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A fictitious domain finite element method for simulations of fluid-structure interactions: The Navier-Stokes equations coupled with a moving solid

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Abstract.

The paper extends a fictitious domain finite element method analyzed for Stokes problem in [30] to the Navier-Stokes equations coupled with a moving solid. The dynamics of the solid is governed by the Newton's laws. The mixed finite element method used cuts element at the interface localized by level-set while preserving an optimal accuracy of the approximation of the normal stress tensor. An algorithm is proposed in order to treat the time evolution of the geometry and numerical tests are given.

Key words: Fluid structure interaction, Navier-Stokes, Fictitious domain, eXtended Finite Element.

1 Introduction

Fluid structure interactions problems remains a challenge as well on a comprehensive study of such problems as on a development of robust numerical methods (see a review in [12]). One class of numerical methods is based on meshes that are conformed to the interface where the physical boundary conditions are imposed [13, 23, 24]. As the geometry of the fluid domain changes through the time, re-meshing is needed, what is excessively time-consuming, in particular for complex systems. An other class of numerical methods is based on non-conforming mesh with a fictitious domain approach where the mesh is cut by the boundary. Most of the non-conforming mesh methods are based upon the framework of the immersed methods where force-equivalent terms are added to the fluid equations in order to represent the fluid structure interaction [20, 15]. Many related numerical methods have been developed, in particular the popular distributed Lagrange multiplier method, introduced for rigid bodies moving in an incompressible flow [8]. In this method, the fluid domain is extended to cover the rigid domain where the fluid velocity is required to be equal to the rigid body velocity.
More recently eXtended Finite Element Method introduced by Moës, Dolbow and Belytschko in [17] (see a review of such methods in [6]) has been adapted to fluid

structure interactions problems in [16, 27, 7, 4]. The idea is similar to the fictitious domain / Lagrange multiplier method mentioned above, but the fluid velocity is no longer extended inside the structure domain and its equality with the structure velocity is enforced by a Lagrange multiplier only on the fluid structure interface. One thus gets rid of unnecessary fluid unknowns and moreover one easily recovers the normal trace of the Cauchy stress tensor on the interface. We note that this method has been originally developed for problems in structural mechanics mostly in the context of cracked domains, see for example [10, 18, 25, 28, 26]. The specificity of the method is that it combines a level-set representation of the geometry of the crack with an enrichment of a finite element space by singular and discontinuous functions.

In the context of fluid structure interactions, the difficulty that present the applications of such techniques lies in the choice of the Lagrange multiplier space used in order to take into account the interface, which is not trivial due to the fact that the interface cuts the mesh [1]. In particular, the natural mesh given by the points of intersection of the interface with the global mesh cannot be used directly. An algorithm to construct a multiplier space satisfying the inf-sup condition is developed in [1], but its implementation can be difficult in practice. The method proposed in the present paper, gives an issue to this difficulty using a stabilization technique proposed in [10]. This method method was adapted to the contact problems in elastostatics in [11] and more recently to the Stokes problem in [30]. An important feature of this method (based on the eXtended Finite Element Method approach, similarly to [7, 4]) is that the Lagrange multiplier is identified with the normal trace of the Cauchy stress tensor $\sigma(\mathbf{u}, p)\mathbf{n}$ at the interface. Moreover, it is possible to obtain a good numerical approximation of $\sigma(\mathbf{u}, p)\mathbf{n}$ (the proof is given in [30] for Stokes problem). This property is crucial in fluid structure interactions since this quantity gives the force exerted by the viscous fluid on the structure. In the present paper, we propose to extend this method to the Navier-Stokes equations coupled with a moving solid. Note that alternative methods based on the Nitsche's work [19] (such as [2, 3] in the context of the Poisson problem and [14] in the context of the Stokes problems) do not introduce the Lagrange multiplier and thus do not necessarily provide a good numerical approximation of this force.

The outline of the paper is as follows. The continuous fluid structure interactions problem is set in section 2 and weak formulation with the introduction of a Lagrange multiplier to impose the boundary condition in the interface is given in subsection 2.1.1. Next, in section 3 the fictitious domain method is recalled with the introduction of the finite element method (subsection 3.1) with time discretization (subsection 3.2). Section 4 is devoted to numerical tests before conclusion.

2 The model

2.1 Fluid structure interactions

We consider a moving solid which occupies a time-depending domain denoted by $\mathcal{S}(t)$. The remaining domain $\mathcal{F}(t) = \mathcal{O} \setminus \overline{\mathcal{S}(t)}$ corresponds to the fluid flow.

