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Stability and optimal convergence of unfitted extended finite element methods with Lagrange multipliers for the Stokes equations

Michel Fournié and Alexei Lozinski

Abstract We study a fictitious domain approach with Lagrange multipliers to discretize Stokes equations on a mesh that does not fit the boundaries. A mixed finite element method is used for fluid flow. Several stabilization terms are added to improve the approximation of the normal trace of the stress tensor and to avoid the inf-sup conditions between the spaces of the velocity and the Lagrange multipliers. We generalize first an approach based on eXtended Finite Element Method due to Haslinger-Renard [12] involving a Barbosa-Hughes stabilization and a robust reconstruction on the badly cut elements. Secondly, we adapt the approach due to Burman-Hansbo [4] involving a stabilization only on the Lagrange multiplier. Multiple choices for the finite elements for velocity, pressure and multiplier are considered. Additional stabilization on pressure (Brezzi-Pitkranta, Interior Penalty) is added, if needed. We prove the stability and the optimal convergence of several variants of these methods under appropriate assumptions. Finally, we perform numerical tests to illustrate the capabilities of the methods.

1 Introduction

Let $\mathcal{D} \subset \mathbb{R}^d$, $d = 2$ or 3, be a bounded polygonal (polyhedral) domain. We are interested in the Stokes equations in a setting motivated by the fluid-structure interaction, especially by simulations of particulate flows. We thus assume that $\mathcal{D}$ is decomposed into the fluid domain $\mathcal{F}$ and the solid one $\mathcal{S}$. The domains $\mathcal{F}$ and $\mathcal{S}$ are separated by the interface Γ, cf. Fig. 1. We also denote $\Gamma_{wall} = \partial\mathcal{D}$ and assume, for simplicity, that Γ and Γ_{wall} are disjoint. Consider the problem

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$$-2\,\mathrm{div}\,D(u)+\nabla p = f \quad \text{in } \mathcal{F}, \tag{1}$$
$$\mathrm{div}\,u = 0 \quad \text{in } \mathcal{F}, \tag{2}$$
$$u = g \quad \text{on } \Gamma, \tag{3}$$
$$u = 0 \quad \text{on } \Gamma_{wall}, \tag{4}$$

for u and p, respectively, the velocity and the pressure of the fluid filling $\mathcal{F}$. Here $D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$ and the viscosity has been set to 1 for simplicity. In applications we have in mind, i.e. simulations of the motion of rigid or elastic particles flowing in the fluid, the interface Γ is moving in time while the outer boundary Γ_{wall} is immobile. In this chapter, we shall study Finite Element (FE) discretizations of the problem above on a mesh fixed on $\mathcal{D}$ which is thus fitted to Γ_{wall} but is cut in an arbitrary manner by interface Γ. The interest of these methods in the context of fluid-structure interaction is that it allows one to avoid remeshing when the interface advances with time.

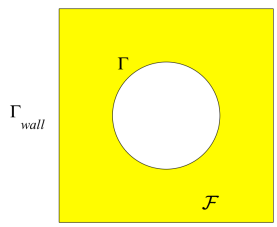

Fig. 1: The fluid domain $\mathcal{F}$, the interface Γ and the outer boundary Γ_{wall}.

Introducing the force exerted by the fluid on the solid at each point of Γ

$$\lambda = -2D(u)n + pn, \text{ on } \Gamma \tag{5}$$

with n the unit normal looking outside from $\mathcal{F}$, and interpreting λ as the Lagrange multiplier associated with the Dirichlet condition (3), we can write the weak formulation of (1)–(4) as

$$\begin{aligned}&\text{Find } (u,p,\lambda) \in H^1_{wall}(\mathcal{F})^d \times L^2_0(\mathcal{F}) \times H^{-\frac{1}{2}}(\Gamma)^d \text{ such that}\\ &\mathcal{A}(u,p,\lambda;v,q,\mu) = \mathcal{L}(v,\mu), \quad \forall (v,q,\mu) \in H^1_{wall}(\mathcal{F})^d \times L^2_0(\mathcal{F}) \times H^{-\frac{1}{2}}(\Gamma)^d\end{aligned} \tag{6}$$

where

$$\begin{aligned}\mathcal{A}(u,p,\lambda;v,q,\mu) &= 2\int_{\mathcal{F}} D(u):D(v) - \int_{\mathcal{F}} (p\,\mathrm{div}\,v + q\,\mathrm{div}\,u) + \int_{\Gamma} (\lambda\cdot v + \mu\cdot u)\\ \mathcal{L}(v,\mu) &= \int_{\mathcal{F}} f\cdot v + \int_{\Gamma} g\cdot\mu\end{aligned}$$

and $H^1_{wall}(\mathcal{F})$ is the space of H^1 functions on $\mathcal{F}$ vanishing on Γ_{wall}. The FE methods studied in this chapter will be based on the variational formulation (6). They shall thus discretize the Lagrange multiplier λ, alongside u and p, thus giving a natural approximation of the force exerted by the fluid on the solid.

As mentioned above, our FE methods will rely on a "background" fixed mesh $\mathcal{T}_h$ that lives on the fluid-structure domain $\mathcal{D} \supset \mathcal{F}$ (the boundary of D is Γ_{wall} and is well fitted by $\mathcal{T}_h$). In the actual computations, the elements of $\mathcal{T}_h$ having no intersection with $\mathcal{F}$ will be discarded and the FE spaces for velocity and pressure will be defined on the mesh $\mathcal{T}_h^e := \mathcal{T}_h^i \cup \mathcal{T}_h^\Gamma$ where $\mathcal{T}_h^\Gamma$ is the union of elements of $\mathcal{T}_h$ that are cut by Γ and $\mathcal{T}_h^i$ is the union of elements of $\mathcal{T}_h$ inside $\mathcal{F}$. The FE space for the Lagrange multiplier will live only on the cut elements $\mathcal{T}_h^\Gamma$, cf. Fig. 2.

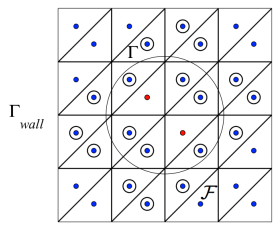

Fig. 2: The meshes $\mathcal{T}_h^e = \mathcal{T}_h^i \cup \mathcal{T}_h^\Gamma$: the triangles of $\mathcal{T}_h^i$ are marked by • and those of $\mathcal{T}_h^\Gamma$ are marked by ⊙; triangles marked by • are not used.

Denoting by $\mathcal{F}_h^e$ (resp. $\mathcal{F}_h^i$, $\mathcal{F}_h^\Gamma$) the domain covered by mesh $\mathcal{T}_h^e$ (resp. $\mathcal{T}_h^i$, $\mathcal{T}_h^\Gamma$) we introduce three FE spaces

$$V_h \subset H^1_{wall}(\mathcal{F}_h^e)^d,\ Q_h \subset L^2(\mathcal{F}_h^e) \cap L^2_0(\mathcal{F}),\ W_h \subset L^2(\mathcal{F}_h^\Gamma)^d \tag{7}$$

to approximate velocity, pressure and Lagrange multiplier respectively. Several choices of FE spaces V_h, Q_h, and W_h will be considered, but we restrict ourselves in this chapter to triangular (tetrahedral) quasi-uniform meshes $\mathcal{T}_h$ and to the standard continuous piecewise polynomial FE-spaces $\mathbb{P}_k$ ($k \geq 1$) or the piecewise constant space $\mathbb{P}_0$ on such a mesh.[1] Our FE spaces will be always based on meshes inherited from $\mathcal{T}_h$: $\mathcal{T}_h^e$ for V_h, Q_h, and $\mathcal{T}_h^\Gamma$ for W_h. Note that velocity and pressure are approximated on a domain $\mathcal{F}_h^e$ slightly larger than $\mathcal{F}$ but all the integrals in the discretized problem will be calculated on $\mathcal{F}$ or Γ. Note also that we choose the FE space for λ on a domain $\mathcal{F}_h^\Gamma$ rather than on the surface Γ to avoid the complicated issue of meshing a surface.

A straightforward Galerkin approximation of (6) is not stable in general (although it often works in practice, as will be seen in the numerical experiments at the end of this chapter). Several stabilization techniques were therefore proposed in

[1] The case of regular non-quasi-uniform meshes can also be easily treated at the expense of some technicalities. However, in applications, one will typically use a simplest possible mesh on **D** (for example, structured Cartesian) so that the quasi-uniformity restriction seems quite acceptable.

the literature, using either Lagrange multipliers [12, 4] or a Nitsche-like method [6] to take into account the boundary conditions on Γ. We shall be concerned in this chapter only with the methods based on Lagrange multipliers. Firstly, we adapt the method of Haslinger-Renard (cf. [12] for the Poisson problem) to Stokes equations. The method is based on a Barbosa-Hughes stabilization [1] on Γ with additional local treatment on badly cut mesh elements. An extension to Stokes equations was already presented in [8] but the analysis there relied on a number of hypotheses, difficult to verify. In this paper, we present a complete theoretical analysis in two cases :

1. LBB-unstable velocity-pressure FE pairs, namely, $\mathbb{P}_1 - \mathbb{P}_1$ or $\mathbb{P}_1 - \mathbb{P}_0$ elements. A stabilization is needed in this case even on a fitted mesh. We shall show, that adding the well known stabilization terms such as Brezzi-Pitkäranta [3] for $\mathbb{P}_1 - \mathbb{P}_1$ elements (or interior penalty for $\mathbb{P}_1 - \mathbb{P}_0$ elements) to a Haslinger-Renard fictitious domain method, as in [8], makes it stable and optimally convergent.
2. LBB-stable velocity-pressure FE pairs, namely, $\mathbb{P}_k - \mathbb{P}_{k-1}$ Taylor-Hood elements. We show that a version of the method above (with and additional pressure stabilization on Γ but avoiding stabilization over the whole domain $\mathcal{F}$) is also stable and optimally convergent. Our proofs are presented here only in the 2D case and under some additional assumptions on the mesh.

We generalize moreover a method by Burman-Hansbo [4] to Stokes equations. This is also a fictitious domain method with Lagrange multipliers. Unlike the method by Haslinger-Renard (where the stabilization comes by enforcing (5) on Γ and thus involves all the variables u, p, λ), one stabilizes here only the multiplier λ by enforcing its continuity in some sense, so that the structure of resulting matrices is simpler. Fortunately, much of the theory outlined above can be reused for the analysis of this method. We are thus able to prove the stability and optimal convergence for the same choices of the FE spaces as above.

The chapter is concluded by numerical experiments aiming at comparing different stabilizations and choices of of FE spaces.

Nomenclature.

Domains: $\mathcal{F}$ is the fluid domain where the problem (1)–(4) is posed while $\mathcal{F}_h^i$, $\mathcal{F}_h^e$, $\mathcal{F}_h^\Gamma$ are the domains occupied by the meshes $\mathcal{T}_h^i$, $\mathcal{T}_h^e$, $\mathcal{T}_h^\Gamma$ respectively. We have thus $\mathcal{F}_h^i \subset \mathcal{F} \subset \mathcal{F}_h^e$ and $\mathcal{F}_h^\Gamma = \mathcal{F}_h^e \setminus \mathcal{F}_h^i$.

Meshes: $\mathcal{T}_h^i$, $\mathcal{T}_h^e$, $\mathcal{T}_h^\Gamma$ are submeshes of a background mesh $\mathcal{T}_h$ so that $\mathcal{T}_h^i = \{T \in \mathcal{T}_h : T \subset \mathcal{F}\}$, $\mathcal{T}_h^\Gamma = \{T \in \mathcal{T}_h : T \cap \Gamma \neq \varnothing\}$ and $\mathcal{T}_h^e := \mathcal{T}_h^i \cup \mathcal{T}_h^\Gamma$.
$\mathcal{E}_h^e$ and $\mathcal{E}_h^\Gamma$ stand for the sets of interior edges of $\mathcal{T}_h^e$ and $\mathcal{T}_h^\Gamma$ respectively.
$\mathcal{F}_T$ (resp. Γ_T) denotes $T \cap \mathcal{F}$ (resp. $T \cap \Gamma$) for any cut element $T \in \mathcal{T}_h^\Gamma$.

Norms: $\|\cdot\|_{k,\omega}$ stands for the norm in $H^k(\omega)$ where ω can be a domain in $\mathbb{R}^d$ or a $(d-1)$-dimensional manifold. We identify $H^0(\omega)$ with $L^2(\omega)$.
$|\cdot|_{k,\omega}$ stands for the semi-norm in $H^k(\omega)$, $k > 0$.
$\|\cdot\|_{\infty,\omega}$ stands for the norm in $L^\infty(\omega)$.

2 Methods à la Haslinger-Renard

The starting point for the construction of the Haslinger-Renard method (proposed in [12] for the Poisson equation) is to add to the variational formulation (6) the Barbosa-Hughes stabilization [1], which enforces the relation $\lambda + 2D(u)n - pn = 0$ on Γ. These terms take the form

$$-\gamma_0 h \int_\Gamma (\lambda + 2D(u)n - pn) \cdot (\mu + 2D(v)n - qn) \tag{8}$$

with a mesh-independent $\gamma_0 > 0$. This idea, at least in the context of the Poisson equation as in [12], produces a stable and optimally convergent approximation provided the mesh elements are cut by Γ in a certain way so that $\mathcal{F} \cap T$ is a big enough portion of T for any $T \in \mathcal{T}_h^\Gamma$. If, for some elements, this is not the case the method can be still cured by replacing the approximating polynomial in such "bad elements" by the polynomial extended from from adjacent "good elements". The relation between bad and good elements is made precise in the following

Assumption A. We fix a threshold $\theta_{\min} \in (0,1]$ and declare any $T \in \mathcal{T}_h^\Gamma$ a good element (resp. bad element) if $\frac{|\mathcal{F}_T|}{|T|} \geq \theta_{\min}$ (resp. $\frac{|\mathcal{F}_T|}{|T|} < \theta_{\min}$). We assume that one can choose for any bad element T a "good neighbor" $T' \in \mathcal{T}_h^e$, $\frac{|T' \cap \mathcal{F}|}{|T'|} \geq \theta_{\min}$, such that T and T' share at least one node, cf. Fig. 3.

Remark 1. Typically, Assumption A will hold true even for $\theta_{\min} = 1$ if the mesh is sufficiently refined. One could also relax the notion of a neighbor (at the expense of some complication of the forthcoming proofs) to the requirement $\mathrm{dist}(T, T') \leq Ch$ with a mesh-independent $C > 0$.

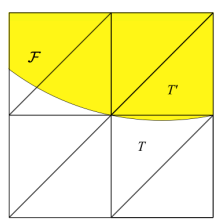

Fig. 3: Good element T' and bad element T

We now define a "robust reconstruction" on $\mathcal{F}_h^\Gamma$ for the FE functions on $\mathcal{F}_h^e$

Definition 1. For any $v_h \in V_h$ set $\widehat{v_h}$ on any $T \in \mathcal{T}_h^\Gamma$ as

- $\widehat{v_h} = v_h$ on T if T is a good element,
- $(\widehat{v_h})_{|T} = (v_h)_{|T'}$ if T is a bad element. Here T' is the good neighbor of T from Assumption A and the relation should be understood in the sense that $\widehat{v_h}$ on T is taken as the same polynomial as the polynomial giving v_h on T'.

For any $q_h \in Q_h$, one constructs $\widehat{q_h}$ in the same way.

We shall show in the subsequent paragraphs that adding stabilization (8) to (6) and replacing u, v in these terms (sometimes also p, q) by their robust reconstructions produces indeed a stable approximation to the Stokes equations. We end this general introduction to the Haslinger-Renard method by a Proposition illustrating the usefulness of the selection criterion for good elements, showing that the L^2 norm on the cut portion of an element T controls L^∞ (and hence any other) norm on the whole element with an equivalence constant depending on the relative measure of the cut portion, followed by a list of interpolation error estimates that shall be needed in the forthcoming analysis.

Proposition 1. *Let p be a polynomial of degree $\leq k$ and $\theta \in (0,1]$. Then for any $T \in \mathcal{T}_h$ and any measurable set $S \subset T$ with $|S| \geq \theta |T|$ one has*

$$\|p\|_{\infty,T} \leq \frac{C}{h^{d/2}} \|p\|_{0,S} \tag{9}$$

with a constant $C > 0$ depending only on θ, k and mesh regularity.

