 2.2 leads to these results. \square

Theorem 2.3 provides equations that relate the thermodynamics of the model through \mathbf{H} , the model itself with possible terms to be closed in $\mathbf{f}(\mathbf{u})$ and $\mathcal{N}(\mathbf{u})$, and the unknown line vector $\mathbf{t}(\mathbf{u})$. Combined with the Definition 2.4, Theorem 2.3 brings out conditions on the model to obtain a supplementary conservative equation given a postulated thermodynamics and it leads to the following corollary.

Corollary 2.2. *Consider a first-order nonlinear system of non-conservation laws (2.7) where $\mathbf{u} \in \Omega \subset \mathbb{R}^p$ is a smooth solution with Ω an open convex of \mathbb{R}^p but*

$\mathbf{f} : \mathbf{u} \in \Omega \mapsto \mathbb{R}^p$ and $\mathcal{N} : \mathbf{u} \in \Omega \mapsto \mathbb{R}^{p \times p}$ are unknown functions to be modelled. Let $H : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ be a scalar function, not necessarily convex, and if we introduce the decomposition as in Definition 2.4, then System (2.7) admits a supplementary conservative equation

$$\partial_t H(\mathbf{u}) + \partial_x G(\mathbf{u}) = 0, \quad (2.26)$$

where $G : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ is a scalar function if and only if the following conditions hold

- (C₁) $\partial_{\mathbf{u}\mathbf{u}}H(\mathbf{u}) \times \partial_{\mathbf{u}}\mathbf{f}(\mathbf{u}) + \partial_{\mathbf{u}}\mathbf{t}(\mathbf{u})$ is symmetric.
- (C₂) $\partial_{\mathbf{u}}H(\mathbf{u})\mathcal{N}(\mathbf{u}) - \mathbf{t}(\mathbf{u}) = \mathbf{0}$.

2.4. Methodology

Corollary 2.2 draws the methodology we have implemented in the MapleTM computer algebra software^b. Our methodology is the following:

- (Step 1) We define the thermodynamics by postulating - if need be - an entropy function $H : \mathbf{u} \in \Omega \mapsto \mathbb{R}$.
- (Step 2) We then use Condition (C₁) and (C₂) of Corollary 2.2 to ensure the existence of an entropy flux $G : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ and solve

$$\begin{cases} \partial_{\mathbf{u}\mathbf{u}}H(\mathbf{u}) \times \partial_{\mathbf{u}}\mathbf{f}(\mathbf{u}) + \partial_{\mathbf{u}}\mathbf{t}(\mathbf{u}) \text{ symmetric,} \\ \partial_{\mathbf{u}}H(\mathbf{u})\mathcal{N}(\mathbf{u}) - \mathbf{t}(\mathbf{u}) = \mathbf{0}. \end{cases} \quad (2.27)$$

In System (2.27), $\mathbf{t}(\mathbf{u})$ is systematically an unknown, $\mathbf{f}(\mathbf{u})$, $\mathcal{N}(\mathbf{u})$ as well as $H(\mathbf{u})$ can include unknown terms for which the variable dependency is specified. MapleTM generates then an exhaustive solution for $\mathbf{t}(\mathbf{u})$ and constraints on all the other unknown terms.

- (Step 3) From that, the software derives the admissible entropy flux $G : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ which gives then the supplementary conservative equation.

3. Application to the Baer-Nunziato model

3.1. Context and presentation of the model

The Baer-Nunziato model has been derived through rational thermodynamics in [14] and describes a two-phase flow out of equilibrium. Extended by the work of

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[32] thanks to the introduction of interfacial quantities, the homogeneous form of the Baer-Nunziato model is

$$\partial_t \mathbf{u} + [\partial_{\mathbf{u}} \mathbf{f}(\mathbf{u}) + \mathcal{N}(\mathbf{u})] \partial_x \mathbf{u} = \mathbf{0},$$

$$\partial_{\mathbf{u}} \mathbf{f}(\mathbf{u}) = \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \partial_{\mathbf{u}_2} \mathbf{f}_2(\mathbf{u}_2) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \partial_{\mathbf{u}_1} \mathbf{f}_1(\mathbf{u}_1) \end{pmatrix}, \quad \mathcal{N}(\mathbf{u}) = \begin{pmatrix} v_I & \mathbf{0} & \mathbf{0} \\ \mathbf{n}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{n}_1 & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad (3.1)$$