The displacement of the deformable solid can be given by a Lagrangian mapping X_S defined by

$$\begin{aligned} \mathcal{S}(t) &= X_S(\mathcal{S}(0),t), \\ X_S(y,t) &= \mathbf{h}(t) + \mathbf{R}(t)X^*(y,t), \quad y \in \mathcal{S}(0), \end{aligned}$$

where $h(t)$ is the gravity center of the solid, $\mathbf{R}(t)$ is the rotation given by $\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ for $c = \cos(\theta(t))$, $s = \sin(\theta(t))$ and $X^*(\cdot,t)$ a mapping that corresponds to the deformation of the solid in its own frame of reference (see Figure 1).

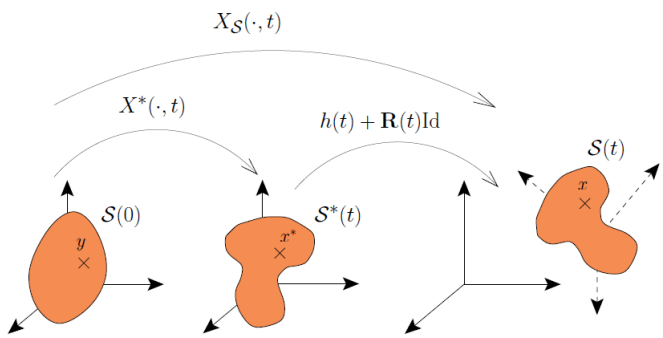


Fig. 1 - Decomposition of the solid movement.

Several deformations of the structure can be considered for the deformation X^* that governs the shape of the solid. However we restrict our study to a rigid structure ($X^*(y,t) = y - \mathbf{h}(0)$) to illustrate the capabilities of the method, so

$$\mathcal{S}(t) = \mathbf{h}(t) + \mathbf{R}(t)\mathcal{S}(0).$$

The velocity of the fluid is represented by $\mathbf{u}$, the pressure by p and ν is the viscosity. We denote by $\mathbf{n}$ the outward unit normal vector to $\partial\mathcal{F}$, and the normal trace on $\Gamma = \partial\mathcal{S}(t)$ of the Cauchy stress tensor is given by

$$\sigma(\mathbf{u},p)\mathbf{n} = 2\nu D(\mathbf{u})\mathbf{n} - p\mathbf{n}, \text{ where } D(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right).$$

The system which is studied consists in finding u, p, $\mathbf{h}(t)$ and the angular velocity $\omega(t) = \theta'(t)$ (a scalar function in 2D) that satisfy the Navier-Stokes problem for the fluid

$$\begin{array}{lll} \dfrac{\partial\mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}.\nabla)\mathbf{u} + \nabla p = 0, & x \in \mathcal{F}(t), & t \in (0,T), \\ \mathrm{div}(\mathbf{u}) = 0, & \mathbf{x} \in \mathcal{F}(t), & t \in (0,T), \\ \mathbf{u} = 0, & \mathbf{x} \in \partial\mathcal{O}, & t \in (0,T), \\ \mathbf{u}(\mathbf{x},t) = \mathbf{h}'(t) + \theta'(t)(\mathbf{x} - \mathbf{h}(t))^\perp = \mathbf{u_\Gamma}, & \mathbf{x} \in \partial\mathcal{S}(t), & t \in (0,T), \end{array}$$

coupled with the Newton's laws for the solid

$$\begin{aligned} m\mathbf{h}''(t) &= -\int_{\partial\mathcal{S}(t)} \sigma(\mathbf{u},p)\mathbf{n}d\Gamma - m\mathbf{g} + m_a\mathbf{g}, \\ I\theta''(t) &= -\int_{\partial\mathcal{S}(t)} (\mathbf{x} - \mathbf{h}(t))^\perp \cdot \sigma(\mathbf{u},p)\mathbf{n}d\Gamma, \end{aligned}$$

where m, I are solid's mass and the inertia moment of the solid, $\mathbf{g}$ the gravity field and m_a comes from Archimdes' principle corresponding to the fluid mass displaced by the solid's volume.

At the interface $\partial\mathcal{S}(t)$ the Dirichlet condition is based on the computation of $\sigma(\mathbf{u}, p)\mathbf{n}$. The main advantage of our numerical method – mathematically justified in [30] – is to return an optimal approximation of $\sigma(\mathbf{u}, p)\mathbf{n}$ at the interface represented by a level set which cuts the global mesh (see Figure 2). A good approximation of this quantity is crucial for the dynamics governed by the two differential equations written above.