Proof. By scaling, it is sufficient to prove (9) on a reference element. We thus fix a simplex $T \in \mathbb{R}^d$ of diameter $h = 1$ and consider for any $p \in \mathbb{P}_k$

$$N_\theta(p) = \inf_{S \subset T, |S| \geq \theta |T|} \|p\|_{0,S}$$

It is easy to see that N_θ is a continuous function on the finite-dimensional space $\mathbb{P}_k$. Consequently, it attains a minimum on the set $\Sigma_1 := \{p \in \mathbb{P}_k, \|p\|_{\infty,T} = 1\}$, i.e. $\exists \alpha \geq 0$ and $p_\alpha \in \Sigma_1$ such that $N_\theta(p) \geq N_\theta(p_\alpha) = \alpha$ for all $p \in \Sigma_1$. It remains to prove $\alpha > 0$. To this end, let $m(\delta) = \text{meas}\{x \in T : |p_\alpha(x)| \leq \delta\}$ for any $\delta \geq 0$. Since $m(\delta)$ is decreasing down to 0 as $\delta \to 0$, one can find $\varepsilon > 0$ s.t. $m(\varepsilon) \leq \frac{\theta}{2}|T|$. We observe now $\|p_\alpha\|_{0,S}^2 \geq \varepsilon^2 \left(|S| - \frac{\theta}{2}|T|\right)$ for any $S \subset T$, hence $\alpha^2 = N_\theta^2(p_\alpha) \geq \varepsilon^2 \frac{\theta}{2}|T| > 0$. By homogeneity, this also proves $N_\theta(p) \geq \alpha \|p\|_{\infty,T}$ for all $p \in \mathbb{P}_k$ entailing (9) with $C = \frac{1}{\alpha}$ (recall that the proof is done on the reference element with $h = 1$). $\square$

We are going to establish interpolation estimates on the cut domain. To this end, we introduce

Assumption B. Ω is a Lipschitz domain and there exist constants $c_\Gamma, C_\Gamma > 0$ such that for any $T \in \mathcal{T}_h^\Gamma$

1. $|\Gamma_T| \leq C_\Gamma h^{d-1}$ with $\Gamma_T := T \cap \Gamma$;
2. there exists a unit vector $\chi_T \in \mathbb{R}^d$ such that $\chi_T \cdot n \geq c_\Gamma$ a.e. on Γ_T where n is the unit normal looking outward from $\mathcal{F}$.

Remark 2. The bound on $|\Gamma_T|$ in the first part of the Assumption B is automatically satisfied on Lipschitz domain. We prefer however to write this bound explicitly in order to emphasize that some of the estimates below will depend on the constant C_Γ, so that Γ should be supposed not too oscillating. The second part of the Assumption B is not too restrictive either. Typically, one can take χ_T as the normal n at the

middle point of Γ_T if Γ_T is smooth or as the average between the two normals if Γ_T is the union of two segments (in the case when Ω is a 2D polygon). Such choices will suffice on a sufficiently refined mesh.

Proposition 2. *Let V_h, Q_h, W_h be (respectively) $\mathbb{P}_{k_u}, \mathbb{P}_{k_p}, \mathbb{P}_{k_\lambda}$ FE spaces on meshes $\mathcal{T}_h^e, \mathcal{T}_h^e, \mathcal{T}_h^\Gamma$ as in (7). Under Assumptions A and B, there exist interpolation operators $I_h^u : H^1_{wall}(\mathcal{F})^d \to V_h$, $I_h^p \in L^2_0(\mathcal{F}) \to Q_h$, $I_h^\lambda : H^{\frac{1}{2}}(\Gamma)^d \to W_h$ s.t. for any sufficiently smooth u, p, λ*

$$\frac{1}{h}\|u - I_h^u u\|_{0,\mathcal{F}} + |u - I_h^u u|_{1,\mathcal{F}} + \frac{1}{\sqrt{h}}\|u - I_h^u u\|_{0,\Gamma} \le Ch^{s_u}|u|_{s_u+1,\mathcal{F}} \tag{10}$$

$$\text{(for all integer } s_u : 0 \le s_u \le k_u)$$

$$\left(\|\nabla u - \nabla I_h^u u\|_{0,\Gamma} + \|\nabla u - \nabla \widehat{I_h^u u}\|_{0,\Gamma}\right) \le Ch^{s_u - \frac{1}{2}}|u|_{s_u+1,\mathcal{F}} \tag{11}$$

$$\text{(for all integer } s_u : 1 \le s_u \le k_u)$$

$$\frac{1}{h}\|p - I_h^p p\|_{0,\mathcal{F}} + |p - I_h^p p|_{1,\mathcal{F}} + \frac{1}{\sqrt{h}}\left(\|p - I_h^p p\|_{0,\Gamma} + \|p - \widehat{I_h^p p}\|_{0,\Gamma}\right) \le Ch^{s_p}|p|_{s_p+1,\mathcal{F}} \tag{12}$$

$$\text{(for all integer } s_p : 0 \le s_p \le k_p)$$

$$\frac{1}{\sqrt{h}}\|\lambda - I_h^\lambda \lambda\|_{0,\Gamma} \le Ch^{s_\lambda}|\lambda|_{s_\lambda+\frac{1}{2},\Gamma} \tag{13}$$

$$\text{(for all integer } s_\lambda : 0 \le s_\lambda \le k_\lambda)$$

with $C > 0$ depending only on the constants in Assumptions A, B, and on the mesh regularity, and $k_u \ge 1$ in the case of estimate (11). Moreover, operator I_h^λ can be extended to $I_h^\lambda : H^1_{wall}(\mathcal{F})^d \to W_h$ s.t. for any $\tilde{\lambda} \in (H^{s_\lambda+1}(\mathcal{F}) \cap H^1_{wall})^d$ and an integer s_λ, $0 \le s_\lambda \le k_\lambda$

$$\frac{1}{h}\|\tilde{\lambda} - I_h^\lambda \tilde{\lambda}\|_{0,\mathcal{F}_h^\Gamma} + |\tilde{\lambda} - I_h^\lambda \tilde{\lambda}|_{1,\mathcal{F}_h^\Gamma} + \frac{1}{\sqrt{h}}\|\tilde{\lambda} - I_h^\lambda \tilde{\lambda}\|_{0,\Gamma} \le Ch^{k_\lambda}|\tilde{\lambda}|_{k_\lambda+1,\mathcal{F}} \tag{14}$$

Proof. We start with the construction of I_h^u. Extension theorems for Sobolev spaces guarantee for any $u \in H^{s_u+1}(\mathcal{F})^d$ existence of $\tilde{u} \in H^{s_u+1}(\mathcal{F}_h^e)^d$ with $\|\tilde{u}\|_{s_u+1,\mathcal{F}_h^e} \le C\|u\|_{s_u+1,\mathcal{F}}$ and $\tilde{u} = u$ on $\mathcal{F}$. Let $\tilde{I}_h : H^1_{wall}(\mathcal{F}_h^e)^d \to V_h$ be a Clément-type interpolation operator [9] satisfying

$$\frac{1}{h}\|\tilde{u} - \tilde{I}_h\tilde{u}\|_{0,T} + |\tilde{u} - \tilde{I}_h\tilde{u}|_{1,T} + \frac{1}{\sqrt{h}}\|\tilde{u} - \tilde{I}_h\tilde{u}\|_{0,\partial T} + \sqrt{h}\|\nabla(\tilde{u} - \tilde{I}_h\tilde{u})\|_{0,\partial T} \le Ch^{s_u}|\tilde{u}|_{s_u+1,\omega_T}$$

on any $T \in \mathcal{T}_h^e$ with ω_T begin the patch of elements of $\mathcal{T}_h^e$ touching T. Let $I_h^u u = \tilde{I}_h\tilde{u}|_{\mathcal{F}}$. Summing the estimates above over all the mesh elements yields immediately the estimates in $L^2(\mathcal{F})$ and $H^1(\mathcal{F})$ in (10). Now, on any element $T \in \mathcal{T}_h^\Gamma$

$$c_\Gamma \|u - I_h^u u\|_{0,\Gamma_T}^2 \leq \int_{\Gamma_T} (\tilde{u} - \tilde{I}_h \tilde{u})^2 \chi_T \cdot n = \int_{\mathcal{F}_T} \mathrm{div}((\tilde{u} - \tilde{I}_h \tilde{u})^2 \chi_T) - \int_{\mathcal{F} \cap \partial T} (\tilde{u} - \tilde{I}_h \tilde{u})^2 \chi_T \cdot n$$

since $\partial \mathcal{F}_T = \Gamma_T \cup (\mathcal{F} \cap \partial T)$. Developing and applying the interpolation estimates above gives

$$\begin{aligned} c_\Gamma \|u - I_h^u u\|_{0,\Gamma_T}^2 &\leq \int_{\mathcal{F}_T} 2(\tilde{u} - \tilde{I}_h \tilde{u}) \nabla(\tilde{u} - \tilde{I}_h \tilde{u}) \cdot \chi_T + \|\tilde{u} - \tilde{I}_h \tilde{u}\|_{0,\mathcal{F} \cap \partial T}^2 \\ &\leq 2\|\tilde{u} - \tilde{I}_h \tilde{u}\|_{0,T} |\tilde{u} - \tilde{I}_h \tilde{u}|_{1,T} + \|\tilde{u} - \tilde{I}_h \tilde{u}\|_{0,\partial T}^2 \leq C h^{2s_u+1} (|\tilde{u}|_{s_u+1,\omega_T}^2) \end{aligned}$$

Summing this over all the elements in $\mathcal{F}_h^\Gamma$ yields the $L^2(\Gamma)$-estimate in (10).

If $s_u \geq 1$, we have moreover on any $T \in \mathcal{T}_h^e$

$$h|\tilde{u} - \tilde{I}_h \tilde{u}|_{2,T} + \sqrt{h} \|\nabla(\tilde{u} - I_h \tilde{u})\|_{0,\partial T} \leq C h^{s_u} |\tilde{u}|_{s_u+1,\omega_T}$$

This, by the same argument as above, gives the $L^2(\Gamma)$ estimate on $\nabla(u - I_h^u u)$ in (11). In order to extend this to $\nabla(u - \widehat{I_h^u u})$ consider a bad element T and its good neighbor T'. Both T and T' belong to the patch $\omega_{T'}$ and examining the derivation of interpolation estimates for the Clément interpolator $\tilde{I}_h$ reveals that the polynomial $(\tilde{I}_h \tilde{u})_{|T'}$ gives actually an optimal approximation of $\tilde{u}$ on the whole $\omega_{T'}$, i.e.

$$|u - \widehat{I_h^u u}|_{1,T} = |\tilde{u} - (\tilde{I}_h \tilde{u})_{|T'}|_{1,T} \leq |\tilde{u} - (\tilde{I}_h \tilde{u})_{|T'}|_{1,\omega_{T'}} \leq C h^{s_u} |u|_{s_u+1,\omega_{T'}}$$

Similarly, $\frac{1}{h}\|u - \widehat{I_h^u u}\|_{0,T} + \frac{1}{\sqrt{h}}\|u - \widehat{I_h^u u}\|_{0,\partial T} \leq C h^{s_u} |u|_{s_u+1,\omega_{T'}}$. Thus, the same argument as above gives the $L^2(\Gamma)$ estimate on $\nabla(u - \widehat{I_h^u u})$ in (11).

The remaining estimates (12), (13) and (14) are proved in a similar manner. We skip the details and make only the following remarks:

- The operator I_h^p should preserve the restriction that pressure is of zero mean on $\mathcal{F}$. We thus define it as $I_h^p p = \tilde{I}_h \tilde{p} - i_h(p)$ where $\tilde{I}_h$ is the Clément interpolation operator on $\mathcal{T}_h^e$, $\tilde{p}$ is an extension of p to $\mathcal{F}_h^e$, and $i_h(p) = \int_{\mathcal{F}} \tilde{I}_h \tilde{p}$. The correction $i_h(p)$ can be bounded as

$$|i_h(p)| = \left| \int_{\mathcal{F}} (\tilde{I}_h \tilde{p} - p) \right| \leq |\mathcal{F}|^{\frac{1}{2}} \|\tilde{p} - \tilde{I}_h \tilde{p}\|_{0,\mathcal{F}_h^e} \leq C h^{s_p+1} |\tilde{p}|_{s_p+1,\mathcal{F}_h^e}$$

  and thus it does not perturb the estimates (12).
- Concerning the interpolation of λ, we note that (13) is in fact an easy corollary to (14). Indeed, for any $\lambda \in H^{k_\lambda + \frac{1}{2}}(\Gamma)^d$ there exists (by the trace theorem) $\tilde{\lambda} \in H^{k_\lambda+1}(\mathcal{F}_h^\Gamma)^d$ satisfying $\tilde{\lambda}|_\Gamma = \lambda$ and $|\tilde{\lambda}|_{k_\lambda+1,\mathcal{F}_h^\Gamma} \leq C|\lambda|_{k_\lambda+\frac{1}{2},\Gamma}$. We can thus define $I_h^\lambda \lambda := I_h^\lambda \tilde{\lambda}$ and observe that (14) entails (13).

□

2.1 $\mathbb{P}_1 - \mathbb{P}_1$ velocity-pressure spaces with Brezzi-Pitkäranta stabilization.

Let us choose $\mathbb{P}_1$ FE spaces for both V_h and Q_h, add Brezzi-Pitkäranta-like stabilization for the pressure and the Barbosa-Hughes-like stabilization on the interface as described above. We choose to introduce the robust reconstruction from Definition 1 in the last terms only for the velocity in this case (on both trial function u_h and test function v_h). The method thus reads

$$\begin{aligned}&\text{Find } (u_h, p_h, \lambda_h) \in V_h \times Q_h \times W_h \text{ such that}\\ &\mathcal{A}^{HR-BP}(u_h, p_h, \lambda_h; v_h, q_h, \mu_h) = \mathcal{L}(v_h, \mu_h), \quad \forall (v_h, q_h, \mu_h) \in V_h \times Q_h \times W_h\end{aligned} \tag{15}$$

where

$$\begin{aligned}\mathcal{A}^{HR-BP}(u, p, \lambda; v, q, \mu) &= \mathcal{A}(u, p, \lambda; v, q, \mu)\\ &\quad - \gamma_0 h \int_\Gamma (\lambda + 2D(\widehat{u})n - pn) \cdot (\mu + 2D(\widehat{v})n - qn) - \theta h^2 \int_{\mathcal{F}_h^e} \nabla p \cdot \nabla q\end{aligned}$$

V_h, Q_h are continuous $\mathbb{P}_1$ FE spaces on mesh $\mathcal{T}_h^e$ and W_h is $\mathbb{P}_1$ or $\mathbb{P}_0$ FE space on mesh $\mathcal{T}_h^\Gamma$, cf. (7).

We remind that the Brezzi-Pitkäranta stabilization (the last term above) should be present on $\mathcal{F}$ to compensate the lack of the discrete inf-sup in P1-P1 velocity-pressure FE spaces. In addition, in our fictitious domain situation, it is extended to the larger domain $\mathcal{F}_h^e$ thus helping to ensure stability near Γ.

In the following propositions, Assumptions A and B are implicitly implied and the constants C may vary from line to line and depend on $c_\Gamma, C_\Gamma > 0$ from Assumption B, $\theta_{\min}$ from Assumption A, and on the mesh regularity.

Proposition 3. *For all $v_h \in V_h$ one has*

$$h\|\nabla \widehat{v_h}\|_{0,\Gamma}^2 \le C|v_h|_{1,\mathcal{F}}^2 \tag{16}$$

Proof. Taking any $T \in \mathcal{T}_h^\Gamma$ and denoting its good neighbor by T' we observe

$$\|\nabla \widehat{v_h}\|_{0,\Gamma_T} \le \sqrt{|\Gamma_T|}\|\nabla \widehat{v_h}\|_{L^\infty(T)} \le C\sqrt{|\Gamma_T|}\|\nabla v_h\|_{L^\infty(T')} \le C\frac{\sqrt{|\Gamma_T|}}{h^{d/2}}\|\nabla v_h\|_{0,\mathcal{F}_{T'}}$$

The last inequality above holds by Proposition 1 with a constant dependent on $\theta_{\min}$. The last but one inequality is easily proven by scaling given that T and T' are neighbors. Using the bound $|\Gamma_T| \le C_\Gamma h^{d-1}$ and summing over all $T \in \mathcal{T}_h^\Gamma$ yields (16). □

Proposition 4. *For all $q_h \in Q_h$ one has*

$$h\|q_h\|_{0,\Gamma}^2 \le C\left(\|q_h\|_{0,\mathcal{F}}^2 + h^2|q_h|_{1,\mathcal{F}_h^e}^2\right)$$

Proof. Using the notations T, T' as in the preceding proof and assuming that these two elements share a node x, we observe

$$\begin{aligned}\|q_h\|_{0,\Gamma_T} &\leq \sqrt{|\Gamma_T|}\|q_h\|_{L^\infty(T)} \leq \sqrt{|\Gamma_T|}(|q_h(x)| + h\|\nabla q_h\|_{L^\infty(T)}) \\ &\leq \sqrt{|\Gamma_T|}(\|q_h\|_{L^\infty(T')} + h\|\nabla q_h\|_{L^\infty(T)}) \leq C\frac{\sqrt{|\Gamma_T|}}{h^{d/2}}(\|q_h\|_{0,\mathcal{F}_{T'}} + h\|\nabla q_h\|_{0,T})\end{aligned}$$

We have used again Proposition 1 on the good element T'. We conclude thanks to $|\Gamma_T| \leq C_\Gamma h^{d-1}$ from Assumption B and the summation over all $T \in \mathcal{T}_h^\Gamma$. □

We shall also need a special interpolation operator adapted to functions vanishing on Γ, the idea of which goes to [13].

Proposition 5. *There exists an interpolation operator $I_h^0 : H_0^1(\mathcal{F})^d \to V_h$ such that*

$$\|v - I_h^0 v\|_{0,\mathcal{F}} \leq Ch|v|_{1,\mathcal{F}}, \quad |I_h^0 v|_{1,\mathcal{F}} \leq C|v|_{1,\mathcal{F}}$$

and $I_h^0 v = 0$ on $\mathcal{F}_h^\Gamma$ (and consequently $I_h^0 v = 0$ on Γ) for any $v \in H_0^1(\mathcal{F})^d$ with a mesh-independent constant $C > 0$.