where the column vector $\mathbf{u} \in \mathbb{R}^7$ is defined by $\mathbf{u}^T = (\alpha_2, \mathbf{u}_2^T, \mathbf{u}_1^T)$, $\mathbf{u}_k^T = (\alpha_k \rho_k, \alpha_k \rho_k v_k, \alpha_k \rho_k E_k)$. The conservative flux $\mathbf{f} : \mathbf{u} \in \Omega \mapsto \mathbb{R}^7$ reads $\mathbf{f}(\mathbf{u})^T = (0, \mathbf{f}_2(\mathbf{u}_2)^T, \mathbf{f}_1(\mathbf{u}_1)^T)$ with $\mathbf{f}_k(\mathbf{u}_k)^T = (\alpha_k \rho_k v_k, \alpha_k (\rho_k v_k^2 + p_k), \alpha_k (\rho_k E_k + p_k) v_k)$. $\mathcal{N} : \mathbf{u} \in \Omega \mapsto \mathbb{R}^{7 \times 7}$ is the matrix containing the non-conservative terms with $\mathbf{n}_2(\mathbf{u})^T = -\mathbf{n}_1(\mathbf{u})^T = (0, -p_I, -p_I v_I)$. Then, α_k is the volume fraction of phase $k \in [1, 2]$, ρ_k the partial density, v_k the phase velocity, p_k the phase pressure, $E_k = \epsilon_k + 1/2 v_k^2$ the total energy per unit of mass, ϵ_k the internal energy, v_I the interfacial velocity and p_I the interfacial pressure.

Two levels of ingredients are still missing for this model. First, the macroscopic set of equations includes the interface dynamics through the interfacial terms v_I and p_I and thus needs closure on these terms. Second the thermodynamics has to be postulated.

The mathematical properties of the model have been studied by [17, 25, 28, 33] among others and many closure have been proposed for the interfacial terms based on wave-type considerations and the entropy inequality.

Regarding the thermodynamics, for non-miscible phases, the entropy $\mathbf{H}(\mathbf{u})$ is commonly defined by Equation (3.2) as in [23, 25],

$$\mathbf{H} = - \sum_{k=1,2} \alpha_k \rho_k s_k, \quad (3.2)$$

with $s_k = s_k(\rho_k, p_k)$ the phase entropy which takes for the Ideal Gas equation of state the form

$$s_k = c_{v,k} \ln \left(\frac{p_k}{\rho_k^{\gamma_k}} \right), \quad (3.3)$$

with $c_{v,k}$ the heat capacity, p_k the pressure, ρ_k the density and γ_k the isentropic coefficient of phase k .

If we were to account for partial miscibility between the two phases, we would have to add a mixing term to the definition of the non-miscible entropy. The mixing term could take the form proposed in [17], so that the entropy rewrites

$$\mathbf{H} = - \sum_{k=1,2} \alpha_k \rho_k [s_k(\rho_k, p_k) - \psi_k(\alpha_k)], \quad (3.4)$$

with ψ_k , $k = [1, 2]$, two strictly convex nonlinear arbitrary functions depending on the volume fraction. Nevertheless, so far in the literature, no explicit expressions of these functions have been proposed. In [17], in order to obtain a supplementary conservative equation using the entropy defined in Equation (3.4), the authors show that the following condition has to be fulfilled

$$\psi_k(\alpha_k) = \psi_{k'}(\alpha_{k'}). \quad (3.5)$$

In this section, we apply to the Baer-Nunziato model the framework introduced in Section 2 by means of computer algebra. We will firstly assume the phases are non-miscible and derive a supplementary conservative equation along with conditions on the interfacial terms. All the closures proposed in the literature will be recovered. Secondly, we will also apply the methodology in the case of a thermodynamics with partial miscibility and derive a supplementary conservative equation together with conditions on both the interfacial terms and the mixing terms of the entropy. Not only all the closures proposed in the literature are recovered but also new ones and we also propose explicit formulations of the mixing terms and show that depending on their expression, the condition expressed in [17] is not necessary.

3.2. Methodology and decomposition

We start without any condition on (v_I, p_I) . We need initially to fix a decomposition of $\partial_{\mathbf{u}}\mathbf{f}(\mathbf{u})$ and $\mathcal{N}(\mathbf{u})$ including a certain degree of freedom as explained in Section 2.3.

Given an entropy $\mathbf{H} : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ of System (3.1), by expressing the entropic variables as $\mathbf{v}(\mathbf{u})^T = (v_\alpha, \mathbf{v}_2^T, \mathbf{v}_1^T)$, we use the decomposition proposed in Definition (2.4). Since we do not want to generate other non-conservative terms, we choose to define the line vector $\mathbf{t} : \mathbf{u} \in \Omega \mapsto \mathbb{R}^p$ by $\mathbf{t}(\mathbf{u}) = (t_\alpha(\mathbf{u}), \mathbf{0}, \mathbf{0})$ where $t_\alpha : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ is the unknown scalar function a priori of all the variables \mathbf{u} . We obtain the following decompositions

$$(\partial_{\mathbf{u}}\mathbf{H}[\mathcal{C}_1 + \mathcal{C}_2])^T = \begin{pmatrix} t_\alpha(\mathbf{u}) \\ \mathbf{v}_2 \cdot \partial_{\mathbf{u}_2}\mathbf{f}_2(\mathbf{u}_2) \\ \mathbf{v}_1 \cdot \partial_{\mathbf{u}_1}\mathbf{f}_1(\mathbf{u}_1) \end{pmatrix}, \quad (3.6a)$$

$$(\partial_{\mathbf{u}}\mathbf{H}[\mathcal{Z}_1 + \mathcal{Z}_2])^T = \begin{pmatrix} -t_\alpha(\mathbf{u}) + v_\alpha v_I + \sum_{k=1,2} \mathbf{v}_k \cdot \mathbf{n}_k \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}. \quad (3.6b)$$

t_α allows fractions of the non-conservative terms to feed the matrix \mathcal{C}_k .