2.1.1 Weak formulation of the problem

We assume that the boundary condition imposed at the interface $\Gamma = \partial\mathcal{S}(t)$ is sufficiently regular to make sense and we introduce the functional spaces

$$\begin{aligned}
\mathbf{V} &= \{\mathbf{v} \in \mathbf{H}^1(\mathcal{F}) \mid \mathbf{v} = 0 \text{ on } \partial\mathcal{O}\}, \\
Q &= \mathrm{L}^2_0(\mathcal{F}) = \left\{ p \in \mathrm{L}^2(\mathcal{F}) \mid \int_{\mathcal{F}} p \, \mathrm{d}\mathcal{F} = 0 \right\}, \\
\mathbf{W} &= \mathbf{H}^{-1/2}(\Gamma) = \left(\mathbf{H}^{1/2}(\Gamma)\right)'.
\end{aligned}$$

Due to the fact that we only consider boundary conditions of Dirichlet type, we impose to the pressure to have null average (this condition is included in Q). The variational problem we consider is the following:

Find $(\mathbf{u}, p, \lambda, \mathbf{h}', \mathbf{h}, \theta', \theta) \in \mathbf{V} \times Q \times \mathbf{W} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$ such that

$$\begin{cases}
\displaystyle\int_{\mathcal{F}} \frac{\partial \mathbf{u}}{\partial t}.\mathbf{v}\mathrm{d}\mathcal{F} + \mathcal{A}((\mathbf{u}, p, \boldsymbol{\lambda}); \mathbf{v}) + \int_{\mathcal{F}} [(\mathbf{u} \cdot \nabla)\mathbf{u}].\mathbf{v}\mathrm{d}\mathcal{F} = 0 & \forall \mathbf{v} \in \mathbf{V}, \\
\mathcal{B}((\mathbf{u}, p, \boldsymbol{\lambda}); q) = 0 & \forall q \in Q, \\
\mathcal{C}((\mathbf{u}, p, \boldsymbol{\lambda}); \boldsymbol{\mu}) = \mathcal{G}(\boldsymbol{\mu}) & \forall \boldsymbol{\mu} \in \mathbf{W}, \\
m\mathbf{h}''(t) = -\displaystyle\int_{\partial\mathcal{S}(t)} \boldsymbol{\lambda} d\Gamma - m\mathbf{g} + m_a\mathbf{g}, \\
I\theta''(t) = -\displaystyle\int_{\partial\mathcal{S}(t)} (\mathbf{x} - \mathbf{h}(t))^{\perp} \cdot \boldsymbol{\lambda} d\Gamma.
\end{cases}$$

where

$$\begin{aligned}
\mathcal{A}((\mathbf{u},p,\boldsymbol{\lambda});\mathbf{v}) &= 2\nu\int_{\mathcal{F}} D(\mathbf{u}):D(\mathbf{v})\mathrm{d}\mathcal{F} - \int_{\mathcal{F}} p\mathrm{div}\ \mathbf{v}\mathrm{d}\mathcal{F} - \int_{\Gamma}\boldsymbol{\lambda}\cdot\mathbf{v}\mathrm{d}\Gamma \\
&\quad -4\nu^2\gamma\int_{\Gamma}(D(\mathbf{u})\mathbf{n})\cdot(D(\mathbf{v})\mathbf{n})\,\mathrm{d}\Gamma + 2\nu\gamma\int_{\Gamma} p\,(D(\mathbf{v})\mathbf{n}\cdot\mathbf{n})\,\mathrm{d}\Gamma + 2\nu\gamma\int_{\Gamma}\boldsymbol{\lambda}\cdot(D(\mathbf{v})\mathbf{n})\,\mathrm{d}\Gamma, \\
\mathcal{B}((\mathbf{u},p,\boldsymbol{\lambda});q) &= -\int_{\mathcal{F}} q\mathrm{div}\ \mathbf{u}\mathrm{d}\mathcal{F} + 2\nu\gamma\int_{\Gamma} q\,(D(\mathbf{u})\mathbf{n}\cdot\mathbf{n})\,\mathrm{d}\Gamma - \gamma\int_{\Gamma} pq\mathrm{d}\Gamma - \gamma\int_{\Gamma} q\boldsymbol{\lambda}\cdot\mathbf{n}\mathrm{d}\Gamma, \\
\mathcal{C}((\mathbf{u},p,\boldsymbol{\lambda});\boldsymbol{\mu}) &= -\int_{\Gamma}\boldsymbol{\mu}\cdot\mathbf{u}\mathrm{d}\Gamma + 2\nu\gamma\int_{\Gamma}\boldsymbol{\mu}\cdot(D(\mathbf{u})\mathbf{n})\mathrm{d}\Gamma - \gamma\int_{\Gamma} p(\boldsymbol{\mu}\cdot\mathbf{n})\mathrm{d}\Gamma - \gamma\int_{\Gamma}\boldsymbol{\lambda}\cdot\mu\mathrm{d}\Gamma, \\
\mathcal{G}(\boldsymbol{\mu}) &= -\int_{\Gamma}\boldsymbol{\mu}\cdot\mathbf{u}_{\boldsymbol{\Gamma}}\mathrm{d}\Gamma = -\int_{\Gamma}\boldsymbol{\mu}\cdot(\mathbf{h}'(t)+\theta'(t)(\mathbf{x}-\mathbf{h}(t))^{\perp})\mathrm{d}\Gamma.
\end{aligned}$$