Proof. The construction of I_h^0 will be based on the interpolator I_h^u from Proposition 2 with $k_u = 1$. For any $v \in H_0^1(\mathcal{F})^d$, let us put $I_h^0 v(x) = I_h^u(x)$ at all the interior nodes x of $\mathcal{T}_h^i$ (i.e. excepting the nodes lying on $\partial\mathcal{T}_h^i$) and $I_h^0 v(x) = 0$ on all the nodes of $\mathcal{T}_h^\Gamma$. Since $I_h^0 v$ is the piecewise linear function on $\mathcal{T}_h^e$, this uniquely defines it everywhere on $\mathcal{F}_h^e$. Moreover, $I_h^0 v = 0$ on $\mathcal{F}_h^\Gamma$.

Let us denote, for a mesh edge E lying on $\partial\mathcal{F}_h^i$, the adjacent element from $\mathcal{T}_h^\Gamma$ by T^Γ and the union of all the elements from $\mathcal{T}_h^i$ sharing at least a node with E by ω_E^i. By scaling

$$\|I_h^u v - I_h^0 v\|_{0,\omega_E^i} \leq C\sqrt{h}\|I_h^u v\|_{0,E} \leq C(\|I_h^u v\|_{0,T^\Gamma} + h|I_h^u v|_{1,T^\Gamma})$$

Summing over all such edges and introducing the extension $\tilde{v}$ to $\mathcal{F}_h^e$ as in the proof of Proposition 2 yields

$$\|I_h^u v - I_h^0 v\|_{0,\mathcal{F}_h^i} \leq C(\|I_h^u v\|_{0,\mathcal{F}_h^\Gamma} + h|I_h^u v|_{1,\mathcal{F}_h^\Gamma}) \leq C(\|\tilde{v} - I_h^u v\|_{0,\mathcal{F}_h^\Gamma} + \|\tilde{v}\|_{0,\mathcal{F}_h^\Gamma} + h|I_h^u v|_{1,\mathcal{F}_h^\Gamma})$$

Since $I_h^0 v = 0$ on $\mathcal{F}_h^\Gamma$ this entails

$$\begin{aligned}\|v - I_h^0 v\|_{0,\mathcal{F}} &\leq \|v\|_{0,\mathcal{F}} + \|v - I_h^u v\|_{0,\mathcal{F}_h^i} + \|I_h^u v - I_h^0 v\|_{0,\mathcal{F}_h^i} \\ &\leq C(\|\tilde{v}\|_{0,\mathcal{F}_h^\Gamma} + \|\tilde{v} - I_h^u v\|_{0,\mathcal{F}_h^e} + h|I_h^u v|_{1,\mathcal{F}_h^\Gamma})\end{aligned}$$

We now employ the bound $\|\tilde{v}\|_{0,\mathcal{F}_h^\Gamma} \leq Ch|\tilde{v}|_{1,\mathcal{F}_h^\Gamma}$, which is valid since $\mathcal{F}_h^\Gamma$ is a band of thickness h around Γ and $\tilde{v} = 0$ on Γ. Moreover,

$$\frac{1}{h}\|\tilde{v} - I_h^u v\|_{0,\mathcal{F}_h^e} + |I_h^u v|_{1,\mathcal{F}_h^e} \leq C|\tilde{v}|_{1,\mathcal{F}_h^e}$$

as follows from the proof of Proposition 2, cf. (10) with $s_u = 0$. Since $|\tilde{v}|_{1,\mathcal{F}_h^e} \leq C|v|_{1,\mathcal{F}}$ by the extension theorem, this proves the announced estimate of $\|v - I_h^0 v\|_{0,\mathcal{F}}$.

The estimate for the H^1 norm of $I_h^0 v$ follows using the inverse inequality and the L^2 error estimates proved above:

$$|I_h^0 v|_{1,\mathcal{F}} = |I_h^0 v|_{1,\mathcal{F}_h^i} \leq |I_h^0 v - I_h^u v|_{1,\mathcal{F}_h^i} + |I_h^u v|_{1,\mathcal{F}_h^i} \leq \frac{C}{h}\|I_h^0 v - I_h^u v\|_{0,\mathcal{F}_h^i} + |I_h^u v|_{1,\mathcal{F}_h^i} \leq C|v|_{1,\mathcal{F}}$$

□

Lemma 1. *Under Assumption A and B, taking $\gamma_0 > 0$ small enough and any $\theta > 0$, there exists a mesh-independent constant $c > 0$ such that*

$$\inf_{(u_h,p_h,\lambda_h)\in V_h\times Q_h\times W_h} \sup_{(v_h,q_h,\mu_h)\in V_h\times Q_h\times W_h} \frac{\mathcal{A}^{HR-BP}(u_h,p_h,\lambda_h;v_h,q_h,\mu_h)}{|||u_h,p_h,\lambda_h|||\ |||v_h,q_h,\mu_h|||} \geq c$$

where the triple norm is defined by

$$|||u,p,\lambda||| = \left(|u|_{1,\mathcal{F}}^2 + \|p\|_{0,\mathcal{F}}^2 + h^2|p|_{1,\mathcal{F}_h^e}^2 + h\|\lambda\|_{0,\Gamma}^2 + \frac{1}{h}\|u\|_{0,\Gamma}^2\right)^{1/2}$$

Proof. We observe, using Proposition 3,

$$\begin{aligned}
&\mathcal{A}^{HR-BP}(u_h,p_h,\lambda_h;u_h,-p_h,-\lambda_h)\\
&\quad= 2\|D(u_h)\|_{0,\mathcal{F}}^2 - 4\gamma_0 h\|D(\widehat{u_h})\|_{0,\Gamma}^2 + \gamma_0 h\|\lambda_h - p_h n\|_{0,\Gamma}^2 + \theta h^2|p_h|_{1,\mathcal{F}_h^e}\\
&\quad\geq 2\|D(u_h)\|_{0,\mathcal{F}}^2 - C\gamma_0|u_h|_{1,\mathcal{F}}^2 + \gamma_0 h\|\lambda_h - p_h n\|_{0,\Gamma}^2 + \theta h^2|p_h|_{1,\mathcal{F}_h^e}\\
&\quad\geq \frac{1}{K}|u_h|_{1,\mathcal{F}}^2 + \gamma_0 h\|\lambda_h - p_h n\|_{0,\Gamma}^2 + \theta h^2|p_h|_{1,\mathcal{F}_h^e}
\end{aligned}$$

We have used in the last line the assumption that γ_0 is sufficiently small and Korn inequality

$$|v|_{1,\mathcal{F}}^2 \leq K\|D(v)\|_{0,\mathcal{F}}^2, \quad \forall v \in H_{wall}^1(\mathcal{F}) \tag{17}$$

Note that the inequality is valid in this form because the functions from $H_{wall}^1(\mathcal{F})$ vanish on Γ_{wall}, i.e. on a part of the boundary $\partial\mathcal{F}$ with non zero measure.

The continuous inf-sup condition [10] implies for all $p_h \in Q_h$ there exists $v_p \in H_0^1(\mathcal{F})^d$ such that

$$-\int_{\mathcal{F}} p_h \operatorname{div} v_p = \|p_h\|_{0,\mathcal{F}}^2 \text{ and } |v_p|_{1,\mathcal{F}} \leq C\|p_h\|_{0,\mathcal{F}}. \tag{18}$$

Recalling that $v_p = I_h^0 v_p = 0$ on Γ we can write

$$
\begin{aligned}
-\int_{\mathcal{F}} p_h \operatorname{div}(I_h^0 v_p) &= \|p_h\|_{0,\mathcal{F}}^2 - \int_{\mathcal{F}} p_h \operatorname{div}(I_h^0 v_p - v_p) \\
&= \|p_h\|_{0,\mathcal{F}}^2 - \int_{\mathcal{F}} \nabla p_h \cdot (v_p - I_h^0 v_p) \geq \|p_h\|_{0,\mathcal{F}}^2 - Ch|p_h|_{1,\mathcal{F}_h^e}|v_p|_{1,\mathcal{F}} \\
&\geq \|p_h\|_{0,\mathcal{F}}^2 - Ch|p_h|_{1,\mathcal{F}_h^e}\|p_h\|_{0,\mathcal{F}} \qquad (19)
\end{aligned}
$$

where we have used the bounds from Proposition 5 and (18). Combining this with Young inequality we obtain

$$
\begin{aligned}
\mathcal{A}^{HR-BP}(u_h, p_h, \lambda_h; I_h^0 v_p, 0, 0) &\geq -\|D(u_h)\|_{0,\mathcal{F}}\|D(I_h^0 v_p)\|_{0,\mathcal{F}} + \|p_h\|_{0,\mathcal{F}}^2 - Ch|p_h|_{1,\mathcal{F}_h^e}\|p_h\|_{0,\mathcal{F}} \\
&\geq \frac{1}{2}\|p_h\|_{0,\mathcal{F}}^2 - C|u_h|_{1,\mathcal{F}}^2 - Ch^2|p_h|_{1,\mathcal{F}_h^e}^2
\end{aligned}
$$

Recall interpolation operator I_h^λ from Proposition 2 and observe, using Proposition 3 with Young inequality,

$$
\begin{aligned}
&\mathcal{A}^{HR-BP}(u_h, p_h, \lambda_h; 0, 0, \frac{1}{h} I_h^\lambda u_h) = \frac{1}{h}\int_\Gamma u_h \cdot I_h^\lambda u_h - \gamma_0 \int_\Gamma (2D(\widehat{u}_h)n - p_h n + \lambda_h) \cdot I_h^\lambda u_h \\
&\geq \frac{1}{2h}\|u_h\|_{0,\Gamma}^2 - \frac{1}{2h}\|u_h - I_h^\lambda u_h\|_{0,\Gamma}^2 - \gamma_0 \left(\frac{C}{\sqrt{h}}|u_h|_{1,\mathcal{F}} + \|\lambda_h - p_h n\|_{0,\Gamma}\right)\left(\|u_h\|_{0,\Gamma} + \|u_h - I_h^\lambda u_h\|_{0,\Gamma}\right) \\
&\geq \frac{1}{4h}\|u_h\|_{0,\Gamma}^2 - \frac{C}{h}\|I_h^\lambda u_h - u_h\|_{0,\Gamma}^2 - C|u_h|_{1,\mathcal{F}}^2 - Ch\|\lambda_h - p_h n\|_{0,\Gamma}^2 \\
&\geq \frac{1}{4h}\|u_h\|_{0,\Gamma}^2 - C|u_h|_{1,\mathcal{F}}^2 - Ch\|\lambda_h - p_h n\|_{0,\Gamma}^2
\end{aligned}
$$

In the last line, we have used the bound $\|u_h - I_h^\lambda u_h\|_{0,\Gamma} \leq C\sqrt{h}|u_h|_{1,\mathcal{F}}$, i.e. (14) with $s_\lambda = 0$.

Combining the above inequalities, we can obtain for any $\kappa, \eta > 0$,

$$
\begin{aligned}
\mathcal{A}^{HR-BP}(u_h, p_h, \lambda_h; u_h + \kappa I_h^0 v_p, -p_h, -\lambda_h + \frac{\eta}{h} I_h^\lambda u_h) &\geq \frac{1}{K}|u_h|_{1,\mathcal{F}}^2 + \frac{\kappa}{2}\|p_h\|_{0,\mathcal{F}}^2 + \frac{\eta}{4h}\|u_h\|_{0,\Gamma}^2 \\
&+ (\theta - C\kappa)h^2|p_h|_{1,\mathcal{F}_h^e} + (\gamma_0 - C\eta)h\|\lambda_h - p_h n\|_{0,\Gamma}^2 - C(\kappa + \eta)|u_h|_{1,\mathcal{F}}^2 \qquad (20)
\end{aligned}
$$

In order to split p_h and λ_h inside $\|\lambda_h - p_h n\|_{0,\Gamma}$ we establish the following bounds with any $t > 0$ and use finally Proposition 4

$$
\begin{aligned}
&\|p_h n - \lambda_h\|_{0,\Gamma}^2 \geq \|p_h\|_{0,\Gamma}^2 + \|\lambda_h\|_{0,\Gamma}^2 - (t+1)\|p_h\|_{0,\Gamma}^2 - \frac{1}{t+1}\|\lambda_h\|_{0,\Gamma}^2 \\
&= \frac{t}{t+1}\|\lambda_h\|_{0,\Gamma}^2 - t\|p_h\|_{0,\Gamma}^2 \geq \frac{t}{t+1}\|\lambda_h\|_{0,\Gamma}^2 - \frac{Ct}{h}\left(\|p_h\|_{0,\mathcal{F}}^2 + h^2|p_h|_{1,\mathcal{F}_h^e}^2\right) \qquad (21)
\end{aligned}
$$

Substituting this into inequality (20) and assuming γ_0, κ, η, t sufficiently small, we obtain finally

$$\mathcal{A}^{HR-BP}(u_h,p_h,\lambda_h;u_h+\kappa I_h^0 v_p,-p_h,-\lambda_h+\frac{\eta}{h}I_h^\lambda u_h) \tag{22}$$
$$\geq c\left(|u_h|^2_{1,\mathcal{F}}+\|p_h\|^2_{0,\mathcal{F}}+h^2|p_h|^2_{1,\mathcal{F}_h^e}+h\|\lambda_h\|^2_{0,\Gamma}+\frac{1}{h}\|u_h\|^2_{0,\Gamma}\right)=c|||u_h,p_h,\lambda_h|||^2.$$

On the other hand, the estimates of Propositions 2 and 5 give immediately

$$|||u_h+\kappa I_h^0 v_p,-p_h,-\lambda_h+\frac{\eta}{h}I_h^\lambda u_h||| \leq C|||u_h,p_h,\lambda_h||| \tag{23}$$

Dividing (22) by (23) yields the result of the Lemma. □

Theorem 1. *Under Assumptions A, B, $\gamma_0>0$ small enough, any $\theta>0$, and $(u,p,\lambda)\in H^2(\mathcal{F})^d\times L^2_0(\mathcal{F})\times H^{\frac{1}{2}}(\Gamma)$, the following a priori error estimates hold for method (15):*

$$|u-u_h|_{1,\mathcal{F}}+\|p-p_h|_{0,\mathcal{F}}+\sqrt{h}\|\lambda-\lambda_h\|_{0,\Gamma}\leq Ch(|u|_{2,\mathcal{F}}+|p|_{1,\mathcal{F}}+|\lambda|_{1/2,\Gamma}) \tag{24}$$

Moreover, assuming the usual elliptic regularity for the Stokes problem in $\mathcal{F}$, i.e. the bound (28) for the solution to (27), one has $\forall\varphi\in H^{3/2}(\Gamma)$

$$\left|\int_\Gamma(\lambda-\lambda_h)\varphi\right|\leq Ch^2(|u|_{2,\mathcal{F}}+|p|_{1,\mathcal{F}}+|\lambda|_{1/2,\Gamma})|\varphi|_{3/2,\Gamma} \tag{25}$$

Proof. Use Galerkin orthogonality (taking $\widehat{u}=u$ for the exact solution u and extending p from $\mathcal{F}$ to $\mathcal{F}_h^e$)

$$\mathcal{A}^{HR-BP}(u_h-u,p_h-p,\lambda_h-\lambda;v_h,q_h,\mu_h)=\theta h^2\int_{\mathcal{F}_h^e}\nabla p\cdot\nabla q_h \tag{26}$$

to conclude

$$\begin{aligned}\mathcal{A}^{HR-BP}(u_h-I_h^u u,p_h-I_h^p p,\lambda_h-I_h^\lambda\lambda;v_h,q_h,\mu_h)&=2\int_{\mathcal{F}}D(u-I_h^u u):D(v_h)\\ -\int_{\mathcal{F}}((p-I_h^p p)\operatorname{div}v_h+q_h\operatorname{div}(u-I_h^u u))&+\int_\Gamma((\lambda-I_h^\lambda\lambda)\cdot v_h+\mu_h\cdot(u-I_h^u u))\\ -\gamma_0 h\int_\Gamma(\lambda-I_h^\lambda\lambda+2D(u-\widehat{I_h^u u})n-(p-I_h^p p)n)&\cdot(\mu_h+2D(\widehat{v_h})n-q_h n)\\ &+\theta h^2\int_{\mathcal{F}_h^e}\nabla I_h^p p\cdot\nabla q_h\end{aligned}$$

All the terms in the right-hand side can be bounded thanks to Proposition 2 with $s_u=1$, $s_p=s_\lambda=0$ so that

$$\mathcal{A}^{HR-BP}(u_h-I_h^u u,p_h-I_h^p p,\lambda_h-I_h^\lambda\lambda;v_h,q_h,\mu_h)\leq Ch(|u|_{2,\mathcal{F}}+|p|_{1,\mathcal{F}}+|\lambda|_{1/2,\Gamma})|||v_h,q_h,\mu_h|||$$

The inf-sup lemma 1 now gives (24).

