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High-Gain Observer Design for a Class of Hyperbolic Systems of Balance Laws

Constantinos Kitsos, Gildas Besançon, and Christophe Prieur

Abstract—Considering a class of hyperbolic systems of balance laws with distributed measurements, and possibly distributed effects of known inputs, a structure suitable for *uniform observability* is first emphasized. Sufficient conditions for an explicit high-gain observer design are then derived for special cases of such systems. The stability of the related observer estimation error is fully established by means of Lyapunov-based techniques, and a numerical example finally illustrates the results.

Keywords: high-gain observers, nonlinear hyperbolic systems, uniform observability.

I. INTRODUCTION

Among the various approaches developed towards state estimation and observer design for finite-dimensional nonlinear continuous-time systems, the so-called *high-gain observer* [8] remains the most popular one, and has motivated many contributions in the literature. It relies on the idea of choosing appropriately large observer gain, such that it can dominate the effect of the nonlinearities on the observer error. This design can be implemented for a class of nonlinear systems in an observable canonical form, typical of *uniform observability* for systems with inputs [7].

Beyond the control theory approaches for finite-dimensional systems, infinite-dimensional ones, and more specifically hyperbolic systems of balance laws, have gained significant attention, focusing on stability analysis and controllability. A large number of distributed physical systems are indeed described by hyperbolic equations (see e.g. [1], Chap. 1). The stability analysis of such systems has been extensively explored via Lyapunov techniques (see again [1], Chap. 4, 6). A few observer techniques have also been investigated, but mostly for cases with boundary measurements, via space discretization and high-gain design, as in [2], or direct infinite-dimension-based Lyapunov techniques, as in [4] (see also [3]), backstepping approach [11], or optimization [12] to cite a few. For distributed measurements, some semigroup-based methods have been investigated (see e.g. [5] and references therein). But to the best of our knowledge, the extension of high-gain observer design to hyperbolic systems of balance laws with distributed measurements has not yet been explored.

Our main contribution here, compared to the existing literature, is to construct a high-gain observer for a class

of nonlinear hyperbolic systems of balance laws with distributed measurements, by exploiting its full dynamics and avoiding discretization techniques. In this way, we extend the fundamental high-gain observers for finite-dimensional systems to the infinite-dimensional case and prove analogous results. Particularly, we provide a method at designing a high-gain observer so as to approach (arbitrarily fast) the states of the system described by hyperbolic laws, by exploiting the knowledge of the input and output functions and the input/output map of the boundary conditions. First, we introduce an appropriate observable form for that, aiming at extending the approaches of [7] and [8] to infinite dimensions. The methodology we follow for the observer design then relies on conventional methods for proving stability for the estimation error via Lyapunov functionals (see [1], Chap. 4 and 6 for extensive presentation of many stabilization methods in various norms for nonlinear hyperbolic systems). We establish well-posedness of the observer system and then we prove asymptotic stability of the estimation error in appropriate norms for sufficiently large high-gain constants.

The paper is organized as follows: In Section II, we introduce an observable form, on which possible observer designs can rely. In Section III we then examine the high-gain observer design for a subclass of the aforementioned form that has specific structural properties. Conditions under which the observer system is well-posed are presented in parallel to the observer convergence in the C^0 -norm (Theorem 3.1). Finally, we illustrate the results of Section III with an example of a two-dimensional nonlinear hyperbolic system, by performing simulations.

Notation: For a given $x \in \mathbb{R}^n$, $|x|$ denotes its usual Euclidean norm. For a given constant matrix $A \in \mathbb{R}^{m \times n}$, A^T denotes its transpose and $|A| := \sup\{|Ax|, |x| = 1\}$ is its induced norm. By $\text{eig}_{\min}(A)$, $\text{eig}_{\max}(A)$ we denote the minimum and maximum eigenvalue, respectively, of a square matrix A . By $B(r)$ we denote the ball in \mathbb{R}^n with center at zero and radius r . For a continuous map $[0, +\infty) \times [0, 1] \ni (t, x) \rightarrow \xi(t, x) \in \mathbb{R}^n$ we adopt the notation $\|\xi(t, \cdot)\|_0 := \|\xi\|_0 := \sup\{|\xi(t, x)|, x \in [0, 1]\}$.

II. A STRUCTURE FOR "HIGH-GAIN OBSERVABILITY"

The famous high-gain observer design available for finite-dimensional nonlinear systems is related to a strong observability property, corresponding to the notion of "uniform observability" for systems with inputs.

Our point here is to extend this notion to a class of hyperbolic systems of balance laws, which is described as