For this problem, the multiplier $\boldsymbol{\lambda}$ is introduced. This unknown is associated to the Dirichlet condition at the interface Γ and presents the advantage to be equal to the normal trace of the Cauchy stress tensor $\sigma(\mathbf{u},p)\mathbf{n}$ (see [9]), what is crucial for the considered model.
The stabilization terms are associated to the constant parameter γ (chosen sufficiently small).

3 Fictitious domain approach

We refer to the article [30] for the details on the fictitious domain approach considered. In the following, we recall the method used for the present work.

3.1 Finite element discretization

The fictitious domain for the fluid is considered on the whole domain $\mathcal{O}$. Let us introduce three discrete finite element spaces, $\tilde{\mathbf{V}}^h \subset \mathbf{H}^1(\mathcal{O})$, $\tilde{Q}^h \subset \mathrm{L}^2_0(\mathcal{O})$ and $\tilde{\mathbf{W}}^h \subset \mathbf{L}^2(\mathcal{O})$. Since $\mathcal{O}$ can be a rectangular domain, this spaces can be defined on the same structured mesh, that can be chosen uniform (see Figure 2). The construction of the mesh is highly simplified (no particular mesh is required). We set

$$\tilde{\mathbf{V}}^h = \left\{\mathbf{v}^h \in C(\overline{\mathcal{O}}) \mid \mathbf{v}^h_{|\partial\mathcal{O}} = 0,\ \mathbf{v}^h_{|T} \in P(T),\ \forall T \in \mathcal{T}^h\right\}, \tag{1}$$

where $P(T)$ is a finite dimensional space of regular functions such that $P(T) \supseteq P_k(T)$ for some integer $k \geq 1$. For more details, see [5] for instance. The mesh parameter stands for $h = \max_{T\in\mathcal{T}^h} h_T$, where h_T is the diameter of T. We define

$$\mathbf{V}^h := \tilde{\mathbf{V}}^h_{|\mathcal{F}}, \quad Q^h := \tilde{Q}^h_{|\mathcal{F}}, \quad \mathbf{W}^h := \tilde{\mathbf{W}}^h_{|\Gamma},$$

which are natural discretizations of $\mathbf{V}$, $\mathrm{L}^2(\mathcal{F})$ and $\mathbf{H}^{-1/2}(\Gamma)$ respectively. This approach is equivalent to eXtended Finite Element Method as proposed in [4] or [7] where the standard Finite Element Method basis functions are multiplied by the Heaviside function ($H(\mathbf{x}) = 1$ for $\mathbf{x} \in \mathcal{F}$ and $H(\mathbf{x}) = 0$ for $\mathbf{x} \in \mathcal{O}\setminus\mathcal{F}$) and the products are substituted in the variational formulation of the problem. Thus the degrees of freedom inside the fluid domain $\mathcal{F}$ are used in the same way as in the standard Finite Element Method, whereas the degrees of freedom in the solid domain $\mathcal{S}$ at

the vertices of the elements cut by the interface (the so called virtual degrees of freedom) do not define the field variable at these nodes, but they are necessary to define the fields on $\mathcal{F}$ and to compute the integrals over $\mathcal{F}$. The remaining degrees of freedom, corresponding to the basis functions with support completely outside of the fluid, are eliminated (see Figure 2). We refer to the papers mentioned above for more details.

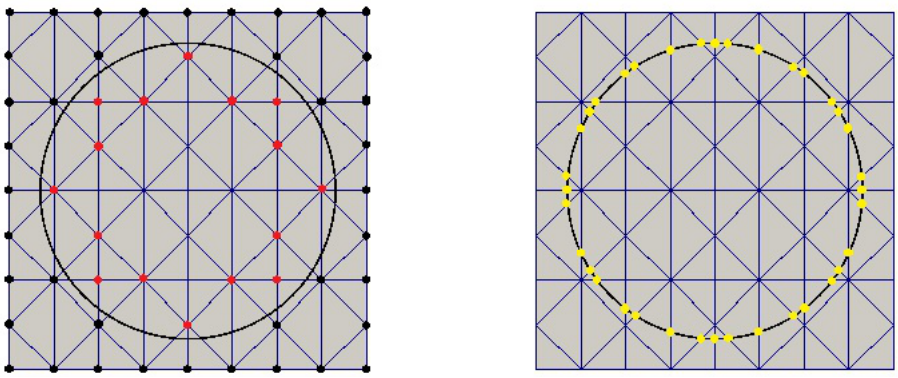


Fig. 2 - Mesh on a fictitious domain.