To prove (25), choose any $\varphi\in H^{3/2}(\Gamma)$ and take v,q solution to

$$-2\,\mathrm{div}\,D(v)+\nabla q=0,\quad \mathrm{div}\,v=0 \text{ on } \mathcal{F},\quad v=\varphi \text{ on } \Gamma \tag{27}$$

as well as $\mu=-(2D(v)n-qn)|_\Gamma$. Integration by parts gives

$$2\int_{\mathcal{F}} D(u-u_h):D(v)-\int_{\mathcal{F}} q\,\mathrm{div}(u-u_h)+\int_\Gamma (u-u_h)\mu=0$$

Subtracting this from Galerkin orthogonality relation (26) gives

$$\begin{aligned}\int_\Gamma(\lambda-\lambda_h)\cdot\varphi=2\int_{\mathcal{F}} D(u-u_h):D(v-v_h)-\int_{\mathcal{F}}((p-p_h)\,\mathrm{div}(v-v_h)+(q-q_h)\,\mathrm{div}(u-u_h))\\ +\int_\Gamma((\lambda-\lambda_h)\cdot(v-v_h)+(\mu-\mu_h)\cdot(u-u_h))\\ -\gamma_0 h\int_\Gamma(\lambda-\lambda_h+2D(u-\widehat{u_h})n-(p-p_h)n)\cdot(\mu-\mu_h+2D(v-\widehat{v_h})n-(q-q_h)n)\\ -\theta h^2\int_{\mathcal{F}_h^e}\nabla p_h\cdot\nabla q_h\end{aligned}$$

Taking $v_h=I_h^u v$, $q_h=I_h^p q$, $\mu_h=I_h^\lambda\mu$, applying Proposition 2 with $s_u=1$, $s_p=s_\lambda=0$ and recalling that

$$(|v|_{2,\mathcal{F}}+|q|_{1,\mathcal{F}}+|\mu|_{1/2,\Gamma})\le C|\varphi|_{3/2,\Gamma} \tag{28}$$

thanks to the elliptic regularity of the Stokes problem, yields (25). □

2.2 $\mathbb{P}_1-\mathbb{P}_0$ *velocity-pressure spaces with interior penalty stabilization.*

Let us now choose $\mathbb{P}_1$ FE for V_h and $\mathbb{P}_0$ for Q_h and add interior penalty (IP) stabilization to the Haslinger-Renard method. The method becomes:

$$\begin{aligned}&\text{Find } (u_h,p_h,\lambda_h)\in V_h\times Q_h\times W_h \text{ such that}\\ &\mathcal{A}^{HR-IP}(u_h,p_h,\lambda_h;v_h,q_h,\mu_h)=\mathcal{L}(v_h,\mu_h),\quad \forall(v_h,q_h,\mu_h)\in V_h\times Q_h\times W_h,\end{aligned} \tag{29}$$

where

$$\begin{aligned}\mathcal{A}^{HR-IP}(u,p,\lambda;v,q,\mu)&=\mathcal{A}(u,p,\lambda;v,q,\mu)\\ &-\gamma_0 h\int_\Gamma(\lambda+2D(\widehat{u})n-pn)\cdot(\mu+2D(\widehat{v})n-qn)-\theta h\sum_{E\in\mathcal{E}_h^e}\int_E[p][q]\end{aligned}$$

V_h is continuous $\mathbb{P}_1$ FE space on mesh $\mathcal{T}_h^e$, Q_h is $\mathbb{P}_0$ FE space on mesh $\mathcal{T}_h^e$, and W_h is $\mathbb{P}_0$ FE space on mesh $\mathcal{T}_h^\Gamma$, cf. (7).

Note that the IP stabilization is applied to the pressure in the interior on $\mathcal{F}$ as well as on the cut elements. The analysis of this method is similar to that of the previous one (15) and we give immediately the final result:

Theorem 2. *Under Assumptions A and B, $\gamma_0 > 0$ small enough, any $\theta > 0$, and $(u,p,\lambda) \in H^2(\mathcal{F})^d \times L^2_0(\mathcal{F}) \times H^{\frac{1}{2}}(\Gamma)$, the a priori error estimates (24) and (25) hold for method (29).*

Proof. We shall not repeat all the technical details but only point out some important changes that should be made in Propositions 3–5 and the inf-sup lemma from the preceding section in order to adapt them to the the analysis of method (29):

- The estimate of Proposition 4 should be changed to

$$h\|q_h\|^2_{0,\Gamma} \leq C\left(\|q_h\|^2_{0,\mathcal{F}} + h\textstyle\sum_{E\in\mathcal{E}^e_h}\int_E [q_h]^2\right)$$

  This can be proved observing on any bad element $T \in \mathcal{T}^\Gamma_h$ sharing an edge E with its good neighbor T'

$$\|q_h\|_{0,\Gamma_T} = \sqrt{|\Gamma_T|}\,|(q_h)_{|T}| \leq \sqrt{|\Gamma_T|}(|[q_h]_E| + |(q_h)_{|T'}|)$$
$$= \sqrt{|\Gamma_T|}\left(\frac{1}{\sqrt{|E|}}\|[q_h]\|_{0,E} + \frac{1}{\sqrt{|T'|}}\|q_h\|_{0,T'}\right) \leq C\left(\|[q_h]\|_{0,E} + \frac{1}{\sqrt{h}}\|q_h\|_{0,T'})\right)$$

  The case of a bad element that does not share an edge with its good neighbor can be treated similarly by introducing a chain of elements connecting T to T'. The case when $T \in \mathcal{T}^\Gamma_h$ is "good" itself is trivial.
- The term $h^2|p|_{1,\mathcal{F}^e_h}$ in the triple norm in Lemma 1 should be replaced by $h\sum_{E\in\mathcal{E}^e_h}\int_E [p]^2$
- The treatment (19) of the velocity-pressure term inside the proof of Lemma 1 is now replaced by

$$-\int_{\mathcal{F}} p_h \operatorname{div} I^0_h v_p = \|p_h\|^2_{0,\mathcal{F}} + \int_{\mathcal{F}} p_h \operatorname{div}(v_p - I^0_h v_p)$$
$$= \|p_h\|^2_{0,\mathcal{F}} + \sum_{E\in\mathcal{E}^e_h}\int_{E\cap\mathcal{F}} [p_h] n\cdot(v_p - I^0_h v_p)$$

  and the bound $\sum_{E\in\mathcal{E}^e_h}\|v_p - I^0_h v_p\|^2_{0,E} \leq Ch|v_p|_{1,\mathcal{F}}$ which is proved as in Proposition 5.

□

2.3 Taylor-Hood spaces.

We now choose $\mathbb{P}_k$ (resp. $\mathbb{P}_{k-1}$) FE space with $k \geq 2$ for V_h (resp. Q_h). These are well known Taylor-Hood spaces which satisfy the discrete inf-sup conditions in the usual

setting and thus no stabilization for pressure "in the bulk" is needed. Intuitively, some extra stabilization should be now added for the pressure on the cut triangles. We thus propose the following modification of the Haslinger-Renard method for Taylor-Hood spaces:

$$\begin{aligned}&\text{Find } (u_h, p_h, \lambda_h) \in V_h \times Q_h \times W_h \text{ such that}\\ &\mathcal{A}^{HR-TH}(u_h, p_h, \lambda_h; v_h, q_h, \mu_h) = \mathcal{L}(v_h, \mu_h), \quad \forall (v_h, q_h, \mu_h) \in V_h \times Q_h \times W_h,\end{aligned} \tag{30}$$

where

$$\begin{aligned}\mathcal{A}^{HR-TH}(u, p, \lambda; v, q, \mu) = {} & \mathcal{A}(u, p, \lambda; v, q, \mu)\\ & -\gamma_0 h \int_\Gamma (\lambda + D(\widehat{u})n - \widehat{p}n) \cdot (\mu + D(\widehat{v})n - \widehat{q}n)\end{aligned}$$

V_h is continuous $\mathbb{P}_k$ FE space on mesh $\mathcal{T}_h^e$, Q_h (resp. W_h) is continuous $\mathbb{P}_{k-1}$ FE space on mesh $\mathcal{T}_h^e$ (resp. $\mathcal{T}_h^\Gamma$) for $k \geq 2$, cf. (7). The notation $\widehat{\cdot}$ stands here again for the "robust reconstruction" from Definition 1. We emphasize that it is applied here not only to the velocity, but also to pressure, unlike versions of the method (15) and (29) studied above.

The analysis of this method will be done under more restrictive assumptions than that of the previous ones:

Assumption C. The dimension is $d = 2$, $\mathcal{F}_h^i$ contains at least 3 triangles, Γ is a curve of class C^2, and the mesh $\mathcal{T}_h$ is sufficiently fine (with respect to the curvature of Γ).

Remark 3. Assumption C covers Assumption B, cf Remark 2.

We shall tacitly assume Assumption C in all the Propositions until the end of this Section. Proposition 3 will be reused in the analysis of the present case but Proposition 4 should be replaced with the following

Proposition 6. *For all $q_h \in Q_h$ one has*

$$h\|\widehat{q}_h\|_{0,\Gamma}^2 \leq C\|q_h\|_{0,\mathcal{F}}^2.$$

The proof is a straight-forward adaptation of Proposition 3 to the pressure space.

Another important ingredient in our analysis will be the discrete inf-sup condition, cf. Proposition 10 below. We recall first a well-known auxiliary result:

Proposition 7. *The exists a mesh independent constant $\beta > 0$ such that for any $q_h \in Q_h$*

$$\beta h |q_h|_{1,\mathcal{F}_h^i} \leq \sup_{v_h \in V_h^i} \frac{\int_{\mathcal{F}} q_h \operatorname{div} v_h}{|v_h|_{1,\mathcal{F}_h^i}} \tag{31}$$

where $V_h^i = V_h \cap (H_0^1(\mathcal{F}_h^i))^d$.

This result is customarily applied to the analysis of FE discretization of the Stokes equations via the Verfürth trick [9]. The proof in the 2D case under the assumption that the mesh contains at least 3 triangles can be found in [2]. We note in passing that a 3D generalization in a similar context is presented in [11].

Let $B_h^\Gamma := \mathcal{F} \setminus \mathcal{F}_h^i$ and note that the boundary of B_h^Γ consists of $\partial\mathcal{F}_h^i$ and Γ.

Proposition 8. *Let $p_h \in Q_h$ and $v \in H^1(B_h^\Gamma)$ vanishing on Γ. Then*

$$\int_{\partial\mathcal{F}_h^i} |p_h v| \le C\|p_h\|_{0,B_h^\Gamma} |v|_{1,B_h^\Gamma} \tag{32}$$

Proof. Take any triangle $T \in \mathcal{T}_h^\Gamma$ such that one of its sides E is an edge on $\partial\mathcal{F}_h^i$. Introduce the polar coordinates (r,φ) centered at the vertex O of T opposite to side E (thus O lies outside $\mathcal{F}$). The part of T inside $\mathcal{F}$ can be represented in these coordinates as

$$\mathcal{F}_T = \{(r,\varphi) \text{ such that } \alpha < \varphi < \beta,\ r_\Gamma(\varphi) < r < r_i(\varphi)\}$$

with $r_\Gamma(\varphi)$ and $r_i(\varphi)$ representing, respectively, Γ and $E \subset \partial\mathcal{F}_h^i$. In view of Assumption C, $r_\Gamma(\varphi)$ is a C^2 function and there are positive numbers $r_{\min}$ and $r_{\max}$ such that $r_{\min} \le r_\Gamma(\varphi) < r_i(\varphi) \le r_{\max}$ for all $\varphi \in [\alpha,\beta]$. There are 2 options: either Γ_T is very close to edge E so that $\frac{r_{\max}}{r_{\min}} \le \rho$, or $\mathcal{F}_T$ covers a significant portion of T so that $|\mathcal{F}_T| \ge \theta|T|$. The positive numbers ρ and θ here can be chosen in a mesh-independent manner.

We start with the first option: $\frac{r_{\max}}{r_{\min}} \le \rho$. Using the notations above and recalling $v = 0$ at $r = r_\Gamma(\varphi)$ gives

$$\begin{aligned}
&\int_E |p_h v| \le C\int_\alpha^\beta (|p_h v| r)_{r=r_i(\varphi)} d\varphi = C\int_\alpha^\beta \int_{r_\Gamma(\varphi)}^{r_i(\varphi)} \frac{\partial |p_h v r|}{\partial r} dr d\varphi \\
&\le C\left(\int_\alpha^\beta \int_{r_\Gamma(\varphi)}^{r_i(\varphi)} \left|\frac{\partial p_h}{\partial r} v\right| r dr d\varphi + \|p_h\|_{0,\mathcal{F}_T}\|\nabla v\|_{0,\mathcal{F}_T} + \frac{1}{r_{\min}}\|p_h\|_{0,\mathcal{F}_T}\|v\|_{0,\mathcal{F}_T}\right)
\end{aligned}$$

We set $l(\varphi) = r_i(\varphi) - r_\Gamma(\varphi)$ and bound the first integral above using, for any φ fixed, an inverse inequality for p_h on the interval $(r_\Gamma(\varphi), r_i(\varphi))$ and Poincaré inequality for v on the same interval (recall that $v = 0$ at $r = r_\Gamma(\varphi)$)

$$\begin{aligned}
\int_\alpha^\beta \int_{r_\Gamma(\varphi)}^{r_i(\varphi)} \left|\frac{\partial p_h}{\partial r} v\right| r dr d\varphi &\le r_{\max}\int_\alpha^\beta \left(\int_{r_\Gamma(\varphi)}^{r_i(\varphi)} \left(\frac{\partial p_h}{\partial r}\right)^2 dr\right)^{\frac12} \left(\int_{r_\Gamma(\varphi)}^{r_i(\varphi)} v^2 dr\right)^{\frac12} d\varphi \\
&\le C r_{\max}\int_\alpha^\beta \frac{1}{l(\varphi)} \left(\int_{r_\Gamma(\varphi)}^{r_i(\varphi)} p_h^2 dr\right)^{\frac12} \times l(\varphi) \left(\int_{r_\Gamma(\varphi)}^{r_i(\varphi)} \left(\frac{\partial v}{\partial r}\right)^2 dr\right)^{\frac12} d\varphi \\
&\le C\frac{r_{\max}}{r_{\min}}\|p_h\|_{0,\mathcal{F}_T}\|\nabla v\|_{0,\mathcal{F}_T}
\end{aligned} \tag{33}$$

Recalling the bound on $\frac{r_{\max}}{r_{\min}}$ (which implies, in particular, $r_{\min} \ge \frac{h}{\rho}$) we conclude

$$\int_E |p_h v| \le C\left(\|p_h\|_{0,\mathcal{F}_T}\|\nabla v\|_{0,\mathcal{F}_T} + \frac{1}{h}\|p_h\|_{0,\mathcal{F}_T}\|v\|_{0,\mathcal{F}_T}\right) \tag{34}$$

On the other hand, if $|\mathcal{F}_T| \ge \theta|T|$, extending v by 0 outside $\mathcal{F}$, applying Proposition 1 and an inverse inequality (valid on the whole triangle T) also yields (34):

$$\begin{aligned}\int_E |p_h v| &\le \sqrt{h}\|p_h v\|_{0,\partial T} \le C(\|p_h v\|_{0,T} + h|p_h v|_{1,T})\\ &\le C(\|p_h\|_{\infty,T}\|v\|_{0,\mathcal{F}_T} + h\|\nabla p_h\|_{\infty,T}\|v\|_{0,\mathcal{F}_T} + h\|p_h\|_{\infty,T}\|\nabla v\|_{0,\mathcal{F}_T})\\ &\le C\left(\|p_h\|_{0,\mathcal{F}_T}\|\nabla v\|_{0,\mathcal{F}_T} + \frac{1}{h}\|p_h\|_{0,\mathcal{F}_T}\|v\|_{0,\mathcal{F}_T}\right)\end{aligned}$$

Summing (34) over all $T \in \mathcal{T}_h^\Gamma$ having a side on $\partial\mathcal{F}_h^i$ yields

$$\int_{\partial\mathcal{F}_h^i} |p_h v| \le C\|p_h\|_{0,B_h^\Gamma}\left(\|\nabla v\|_{0,B_h^\Gamma} + \frac{1}{h}\|v\|_{0,B_h^\Gamma}\right)$$

Recall that $v = 0$ on Γ and the width of B_h^Γ is of order h, so that $\|v\|_{0,B_h^\Gamma} \le Ch\|\nabla v\|_{0,B_h^\Gamma}$ by a Poincaré inequality. We have thus proved (32). □

Proposition 9. *Under Assumption C, there exists a continuous piecewise linear vector-valued function ψ_h on mesh $\mathcal{T}_h^e$ such that $\psi_h \cdot n \ge 0$ on Γ, $\operatorname{div}\psi_h \ge \delta_0$ on all the triangles of $\mathcal{T}_h^\Gamma$, and $\operatorname{div}\psi_h \ge -\delta_1 h$ on all the triangles of $\mathcal{T}_h^i$ with some constants $\delta_0, \delta_1 > 0$. Moreover, there is a constant $C > 0$ such that for any $p_h \in Q_h$*

$$|p_h\psi_h|_{1,\mathcal{F}} + \frac{1}{\sqrt{h}}\|p_h\psi_h\|_{0,\Gamma} \le C\|p_h\|_{0,\mathcal{F}} \tag{35}$$

Proof. Let $\mathcal{B}_\zeta = \{x \in \mathbb{R}^2 / \operatorname{dist}(x,\Gamma) < \zeta\}$ for any $\zeta > 0$. Thanks to the smoothness of Γ, one can introduce orthogonal coordinates (ξ_1, ξ_2) on $\mathcal{B}_\eta$ with some mesh-independent $\eta > 0$ such that $\xi_2 = 0$ on Γ and $\xi_2 < 0$ on $\mathcal{F} \cap \mathcal{B}_\eta$. Let e_i denote the basis vectors of these coordinates ($e_i = \partial\mathbf{r}/\partial\xi_i$). Assuming $\eta > h$, let us introduce the vector-valued function given on $\mathcal{B}_\eta$ by $\psi = \xi_2 e_2$ for $|\xi_2| < h$, $\psi = -h\frac{\xi_2+\eta}{\eta-h}e_2$ for $-\eta < \xi_2 < -h$, left undefined for $h < \xi_2 < \eta$, and extended by 0 on $\mathcal{F} \setminus \mathcal{B}_\eta$. Let $\psi_h = I_h\psi$ where I_h is the standard nodal interpolation operator to continuous $\mathbb{P}_1$ FE space on $\mathcal{T}_h^e$.