Given this decomposition, we use the methodology proposed in Section 2.4. (Step 2) will be split here into two sub-steps.

- (Step 2.a) Condition (C_1) on the symmetry of the matrix $\partial_{\mathbf{u}\mathbf{u}}\mathbf{H}(\mathbf{u}) \times \partial_{\mathbf{u}}\mathbf{f}(\mathbf{u}) + \partial_{\mathbf{u}}\mathbf{t}(\mathbf{u})$ ensures the existence of an entropy flux $\mathbf{G}(\mathbf{u})$. It will determine $\mathbf{t}(\mathbf{u})$.
- (Step 2.b) Knowing $\mathbf{t}(\mathbf{u})$, Condition (C_2) , $\partial_{\mathbf{u}}\mathbf{H}(\mathbf{u})\mathcal{N}(\mathbf{u}) - \mathbf{t}(\mathbf{u}) = \mathbf{0}$, will return an equation linking (v_I, p_I) and also ψ_k when miscibility is accounted for.

3.3. Non-miscible phases entropy

Thus, we start applying our method (Step 1) by postulating \mathbf{H} as in Equation (3.2). The thermodynamics is entirely known and we use the Ideal Gas EOS. The entropic variables \mathbf{v} are then

$$\mathbf{v} = \begin{pmatrix} v_\alpha \\ \mathbf{v}_2 \\ \mathbf{v}_1 \end{pmatrix} \text{ with } v_\alpha = \frac{p_1}{T_1} - \frac{p_2}{T_2} \text{ and } \mathbf{v}_k = \frac{1}{T_k} \begin{pmatrix} g_k - 1/2v_k^2 \\ v_k \\ -1 \end{pmatrix}, \quad (3.7)$$

with g_k the Gibbs free energy, $g_k = \epsilon_k + p_k/\rho_k - T_k s_k$. We now apply the conditions to determine $t_\alpha(\mathbf{u})$ and derives the equation that links the interfacial quantities v_I and p_I .

Theorem 3.1. Consider System (3.1). If the mixture entropy is defined as $\mathbf{H} = -\sum_{k=1,2} \alpha_k \rho_k s_k$ then with the decomposition proposed in Equations (3.6)

$$\partial_{\mathbf{u}\mathbf{u}}\mathbf{H}(\mathbf{u}) \times \partial_{\mathbf{u}}\mathbf{f}(\mathbf{u}) + \partial_{\mathbf{u}}\mathbf{t}(\mathbf{u}) \text{ symmetric} \Leftrightarrow t_\alpha(\mathbf{u}) = F(\alpha_2) + \frac{p_1}{T_1}u_1 - \frac{p_2}{T_2}u_2, \quad (3.8)$$

with F a strictly convex arbitrary function depending on the volume fraction α_2 . As a consequence the condition on $\partial_{\mathbf{u}}\mathbf{H}(\mathbf{u}) [\mathcal{Z}_1(\mathbf{u}) + \mathcal{Z}_2(\mathbf{u})]$ gives

$$\begin{aligned} & \partial_{\mathbf{u}}\mathbf{H}(\mathbf{u}) [\mathcal{Z}_1(\mathbf{u}) + \mathcal{Z}_2(\mathbf{u})] = \mathbf{0} \\ \Leftrightarrow & -F(\alpha_2) + \sum_{k=1,2} \frac{(-1)^k}{T_k} (p_I - p_k)(v_k - v_I) = 0. \end{aligned} \quad (3.9)$$

Proof. The function t_α is found relying on symbolic computation and it holds as a proof. \square

As explained in (Step 2.a), Equation (3.8) guaranties the existence of an entropy flux \mathbf{G} associated with the mixture entropy \mathbf{H} chosen as in Equation (3.2) by defining the unknown function $t_\alpha(\mathbf{u})$.