In the first figure: Standard degrees of freedom (black), virtual ones (red), remaining ones are removed. In the second figure: Bases nodes used for the multiplier space (yellow).

The discrete problem corresponds to finding $(\mathbf{u}^h, p^h, \boldsymbol{\lambda}^h, \mathbf{h}', \mathbf{h}, \theta', \theta) \in \mathbf{V}^h \times Q^h \times \mathbf{W}^h \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$
such that

$$\begin{cases} \displaystyle\int_{\mathcal{F}} \frac{\partial \mathbf{u}^h}{\partial t}.\mathbf{v}^{\mathbf{h}} \mathrm{d}\mathcal{F} + \mathcal{A}((\mathbf{u}^h, p^h, \boldsymbol{\lambda}^h); \mathbf{v}^h) + \int_{\mathcal{F}} [(\mathbf{u}^h \cdot \nabla)\mathbf{u}^h].\mathbf{v}^h \mathrm{d}\mathcal{F} = 0 & \forall \mathbf{v}^h \in \mathbf{V}^h, \\ \mathcal{B}((\mathbf{u}^h, p^h, \boldsymbol{\lambda}^h); q^h) = 0 & \forall q^h \in Q^h, \\ \mathcal{C}((\mathbf{u}^h, p^h, \boldsymbol{\lambda}^h); \boldsymbol{\mu}^h) = \mathcal{G}(\boldsymbol{\mu}^h) & \forall \boldsymbol{\mu}^h \in \mathbf{W}^h, \\ m\mathbf{h}''(t) = -\displaystyle\int_{\partial\mathcal{S}(t)} \boldsymbol{\lambda}^h \mathrm{d}\Gamma - m\mathbf{g} + m_a\mathbf{g}, \qquad I\theta''(t) = -\int_{\partial\mathcal{S}(t)} (\mathbf{x} - \mathbf{h}(t))^{\perp} \cdot \boldsymbol{\lambda}^h \mathrm{d}\Gamma. \end{cases}$$

The problem is a system of nonlinear differential algebraic equations that can be formulated into a compact form where $\boldsymbol{U}$, $\boldsymbol{P}$ and $\boldsymbol{\Lambda}$ are the degrees of freedom of $\mathbf{u}^h$, p^h and $\boldsymbol{\lambda}^h$ respectively. It is given by

$$\begin{aligned} M_{\mathbf{uu}} \frac{\mathrm{d}\boldsymbol{U}(\boldsymbol{t})}{\mathrm{d}t} + A_{\mathbf{uu}}\boldsymbol{U}(t) + N(\boldsymbol{U}(t))\boldsymbol{U}(t) + A_{\mathbf{u}p}\boldsymbol{P}(t) + A_{\mathbf{u}\lambda}\boldsymbol{\Lambda}(t) &= 0, \\ A^T_{\mathbf{u}p}\boldsymbol{U}(t) + A_{pp}\boldsymbol{P}(t) + A_{p\lambda}\boldsymbol{\Lambda}(t) &= 0, \\ A^T_{\mathbf{u}\lambda}\boldsymbol{U}(t) + A^T_{p\lambda}\boldsymbol{P}(t) + A_{\lambda\lambda}\boldsymbol{\Lambda}(t) &= \boldsymbol{G}, \\ m\mathbf{h}''(t) &= M_\lambda \boldsymbol{\Lambda}(t) - m\mathbf{g} + m_a\mathbf{g}, \\ I\theta''(t) &= M_\lambda \left[(\mathbf{x} - \mathbf{h}(t))^{\perp} \cdot \boldsymbol{\Lambda}(t)\right], \end{aligned}$$

where these matrices are discretizations (from the standard finite element approach) of the following functions