Clearly, $\operatorname{div}\psi = 1$ on Γ, hence $\operatorname{div}\psi \ge \frac{1}{2}$ on $\mathcal{B}_h$ by continuity for sufficiently small h. Since $\mathcal{B}_h \supset \mathcal{F}_h^\Gamma$, one observes on all the triangles of $\mathcal{T}_h^\Gamma$

$$\operatorname{div}\psi_h \ge \frac{1}{2} - \operatorname{div}(\psi - \psi_h) \ge \frac{1}{2} - Ch\|\psi\|_{W^{2,\infty}(\mathcal{B}_h)} = \delta_0 > 0$$

since h is sufficiently small and ψ is sufficiently smooth thanks to the hypothesis on Γ. Turning to the triangles of $\mathcal{T}_h^i$ we make the following observation: if Γ were a straight line, $\psi \cdot e_1$ would vanish, and $\psi \cdot e_2$ would be linear on $-\eta < \xi_2 < -h$ with the slope $-\frac{h}{\eta-h}$ so that $\operatorname{div}\psi_h \ge -\frac{h}{\eta-h}$ on the triangles of $\mathcal{T}_h^i$. The actual geometry

of Γ introduces the corrections of order $O(h^2)$ into this argument so that one still has $\operatorname{div}\psi_h \geq -\delta_1 h$ on these triangles. We also observe, by construction of ψ_h,

$$\|\psi_h\|_{\infty,\mathcal{F}_h^e} \leq Ch \text{ and } \|\nabla\psi_h\|_{\infty,\mathcal{F}_h^e} \leq C$$

Let us now take any $p_h \in Q_h$. By the bounds on ψ_h proven above and by an inverse inequality we deduce on any triangle $T \in \mathcal{T}_h^i$

$$|p_h\psi_h|_{1,T} \leq Ch|p_h|_{1,T} + C\|p_h\|_{0,T} \leq C\|p_h\|_{0,T} \tag{36}$$

A similar bound also holds on any cut triangle $T \in \mathcal{T}_h^\Gamma$. One cannot use a straightforward inverse inequality in this case, since the width of the cut portion $\mathcal{F}_T$, say ε, can be much smaller than h. However, the construction of ψ_h implies in such a situation $\|\psi_h\|_{\infty,\mathcal{F}_T} \leq C\varepsilon$. Combining this with the inverse inequality $|p_h|_{1,\mathcal{F}_T} \leq \frac{C}{\varepsilon}\|p_h\|_{0,\mathcal{F}_T}$ one arrives at (36). Summing this over all the triangles $T \in \mathcal{T}_h^i$ yields $|p_h\psi_h|_{1,\mathcal{F}} \leq C\|p_h\|_{0,\mathcal{F}}$. Finally, in order to bound $p_h\psi_h$ in $L^2(\Gamma)$ we recall. Hence

$$\|p_h\psi_h\|_{1,\Gamma} \leq \|p_h\psi_h\|_{0,\partial\mathcal{F}_h^i} + C\sqrt{h}|p_h\psi_h|_{1,B_h^\Gamma} \leq Ch\|p_h\|_{0,\partial\mathcal{F}_h^i} + C\sqrt{h}\|p_h\|_{0,\mathcal{F}}$$

By scaling, $\|p_h\|_{0,E} \leq \frac{C}{\sqrt{h}}\|p_h\|_{0,T}$ for any edge $E \in \partial\mathcal{F}_h$ adjacent to a triangle $T \in \mathcal{T}_h^i$. The summation over all such edges yields $\|p_h\|_{0,\partial\mathcal{F}_h^i} \leq \frac{C}{\sqrt{h}}\|p_h\|_{0,\mathcal{F}_h^i}$ and consequently $\|p_h\psi_h\|_{1,\Gamma} \leq C\sqrt{h}\|p_h\|_{0,\mathcal{F}}$. $\square$

Proposition 10. *For any $p_h \in Q_h$ there exists $v_h^p \in V_h$ such that*

$$-\int_{\mathcal{F}} p_h \operatorname{div} v_h^p = \|p_h\|_{0,\mathcal{F}}^2 \text{ and } |v_h^p|_{1,\mathcal{F}} + \frac{1}{\sqrt{h}}\|v_h^p\|_{0,\Gamma} \leq C\|p_h\|_{0,\mathcal{F}} \tag{37}$$

Proof. The continuous inf-sup condition implies that for all $p_h \in Q_h$ there exists $v_p \in (H_0^1(\mathcal{F}))^d$ satisfying (18). Recalling the interpolation operator I_h^0 from Proposition 5, we observe

$$\begin{aligned}
-\int_{\mathcal{F}} p_h \operatorname{div} I_h^0 v_p &= -\int_{\mathcal{F}_h^i} p_h \operatorname{div} I_h^0 v_p = \|p_h\|_{0,\mathcal{F}_h^i}^2 + \int_{\mathcal{F}_h^i} p_h \operatorname{div}(v_p - I_h^0 v_p) \\
&= \|p_h\|_{0,\mathcal{F}_h^i}^2 + \int_{\partial\mathcal{F}_h^i} p_h n \cdot v_p - \int_{\mathcal{F}_h^i} \nabla p_h \cdot (v_p - I_h^0 v_p) \\
&\geq \|p_h\|_{0,\mathcal{F}_h^i}^2 - C\|p_h\|_{0,B_h^\Gamma}|v_p|_{1,B_h^\Gamma} - Ch|p_h|_{1,\mathcal{F}_h^i}|v_p|_{1,\mathcal{F}_h^i} \\
&\geq \|p_h\|_{0,\mathcal{F}_h^i}^2 - C|v_p|_{1,\mathcal{F}_h^i}\left(\|p_h\|_{0,B_h^\Gamma}^2 + h^2|p_h|_{1,\mathcal{F}_h^i}^2\right)^{\frac{1}{2}} \\
&\geq \frac{1}{2}\|p_h\|_{0,\mathcal{F}_h^i}^2 - C\|p_h\|_{0,B_h^\Gamma}^2 - Ch^2|p_h|_{1,\mathcal{F}_h^i}^2
\end{aligned} \tag{38}$$

We have used Proposition 8, the interpolation estimate from Proposition 5, Young inequality and $|v_p|_{1,B_h^\Gamma}^2 + |v_p|_{1,\mathcal{F}_h^i}^2 = |v_p|_{1,\mathcal{F}}^2 \leq C\|p_h\|_{0,\mathcal{F}}^2 = C(\|p_h\|_{0,\mathcal{F}_h^i}^2 + \|p_h\|_{0,\mathcal{F}_h^\Gamma}^2)$.

Moreover, thanks to Proposition 7 and the inverse inequality there exists $v_h^{p,i} \in V_h^i$ such that

$$-\int_{\mathcal{F}} p_h \operatorname{div} v_h^{p,i} = h^2 |p_h|^2_{1,\mathcal{F}_h^i} \quad \text{and} \quad |v_h^{p,i}|_{1,\mathcal{F}} \leq Ch|p_h|_{1,\mathcal{F}_h^i} \leq C\|p_h\|_{0,\mathcal{F}_h^i} \tag{39}$$

In order to control p_h on B_h^Γ, we introduce $v_h^{p,\Gamma} = -p_h \psi_h$ with ψ_h from Proposition 9. Then

$$\begin{aligned} -\int_{\mathcal{F}} p_h \operatorname{div} v_h^{p,\Gamma} &= \int_{\mathcal{F}} p_h \nabla p_h \cdot \psi_h + \int_{\mathcal{F}} p_h^2 \operatorname{div} \psi_h \\ &= \frac{1}{2}\int_\Gamma p_h^2 n \cdot \psi_h + \frac{1}{2}\int_{\mathcal{F}} p_h^2 \operatorname{div} \psi_h \geq \frac{\delta_0}{2}\|p_h\|^2_{0,B_h^\Gamma} - \delta_1 h \|p_h\|^2_{0,\mathcal{F}_h^i} \end{aligned} \tag{40}$$

thanks to $n \cdot \psi_h \geq 0$ on Γ and the bounds on $\operatorname{div} \psi_h$.

Let $v_h^p = I_h^0 v_p + \kappa v_h^{p,i} + \kappa v_h^{p,\Gamma}$ with sufficiently big $\kappa > 0$. Then, taking the sum of (38), (39), (40), and reminding $\|p_h\|^2_{0,\mathcal{F}} = \|p_h\|^2_{0,\mathcal{F}_h^i} + \|p_h\|^2_{0,B_h^\Gamma}$ yields

$$-\int_{\mathcal{F}} p_h \operatorname{div} v_h^p \geq \frac{1}{2}\|p_h\|^2_{0,\mathcal{F}}$$

Turning to the second estimate in (37), we recall

$$|I_h^0 v_p|_{1,\mathcal{F}} + |v_h^{p,i}|_{1,\mathcal{F}} \leq C\|p_h\|_{0,\mathcal{F}}$$

and $I_h^0 v_p = v_h^{p,i} = 0$ on Γ. Moreover, $v_h^{p,\Gamma}$ is bounded in $H^1(\mathcal{F})$ and $L^2(\Gamma)$ by (35). Thus

$$|v_h^p|_{1,\mathcal{F}} + \frac{1}{\sqrt{h}}\|v_h^p\|_{0,\Gamma} \leq C\|p_h\|_{0,B_h^\Gamma}$$

This entails (37). □

Lemma 2. *Under Assumption C, taking γ_0 small enough, there exists a mesh-independent constant $c > 0$ such that*

$$\inf_{(u_h,p_h,\lambda_h)\in V_h\times Q_h\times W_h} \sup_{(v_h,q_h,\mu_h)\in V_h\times Q_h\times W_h} \frac{\mathcal{A}^{HR-TH}(u_h,p_h,\lambda_h;v_h,q_h,\mu_h)}{|||u_h,p_h,\lambda_h|||\ |||v_h,q_h,\mu_h|||} \geq c$$

where the triple norm is defined by

$$|||u,p,\lambda||| = \left(|u|^2_{1,\mathcal{F}} + \|p\|^2_{0,\mathcal{F}} + h\|\lambda\|^2_{0,\Gamma} + \frac{1}{h}\|u\|^2_{0,\Gamma}\right)^{\frac{1}{2}}$$

Proof. As in the proof of Lemma 1, we observe that

$$\mathcal{A}^{HR-TH}(u_h,p_h,\lambda_h;u_h,-p_h,-\lambda_h) \geq \frac{1}{K}|u_h|^2_{1,\mathcal{F}} + \gamma_0 h\|\lambda_h - \widehat{p}_h n\|^2_{0,\Gamma}$$

thanks to Korn inequality (17) and the smallness of γ_0. Moreover, employing v_h^p from Proposition 10 and the estimates from Propositions 3 and 6,

$$\begin{aligned}\mathcal{A}^{HR-TH}(u_h,p_h,\lambda_h;v_h^p,0,0) &= 2\int_{\mathcal{F}} D(u_h):D(v_h^p)+\|p_h\|_{0,\mathcal{F}}^2+\int_\Gamma \lambda\cdot v_h^p \\ &\quad -\gamma_0 h\int_\Gamma (D(\widehat{u_h})n-\widehat{p_h}n+\lambda_h)\cdot D(\widehat{v_h^p})n \\ &\geq \frac{1}{2}\|p_h\|_{0,\mathcal{F}}^2 - C|u_h|_{1,\mathcal{F}}^2 - Ch\|\lambda_h\|_{0,\Gamma}^2 - \frac{\gamma_0}{2}h\|\lambda_h-\widehat{p_h}n\|_{0,\Gamma}^2\end{aligned}$$

We proceed as in the proof of Lemma 1 and arrive at, cf. (20),

$$\begin{aligned}&\mathcal{A}^{HR-TH}(u_h,p_h,\lambda_h;u_h+\kappa v_h^p,-p_h,-\lambda_h+\frac{\eta}{h}I_h^\lambda u_h) \geq \frac{1}{K}|u_h|_{1,\mathcal{F}}^2+\frac{\kappa}{2}\|p_h\|_{0,\mathcal{F}}^2 \\ &+\frac{\eta}{2h}\|u_h\|_{0,\Gamma}^2+\left(\frac{\gamma_0}{2}-C\eta\right)h\|\lambda_h-\widehat{p_h}n\|_{0,\Gamma}^2 - C(\kappa+\eta)|u_h|_{1,\mathcal{F}} - C\kappa h\|\lambda_h\|_{0,\Gamma}^2\end{aligned}$$

The rest of the proof follows again that of Lemma 1, with the only modification that $\|\widehat{p_h}n\|_{0,\Gamma}^2$ rather than $\|p_h n\|_{0,\Gamma}^2$ will appear in the calculation (21). This gives now

$$\|\widehat{p_h}n-\lambda_h\|_{0,\Gamma}^2 \geq \frac{t}{t+1}\|\lambda_h\|_{0,\Gamma}^2 - \frac{Ct}{h}\|p_h\|_{0,\mathcal{F}}^2$$

which is established using Proposition 6 rather than Proposition 4. Substituting this into the bound above and taking t,κ,η sufficiently small leads to

$$\mathcal{A}^{HR-TH}(u_h,p_h,\lambda_h;u_h+\kappa v_h^p,-p_h,-\lambda_h+\frac{\eta}{h}I_h^\lambda u_h) \geq c|||u_h,p_h,\lambda_h|||^2$$

Finally, the test function $(u_h+\kappa v_h^p,-p_h,-\lambda_h+\frac{\eta}{h}I_h^\lambda u_h)$ can be bounded in the triple norm via (u_h,p_h,λ_h). This ends the proof in the same way as as in the case of Lemma 1. □

Theorem 3. *The following a priori error estimate hold under Assumption C for method (30) with $\mathbb{P}_k$ FE for v and $\mathbb{P}_{k-1}$ FE for p and λ ($k\geq 2$):*

$$\begin{aligned}&|u-u_h|_{1,\mathcal{F}}+\|p-p_h\|_{0,\mathcal{F}}+\sqrt{h}\|\lambda-\lambda_h\|_{0,\Gamma} \\ &\qquad\qquad \leq Ch^k(|u|_{k+1,\mathcal{F}}+|p|_{k,\mathcal{F}}+|\lambda|_{k-1/2,\Gamma})\end{aligned} \tag{41}$$

and, assuming the usual elliptic regularity (28) for the Stokes problem (27),

$$\left|\int_\Gamma(\lambda-\lambda_h)\varphi\right| \leq Ch^{k+1}(|u|_{k+1,\mathcal{F}}+|p|_{k,\mathcal{F}}+|\lambda|_{k-1/2,\Gamma})|\varphi|_{3/2,\Gamma} \tag{42}$$

for all $\varphi\in H^{3/2}(\Gamma)$.

Proof. The proof follows the same lines as that of Theorem 1. □

3 Methods à la Burman-Hansbo.

We turn now to alternative methods generalizing that of [4, 5] to the Stokes equations, cf. (6). The meshes and FE spaces follow the same pattern as before, cf. (7). We shall employ either $\mathbb{P}_0$ or $\mathbb{P}_1$ FE for λ and several choices for velocity and pressure. The method reads:

$$\begin{aligned}&\text{Find } (u_h, p_h, \lambda_h) \in V_h \times Q_h \times W_h \text{ such that}\\ &\mathcal{A}^{BH-l-var}((u_h, p_h, \lambda_h; v_h, q_h, \mu_h) = \mathcal{L}(v_h, \mu_h), \quad \forall (v_h, q_h, \mu_h) \in V_h \times Q_h \times W_h,\end{aligned} \tag{43}$$

where

$$\mathcal{A}^{BH-l-var}(u, p, \lambda; v, q, \mu) = \mathcal{A}(u, p, \lambda; v, q, \mu) + \mathcal{S}^l_\lambda(\lambda, \mu) + \mathcal{S}^{var}_p(p, q)$$

Here, $\mathcal{S}^l_\lambda(\lambda, \mu)$ with $l \in \{0, 1\}$ is the stabilization term for Lagrange multiplier discretized by $\mathbb{P}_l$ FE. We set

$$\mathcal{S}^0_\lambda(\lambda, \mu) = -\gamma h \sum_{E \in \mathcal{E}^\Gamma_h} \int_E [\lambda] \cdot [\mu] \quad \text{and} \quad \mathcal{S}^1_\lambda(\lambda, \mu) = -\gamma h^2 \int_{\mathcal{F}^\Gamma_h} \nabla\lambda : \nabla\mu$$

Moreover, $\mathcal{S}^{var}_p(p, q)$ with $var \in \{BP, IP, TH\}$ is the stabilization term for pressure chosen accordingly to the velocity-pressure FE-pair as in the following table

Velocity FE	Pressure FE	Acronym	Stabilization
$\mathbb{P}_1$	$\mathbb{P}_1$	BP	$\mathcal{S}^{BP}_p(p,q) = -\theta h^2 \int_{\mathcal{F}^e_h} \nabla p \cdot \nabla q$
$\mathbb{P}_1$	$\mathbb{P}_0$	IP	$\mathcal{S}^{IP}_p(p,q) = -\theta h \sum^e_{E \in \mathcal{E}_h} \int_E [p][q]$
$\mathbb{P}_2$	$\mathbb{P}_1$	TH	$\mathcal{S}^{TH}_p(p,q) = 0$

Note that higher order Taylor-Hood spaces ($\mathbb{P}_k$–$\mathbb{P}_{k-1}$ for $k > 2$) can also be used but one should choose then another stabilization for λ which we do not develop here.
Remark: The stabilization term $\mathcal{S}^1_\lambda(\lambda, \mu)$ does not correspond precisely to the one introduced in [5] where the jumps in derivatives over elements sides are used. Adaptation requires future investigations.