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follows:

$$\begin{aligned} \xi_t + D(\xi)\xi_x + u_1 E(\xi)\xi_x &= A\xi + f(\xi) + u_2 g(\xi), \\ \forall t \in [0, +\infty), \forall x \in [0, 1] \end{aligned} \quad (2.1a)$$

with distributed measurement

$$y = C\xi \quad (2.1b)$$

where $\xi = (\xi_1, \dots, \xi_n)^T : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}^n$ is the vector of state variables, $y : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}$ is the output, $u_1, u_2 : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}$ are input functions, and the system satisfies some "triangular" form as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ \vdots & & & & 1 \\ 0 & \cdots & & & 0 \end{bmatrix}, C = [1 \quad 0 \quad \cdots \quad 0],$$

$$f(\xi) = [f_1(\xi_1) \quad f_2(\xi_1, \xi_2) \quad \cdots \quad f_n(\xi_1, \dots, \xi_n)]^T,$$

$$g(\xi) = [g_1(\xi_1) \quad g_2(\xi_1, \xi_2) \quad \cdots \quad g_n(\xi_1, \dots, \xi_n)]^T,$$

$$D(\xi) = \begin{bmatrix} d_{11} & 0 & \cdots & & 0 \\ d_{21} & d_{22} & 0 & \cdots & 0 \\ \vdots & \ddots & & & \\ d_{n-1,1} & \cdots & & d_{n-1,n-1} & 0 \\ d_{n1} & d_{n2} & \cdots & & d_{nn} \end{bmatrix},$$

$$E(\xi) = \begin{bmatrix} e_{11} & 0 & \cdots & & 0 \\ e_{21} & e_{22} & 0 & \cdots & 0 \\ \vdots & \ddots & & & \\ e_{n-1,1} & \cdots & & e_{n-1,n-1} & 0 \\ e_{n1} & e_{n2} & \cdots & & e_{nn} \end{bmatrix},$$

with $d_{ij} = d_{ij}(\xi_1, \dots, \xi_i), e_{ij} = e_{ij}(\xi_1, \dots, \xi_i), i, j = 1, \dots, n$. We further consider initial and boundary conditions of the following form:

$$\xi(0, x) = \xi^0(x) \quad (2.2a)$$

$$\mathcal{B}(t, \xi(t, 0), \xi(t, 1)) = 0 \quad (2.2b)$$

with $\xi^0 : [0, 1] \rightarrow \mathbb{R}^n, \mathcal{B} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.

For well-posedness purposes of the Cauchy problem described by (2.1a), (2.2), we assume that $D(\xi) + u_1 E(\xi) \neq 0, \forall \xi \in \mathbb{R}^n, t \in [0, +\infty), x \in [0, 1]$, with $D, E, f, g, \xi^0, \mathcal{B}$ and inputs u_1, u_2 sufficiently smooth in their arguments. More precise well-posedness conditions for such nonlinear inhomogeneous systems of conservation laws with source terms are investigated for instance in [9] and [14].

Let us now show that the form (2.1) is an observable one, canonical in some sense, analogously to the case of finite-dimensional systems. To that end, let us consider a more general hyperbolic system of the form

$$\begin{aligned} \bar{\xi}_t + \bar{D}(\bar{\xi})\bar{\xi}_x + u_1 \bar{E}(\bar{\xi})\bar{\xi}_x &= \bar{f}(\bar{\xi}) + u_2 \bar{g}(\bar{\xi}) = 0, \\ \forall t \in [0, +\infty), \forall x \in [0, 1] \end{aligned} \quad (2.3a)$$

under distributed measurements throughout the domain given by

$$y = h(\bar{\xi}) \quad (2.3b)$$

where $h : \mathbb{R}^n \rightarrow \mathbb{R}$. We also consider initial and boundary conditions of the form

$$\bar{\xi}(0, x) = \bar{\xi}^0(x) \quad (2.4a)$$

$$\bar{\mathcal{B}}(t, \bar{\xi}(t, 0), \bar{\xi}(t, 1)) = 0 \quad (2.4b)$$

where $\bar{\xi}^0 : [0, 1] \rightarrow \mathbb{R}^n, \bar{\mathcal{B}} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. We assume, similarly to the case of system (2.1), that system (2.3) with boundary and initial conditions (2.4) possesses regularity properties that allow it (along with other properties described in [9]) to be well-posed, i.e., $\bar{D}, \bar{E}, \bar{f}, \bar{g}, h, \bar{\xi}^0$ and $\bar{\mathcal{B}}$ are sufficiently smooth in their arguments and it holds $\bar{D}(\bar{\xi}) + u_1 \bar{E}(\bar{\xi}) \neq 0, \forall \bar{\xi} \in \mathbb{R}^n, t \in [0, +\infty), x \in [0, 1]$. By extension of the notion available for finite-dimensional nonlinear systems, we can now provide a definition for *uniform observability* of system (2.3)-(2.4) as follows:

Definition 2.1: Given $T > 0$, system (2.3)-(2.4) is said to be *uniformly observable on $[0, T]$* if for any pair of different initial states ξ^0 and $\tilde{\xi}^0$ and any pair of inputs u_1, u_2 defined on $[0, T] \times [0, 1]$ for which solutions $\xi, \tilde{\xi}$ exist, there exists t in $[0, T]$ and $x \in [0, 1]$ such that: $h(\xi(t, x)) \neq h(\tilde{\xi}(t, x))$. The system is *uniformly observable* if the above holds for any T .

We can then state the following result:

Proposition 2.1: Consider a system of described by (2.3)-(2.4) with $h(\bar{\xi}) = C\bar{\xi}, \bar{f} = A\bar{\xi} + f$ and \bar{D} of the form D for A, C, f, D as in (2.1). Then the system is uniformly observable if and only if \bar{E} and \bar{g} satisfy the same triangular structures as E and g in (2.1).