$$
\begin{aligned}
\mathcal{M}_{\mathbf{uu}} : (\mathbf{u},\mathbf{v}) &\longmapsto \int_{\mathcal{F}} \mathbf{u}.\mathbf{v}\mathrm{d}\mathcal{F}, \qquad \mathcal{M}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \longmapsto -\int_{\partial\mathcal{S}(t)} \boldsymbol{\lambda}\mathrm{d}\Gamma,\\
\mathcal{A}_{\mathbf{uu}} : (\mathbf{u},\mathbf{v}) &\longmapsto 2\nu \int_{\mathcal{F}} D(\mathbf{u}) : D(\mathbf{v})\mathrm{d}\mathcal{F} - 4\nu^2\gamma \int_{\Gamma} (D(\mathbf{u})\mathbf{n}) \cdot (D(\mathbf{v})\mathbf{n})\,\mathrm{d}\Gamma,\\
\mathcal{A}_{\mathbf{u}p} : (\mathbf{v},p) &\longmapsto -\int_{\mathcal{F}} p\mathrm{div}\ \mathbf{v}\mathrm{d}\mathcal{F} + 2\nu\gamma \int_{\Gamma} p\,(D(\mathbf{v})\mathbf{n}\cdot\mathbf{n})\,\mathrm{d}\Gamma,\\
\mathcal{A}_{\mathbf{u}\boldsymbol{\lambda}} : (\mathbf{u},\boldsymbol{\lambda}) &\longmapsto -\int_{\Gamma} \boldsymbol{\lambda}\cdot\mathbf{v}\mathrm{d}\Gamma + 2\nu\gamma \int_{\Gamma} \boldsymbol{\lambda}\cdot(D(\mathbf{v})\mathbf{n})\,\mathrm{d}\Gamma,\\
\mathcal{A}_{pp} : (p,q) &\longmapsto -\gamma \int_{\Gamma} pq\mathrm{d}\Gamma, \qquad \mathcal{A}_{p\boldsymbol{\lambda}} : (q,\boldsymbol{\lambda}) \longmapsto -\gamma \int_{\Gamma} q\boldsymbol{\lambda}\cdot\mathbf{n}\mathrm{d}\Gamma,\\
\mathcal{A}_{\boldsymbol{\lambda}\boldsymbol{\lambda}} : (\boldsymbol{\lambda},\boldsymbol{\mu}) &\longmapsto -\gamma \int_{\Gamma} \boldsymbol{\lambda}\cdot\boldsymbol{\mu}\mathrm{d}\Gamma,
\end{aligned}
$$

and the vector $\boldsymbol{G}$ is the discretization of $\mathcal{G}$ (for example matrix $M_{\mathbf{uu}}$ comes from $\mathcal{M}_{\mathbf{uu}}$, $\cdots$). The term $N(\boldsymbol{U}(t))\boldsymbol{U}(t)$ is a matrix depending on the velocity corresponding of the nonlinear convective term $\int_{\mathcal{F}} [(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v}\mathrm{d}\mathcal{F}$.

3.2 Time discretization and treatment of the nonlinearity

Classical methods like $\theta-$methods can be used to carry out the time discretization. To assume unconditional stability of the scheme, we consider an implicit discretization in time based on the backward Euler method. We denote by $\boldsymbol{U}^{n+1}$ the solution at the time level t^{n+1} and $dt = t^{n+1} - t^n$ is the time step. Particular attention must be done for a moving particle problem. Indeed, at the time level t^{n+1} the solid occupies $\mathcal{S}(t^{n+1})$ which is different from the previous time level t^n. So, the field variable at the time level t^{n+1} can become undefined near the interface since there was no fluid flow at the time level t^n ($\mathcal{S}(t^{n+1}) \neq \mathcal{S}(t^n)$ for the solid and $\mathcal{F}(t^{n+1}) \neq \mathcal{F}(t^n)$ for the fluid). In other words, some degrees of freedom (inside the solid) which are not considered at the time level t^n must be taken into account at the time level t^{n+1}. In particular the velocity field must be known in such nodes. Different strategies exist, in the present work, we impose the velocity to be equal to the motion of the solid.

In the following we present the algorithm to compute at the time level t^{n+1} the solution $(\boldsymbol{U}^{n+1}, \boldsymbol{P}^{n+1}, \boldsymbol{\Lambda}^{n+1}, \mathbf{h}'^{n+1}, \mathbf{h}^{n+1}, \theta'^{n+1}, \theta^{n+1})$ on $\mathcal{F}(t^{n+1})$.
At the time level t^n we have access to $(\boldsymbol{U}^n, \boldsymbol{P}^n, \boldsymbol{\Lambda}^n, \mathbf{h}'^n, \mathbf{h}^n, \theta'^n, \theta^n)$ on $\mathcal{F}(t^n)$.

1– We compute $(\mathbf{h}'^{n+1}, \theta'^{n+1})$ such that

$$
\begin{aligned}
m\frac{\mathbf{h}'^{n+1} - \mathbf{h}'^n}{dt} &= M_\lambda \boldsymbol{\Lambda}^n - m\mathbf{g} + m_a\mathbf{g},\\
I\frac{\theta'^{n+1} - \theta'^n}{dt} &= M_\lambda \left[(\mathbf{x} - \mathbf{h}^n)^\perp \cdot \boldsymbol{\Lambda}^n\right].
\end{aligned}
$$

2– We complete the velocity $\boldsymbol{U}^n$ defined on $\mathcal{F}(t^n)$ to the full domain by imposing the velocity on each node of $\mathcal{S}(t^n)$ equal to $\mathbf{h}'^{n+1} + \theta'^{n+1}(\mathbf{x} - \mathbf{h}^n)^\perp$.