Then as described in (Step 2.b), Equation (3.9) relates the interfacial terms (v_I, p_I) . By choosing $F(\alpha_2) = 0$, the condition on $\partial_{\mathbf{u}}\mathbf{H} \times [\mathcal{Z}_1 + \mathcal{Z}_2]$ writes

$$\sum_{k=1,2} \frac{1}{T_k} (p_k - p_I) (v_I - v_k) = 0. \quad (3.10)$$

So now, to obtain a closed model along with a supplementary conservative equation, we can postulate an interfacial velocity v_I and derive the corresponding p_I . We will limit ourselves to defining v_I such that the field associated to v_I is linearly degenerate. In that case, the only admissible interfacial velocities are $v_I = \beta u_1 + (1 - \beta)u_2$ with $\beta \in [0, 1, \alpha_1 \rho_1 / \rho]$ [25]. We will focus on the particular case where $F(\alpha_2) = 0$. We obtain the following results:

- If $v_I = v_k$, then Equation (3.10) returns $p_I = p_{k'}$. $(v_k, p_{k'})$ is the closure proposed first by [14], [21] [22], in the context of deflagration-to-detonation.
- If $v_I = \beta u_1 + (1 - \beta)u_2$ with $\beta = \alpha_1 \rho_1 / \rho$, then Equation (3.10) returns $p_I = \mu p_1 + (1 - \mu)p_2$ with $\mu(\beta) = (1 - \beta)T_2 / (\beta T_1 + (1 - \beta)T_2)$. It is the closure found in [23] among others.

We see that first these closures are a specific case where $F(\alpha_2)$ is chosen to be zero in Equation (3.9). Second, one could have chosen another interfacial velocity v_I and it would have led to another interfacial pressure p_I compatible with an entropy pair.

Remark 3.1. If we had used the extended condition expressed in Equation (2.15), then the condition on $\partial_{\mathbf{u}}\mathbf{H}[\mathcal{Z}_1 + \mathcal{Z}_2]$ would be

$$\sum_{k=1,2} \frac{1}{T_k} [p_k - p_I(\mathbf{u}, \partial_x \mathbf{u})] [v_I(\mathbf{u}, \partial_x \mathbf{u}) - v_k] \partial_x \alpha_k \leq 0 \quad (3.11)$$

$$\Leftrightarrow - \sum_{k=1,2} \frac{1}{T_k} \frac{Z_k}{(Z_1 + Z_2)^2} [p_{k'} - p_k + \text{sgn}(\partial_x \alpha_1) (u_{k'} - v_k) Z_{k'}]^2 \leq 0, \quad (3.12)$$

where Z_k is defined by $Z_k = \rho_k a_k$ with the phase sound speed $a_k^2 = \partial p_k / \partial \rho_k|_{s_k}$. From Equation (3.11), one sees that the dependency on $\partial_x \mathbf{u}$ reduces to $\partial_x \alpha_2$ otherwise some terms would not be signable. Then closures such as the one found through Discrete Element Method (DEM) [24] are obtained

$$v_I = \frac{Z_1 u_1 + Z_2 u_2}{Z_1 + Z_2} + \text{sgn}(\partial_x \alpha_1) \frac{p_2 - p_1}{Z_1 + Z_2}, \quad (3.13)$$

$$p_I = \frac{Z_2 p_1 + Z_1 p_2}{Z_1 + Z_2} + \text{sgn}(\partial_x \alpha_1) \frac{Z_1 Z_2}{Z_1 + Z_2} (u_2 - u_1). \quad (3.14)$$

3.4. Partially miscible phases entropy

Now, let us add a degree of freedom in the thermodynamics by introducing mixing terms in the definition of the entropy \mathbf{H} as in Equation (3.4) to account for partial miscibility of the phases. The added terms, ψ_k , functions of the volume fraction α_k only, are to be determined.

The entropic variables \mathbf{v} are

$$\mathbf{v} = \begin{pmatrix} \sum_{k=1,2} (-1)^{k+1} \frac{p_k}{T_k} \left[1 - \frac{\alpha_k}{r_k} \psi'_k(\alpha_k) \right] \\ \mathbf{v}_2 \\ \mathbf{v}_1 \end{pmatrix} \quad \text{with } \mathbf{v}_k = \frac{1}{T_k} \begin{pmatrix} g_k - 1/2v_k^2 \\ v_k \\ -1 \end{pmatrix} \quad (3.15)$$

Theorem 3.2. *Consider System (3.1). If the mixture entropy is defined as $\mathbf{H} = -\sum_{k=1,2} \alpha_k \rho_k [s_k - \psi_k(\alpha_k)]$ with ψ_k , $k = [1, 2]$, two strictly convex arbitrary functions depending on the volume fraction, then with the decomposition proposed in Equations (3.6), we have*

$$\begin{aligned} & \partial_{\mathbf{u}\mathbf{u}}\mathbf{H} \times \partial_{\mathbf{u}}\mathbf{f} + \partial_{\mathbf{u}}\mathbf{t} \text{ symmetric} \\ \Leftrightarrow t_\alpha(\mathbf{u}) &= F(\alpha_2) + \frac{p_1}{T_1} u_1 \left[1 - \frac{\alpha_1}{r_1} \psi'_1(\alpha_1) \right] - \frac{p_2}{T_2} u_2 \left[1 - \frac{\alpha_2}{r_2} \psi'_2(\alpha_2) \right] \end{aligned} \quad (3.16)$$

with F a strictly convex arbitrary function depending on the volume fraction. As a consequence the condition on $\partial_{\mathbf{u}}\mathbf{H}[\mathcal{Z}_1 + \mathcal{Z}_2]$ gives

$$\begin{aligned} \mathbf{0} &= \partial_{\mathbf{u}}\mathbf{H}(\mathbf{u}) [\mathcal{Z}_1(\mathbf{u}) + \mathcal{Z}_2(\mathbf{u})] \\ \Leftrightarrow 0 &= -F(\alpha_2) + \sum_{k=1,2} (-1)^{k+1} \alpha_k \rho_k \psi'_k(\alpha_k) (u_k - v_I) \\ &+ \sum_{k=1,2} \frac{(-1)^k}{T_k} (p_I - p_k) (v_k - v_I) \end{aligned} \quad (3.17)$$