One can show that all the choices above lead to inf-sup stable methods. We provide here a detailed proof for the case $\mathcal{A}^{BH-1-BP}$ (thus employing $\mathbb{P}_1$ FE for all the 3 variables) and comment briefly on other cases below.

Proposition 11. *Let V_h, Q_h, W_h in (7) be $\mathbb{P}_1$ FE spaces on respective meshes. Under Assumption B, there exists an interpolation operator $I^\Gamma_h : H^1(\mathcal{F})^d \to V_h$ such that*

$$|I^\Gamma_h v|_{1,\mathcal{F}} + \frac{1}{\sqrt{h}} \|v - I^\Gamma_h v\|_{0,\Gamma} \le C|v|_{1,\mathcal{F}} \tag{44}$$

and for all $\lambda_h \in W_h$

$$\int_\Gamma \lambda_h \cdot (v - I^\Gamma_h v) \le Ch|v|_{1,\mathcal{F}} |\lambda_h|_{1,\mathcal{F}^\Gamma_h} \tag{45}$$

Proof. One can cover $\mathcal{F}_h^\Gamma$ by patches Π_k, $k = 1,\dots,N_\Pi$ such that

1. Each Π_k is a connected union of elements of $\mathcal{T}_h^\Gamma$ and the number of elements in Π_k is uniformly bounded;
2. There exist sub-patches $\Pi_k^{00} \subset \Pi_k^0 \subset \Pi_k$, $k = 1,\dots,N_\Pi$, which are also connected unions of elements of $\mathcal{T}_h^\Gamma$ and have the following properties: Π_k^0 and Π_l^0 are disjoint if $k \neq l$; Π_k^0 contains all the neighbors of elements in Π_k^{00}; $|\Gamma \cap \Pi_k^{00}| \geq ch^{d-1}$ with a mesh-independent constant $c > 0$.

One can then construct a partition of unity $\{\varphi^{(k)}\}$ subordinate to $\{\Pi_k\}$, i.e. $\sum_{k=1}^{N_\Pi} \varphi^{(k)} = 1$ and $\varphi^{(k)} \geq 0$ on $\mathcal{F}_h^\Gamma$, $\operatorname{supp}\varphi^{(k)} \subset \Pi_k$, and moreover $\varphi^{(k)} = 1$ on Π_k^0.

Let $v_h^{(k)}$ be a $\mathbb{P}_1$ finite element function equal to 1 on Π_k^{00} and vanishing at all the nodes of $\mathcal{T}_h^e$ outside $\overline{\Pi_k^{00}}$. Recall the interpolation operator I_h^u from Lemma 2 ($k_u = 1$) and set

$$I_h^\Gamma v = I_h^u v + \sum_{k=1}^{N_\Pi} \frac{\int_\Gamma \varphi^{(k)}(v - I_h^u v)}{\int_\Gamma \varphi^{(k)} v_h^{(k)}} v_h^{(k)}$$

By construction, $\operatorname{supp} v_h^{(k)} \subset \Pi_k^0$ so that $\int_\Gamma \varphi^{(k)} v_h^{(l)} = 0$ for $k \neq l$. Thus, the definition of I_h^Γ entails

$$\int_\Gamma \varphi^{(k)}(v - I_h^\Gamma v) = 0, \quad \forall k = 1,\dots,N_\Pi$$

so that, for any $\lambda_h \in W_h$,

$$\begin{aligned}\int_\Gamma \lambda_h \cdot (v - I_h^\Gamma v) &= \sum_{k=1}^{N_\Pi} \int_\Gamma \varphi^{(k)}(\lambda_h - \bar{\lambda}_h^{(k)}) \cdot (v - I_h^\Gamma v) \\ &\leq \sum_{k=1}^{N_\Pi} \|\lambda_h - \bar{\lambda}_h^{(k)}\|_{0,\Gamma\cap\Pi_k} \|v - I_h^\Gamma v\|_{0,\Gamma\cap\Pi_k} \qquad (46)\end{aligned}$$

with $\bar{\lambda}_h^{(k)} = \frac{1}{|\Pi_k|}\int_{\Pi_k} \lambda_h$. By scaling

$$\|\lambda_h - \bar{\lambda}_h^{(k)}\|_{0,\Gamma\cap\Pi_k} \leq \sqrt{|\Gamma \cap \Pi_k|}\, \|\lambda_h - \bar{\lambda}_h^{(k)}\|_{\infty,\Pi_k} \leq C\sqrt{h}|\lambda_h|_{1,\Pi_k} \qquad (47)$$

Moreover, since $0 \leq v_h^{(k)}, \varphi^{(k)} \leq 1$ and $\int_\Gamma \varphi^{(k)} v_h^{(k)} \geq \int_{\Gamma\cap\Pi_k^{00}} \varphi^{(k)} \geq ch$ for all k,

$$\begin{aligned}\|v - I_h^\Gamma v\|_{0,\Gamma\cap\Pi_k} &\leq \|v - I_h^u v\|_{0,\Gamma\cap\Pi_k} + \sum_{l=1}^{N_\Pi} \frac{\|\varphi^{(l)}\|_{0,\Gamma\cap\Pi_k} \|v - I_h^u v\|_{0,\Gamma\cap\Pi_k}}{ch^{d-1}} \|v_h^{(l)}\|_{0,\Gamma\cap\Pi_k} \\ &\leq C\|v - I_h^u v\|_{0,\Gamma\cap\Pi_k} \leq C\sqrt{h}|v|_{1,\tilde{\Pi}_k} \qquad (48)\end{aligned}$$

where $\tilde{\Pi}_k$ is the set of mesh elements touching Π_k. We have used here Lemma 2 with $s_u = 0$ and the fact that the number of patches Π_l intersecting a given patch Π_k is uniformly bounded. Substituting (47) and (48) into (46) yields (45). Incidentelly,

(48) also proves the $L^2(\Gamma)$ bound in (44). The $H^1(\mathcal{F})$ bound in (44) is established in the same way as in (48) noting that $|v_h^{(k)}|_{1,\mathcal{F}} \leq Ch^{\frac{d}{2}-1}$ for all k. $\square$

Lemma 3. *Under Assumption B, for any $\gamma, \theta > 0$ there exists a mesh-independent constant $c > 0$ such that*

$$\inf_{(u_h,p_h,\lambda_h)\in V_h\times Q_h\times W_h} \sup_{(v_h,q_h,\mu_h)\in V_h\times Q_h\times W_h} \frac{\mathcal{A}^{BH-1-BP}(u_h,p_h,\lambda_h;v_h,q_h,\mu_h)}{|||u_h,p_h,\lambda_h|||\ |||v_h,q_h,\mu_h|||} \geq c$$

where the triple norm is defined by

$$|||u,p,\lambda||| = \left(|u|^2_{1,\mathcal{F}} + \|p\|^2_{0,\mathcal{F}} + h^2|p|^2_{1,\mathcal{F}_h^e} + h\|\lambda\|^2_{0,\Gamma} + h^2|\lambda|^2_{1,\mathcal{F}_h^\Gamma} + \frac{1}{h}\|u\|^2_{0,\Gamma}\right)^{1/2}$$

Proof. Take $\lambda_h \in W_h$ and consider $v_\lambda \in H^1(\mathcal{F})^d$ the solution to

$$\int_{\mathcal{F}} \nabla v_\lambda : \nabla w + \frac{1}{h}\int_\Gamma v_\lambda \cdot w = \int_\Gamma \lambda_h \cdot w \quad \forall w \in H^1(\mathcal{F})^d \tag{49}$$

Taking $w = v_\lambda$ here yields

$$\int_\Gamma \lambda_h \cdot v_\lambda = |v_\lambda|^2_{1,\mathcal{F}} + \frac{1}{h}\|v_\lambda\|^2_{0,\Gamma} \tag{50}$$

Hence, with the aid of the interpolation operator I_h^Γ introduced in Proposition 11,

$$\begin{aligned}\int_\Gamma \lambda_h \cdot I_h^\Gamma v_\lambda = \int_\Gamma \lambda_h \cdot v_\lambda - \int_\Gamma \lambda_h \cdot (v_\lambda - I_h^\Gamma v_\lambda) &\geq |v_\lambda|^2_{1,\mathcal{F}} + \frac{1}{h}\|v_\lambda\|^2_{0,\Gamma} - Ch|\lambda_h|_{1,\mathcal{F}_h^\Gamma}|v_\lambda|_{1,\mathcal{F}} \\ &\geq \frac{1}{2}|v_\lambda|^2_{1,\mathcal{F}} + \frac{1}{h}\|v_\lambda\|^2_{0,\Gamma} - Ch^2|\lambda_h|^2_{1,\mathcal{F}_h^\Gamma}\end{aligned} \tag{51}$$

Let $\widetilde{\lambda}_h$ be the $\mathbb{P}_1$ FE function on $\mathcal{F}_h^e$ that vanishes at all the interior nodes of $\mathcal{F}_h^i$ and coincides with λ_h on $\mathcal{F}_h^\Gamma$. Taking $w = \widetilde{\lambda}_h$ in (49), we observe for any $\alpha > 0$

$$\begin{aligned}\|\lambda_h\|^2_{0,\Gamma} &= \int_{\mathcal{F}} \nabla v_\lambda : \nabla \widetilde{\lambda}_h + \frac{1}{h}\int_\Gamma v_\lambda \cdot \lambda_h \\ &\leq \frac{1}{2\alpha h}|v_\lambda|^2_{1,\mathcal{F}} + \frac{\alpha h}{2}|\widetilde{\lambda}_h|^2_{1,\mathcal{F}} + \frac{1}{h^2}\|v_\lambda\|^2_{0,\Gamma} + \frac{1}{4}\|\lambda_h\|^2_{0,\Gamma}\end{aligned} \tag{52}$$

Using a scaling argument and the fact that $B_h^\Gamma = \mathcal{F} \setminus \mathcal{F}_h^i$ is a strip of width h, we get

$$\begin{aligned}|\widetilde{\lambda}_h|^2_{1,\mathcal{F}} &\leq |\widetilde{\lambda}_h|^2_{1,\mathcal{F}_h^i} + |\lambda_h|^2_{1,\mathcal{F}_h^\Gamma} \leq \frac{C}{h}\|\lambda_h\|^2_{0,\partial\mathcal{F}_h^i} + |\lambda_h|^2_{1,\mathcal{F}_h^\Gamma} \\ &\leq \frac{C}{h}(\|\lambda_h\|^2_{0,\Gamma} + h|\lambda_h|^2_{1,\mathcal{F}_h^\Gamma}) + |\lambda_h|^2_{1,\mathcal{F}_h^\Gamma} \leq \frac{C}{h}\|\lambda_h\|^2_{0,\Gamma} + C|\lambda_h|^2_{1,\mathcal{F}_h^\Gamma}\end{aligned} \tag{53}$$

Substituting this into the preceding estimate and taking α sufficiently small yields

$$\|\lambda_h\|_{0,\Gamma}^2 \leq \frac{1}{2\alpha h}|v_\lambda|_{1,\mathcal{F}}^2 + \frac{1}{h^2}\|v_\lambda\|_{0,\Gamma}^2 + \frac{1}{2}\|\lambda_h\|_{0,\Gamma}^2 + \frac{C\alpha h}{2}|\lambda_h|_{1,\mathcal{F}_h^\Gamma}^2 \tag{54}$$

so that

$$|v_\lambda|_{1,\mathcal{F}}^2 + \frac{1}{h}\|v_\lambda\|_{0,\Gamma}^2 \geq Ch\|\lambda_h\|_{0,\Gamma}^2 - Ch^2|\lambda_h|_{1,\mathcal{F}_h^\Gamma}^2$$

and, by (51),

$$\int_\Gamma \lambda_h \cdot I_h^\Gamma v_\lambda \geq Ch\|\lambda_h\|_{0,\Gamma}^2 - Ch^2|\lambda_h|_{1,\mathcal{F}_h^\Gamma}^2$$

Recall that we have a similar bound (19) involving v_p defined by (18) and interpolation operator I_h^0 from Lemma 5. This allows us to control the pressure. Thus, for any $\kappa, \rho, \eta > 0$

$$\begin{aligned}
&\mathcal{A}^{BH-1-BP}(u_h, p_h, \lambda_h; v_h + \kappa I_h^0 v_p + \rho I_h^\Gamma v_\lambda, -p_h, -\lambda_h + \frac{\eta}{h} I_h^\lambda u_h) \geq \\
&\qquad 2\|D(u_h)\|_{0,\mathcal{F}}^2 - C|u_h|_{1,\mathcal{F}}(\kappa|I_h^0 v_p|_{1,\mathcal{F}} + \rho|I_h^\Gamma v_\lambda|_{1,\mathcal{F}}) \\
&\qquad + \kappa\|p_h\|_{0,\mathcal{F}}^2 - C\|p_h\|_{0,\mathcal{F}}(\kappa h|p_h|_{1,\mathcal{F}_h^e} + \rho|I_h^\Gamma v_\lambda|_{1,\mathcal{F}}) \\
&+ C\rho h\|\lambda_h\|_{0,\Gamma}^2 - C\rho h^2|\lambda_h|_{1,\mathcal{F}_h^\Gamma}^2 + \frac{\eta}{2h}\|u_h\|_{0,\Gamma}^2 - \frac{\eta}{2h}\|u_h - I_h^\lambda u_h\|_{0,\Gamma}^2 \\
&\qquad + \gamma h^2|\lambda_h|_{1,\mathcal{F}_h^\Gamma}^2 - \gamma\eta h|\lambda_h|_{1,\mathcal{F}_h^\Gamma}|I_h^\lambda u_h|_{1,\mathcal{F}_h^\Gamma} + \theta h^2|p_h|^2
\end{aligned}$$

with a constant $C > 0$ independent form the mesh and from the parameters κ, ρ, η, γ, θ. We now apply Korn inequality (17), the Young inequality and the bounds similar to those used in the proof of Lemma 1, such as $|I_h^0 v_p|_{1,\mathcal{F}} \leq C\|p_h\|_{0,\mathcal{F}}$, $|I_h^\Gamma v_\lambda|_{1,\mathcal{F}} \leq C|v_\lambda|_{1,\mathcal{F}} \leq C\sqrt{h}\|\lambda_h\|_{0,\Gamma}$, $\|u_h - I_h^\lambda u_h\|_{0,\Gamma} \leq C\sqrt{h}|u_h|_{1,\mathcal{F}}$, $|I_h^\lambda u_h|_{1,\mathcal{F}_h^\Gamma} \leq C|u_h|_{1,\mathcal{F}}$. In particular, the bound $|v_\lambda|_{1,\mathcal{F}} \leq \sqrt{h}\|\lambda_h\|_{0,\Gamma}$ follows from (50) and Cauchy-Schwarz inequality. This yields

$$\begin{aligned}
&\mathcal{A}^{BH-1-BP}(u_h, p_h, \lambda_h; v_h + \kappa I_h^0 v_p + \rho I_h^\Gamma v_\lambda, -p_h, -\lambda_h + \frac{\eta}{h} I_h^\lambda u_h) \\
&\quad \geq (K - C\eta)|u_h|_{1,\mathcal{F}} + \left(\frac{\kappa}{2} - C\kappa^2\right)\|p_h\|_{0,\mathcal{F}}^2 + C\left(\rho - \rho^2 - \frac{\rho^2}{\kappa}\right)h\|\lambda_h\|_{0,\Gamma}^2 \\
&+ (\theta - C\kappa)h^2|p_h|_{1,\mathcal{F}_h^e}^2 + (\gamma - C\rho - \gamma^2\eta)h^2|\lambda_h|_{1,\mathcal{F}_h^\Gamma}^2 + \frac{\eta}{2h}\|u_h\|_{0,\Gamma}^2 \geq c|\!|\!|u_h, p_h, \lambda_h|\!|\!|^2
\end{aligned}$$

if $\kappa, \rho, \eta > 0$ are chosen sufficiently small. In particular, ρ should be small with respect to κ.