Proof: Let us denote $e_{ij}(\xi) := e_{ij}(\xi_1, \dots, \xi_i), g_i(\xi) := g_i(\xi_1, \dots, \xi_i)$. To prove the sufficiency part, let us consider two initial conditions $\xi^0, \tilde{\xi}^0$ such that $\forall i = 1, \dots, i^0 : \xi_i^0(x) = \tilde{\xi}_i^0(x), \forall x \in [0, 1]$, where $i^0 < n$ and also $\xi_{i^0+1}^0(x) \neq \tilde{\xi}_{i^0+1}^0(x)$ for some $x \in [0, 1]$. From the canonical form (2.1), we have for all $x \in [0, 1], g_j(\xi^0(x)) = g_j(\tilde{\xi}^0(x))$ and $e_{jk}(\xi^0(x)) = e_{jk}(\tilde{\xi}^0(x))$ for $j, k = 1, \dots, i^0$. For any u_1, u_2 we obtain from (2.1): $\partial_t \xi_{i^0}(t, x) - \partial_t \tilde{\xi}_{i^0}(t, x) = \xi_{i^0+1}(t, x) - \tilde{\xi}_{i^0+1}(t, x)$, thus, by integrating, $\xi_{i^0}(t, x) - \tilde{\xi}_{i^0}(t, x) = \xi_{i^0}^0(x) - \tilde{\xi}_{i^0}^0(x) + \int_0^t (\xi_{i^0+1}(s, x) - \tilde{\xi}_{i^0+1}(s, x)) ds$. Consequently, there exists sufficiently small time T such that $\xi_{i^0}(t, x) \neq \tilde{\xi}_{i^0}(t, x), \forall t \in [0, T]$, for some $x \in [0, 1]$. The iteration of this approach leads to the conclusion that the same applies for ξ_1 and $\tilde{\xi}_1$. We conclude that for any u_1, u_2 , different initial conditions produce different outputs (the initial conditions are distinguished by any set u_1, u_2) and, therefore, system (1.1) is uniformly observable.

Necessity is proved by contradiction: assume that \bar{g} or \bar{E} do not satisfy the appropriate triangular form, then we can show that uniform observability is not satisfied. Consider for instance the first case when \bar{g} does not satisfy the form of g (the other case is omitted since it can be treated in the same way). This means that there exists $i^0 < n$, such that $\bar{g}_{i^0} = \bar{g}_{i^0}(\xi_1, \dots, \xi_{i^0}, \xi_{j^0})$ for some $j^0 > i^0$. One can then

select initial conditions ξ^0 and $\tilde{\xi}^0$ such that $\xi_k^0(x) = \tilde{\xi}_k^0(x)$ for $k \leq i^0$ and $\xi_{j^0}^0(x) \neq \tilde{\xi}_{j^0}^0(x)$ for any $x \in [0, 1]$. Then, one can set $u_1 := 0$ and define u_2 as:

$$u_2(t, x) = -\frac{\xi_{i^0+1}(t, x)}{\Delta g_{i^0}(t, x)}, t \geq 0, x \in [0, 1] \quad (2.5)$$

where $\Delta g_{i^0}(t, x) := \bar{g}_{i^0}(\xi_1(t, x), \dots, \xi_{j^0}(t, x)) - \bar{g}_{i^0}(\tilde{\xi}_1(t, x), \dots, \tilde{\xi}_{j^0}(t, x))$. Since we have $\bar{g}_{i^0}(\xi_1^0(x), \dots, \xi_{j^0}^0(x)) \neq \bar{g}_{i^0}(\tilde{\xi}_1^0(x), \dots, \tilde{\xi}_{j^0}^0(x))$, there is a sufficiently small time T such that u_2 is well-defined on $[0, T] \times [0, 1]$. Under the previous hypothesis and with these inputs, we see that for $t \in [0, T]$, $\partial_t(\xi_i(t, x) - \tilde{\xi}_i(t, x)) = 0$ and $\xi_i^0(x) - \tilde{\xi}_i^0(x) = 0, \forall x \in [0, 1]$ for $i = 1, \dots, i^0$. Therefore, $\xi_1(t, x) = \tilde{\xi}_1(t, x), \forall (t, x) \in [0, T] \times [0, 1]$. The system is not uniformly observable, since for the set of the previous inputs (u_1, u_2) defined on $[0, T] \times [0, 1]$, different initial conditions provide the same outputs. ■

III. A RESULT ON HIGH-GAIN OBSERVER DESIGN

Although system (2.1) is in appropriate form for uniform observability, designing an observer for it remains a tricky problem. In the present section, we propose a solution for the special case where matrices D, E take the form of diagonal matrices of identical entries only depending on $C\xi$. In addition, just in order to simplify the presentation (and with no restriction on the methodology) we propose to omit terms $D(\xi)\xi_x$ and $f(\xi)$ in the system description. This means that we consider systems of the following form:

$$\xi_t + u_1 \Lambda(\xi_1)\xi_x = A\xi + u_2 f(\xi), \forall t \in [0, +\infty), \forall x \in [0, 1] \quad (3.1a)$$

with distributed measurement

$$y = C\xi \quad (3.1b)$$

where Λ satisfies:

$$\Lambda(\xi_1) = \lambda(\xi_1)I_{n \times n} \quad (3.1c)$$

for some function $\lambda(\cdot)$, and $\xi : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}^n$ is the state, $y : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}$ is the output, $u_1, u_2 : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}$ are the inputs, while matrices A and C are as in (2.1). Also, f is of the form $f(\xi) = [f_1(\xi_1) \ f_2(\xi_1, \xi_2) \ \dots \ f_n(\xi_1, \dots, \xi_n)]^T$. We consider boundary conditions of the form

$$\xi(t, 0) = \mathcal{H}(t, \xi(t, 1)) \quad (3.2)$$

where $\mathcal{H} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. We denote by $\xi^0(x) := \xi(0, x), x \in [0, 1]$ the unknown initial condition. Primarily, we make an assumption on the well-posedness of the system under consideration (3.1), i.e., the existence of unique classical solutions for all times $t \geq 0$ for certain initial conditions ξ^0 .