Again, Equation (3.16) guaranties the existence of an entropy flux $\mathbf{G}(\mathbf{u})$ conditioning the function $t_\alpha(\mathbf{u})$ (Step 2.a). The interfacial quantities (v_I, p_I) and ψ_k are linked by Equation (3.17) (Step 2.b).

The difference with the previous case for immiscible phases is that there are two supplementary unknowns ψ_k , $k = 1, 2$. We thus are free to either postulate first an interfacial velocity v_I and then derive the corresponding p_I and ψ_k or postulate first the functions ψ_k and see what choices we have for the interfacial terms. In the following we investigate the two approaches.

3.4.1. Interfacial closures impacting thermodynamics

Let us postulate v_I and limit ourselves to the case $F(\alpha_2) = 0$. We will again seek a linearly degenerate field for v_I . In such case, the results in Table 1 are obtained.

In Case 1 of Table 1, ψ_k can be interpreted as a configuration energy of phase k as in [14], [21] [22], in the context of deflagration-to-detonation. It is a term defining an interaction of one phase with itself only. More importantly, Equation (3.17) shows that it is not possible to include a configuration energy for each phase when choosing the closure $(v_I, p_I) = (v_k, p_{k'})$.

Table 1: Admissible thermodynamics and model closures obtained by postulating v_I

	v_I	p_I	$(\psi_k, \psi_{k'})$
Case 1	v_k	$p_{k'}$	$(\psi_k, 0)$
Case 2	$\beta u_1 + (1 - \beta)u_2$ $\beta = \alpha_1 \rho_1 / \rho$	$\mu p_1 + (1 - \mu)p_2$ $\mu(\beta) = \frac{(1-\beta)T_2}{\beta T_1 + (1-\beta)T_2}$	$\psi_k(\alpha_k) = \psi_{k'}(\alpha_{k'})$

In Case 2 of Table 1, the condition on the mixing term introduced in Equation (3.5) by [17] is recovered and the closures are the one stated in [25]. However, the condition on the mixing terms imposes a constraint on the volume fraction and thus on the flow topology. Since mixing of the phases should be able to occur disregarding the flow topology, these terms fail to introduce free mixing among the phases.

3.4.2. Thermodynamics impacting interfacial term closures

Since Case 1 and Case 2 of Table 1 do not allow the phases to mix, let us choose first the thermodynamics of the system and induce the admissible interfacial terms.

It has been shown that the mixing entropy of an ideal compressible binary mixture is of the form $\sum_{k=1,2} \alpha_k \ln(\alpha_k)$. Therefore, we choose to define the functions ψ_k by $\psi_k(\alpha_k) = r_k \ln(\alpha_k)$. In this case, the entropy writes

$$H = - \sum_{k=1,2} \alpha_k \rho_k [s_k - r_k \ln(\alpha_k)], \quad (3.18)$$

we now account for quasi-miscibility between the phases.

The condition on t_α degenerates, $t_\alpha = F(\alpha_2)$ and the condition on $\partial_{\mathbf{u}} H[\mathcal{Z}_1 + \mathcal{Z}_2]$ is now

$$-F(\alpha_2) + p_I \left(\frac{u_1 - v_I}{T_1} - \frac{u_2 - v_I}{T_2} \right) = 0. \quad (3.19)$$

It is no more possible to obtain the classic definition on v_I and p_I . In the case $F(\alpha_2) = 0$ two choices are possible to verify Equation (3.19) and summarized in Table 2.

Case 3 of Table 2 proposes a temperature-based averaged velocity for v_I , which does not seem to be physically reasonable. In Case 4, the interfacial pressure must vanish for the system to admit a supplementary conservation equation and the Baer-Nunziato model becomes a conservative system if one assumes the field associated to v_I to be linearly degenerate. One knows how much it simplifies the problem in terms of numerical implementation. This result can be interpreted as an incompatibility

Table 2: Admissible thermodynamics and model closures obtained by postulating ψ_k

	v_I	p_I
Case 3	$\beta u_1 + (1 - \beta)u_2$ with $\beta = T_2/(T_2 - T_1)$	no constraint
Case 4	no constraint	0

between the existence of a mixing process in the thermodynamics of the mixture and an interfacial pressure, that stays meaningful as long as there is an interface between the two phases.

3.4.3. Link with dispersed phase flow

When the thermodynamics accounts for mixing (Case 4 Table 2), the existence of a supplementary conservation equation is incompatible with the interfacial pressure, and thus the nozzling terms $p_I \partial_x \alpha_k$ vanish.