3– We update the geometry to determine $\mathcal{F}(t^{n+1})$ by computing $(\mathbf{h}^{n+1}, \theta^{n+1})$ such that

$$m\frac{\mathbf{h}^{n+1} - 2\mathbf{h}^n + \mathbf{h}^{n-1}}{dt^2} = M_\lambda \mathbf{\Lambda}^n - m\mathbf{g} + m_a\mathbf{g},$$
$$I\frac{\theta^{n+1} - 2\theta^n + \theta^{n-1}}{dt^2} = M_\lambda \left[(\mathbf{x} - \mathbf{h}^n)^\perp \cdot \mathbf{\Lambda}^n\right].$$

4– We compute the Dirichlet condition for the velocity at the new interface $\Gamma = \partial\mathcal{S}(t^{n+1})$. So we determine $\boldsymbol{G}^{n+1}$ from $\mathbf{u}_\Gamma^{n+1} = \mathbf{h}'^{n+1} + \theta'^{n+1}(\mathbf{x} - \mathbf{h}^{n+1})^\perp$.

5– Finally, we find $(\boldsymbol{U}^{n+1}, \boldsymbol{P}^{n+1}, \mathbf{\Lambda}^{n+1})$ such that

$$M_{\mathbf{uu}}\frac{\boldsymbol{U}^{n+1} - \boldsymbol{U}^n}{dt} + A_{\mathbf{uu}}\boldsymbol{U}^{n+1} + N(\boldsymbol{U}^{n+1})\boldsymbol{U}^{n+1} + A_{\mathbf{u}p}\boldsymbol{P}^{n+1} + A_{\mathbf{u}\lambda}\mathbf{\Lambda}^{n+1} = 0,$$
$$A_{\mathbf{u}p}^T\boldsymbol{U}^{n+1} + A_{pp}\boldsymbol{P}^{n+1} + A_{p\lambda}\mathbf{\Lambda}^{n+1} = 0,$$
$$A_{\mathbf{u}\lambda}^T\boldsymbol{U}^{n+1} + A_{p\lambda}^T\boldsymbol{P}^{n+1} + A_{\lambda\lambda}\mathbf{\Lambda}^{n+1} = \boldsymbol{G}^{n+1}.$$

At this stage, the solution of the resulting nonlinear algebraic system is achieved by a Newton technique.

Notice that in steps 2 and 4, the expressions of the rigid velocities are not the same.

4 Numerical tests: Free fall of an ellipse

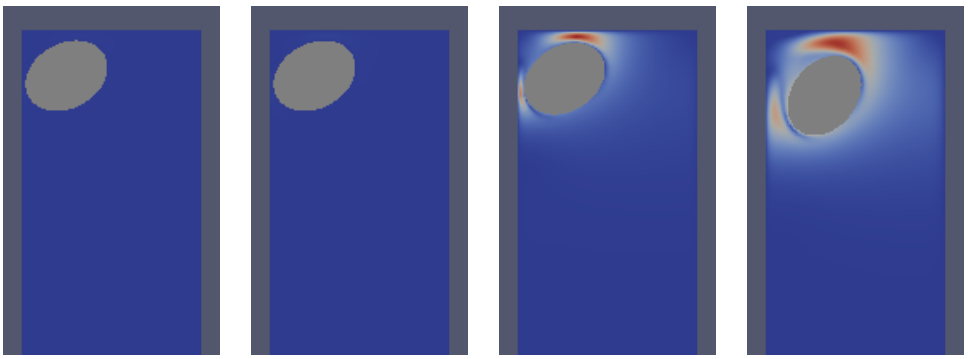
In order to illustrate our approach, we propose the simulation of the free fall of an homogeneous rigid ellipse, in 2D. It is dropped without initial velocity in a boxed $[0, 1] \times [0, 3]$. The dimension of its semi-major axis and semi-minor axis are respectively:

$$a = 0.24, \qquad b = 0.18.$$

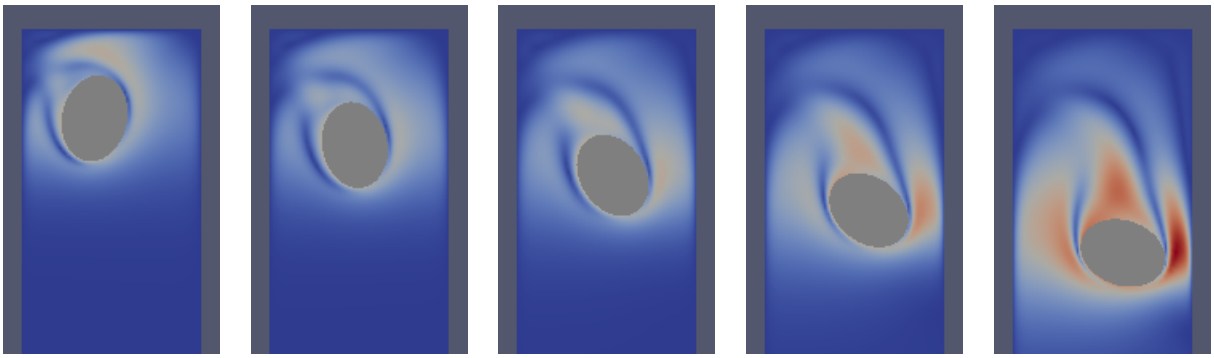
The viscosity of the fluid and the mass of the solid are respectively chosen as

$$\nu = 1.00, \qquad m = 0.13.$$

The triangular mesh on the full domain is based on 40 subdivisions in horizontal direction and 120 subdivisions in vertical direction (so the characteristic size of the mesh $h \approx 0.01$). For $(\mathbf{u}, p, \boldsymbol{\lambda})$, we choice the classical finite element P2/P1/P0 and the stabilization parameter $\gamma = h \times \gamma_0$ for $\gamma_0 = 0.05$.
The ellipse is dropped while being inclined of an angle θ_0 equal to 0.50 rad (see Figure 3 at $t = 0$).



$t = 0$ $t = 123$ $t = 223$ $t = 323$



$t = 423$ $t = 523$ $t = 623$ $t = 723$ $t = 823$

Fig. 3 - Fall of the solid.

The evolution of the ellipse is represented in Figure 4 with the position of its center and its orientation. Observe in Figure 3 that the ellipse starts with straightening up and recentering in the channel, before turning over.

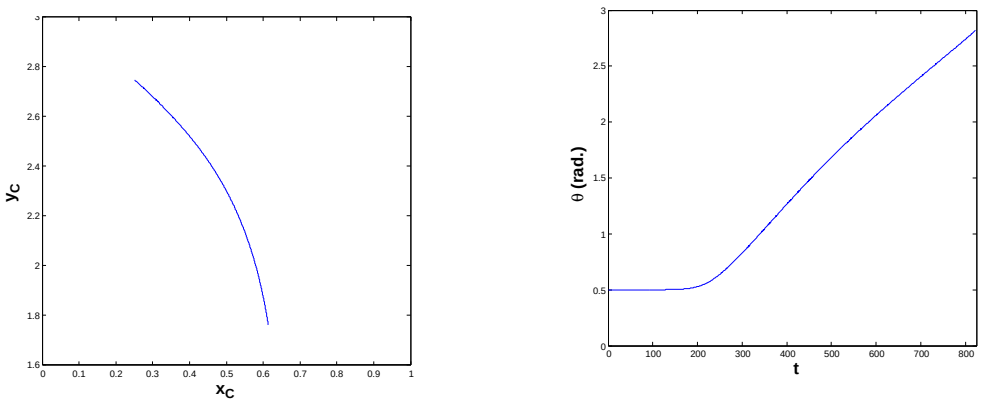


Fig. 4: Trajectory of the ellipse (left) and its orientation θ (right).

4.0.1 Practical remarks on the numerical implementation.

- All numerical simulations were performed with the free generic library Getfem++ [22] (same source code for 2D and 3D) and implemented on High Performing Computers (parallel aspect).
- To compute properly the integrals over elements at the interface (during assembling) external call to QHULL Library [29] is realized.
- The method is very efficient in time computation requiring an update of the assembling matrices only locally near the interface.
- As mentioned in the paper [10], it is possible to define a reinforced stability to prevent difficulties that can occur when the intersection of the solid and the mesh over the whole domain introduce "very small" elements. The technique is based on a strategy to select elements which are better to deduce the normal derivative on Γ. A similar approach is given in [21].
- The parameter γ has to respond to a compromise between the coercivity of the system and the weight of the stabilization term. From numerical tests, a good choice for γ is closed to $h \times \gamma_0$ where $\gamma_0 = 0.05$ (see paper [30]).

5 Conclusions

In this paper, we considered a new fictitious domain method based on extended finite element with stabilized term applied to the Navier-Stokes equations coupled with a moving solid. This method is quite simple to implement since all the variables (multipliers and primal variables) are defined on a single mesh independent of the computational domain. The algorithm presented leads to a robust method (good computation of the normal Cauchy stress tensor) whatever is the intersection of the domain with the mesh. Applications in 3D are actually in progress in particular for control flow by acting on the boundary of the solid.

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