A1. There exist initial conditions $\xi^0 \in C^1([0, 1]; \mathbb{R}^n)$ satisfying zero-order and one-order compatibility conditions, for which the Cauchy problem (3.1a)-(3.2) has a unique classical solution in $[0, +\infty) \times [0, 1]$.

To proceed to the observer design, we need to make the following crucial assumptions, the first two of which guarantee the well-posedness property of the observer system

(along with Assumption A1), while the last one is crucial in the Lyapunov analysis.

A2. Functions u_1 and u_2 are continuous, u_1 and u_2 are C^1 in x , λ is C^1 in its argument, and f and \mathcal{H} are of class C^1 in their arguments.

A3. We have the following bounds: $\lambda(y) > 0, \forall y \in \mathbb{R}$ and $u_1(t, x) > 0, \forall (t, x) \in [0, +\infty) \times [0, 1]$. Also, $\nabla_\xi f(\cdot)$ is bounded on \mathbb{R}^n by L_f , $\nabla_\xi \mathcal{H}(\cdot, \cdot)$ is bounded on $\mathbb{R} \times \mathbb{R}^n$ by $L_{\mathcal{H}}$ and $u_2(\cdot, \cdot)$ is bounded on $[0, +\infty) \times [0, 1]$, i.e., there exists $\bar{u}_2 > 0$, such that $|u_2(t, x)| \leq \bar{u}_2, \forall t \in [0, +\infty), x \in [0, 1]$.

A4. For initial conditions for which A1 holds, $\xi_1(\cdot, \cdot)$ and $\partial_x \xi_1(\cdot, \cdot)$ are bounded on $[0, +\infty) \times [0, 1]$. Namely, there exist constants $\bar{y}, \bar{y}' > 0$, such that $|\xi_1| \leq \bar{y}, |\partial_x \xi_1(t, x)| \leq \bar{y}', \forall t \in [0, +\infty), x \in [0, 1]$. Furthermore, there exist constants $\underline{u}_1, \bar{u}_1, \bar{u}'_1 > 0$, such that $\forall t \in [0, +\infty), x \in [0, 1], \underline{u}_1 \leq u_1(t, x) \leq \bar{u}_1$ and $|\partial_x u_1(t, x)| \leq \bar{u}'_1$.

The assumed strict positiveness of both λ and u_1 in Assumption A2 can be alternated randomly, so to guarantee strict positiveness or negativeness of the product λu_1 . The results of the present section remain unchanged for any of these cases. Furthermore, by virtue of Assumptions A2, A3 and A4 (continuity of λ and of its derivative λ' and boundedness of y), there exist constants $\underline{\lambda}, \bar{\lambda}, \bar{\lambda}' > 0$, such that $\forall y \in B(\bar{y}), \underline{\lambda} \leq \lambda(y) \leq \bar{\lambda}, |\lambda'(y)| \leq \bar{\lambda}'$.

We now state our main result, concerning the high-gain observer design problem.

Theorem 3.1: Consider system (3.1)-(3.2) with initial conditions as in Assumption A1 and suppose that Assumptions A2-A4 hold. Let also P be a positive definite symmetric matrix and $K = [k_1 \ \dots \ k_n]^T, k_i \in \mathbb{R}, i = 1, \dots, n$ satisfying¹

$$(A + KC)^T P + P(A + KC) = -I_{n \times n} \quad (3.3)$$

Then, the system

$$\hat{\xi}_t + u_1 \Lambda(y)\hat{\xi}_x = A\hat{\xi} + u_2 f(\hat{\xi}) - \Theta K(y - C\hat{\xi}), t \geq 0, x \in [0, 1] \quad (3.4)$$

with $\Theta = \text{diag}\{\theta^i, i = 1, \dots, n\}, \theta > 1$, boundary conditions

$$\hat{\xi}(t, 0) = \mathcal{H}(t, \hat{\xi}(t, 1)) \quad (3.5)$$

and initial condition $\hat{\xi}^0 \in C^1([0, 1]; \mathbb{R}^n)$, with $\hat{\xi}(0, x) = \xi^0(x)$, satisfying zero-order and one-order compatibility conditions, is a well-posed high-gain observer, in the sense that it admits a unique classical solution in $[0, +\infty) \times [0, 1]$ on one hand, providing an estimate for the state of (3.1a) for θ large enough on the other hand. More precisely, there exists a constant $\theta_0 \geq 1$, such that for every $\theta > \theta_0$, there exist constants $l, \kappa > 0$, such that: $\|\xi(t, \cdot) - \hat{\xi}(t, \cdot)\|_0 \leq l e^{-\kappa t} \|\xi^0(\cdot) - \hat{\xi}^0(\cdot)\|_0$. The observer convergence rate κ is adjustable by the choice of the high-gain constant θ and can become arbitrarily large.