In separated two-phase flows, these terms are known to be necessary to preserve uniformity in velocity and pressure of the flow during its temporal evolution [29] and are usually compared to the terms obtained in a single gas with a variable section [34]. Whereas these arguments seem valid for separated two-phase flows, one may question the role these terms play in a dispersed phase flows.

Taking the particular case $p_I = 0$ and $p_2 = 0$ in the Baer-Nunziato model seems to lead to a system of equations similar to one that would describe a flow of incompressible suspended particles, where 1 would denote the carrier phase and 2 the dispersed phase. Doing so, one recovers not only the Marble model [35], which proposes a pressureless gas dynamic equations for the particle phase, valid in the limit where $\alpha_2 < 10^{-3}$, but also the model obtained by Sainsaulieu [36] in the asymptotic limit where the volume fraction of the particles $\alpha_2 \rightarrow 0$.

Nevertheless, even if the partial differential equations are alike, the thermodynamics associated to Marble and Sainsaulieu models differ from the one we propose for the Baer-Nunziato model. The latter accounts for compressibility of the two phases and partial miscibility whereas the thermodynamics of the Marble model assumes incompressibility of the particles and non-miscibility between the two phases.

To conclude, if one aims at unifying the description of both separated phases and dispersed flow through a unique model, the thermodynamics must be treated together with the system modelling.

4. Application to the plasma model

The multicomponent fluid modeling of plasmas flows out of thermal equilibrium has been derived rigorously from kinetic theory using a multi-scale Chapman-Enskog expansion mixing a hyperbolic scaling for the heavy species with a parabolic scaling for the electrons [15]. The system takes the form

$$\partial_t \mathbf{u} + [\partial_{\mathbf{u}} \mathbf{f}(\mathbf{u}) + \mathcal{N}(\mathbf{u})] \partial_x \mathbf{u} = \partial_x (\mathcal{D}(\mathbf{u}) \partial_x \mathbf{u}), \quad (4.1)$$

with

$$\partial_{\mathbf{u}} \mathbf{f}(\mathbf{u}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ (\kappa/2 - 1)v^2 & (2 - \kappa)v & \kappa & 0 & 0 \\ (\kappa/2 v^2 - \frac{h^{tot}}{\rho_h})v & \frac{h^{tot}}{\rho_h} - \kappa v^2 & (1 + \kappa)v & 0 & 0 \\ -\frac{\rho_e}{\rho_h} v & \frac{\rho_e}{\rho_h} & 0 & v & 0 \\ -\frac{\rho_e \epsilon_e}{\rho_h} v & \frac{\rho_e \epsilon_e}{\rho_h} & 0 & 0 & v \end{pmatrix}, \quad (4.2)$$

$$\mathcal{N}(\mathbf{u}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{\rho_e \epsilon_e}{\rho_h} \kappa v & \frac{\rho_e \epsilon_e}{\rho_h} \kappa & 0 & 0 & 0 \end{pmatrix}, \quad (4.3)$$

$$\mathcal{D}(\mathbf{u}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\lambda \kappa \epsilon_e}{\rho_e} & \frac{\lambda \kappa \epsilon_e}{\rho_e} + \gamma D \\ 0 & 0 & 0 & 0 & \frac{D \kappa}{T_e} \\ 0 & 0 & 0 & -\frac{\lambda \kappa \epsilon_e}{\rho_e} & \frac{\lambda \kappa \epsilon_e}{\rho_e} + \gamma D \end{pmatrix}, \quad (4.4)$$

where the column vector $\mathbf{u} \in \mathbb{R}^5$ is defined by $\mathbf{u}^T = (\rho_h, \rho_h v, E, \rho_e, \rho_e \epsilon_e)$ with ρ_h is the density of the heavy particles, v the hydrodynamic velocity, E the total energy defined by $E = 1/2 \rho_h v^2 + \rho_h \epsilon_h + \rho_e \epsilon_e$, ϵ_h the internal energy of the heavy particles, ρ_e the density of the electrons, ϵ_e the internal energy of the electrons, h^{tot} the total enthalpy defined by $h^{tot} = E + p$ with $p = p_h + p_e$, T_e the temperature of the electrons, the constant κ defined by $\kappa = \gamma - 1$ with γ the isentropic coefficient, p_h is the pressure of the heavy particles and p_e is the pressure of the electrons. In the diffusive terms, λ is the electron thermal conductivity, D the electron diffusion coefficient.

Concerning the thermodynamics, it can be obtained from kinetic theory. The electrons and the heavy particles thermodynamics are defined by a ideal gas equation of state, and they share both the same isentropic coefficient: $p_h = \kappa \rho_h \epsilon_h$, $p_e = \kappa \rho_e \epsilon_e$ where p_h is the pressure of the heavy particles and p_e is the pressure of the electrons, r is the constant of the gas $r = c_v \kappa$ with c_v the calorific heat at constant volume, the model being adimensionalized $r = c_v (\gamma - 1) = 1$.