On the other hand, the test function $(u_h + \kappa I_h^0 v_p + \rho I_h^\Gamma v_\lambda, -p_h, -\lambda_h + \frac{\eta}{h} I_h^\lambda u_h)$ can also be bound from above in the triple norm by $|\!|\!|u_h, p_h, \lambda_h|\!|\!|$ thanks to the bounds listed above and to the estimate derived from (44) and (50): $\frac{1}{\sqrt{h}}\|I_h^\Gamma v_\lambda\|_{0,\Gamma} \leq \frac{1}{\sqrt{h}}\|v_\lambda\|_{0,\Gamma} + C|v_\lambda|_{1,\mathcal{F}} \leq C\sqrt{h}\|\lambda_h\|_{0,\Gamma}$. This leads to the announced inf-sup estimate. □

Analogous inf-sup lemmas can be proved for all the other variants of method (43) introduced above. in particular, the adaptation to the case $BH-1-TH$ is very

simple: one should just use the velocity-pressure inf-sup Lemma 10 (valid under Assumption C). The adaptation to the case $BH-0-IP$ requires some more substantial changes in the proofs that we outline below:

- The term $h|\lambda_h|_{1,\mathcal{F}_h^\Gamma}$ should be replaced by $\sqrt{h}|\lambda_h|_{\mathcal{E}_h^\Gamma} := \sqrt{h}\sum_{E\in\mathcal{E}_h^\Gamma}\|[\lambda_h]\|_{0,E}^2$ in both (45) and in the definition of the triple norm in Lemma 3 . The proof of Proposition 11 follows then the same lines as before. Note in particular that inequality (47) remains valid if one changes its right-hand side to $C|\lambda_h|_{\mathcal{E}_h^\Gamma\cap\Pi_k} := C\sum_{E\in\mathcal{E}_h^\Gamma\cap\Pi_k}\|[\lambda_h]\|_{0,E}^2$.
- The estimates in (52)–(54) are easily adapted to the case of $\mathbb{P}_0$ FE for λ_h if one redefines there $\widetilde{\lambda}_h$ as an interpolation of λ_h to the $\mathbb{P}_1$ FE space on $\mathcal{F}_h^\Gamma$, extended to $\mathcal{F}_h^e$ by setting to 0 its values at all the interior nodes of $\mathcal{F}_h^i$. In particular, $\widetilde{\lambda}_h$ can be constructed so that

$$\|\lambda_h-\widetilde{\lambda}_h\|_{0,\Gamma}+\sqrt{h}|\widetilde{\lambda}_h|_{1,\mathcal{F}_h^\Gamma}\le C|\lambda_h|_{\mathcal{E}_h^\Gamma}$$

  Using this bound, one arrives at (52) with $|\lambda_h|_{1,\mathcal{F}_h^\Gamma}$ replaced (here and in all the subsequent inequalities) by $\frac{1}{\sqrt{h}}|\lambda_h|_{\mathcal{E}_h^\Gamma}$.

Having at our disposale the inf-sup lemmas of the type 3, it is easy to establish the convergence theorems completely analogous to Theorems 1, 2, and 3.

Theorem 4. *Consider the three variants of method (43): $BH-1-BP$ under Assumption B with $\mathbb{P}_1$ FE for v, p and λ; $BH-0-IP$ under Assumption B with $\mathbb{P}_1$ FE for v and $\mathbb{P}_0$ FE for p, λ; $BH-1-TH$ under Assumption C with $\mathbb{P}_2$ FE for v and $\mathbb{P}_1$ FE for p, λ. The following a priori error estimates hold for these methods with k denoting the degree of FE space V_h*

$$|u-u_h|_{1,\mathcal{F}}+\|p-p_h\|_{0,\mathcal{F}}+\sqrt{h}\|\lambda-\lambda_h\|_{0,\Gamma} \le Ch^k(|u|_{k+1,\mathcal{F}}+|p|_{k,\mathcal{F}}+|\lambda|_{k-1/2,\Gamma})$$

and

$$\left|\int_\Gamma(\lambda-\lambda_h)\varphi\right|\le Ch^{k+1}(|u|_{k+1,\mathcal{F}}+|p|_{k,\mathcal{F}}+|\lambda|_{k-1/2,\Gamma})|\varphi|_{3/2,\Gamma}$$

for all $\varphi\in H^{3/2}(\Gamma)$

Proof. The proof follows the same lines as that of Theorem 1. In particular, all the necessary interpolation estimates can be taken from Proposition 2. Note that wa no longer require Assumption A there since it is only necessary for the estimates involving $\widehat{I_h^u u}$ and $\widehat{I_h^p p}$. □

4 Numerical experiments

In this section we present some numerical tests. The fluid-structure domain $D \supset \mathcal{F}$ is $[0,1]^2$. The exact solution for the velocity and the pressure is

$$\begin{aligned} u_{ex}(x,y) &= (\cos(\pi x)\sin(\pi y), -\sin(\pi x)\cos(\pi y)), \\ p_{ex}(x,y) &= (y-0.5)\cos(2\pi x) + (x-0.5)\sin(2\pi y). \end{aligned}$$

Dirichlet boundary conditions are imposed on Γ_{wall} and Γ using u_{ex} (contrary to (4) we have non homogenous conditions but the analysis presented in the paper remains valid). The domain is represented in Fig. 1 to describe fluid part $\mathcal{F}$ outside the structure defined by a disk centered in $[0.5, 0.5]$ of radius $R = 0.21$, see Fig. 2. In practice, the boundary of the disk Γ is defined by level-set. For all tests, the threshold ratio θ_{min} (Definition 1) for the "robust reconstruction" is fixed to 0.01 and we use $\gamma_0 = \theta_0 = \gamma = 0.05$.

In the following, U, P and Λ are the degrees of freedom of u_h, p_h and λ_h respectively, with $\{\phi_{i_u}\}, \{\psi_{i_p}\}, \{\zeta_{i_\lambda}\}$ being the selected basis functions of V_h, Q_h and W_h (j_u, j_p, j_λ are indexes similar to i_u, i_p, i_λ). In the experiments, the system is solved using direct solver Mumps.In the numerical tests we compute the error on the resulting force exerted by the fluid on Γ by plotting the quantity $|\int_\Gamma (\lambda - \lambda_h)|$. Rates of convergence are computed using regular meshes based on uniform repartition of N points ($N = 10, 20, 40, 80, 160$) on each side of the boundary Γ_{wall}. So for the tested configuration (and fixed threshold), three meshes require to use "robust reconstruction" (Definition 1). More precisely, for $N = 20, 40$ and 160 we have $8, 8$ and 56 triangles concerned respectively.

4.1 Fictitious domain without any stabilization.

First, we present numerical tests without any stabilization corresponding to (6). The linear system to solve is of the form

$$\begin{pmatrix} K & B^T & C^T \\ B & 0 & 0 \\ C & 0 & 0 \end{pmatrix} \begin{pmatrix} U \\ P \\ \Lambda \end{pmatrix} = \begin{pmatrix} F \\ 0 \\ G \end{pmatrix} \tag{55}$$

where K, B, C, F and G are

$$(K)_{i_u j_u} = 2\int_{\mathcal{F}} D(\phi_{i_u}) : D(\phi_{j_u}), (B)_{i_u j_p} = -\int_{\mathcal{F}} \psi_{j_p} div(\phi_{i_u}), (C)_{i_u j_\lambda} = \int_\Gamma \zeta_{j_\lambda} \phi_{i_u}$$

$$(F)_{i_u} = \int_{\mathcal{F}} f\phi_{i_u}, \quad (G)_{i_\lambda} = \int_\Gamma g\zeta_{i_\lambda}$$

Rates of convergence are presented in Fig. 4 for the couples of spaces $P_2 - P_1 - P_1$, $P_2 - P_1 - P_0$ (velocity-pressure-multiplier) and the choice $P_1 - P_1$ for velocity-

pressure which suffers of course from the non-satisfaction of a mesh-independent inf-sup condition. It has to be stressed that in all the experiments without stabilization, and particularly for the $P_1 - P_1$ case, a singular linear system can be obtained. However, if we consider P_1 (see Figure 4) for the multiplier, the resulting system is non-singular contrary to lower degree P_0 and as expected the solution is not good. For $P_2 - P_1$ (u, p) couple, optimal convergence is observed for all unknowns. However, some problems could remain with too small intersections of elements with $\mathcal{F}$. In the present configuration, this situation is limited. We refer to [7] where this aspect is addressed.

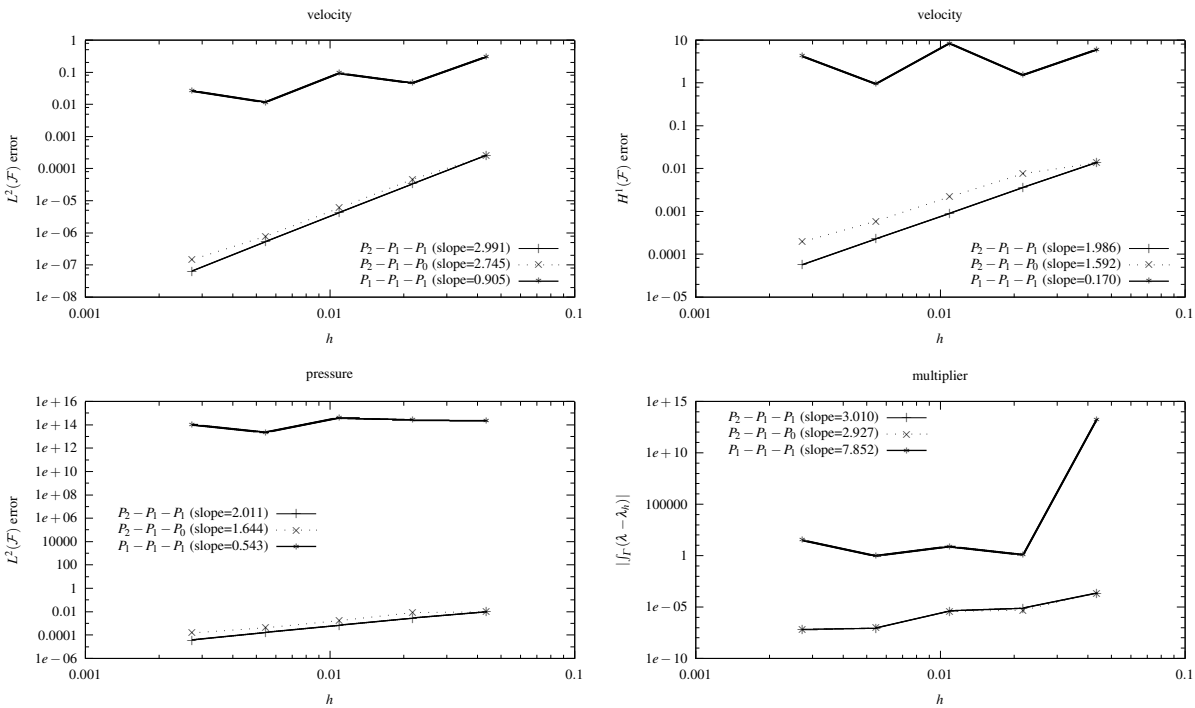

Fig. 4: Rates of convergence without stabilization for $\|u - u_h\|_{0,\mathcal{F}}$, $\|u - u_h\|_{1,\mathcal{F}}$, $\|p - p_h\|_{0,\mathcal{F}}$ and $|\int_\Gamma(\lambda - \lambda_h)|$

4.2 Methods à la Barbosa-Hughes.

For the methods à la Barbosa-Hughes, some stabilization terms (multiplied by $\gamma_0 h$) are added to the system (55) to have

$$\begin{pmatrix} K+S_{uu}^{\gamma_0} & B^T+S_{up}^{\gamma_0 T} & C^T+S_{u\lambda}^{\gamma_0 T} \\ B+S_{up}^{\gamma_0} & S_{pp}^{\gamma_0} & S_{p\lambda}^{\gamma_0 T} \\ C+S_{u\lambda}^{\gamma_0} & S_{p\lambda}^{\gamma_0} & S_{\lambda\lambda}^{\gamma_0} \end{pmatrix} \begin{pmatrix} U \\ P \\ \Lambda \end{pmatrix} = \begin{pmatrix} F \\ 0 \\ G \end{pmatrix} \tag{56}$$

where

$$(S_{uu}^{\gamma_0})_{i_u j_u} = -4\gamma_0 h \int_\Gamma D(\phi_{i_u})n.D(\phi_{j_u})n,\ (S_{up}^{\gamma_0})_{i_u j_p} = 2\gamma_0 h \int_\Gamma D(\phi_{i_u})n.\psi_{j_p}n,\ (S_{u\lambda}^{\gamma_0})_{i_u j_\lambda} = -2\gamma_0 h \int_\Gamma D(\phi_{i_u})n.\zeta_{j_\lambda}$$

$$(S_{pp}^{\gamma_0})_{i_p j_p} = -\gamma_0 h \int_\Gamma \psi_{i_p}\psi_{j_p},\ \left(S_{p\lambda}^{\gamma_0}\right)_{i_p j_\lambda} = \gamma_0 h \int_\Gamma \psi_{i_p} n.\zeta_{j_\lambda},\ (S_{\lambda\lambda}^{\gamma_0})_{i_\lambda j_\lambda} = -\gamma_0 h \int_\Gamma \zeta_{i_\lambda}.\zeta_{j_\lambda}$$

Notice that no "robust reconstruction" is done using "good elements" (Definition 1). Small intersections with the domain occur and no particular treatment are realized. We report in Fig. 5 the rates of convergence that are already observed in [8]. The spaces considered are the same as the previous tests (without stabilization). Results for velocity and pressure are similar with rates of convergence of same optimal order. The improvement is clear for $P_1-P_1-P_1$ where the force on Γ is well computed with optimal convergence.

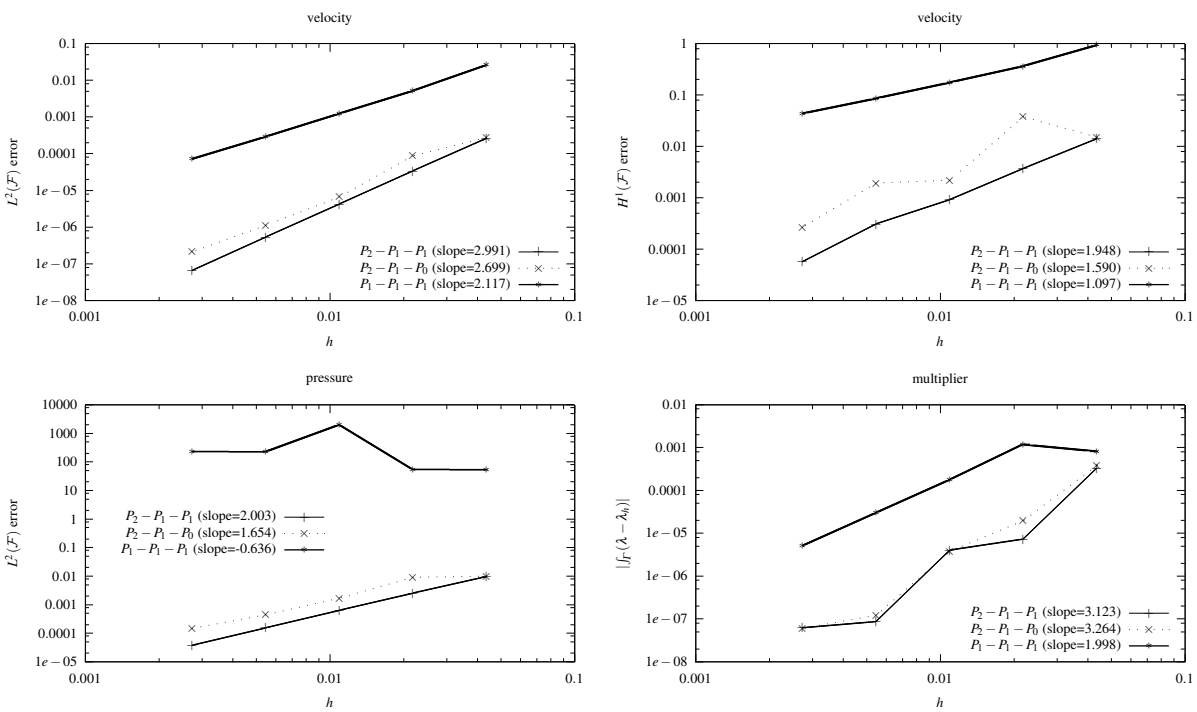

Fig. 5: Rates of convergence with Barbosa-Hughes stabilization for $\|u-u_h\|_{0,\mathcal{F}}$, $\|u-u_h\|_{1,\mathcal{F}}$, $\|p-p_h\|_{0,\mathcal{F}}$ and $|\int_\Gamma(\lambda-\lambda_h)|$

4.3 $\mathbb{P}_1 - \mathbb{P}_1$ velocity-pressure spaces with Brezzi-Pitkäranta stabilization.

Here, the system (56) is modified using Haslinger-Renard strategy with good element (Definition 1) for u only (specified with $\hat{u}$ instead of u) and adding one term $S^{\theta_0}_{pp}$ defined by

$$\left(S^{\theta_0}_{pp}\right)_{i_p j_p} = -\theta_0 h^2 \int_{\mathcal{F}^e_h} \nabla\psi_{i_p}.\nabla\psi_{j_p}$$

For that choice, the method is robust. The convergence is clear with optimal rates of convergence. Notice better results as expected by theory are observed for pressure which is more concerned by the stabilization. The difference between $P_1 - P_0 - P_1$ and $P_1 - P_0 - P_0$ is very small (see the slopes). Considering low degree for the multiplier doesn't affect the global accuracy of the solution.

$$\begin{pmatrix} K + S^{\gamma_0}_{\hat{u}\hat{u}} & B^T + S^{\gamma_0 T}_{\hat{u}p} & C^T + S^{\gamma_0 T}_{\hat{u}\lambda} \\ B + S^{\gamma_0}_{\hat{u}p} & S^{\gamma_0}_{pp} + S^{\theta_0}_{pp} & S^{\gamma_0 T}_{p\lambda} \\ C + S^{\gamma_0}_{\hat{u}\lambda} & S^{\gamma_0}_{p\lambda} & S^{\gamma_0}_{\lambda\lambda} \end{pmatrix} \begin{pmatrix} U \\ P \\ \Lambda \end{pmatrix} = \begin{pmatrix} F \\ 0 \\ G \end{pmatrix} \tag{57}$$

4.4 $\mathbb{P}_1 - \mathbb{P}_0$ velocity-pressure spaces with interior penalty stabilization.