Proof: We first note that from Assumption A1 there exist unique solutions in $[0, +\infty) \times [0, 1]$ for system (3.1a)

¹This is always possible since (A, C) is observable.

with initial condition $\xi^0(x) = \xi^0(0, x)$ (satisfying zero-order and one-order compatibility conditions) and with boundary conditions satisfying (3.2). To show the existence and uniqueness of global classical solutions of the observer system, we invoke Theorem 2.1 in [10]. The observer system (3.4) under boundary conditions (3.5) and for any initial condition $\hat{\xi}^0 \in C^1([0, 1]; \mathbb{R}^n)$ satisfying zero-order and one-order compatibility conditions, with $\xi^0(x) := \hat{\xi}^0(0, x)$, belongs to the same class of systems described in [10] and under Assumptions A2 and A3 we apply Theorem 2.1 therein, which states that there exists a unique classical C^1 solution $\hat{\xi}$ in $[0, +\infty) \times [0, 1]$.

We now define the linearly transformed observation error by

$$\epsilon = \Theta^{-1}(\hat{\xi} - \xi) \quad (3.6)$$

where $\Theta = \text{diag}\{\theta^i, i = 1, \dots, n\}$, $\theta > 1$. The observer error satisfies the equation

$$\epsilon_t + u_1 \Lambda(y) \epsilon_x = \theta(A + KC)\epsilon + u_2 \delta f \quad (3.7)$$

where $\delta f := \Theta^{-1}(f(\hat{\xi}) - f(\xi))$. Further, the following equation is satisfied on the boundaries, as a consequence of (3.2) and (3.5):

$$\epsilon(t, 0) = \Theta^{-1}(\mathcal{H}(t, \hat{\xi}(t, 1)) - \mathcal{H}(t, \xi(t, 1))) \quad (3.8)$$

We now derive exponential stability of the estimation error in the C^0 -norm. For all continuously differentiable functions $\epsilon : [0, 1] \rightarrow \mathbb{R}^n$, let us consider the non-negative functional for $p \in \mathbb{N}$:

$$W_p[\epsilon] := \left(\int_0^1 q(x) (e^{\mu x} \epsilon^T P \epsilon)^p dx \right)^{1/p} \quad (3.9)$$

for given positive definite symmetric non-diagonal matrix P , $q(\cdot)$ given by

$$q(x) := \left(\frac{\bar{\lambda} \bar{u}_1}{\lambda \underline{u}_1} - 1 \right) x + 1, x \in [0, 1] \quad (3.10)$$

and $\mu \in \mathbb{R}$ given by

$$\mu = (2n - 2) \ln \theta + 2 \ln L_{\mathcal{H}} + \ln \frac{\text{eig}_{\max}(P)}{\text{eig}_{\min}(P)} \quad (3.11)$$

The choice of the above-mentioned functional is inspired by [6], Theorem 3.3, where an analogous p -functional is chosen and then by letting $p \rightarrow +\infty$, C^0 stability is proven.

Calculating the time derivative \dot{W}_p along the classical C^1 solutions of (3.7), (3.8) yields the following:

$$\begin{aligned} \dot{W}_p &= \frac{1}{p} W_p^{1-p} \int_0^1 p q(x) e^{\mu x} (e^{\mu x} \epsilon^T P \epsilon)^{p-1} \\ &\quad \times (\epsilon_t^T P \epsilon + \epsilon^T P \epsilon_t) dx \\ &= \frac{1}{p} W_p^{1-p} \int_0^1 (-q(x) e^{p\mu x} \lambda(y) u_1 \partial_x (\epsilon^T P \epsilon))^p \\ &+ p q(x) e^{\mu x} (e^{\mu x} \epsilon^T P \epsilon)^{p-1} \theta \epsilon^T ((A + KC)^T P + P(A + KC)) \epsilon \\ &\quad + 2p u_2 q(x) e^{\mu x} (e^{\mu x} \epsilon^T P \epsilon)^{p-1} \epsilon^T P \delta f dx \quad (3.12) \end{aligned}$$

Then, using integration by parts, \dot{W}_p is written as follows:

$$\dot{W}_p = W_p^{1-p} \left(\frac{1}{p} \mathcal{T}_{1,p} + \frac{1}{p} \mathcal{T}_{2,p} + \mathcal{T}_{3,p} \right) \quad (3.13)$$

where

$$\begin{aligned} \mathcal{T}_{1,p} &:= -q(1) \lambda(y(t, 1)) u_1(t, 1) (e^{\mu} \epsilon(t, 1))^T P \epsilon(t, 1))^p \\ &\quad + q(0) \lambda(y(t, 0)) u_1(t, 0) (\epsilon(t, 0))^T P \epsilon(t, 0))^p \quad (3.14a) \end{aligned}$$

$$\mathcal{T}_{2,p} := \int_0^1 \partial_x (q(x) e^{p\mu x} \lambda(y) u_1) (\epsilon^T P \epsilon)^p dx \quad (3.14b)$$

$$\begin{aligned} \mathcal{T}_{3,p} &:= \int_0^1 (-\theta q(x) e^{p\mu x} (\epsilon^T P \epsilon)^{p-1} |\epsilon|^2 \\ &\quad + 2u_2 q(x) e^{\mu x} (e^{\mu x} \epsilon^T P \epsilon)^{p-1} \epsilon^T P \delta f) dx \quad (3.14c) \end{aligned}$$