The model is naturally hyperbolic [15] and also involves second-order terms and eventually source terms [18]. Here we considered the homogeneous form.

In this section, we would like to derive the usual supplementary conservative equation found by [15] and show that it is unique, to attest the effectiveness of the theory.

4.1. Decomposition

We need to proceed to the decomposition of the conservative and non conservative terms of System (4.1). We restrict ourselves again to the decomposition proposed in Definition (2.4) and we add a degree of liberty to each non-null non-conservative components by defining $\mathbf{t} : \mathbf{u} \in \Omega \mapsto \mathbb{R}^5$ as $\mathbf{t}(\mathbf{u})^T = (t_1(\mathbf{u}), t_2(\mathbf{u}), 0, 0, 0)$ such that the following decompositions are obtained

$$(\partial_{\mathbf{u}}\mathbf{H}(\mathbf{u}) [\mathcal{C}_1(\mathbf{u}) + \mathcal{C}_2(\mathbf{u})])^T = \mathbf{v}(\mathbf{u}) \cdot \partial_{\mathbf{u}}\mathbf{f}(\mathbf{u}) + \begin{pmatrix} t_1(\mathbf{u}) \\ t_2(\mathbf{u}) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (4.5)$$

$$(\partial_{\mathbf{u}}\mathbf{H}(\mathbf{u}) [\mathcal{Z}_1(\mathbf{u}) + \mathcal{Z}_2(\mathbf{u})])^T = \begin{pmatrix} -t_1(\mathbf{u}) - \frac{\rho_e}{\rho_h} \left(1 - \frac{T_e}{T_h}\right) v \\ -t_2(\mathbf{u}) + \frac{\rho_e}{\rho_h} \left(1 - \frac{T_e}{T_h}\right) \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (4.6)$$

The unknown scalar functions $\mathbf{t}_k(\mathbf{u})$ give the possibility to fractions of the non-conservative terms to be given to the matrix \mathcal{C}_k .

4.2. Ideal Gas entropy

The entropy $\mathbf{H} : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ for two perfect gas is defined as

$$\mathbf{H} = -\rho_h s_h - \rho_e s_e, \quad (4.7)$$

with the partial entropies defined by

$$s_h = c_v \ln \left(\frac{p_h}{\kappa \rho_h^{\frac{1}{\gamma_h}}} \right), \quad s_e = c_v \ln \left(\frac{p_e}{\kappa \rho_e^{\frac{1}{\gamma_e}}} \right). \quad (4.8)$$

This entropy includes mixing between the electrons and the heavy particles. Thus, we start applying our method (*Step 1*) by postulating \mathbf{H} as in Equation (4.7). The

entropic variables \mathbf{v} are then

$$\mathbf{v} = \begin{pmatrix} \frac{1}{T_h} (g_h - 1/2v^2) \\ \frac{1}{T_h} v \\ -\frac{1}{T_h} \\ \frac{1}{T_e} g_e \\ \frac{1}{T_h} - \frac{1}{T_e} \end{pmatrix}, \quad (4.9)$$

with g_k the Gibbs free energy, $g_k = \epsilon_k + p_k/\rho_k - T_k s_k$.

Remark 4.1. In the fourth component of the entropic variable, the kinetic energy of the electrons has vanished. This is due to the Low Mach assumption made for the electrons.

We now apply the conditions to determine $t_k(\mathbf{u})$.

Theorem 4.1. Consider System (4.1). If the mixture entropy is defined as $\mathbf{H} = -\rho_h s_h - \rho_e s_e$, then with the decomposition proposed in Equations (4.5)

$$\begin{aligned} & \partial_{\mathbf{u}\mathbf{u}}\mathbf{H}(\mathbf{u}) \times \partial_{\mathbf{u}}\mathbf{f}(\mathbf{u}) + \partial_{\mathbf{u}}\mathbf{t}(\mathbf{u}) \text{ symmetric} \\ \Leftrightarrow t_1(\mathbf{u}) &= \frac{\rho_e}{\rho_h} \left(1 - \frac{T_e}{T_h}\right) v \text{ and } t_2(\mathbf{u}) = -\frac{\rho_e}{\rho_h} \left(1 - \frac{T_e}{T_h}\right), \end{aligned} \quad (4.10)$$

and the condition on $\partial_{\mathbf{u}}\mathbf{H}(\mathbf{u}) [\mathcal{Z}_1(\mathbf{u}) + \mathcal{Z}_2(\mathbf{u})]$ is

$$\partial_{\mathbf{u}}\mathbf{H}(\mathbf{u}) [\mathcal{Z}_1(\mathbf{u}) + \mathcal{Z}_2(\mathbf{u})] = (0, 0, 0, 0, 0) \quad (4.11)$$