The system corresponds to (57) where $S^{\theta_0}_{pp}$ is replaced by $S^{\theta_0}_{[p][p]}$ where

$$\left(S^{\theta_0}_{[p][p]}\right)_{i_p j_p} = -\theta_0 h \sum_{E \in \mathcal{E}^e_h} \int_E [\psi_{i_p}][\psi_{j_p}]$$

Then the systems is given by

$$\begin{pmatrix} K + S^{\gamma_0}_{\hat{u}\hat{u}} & B^T + S^{\gamma_0 T}_{\hat{u}p} & C^T + S^{\gamma_0 T}_{\hat{u}\lambda} \\ B + S^{\gamma_0}_{\hat{u}p} & S^{\gamma_0}_{pp} + S^{\theta_0}_{[p][p]} & S^{\gamma_0 T}_{p\lambda} \\ C + S^{\gamma_0}_{\hat{u}\lambda} & S^{\gamma_0}_{p\lambda} & S^{\gamma_0}_{\lambda\lambda} \end{pmatrix} \begin{pmatrix} U \\ P \\ \Lambda \end{pmatrix} = \begin{pmatrix} F \\ 0 \\ G \end{pmatrix} \tag{58}$$

The spaces used for that stabilization are similar as Brezzi-Pitkaranta stabilization (and Haslinger-Renard strategy for u only). The results can be compared expect for the pression which less accurate. Here again, the difference between $P_1 - P_0 - P_1$ and $P_1 - P_0 - P_0$ is very small (see the slopes).

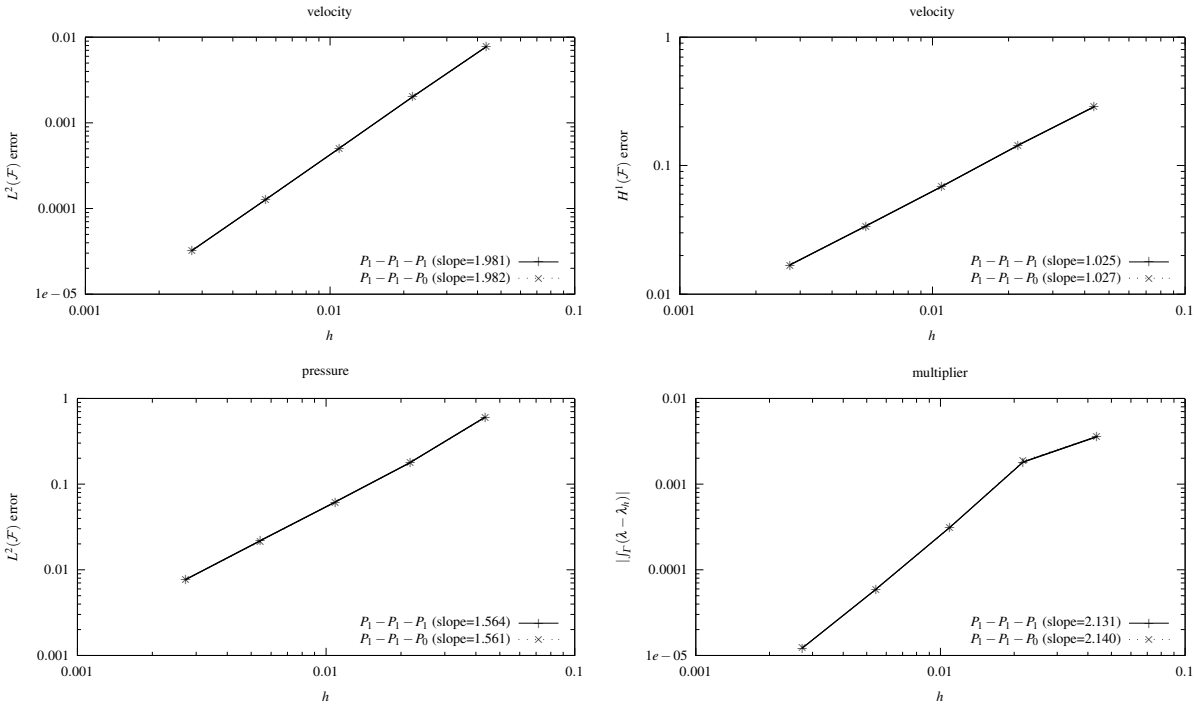

Fig. 6: Rates of convergence with Brezzi-Pitkaranta stabilization for $\|u-u_h\|_{0,\mathcal{F}}$, $\|u-u_h\|_{1,\mathcal{F}}$, $\|p-p_h\|_{0,\mathcal{F}}$ and $|\int_\Gamma(\lambda-\lambda_h)|$

4.5 Taylor-Hood spaces.

Here, the system (56) is modified using good elements (Haslinger-Renard, Definition 1) for u and p (specified with $\hat{u}$ and $\hat{p}$ instead of u and p).

$$\begin{pmatrix} K+S^{\gamma_0}_{\hat{u}\hat{u}} & B^T+S^{\gamma_0 T}_{\hat{u}\hat{p}} & C^T+S^{\gamma_0 T}_{\hat{u}l} \\ B+S^{\gamma_0}_{\hat{u}\hat{p}} & S^{\gamma_0}_{\hat{p}\hat{p}} & S^{\gamma_0 T}_{\hat{p}\lambda} \\ C+S^{\gamma_0}_{\hat{u}\lambda} & S^{\gamma_0}_{\hat{p}\lambda} & S^{\gamma_0}_{\lambda\lambda} \end{pmatrix} \begin{pmatrix} U \\ P \\ \Lambda \end{pmatrix} = \begin{pmatrix} F \\ 0 \\ G \end{pmatrix} \tag{59}$$

Comparing to the results presented in section 4.2 and Fig. 5, the elements having a too small intersection with the domain are selected to apply "robust reconstruction" for velocity and pressure. However the effect is not important due to the simple configuration considered in the present study and we refer to the paper [7] for more considerations. So it is natural to observed some results very close to the ones given in Fig. 5.

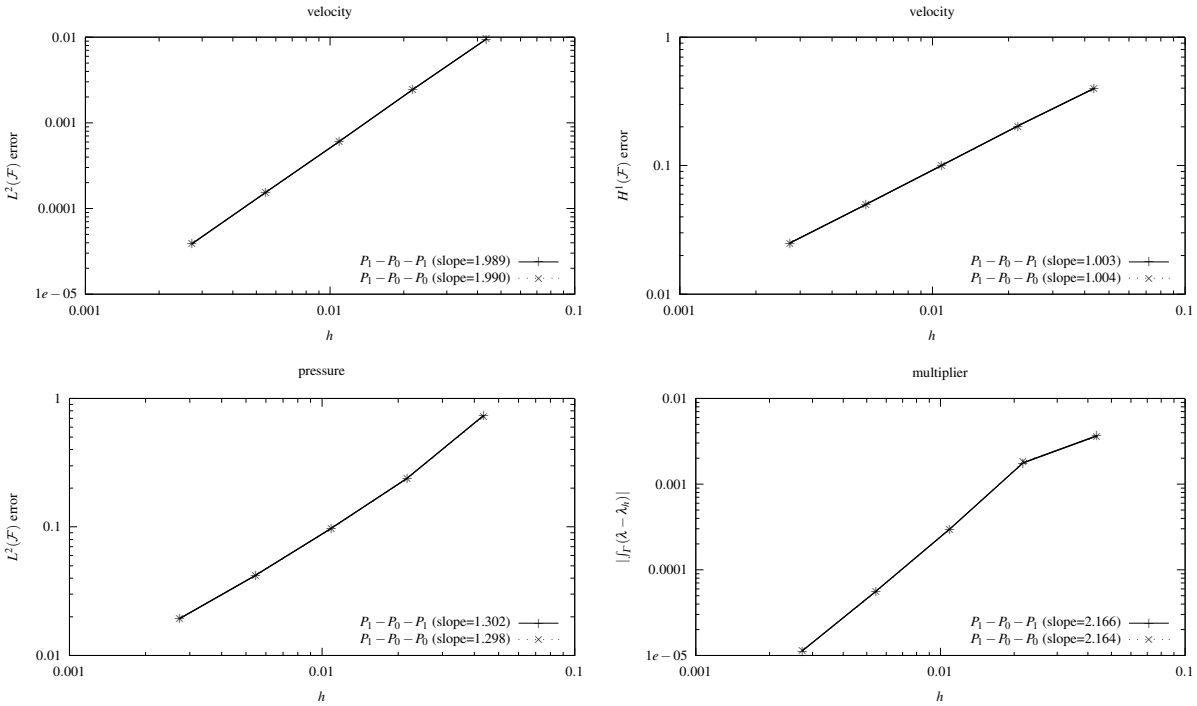

Fig. 7: Rates of convergence with Interior Penalty stabilization for $\|u-u_h\|_{0,\mathcal{F}}$, $\|u-u_h\|_{1,\mathcal{F}}$, $\|p-p_h\|_{0,\mathcal{F}}$ and $|\int_\Gamma(\lambda-\lambda_h)|$

4.6 Methods à la Burman-Hansbo.

For the methods à la Burman-Hansbo, some stabilization terms (multiplied by γ) are added to the system (55) to have

$$\begin{pmatrix} K & B^T & C^T \\ B & S_{l_p}^{\theta_0} & 0 \\ C & 0 & S_{l_\lambda}^{\gamma} \end{pmatrix} \begin{pmatrix} U \\ P \\ \Lambda \end{pmatrix} = \begin{pmatrix} F \\ 0 \\ G \end{pmatrix} \tag{60}$$

for $l_\lambda = 0,1$ and $l_p = 0,1,2$ with
$\left(S_0^\gamma\right)_{i_\lambda j_\lambda} = \left(S_{[\lambda][\lambda]}^{\gamma,0}\right)_{i_\lambda j_\lambda} = -\gamma h \sum_{E\in\mathcal{E}_h^\Gamma} \int_E [\zeta_{i_\lambda}]\cdot[\zeta_{j_\lambda}], \quad \left(S_1^\gamma\right)_{i_\lambda j_\lambda} = \left(S_{\lambda\lambda}^{\gamma,1}\right)_{i_\lambda j_\lambda} = -\gamma h^2 \int_{\mathcal{F}_h^\Gamma} \nabla\zeta_{i_\lambda}.\nabla\zeta_{j_\lambda}$
$S_0^{\theta_0} = S_{pp}^{\theta_0}, \quad S_1^{\theta_0} = S_{[p][p]}^{\theta_0}, \quad S_2^{\theta_0} = 0$

No "robust reconstruction" are used and the orders for the finite element spaces govern the stabilization terms to introduce. Depending on the space for the multiplier, we use S_0^γ and S_1^γ for P_0 and P_1 respectively. Associated to the velocity, we

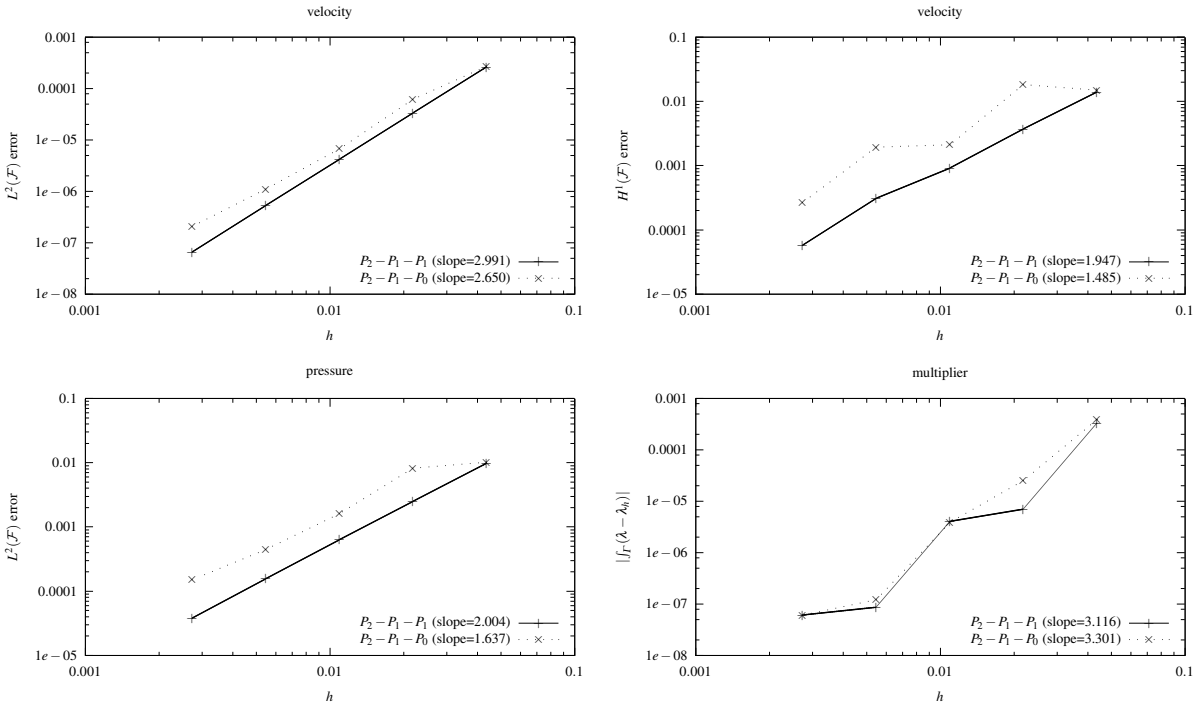

Fig. 8: Rates of convergence with Taylor-Hood stabilization for $\|u-u_h\|_{0,\mathcal{F}}$, $\|u-u_h\|_{1,\mathcal{F}}$, $\|p-p_h\|_{0,\mathcal{F}}$ and $|\int_\Gamma(\lambda-\lambda_h)|$

don't add more stabilization term ($l_p=2$, only $S^{\gamma}_{l_\lambda}$ is considered) for P_2 contrary to P_1 for velocity coupled with P_{l_p} for pressure that requires the introduction of $S^{\theta_0}_{l_p}$ for $l_p=0,1$. We recover the optimal rates of converges with classical improvement using higher degree. The results for $P_2-P_1-P_0$ can be compared to the ones obtained using the Taylor-Hood stabilization (see Fig. 8).

5 Conclusion

In this paper, we propose methods that are very interesting to consider in the context of fluid-structure interaction with complex interface. We combined the Barbosa-Hughes approach with several stabilization strategies involving a "robust reconstruction" (Haslinger-Renard) when there are small intersections with the domain. The error estimates proven theoretically under non-restrictive assumptions are observed numerically. Alternative methods à la Burman-Hansbo are considered theoretically and numerically for Stokes problem and allow to recover similar results.

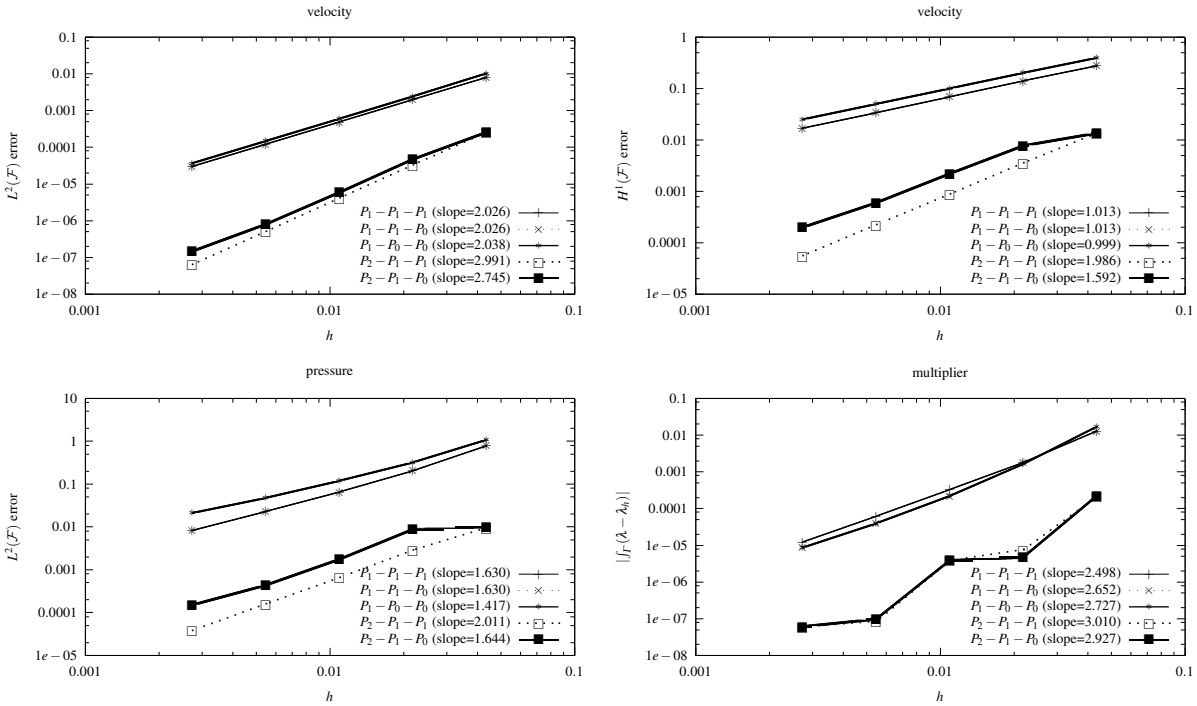

Fig. 9: Rates of convergence with Burman-Hansbo stabilization for $\|u-u_h\|_{0,\mathcal{F}}$, $\|u-u_h\|_{1,\mathcal{F}}$, $\|p-p_h\|_{0,\mathcal{F}}$ and $|\int_\Gamma(\lambda-\lambda_h)|$

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