In (3.14c), (3.3) has been used. By virtue of (3.8), (3.10) and Assumptions A2, A3 and A4 (in particular Lipschitzness of \mathcal{H}), we obtain the following:

$$\begin{aligned} \mathcal{T}_{1,p} &\leq -q(1) \lambda \underline{u}_1 (e^{\mu} \epsilon(t, 1))^T P \epsilon(t, 1))^p + q(0) \bar{\lambda} \bar{u}_1 \\ &\times (\delta \mathcal{H}^T \Theta^{-1} P \Theta^{-1} \delta \mathcal{H})^p \leq -q(1) \lambda \underline{u}_1 (e^{\mu} \epsilon(t, 1))^T P \epsilon(t, 1))^p \\ &\quad + q(0) \bar{\lambda} \bar{u}_1 (|\epsilon(t, 1)|^2 |\Theta^{-1}|^2 L_{\mathcal{H}}^2 \text{eig}_{\max}(P))^p \\ &\leq \bar{\lambda} \bar{u}_1 |\epsilon(t, 1)|^{2p} (-e^{\mu} \text{eig}_{\min}(P))^p \\ &\quad + (\theta^{2n-2} L_{\mathcal{H}}^2 \text{eig}_{\max}(P))^p \quad (3.15) \end{aligned}$$

where $\delta \mathcal{H} := \mathcal{H}(t, \hat{\xi}(t, 1)) - \mathcal{H}(t, \xi(t, 1))$. Using (3.11), we easily get

$$\mathcal{T}_{1,p} \leq 0 \quad (3.16)$$

The term $\mathcal{T}_{2,p}$ is written as follows:

$$\begin{aligned} \mathcal{T}_{2,p} &= \int_0^1 ((q(1) - 1) \lambda(y) u_1 + p \mu q(x) \lambda(y) u_1 \\ &\quad + q(x) \lambda'(y) y_x u_1 + q(x) \lambda(y) \partial_x u_1) (e^{\mu x} \epsilon^T P \epsilon)^p dx \quad (3.17) \end{aligned}$$

Exploiting the fact that $q(x) \geq 1, \forall x \in [0, 1]$, (3.17) yields the following:

$$\begin{aligned} \mathcal{T}_{2,p} &\leq ((q(1) - 1) \bar{\lambda} \bar{u}_1 + \bar{\lambda}' \bar{y}' \bar{u}_1 + \bar{\lambda} \bar{u}_1') \\ &\quad \times \int_0^1 (q(x) (e^{\mu x} \epsilon^T P \epsilon)^p dx \\ &\quad + p |\mu| \bar{\lambda} \bar{u}_1 \int_0^1 q(x) (e^{\mu x} \epsilon^T P \epsilon)^p dx \\ &= ((q(1) - 1) \bar{\lambda} \bar{u}_1 + \bar{\lambda}' \bar{y}' \bar{u}_1 + \bar{\lambda} \bar{u}_1' + p |\mu| \bar{\lambda} \bar{u}_1) W_p^p \quad (3.18) \end{aligned}$$

By exploiting the Lipschitzness of f , $\mathcal{T}_{3,p}$ is bounded above as follows:

$$\begin{aligned} \mathcal{T}_{3,p} &\leq -\theta \int_0^1 q(x) (e^{\mu x} \epsilon^T P \epsilon)^p \frac{1}{\text{eig}_{\max}(P)} dx \\ &\quad + 2\bar{u}_2 \sqrt{n} L_f \frac{\text{eig}_{\max}(P)}{\text{eig}_{\min}(P)} \int_0^1 q(x) (e^{\mu x} \epsilon^T P \epsilon)^p dx \\ &\leq \left(-\frac{\theta}{\text{eig}_{\max}(P)} + b \right) W_p^p \quad (3.19) \end{aligned}$$

where

$$b := 2\bar{u}_2\sqrt{n}L_f \frac{\text{eig}_{\max}(P)}{\text{eig}_{\min}(P)} \quad (3.20)$$

Invoking (3.13) in conjunction with (3.16), (3.18), (3.19) and the fact that $p \geq 1$, \dot{W}_p is bounded above as follows:

$$\dot{W}_p \leq \left(-\frac{\theta}{\text{eig}_{\max}(P)} + a + b\right)W_p, \forall t \geq 0 \quad (3.21)$$

where

$$a := (q(1) - 1)\bar{\lambda}\bar{u}_1 + \bar{\lambda}'\bar{y}'\bar{u}_1 + \bar{\lambda}\bar{u}'_1 + |\mu|\bar{\lambda}\bar{u}_1 \quad (3.22)$$

As a direct consequence of the comparison lemma, (3.21) implies that

$$W_p(t) \leq e^{-\left(\frac{\theta}{\text{eig}_{\max}(P)} - a - b\right)t}W_p(0), \forall t \geq 0 \quad (3.23)$$