Proof. Using MapleTM, we find

$$\begin{aligned} & \partial_{\mathbf{u}\mathbf{u}}\mathbf{H}(\mathbf{u}) \times \partial_{\mathbf{u}}\mathbf{f}(\mathbf{u}) + \partial_{\mathbf{u}}\mathbf{t}(\mathbf{u}) \text{ symmetric} \\ \Leftrightarrow t_1(\mathbf{u}) &= \frac{\rho_e}{\rho_h} \left(1 - \frac{T_e}{T_h}\right) v + \int [-v\partial_v F_1(\rho_h, v) + \rho_h \partial_{\rho_h} F_1(\rho_h, v)] dv + F_2(\rho_h) \\ \text{and } t_2(\mathbf{u}) &= -\frac{\rho_e}{\rho_h} \left(1 - \frac{T_e}{T_h}\right) + F_1(\rho_h, v) \end{aligned}$$

with F_1, F_2 two arbitrary functions and the condition on $\partial_{\mathbf{u}}\mathbf{H}(\mathbf{u}) [\mathcal{Z}_1(\mathbf{u}) + \mathcal{Z}_2(\mathbf{u})]$ is

$$\begin{aligned} (\partial_{\mathbf{u}}\mathbf{H}(\mathbf{u}) [\mathcal{Z}_1(\mathbf{u}) + \mathcal{Z}_2(\mathbf{u})])^T &= \begin{pmatrix} -\int [-v\partial_v F_1(\rho_h, v) + \rho_h \partial_{\rho_h} F_1(\rho_h, v)] dv - F_2(\rho_h) \\ -F_1(\rho_h, v) \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \mathbf{0}. \end{aligned}$$

One sees that the last equation imposes first $F_1 = 0$ and thus $F_2 = 0$. Reinjecting these terms into the first equation gives the result. \square

As explained in (*Step 2.a*), the Equation (4.10) guaranties the existence of an entropy flux $\mathbf{G} : \mathbf{u} \in \Omega \mapsto \mathbb{R}$ associated with the entropy \mathbf{H} defined in Equation (4.7) by solving the unknown functions $t_1(\mathbf{u})$ and $t_2(\mathbf{u})$.

Therefore, for the entropy \mathbf{H} defined in Equation (4.7), there is a unique decomposition which ensures the existence of a supplementary conservative equation which is given by

$$(\partial_{\mathbf{u}}\mathbf{H}[\mathcal{C}_1 + \mathcal{C}_2])^T = \mathbf{v}^T \cdot \partial_{\mathbf{u}}\mathbf{f}(\mathbf{u}) + \begin{pmatrix} \frac{\rho_e}{\rho_h} \left(1 - \frac{T_e}{T_h}\right) v \\ \frac{\rho_e}{\rho_h} \left(1 - \frac{T_e}{T_h}\right) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (4.12)$$

$$\partial_{\mathbf{u}}\mathbf{H}[\mathcal{Z}_1 + \mathcal{Z}_2] = \mathbf{0}. \quad (4.13)$$

It leads to the following entropy flux couple

$$\mathbf{H} = -\rho_h s_h - \rho_e s_e, \quad (4.14)$$

$$\mathbf{G} = -(\rho_h s_h + \rho_e s_e) v. \quad (4.15)$$

The theory recovers the supplementary conservative equation already found in the literature from the kinetic theory [15].

5. Conclusion

In the present contribution, we have extended the theory on the existence of a supplementary conservative equation to first-order nonlinear system of partial differential equation including non-conservative terms.

Given a reasonable choice in the combination of the conservative and non-conservative terms, we have been able to show how to use the theory to design or analyze systems by means of computer algebra on two applications chosen for their numerous differences in terms of model and thermodynamics closure as well as the nature of the waves impacted by the non-conservative terms.

Firstly, applied to the Baer-Nunziato two-phase flow model derived from rational thermodynamics, the theory has brought about supplementary conservative equations together with constraints on the interfacial quantities and the definition of the thermodynamics for non-miscible fluids and also when accounting for some level of mixing of the two phases. A new closure for the interfacial quantities has

been proposed and leads to a conservative system. Secondly, for a plasma model obtained rigorously from the kinetic theory of gases, where the thermodynamics is also provided, the approach allows to recover as unique the supplementary conservation equation related to the kinetic entropy and is thus assessed.

A strongly connected question for such systems is the ability to derive an entropic symmetrization in the sense of Godunov-Mock and the related constraints on the decomposition, as well as the study of the spectrum and hyperbolicity. The proposed framework introduced in this paper allows to shed some light on these questions and the two systems, we have applied the theory to, are especially suited for such a purpose. This is the subject of our current research.

Acknowledgment

The authors would like to acknowledge the support of a CNES/ONERA PhD Grant for P. Cordesse and the help of M. Théron (CNES). They would like to express their special thanks to F. Coquel, S. Kokh, V. Giovangigli and A. Murrone for their invaluable help and numerous pieces of advice during the writing of the paper. Part of this work was conducted during the Summer Program 2018 at NASA Ames Research Center and the support and help of Nagi N. Mansour is also gratefully acknowledged.

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