Taking into account that for all $x \in [0, 1]$, $1 \leq q(x) \leq \frac{\bar{\lambda}\bar{u}}{\bar{\lambda}\bar{u}}$, we get that

$$\begin{aligned} \lim_{p \rightarrow \infty} W_p &= \lim_{p \rightarrow \infty} \|q(x)^{\frac{1}{p}} e^{\mu x} \epsilon^T P \epsilon\|_p \\ &= \|e^{\mu x} \epsilon^T P \epsilon\|_0 \end{aligned} \quad (3.24)$$

We now define the Lyapunov functional for $\epsilon : [0, 1] \rightarrow \mathbb{R}^n$:

$$V[\epsilon] := \|e^{\mu x} \epsilon^T P \epsilon\|_0 \quad (3.25)$$

Equation (3.24) in conjunction with (3.23) implies that

$$V(t) \leq e^{-\left(\frac{\theta}{\text{eig}_{\max}(P)} - a - b\right)t}V(0), \forall t \geq 0 \quad (3.26)$$

From (3.26) and by using the inequality $e^{-\frac{\mu+|\mu|}{2}} \text{eig}_{\min}(P) \|\epsilon\|_0^2 \leq V \leq e^{-\frac{\mu-|\mu|}{2}} \text{eig}_{\max}(P) \|\epsilon\|_0^2$, we obtain

$$\|\epsilon\|_0 \leq e^{|\mu|/2} \sqrt{\frac{\text{eig}_{\max}(P)}{\text{eig}_{\min}(P)}} e^{-\frac{1}{2}\left(\frac{\theta}{\text{eig}_{\max}(P)} - a - b\right)t} \|\epsilon^0\|_0 \quad (3.27)$$

where $\epsilon^0(x) := \epsilon(0, x)$. Next, we select the high-gain constant to satisfy the following:

$$\theta > \max\{1, \text{eig}_{\max}(P)(a + b)\} := \theta_0 \quad (3.28)$$

We note that b is independent of θ but, according to (3.11), a is dependent on $\ln \theta$. An inequality of the form (3.28) is feasible for sufficiently large θ . From (3.28), there exists $\kappa > 0$, such that $\frac{\theta}{\text{eig}_{\max}(P)} - a - b \geq 2\kappa$. This renders the transformed error ϵ exponentially stable in the C^0 -norm with convergence rate κ , which can be arbitrarily large, depending on the selection of high-gain θ . Simultaneously, the observation error $\xi - \hat{\xi}$ is also exponentially stable in the C^0 -norm satisfying the following:

$$\begin{aligned} \|\xi - \hat{\xi}\|_0 &= \|\Theta \epsilon\|_0 \leq \theta^n e^{|\mu|/2} \sqrt{\frac{\text{eig}_{\max}(P)}{\text{eig}_{\min}(P)}} \\ &\quad \times e^{-\frac{1}{2}\left(\frac{\theta}{\text{eig}_{\max}(P)} - a - b\right)t} \|\epsilon^0\|_0 \\ &\leq \theta^{n-1} e^{|\mu|/2} \sqrt{\frac{\text{eig}_{\max}(P)}{\text{eig}_{\min}(P)}} e^{-\frac{1}{2}\left(\frac{\theta}{\text{eig}_{\max}(P)} - a - b\right)t} \|\xi^0 - \hat{\xi}^0\|_0 \\ &\leq \theta^{n-1} e^{|\mu|/2} \sqrt{\frac{\text{eig}_{\max}(P)}{\text{eig}_{\min}(P)}} e^{-\kappa t} \|\xi^0 - \hat{\xi}^0\|_0, t \geq 0 \end{aligned} \quad (3.29)$$

We conclude that we achieve to construct an exponential high-gain observer of adjustable convergence rate κ , dependent on the selection of θ . The higher the values θ attains, the faster the observation error converges to zero. ■

Remark 3.1: We have chosen P in (3.3) and (3.9) to be non-diagonal, so as to make (3.3) feasible. In contrast to this, in the existing literature on stability approaches (see for instance [1], Chap. 4 and [6]) a diagonal positive definite Lyapunov matrix P is used. Our particular diagonal form of Λ allows commutativity of Λ and P , which is essential in the integration by parts, while calculating the time-derivative of W_p . An assumption of such commutativity property for non-diagonal Lyapunov matrix is made in Prop. 2.1 of [13].

To conclude the section, let us present an application of Theorem 3.1 to a numerical example.

Example 3.1: Consider system

$$\partial_t \xi_1 + 0.1(2 + \cos(\xi_1))\partial_x \xi_1 = \xi_2 + 10 \sin(\xi_1), \quad (3.30a)$$

$$\partial_t \xi_2 + 0.1(2 + \cos(\xi_1))\partial_x \xi_2 = -\sin(\xi_1 - \xi_2), \quad (3.30b)$$

$\forall t \in [0, +\infty), \forall x \in [0, 1]$, with distributed measurement

$$y = \xi_1 \quad (3.30c)$$

and boundary conditions of the form

$$\xi_1(t, 0) = -\frac{1}{7}\xi_1(t, 1) - \xi_2(t, 1) \quad (3.30d)$$

$$\xi_2(t, 0) = \xi_2(t, 1) \quad (3.30e)$$

Consider initial conditions $\xi_1^0(x) = \pi(1 - x)$, $\xi_2^0(x) = -\pi$. System (3.30) is of the form (3.1) with boundary conditions described by (3.2). All Assumptions A1-A4 that we have assumed for system (3.1) are satisfied. The high-gain observer for system (3.30) is of the form given by (3.4), (3.5). After calculating all the essential constants that are used in Theorem 3.1, we can proceed to the observer design. We appropriately choose θ being equal to 5. With $K = (-1, -2)^T$, the high-gain observer is given by the following equations:

$$\partial_t \hat{\xi}_1 + 0.1(2 + \cos(y))\partial_x \hat{\xi}_1 = \hat{\xi}_2 + 10 \sin(\hat{\xi}_1) + 5(y - \hat{\xi}_1), \quad (3.31a)$$

$$\partial_t \hat{\xi}_2 + 0.1(2 + \cos(y))\partial_x \hat{\xi}_2 = -\sin(\hat{\xi}_1 - \hat{\xi}_2) + 50(y - \hat{\xi}_1), \quad (3.31b)$$

$\forall t \in [0, +\infty), \forall x \in [0, 1]$,

$$\hat{\xi}_1(t, 0) = -\frac{1}{7}\hat{\xi}_1(t, 1) - \hat{\xi}_2(t, 1), \hat{\xi}_2(t, 0) = \hat{\xi}_2(t, 1) \quad (3.31c)$$

We choose observer initial conditions (in accordance with the compatibility conditions) $\hat{\xi}_1^0(x) = 2\pi(1 - x)$, $\hat{\xi}_2^0(x) = -2\pi$. In Figure 1 the solution ξ_1 is shown and in Figures 2 and 3 we illustrate the estimation error functions for both states, which exhibit exponential convergence to zero, as predicted by Theorem 3.1.

IV. CONCLUSION AND PERSPECTIVES

We introduce an observable canonical form for a class of nonlinear hyperbolic systems with distributed measurements. Sufficient conditions for the construction of a well-posed high-gain observer system for a particular class of hyperbolic systems in the canonical form are derived. Exponential stability of the estimation error is proven by means of an appropriately chosen Lyapunov functional and for observer gain sufficiently large. We illustrate our results via a simulation. Further exploration of the distributed observability and canonical form properties will be part of future studies. Extending the above-described observer design results to more general classes of nonlinear hyperbolic systems in a canonical form and weakening of some of our assumptions will also be considered in future developments.

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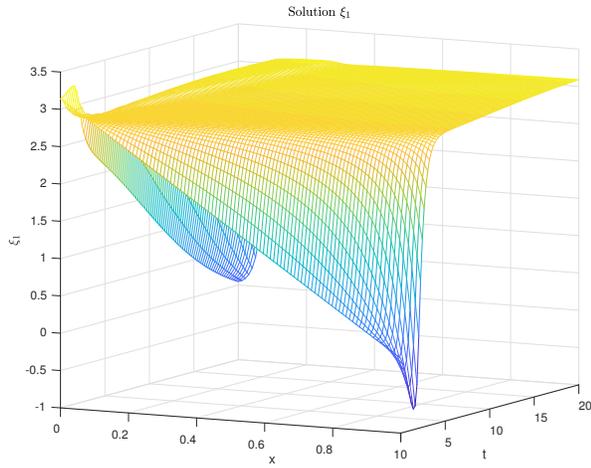


Fig. 1. Solution ξ_1

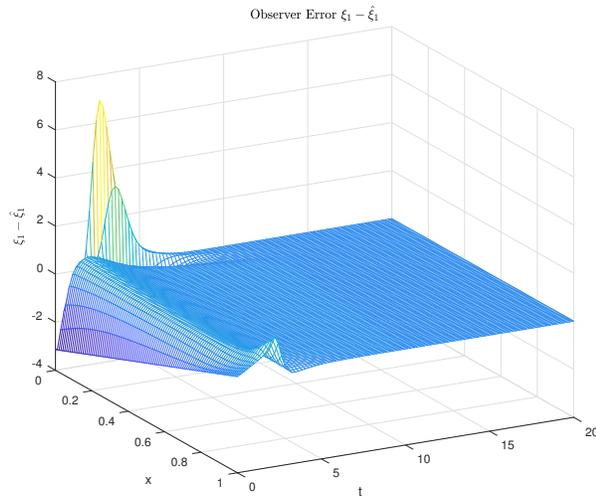


Fig. 2. Observer Error $\xi_1 - \hat{\xi}_1$

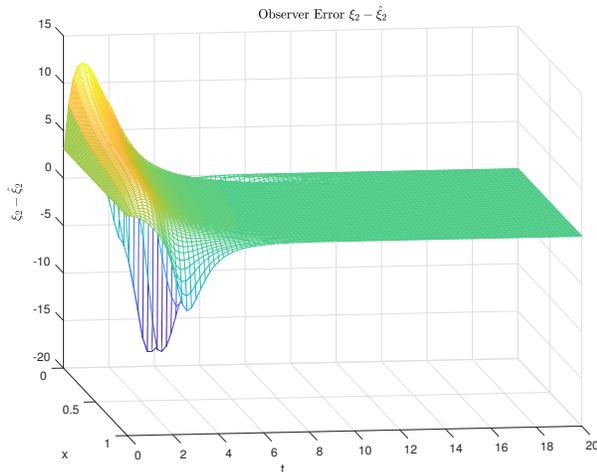


Fig. 3. Observer Error $\xi_2 - \hat{\xi}_2$