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# LOCAL TRIVIALITY FOR $G$ -TORSORS

P. GILLE, R. PARIMALA, AND V. SURESH

**ABSTRACT.** Let  $C \rightarrow \text{Spec}(R)$  be a relative proper flat curve over a henselian base. Let  $G$  be a reductive  $C$ -group scheme. Under mild technical assumptions, we show that a  $G$ -torsor over  $C$  which is trivial on the closed fiber of  $C$  is locally trivial for the Zariski topology.

**Keywords:** Reductive group scheme, torsor, deformation.

**MSC 2000:** 14D23, 14F20.

## 1. INTRODUCTION

The purpose of the paper is to study local triviality for  $G$ -torsors over a *relative curve* over an affine base  $\text{Spec}(R)$ , that is a flat proper morphism  $C \rightarrow \text{Spec}(R)$  of finite presentation whose fibers have dimension  $\leq 1$ . We deal here with semisimple  $C$ -group schemes which are not necessarily extended from  $R$ . Our main result can be stated as follows.

**Theorem 1.1.** *Let  $R$  be a henselian local noetherian ring with residue field  $\kappa$ . Let  $f : C \rightarrow \text{Spec}(R)$  be a relative curve of relative dimension 1 and denote by  $C^{\text{sm}}$  the smooth locus of  $f$ . We assume that one of the following holds:*

- (I)  $C_{\kappa}^{\text{sm}}$  is dense in  $C_{\kappa}$ ;
- (II)  $R$  is a DVR and  $C$  is integral regular.

*Let  $G$  be a semisimple  $C$ -group scheme and denote by  $q : G^{\text{sc}} \rightarrow G$  its simply connected covering. We assume that the fundamental group  $\mu/C = \ker(q)$  of  $G$  is étale over  $C$ . Let  $E, E'$  be two  $G$ -torsors over  $C$  such that  $E \times_C C_{\kappa}$  is isomorphic to  $E' \times_C C_{\kappa}$ . Then  $E$  and  $E'$  are locally isomorphic for the Zariski topology.*

We recall that relative dimension 1 means that all nonempty fibers  $C_s$  are equidimensional of dimension 1 [49, Tag 02NJ]. A related result is that of Drinfeld and Simpson [14, th. 2]. In the case  $G$  is semisimple split and  $R$  is strictly henselian they showed in the smooth case (I) that a  $G$ -torsor over  $C$  is locally trivial for the Zariski topology; according to one of the referees, inspection of the proof shows this extends to the case of a henselian ring  $R$  in the case  $C(R) \neq \emptyset$ . Drinfeld-Simpson's result has been generalized recently by Belkale-Fakhruddin to a wider setting [3, 4]. We provide a variant

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in Theorem 7.2 in the case of a henselian base and without any splitting assumptions; more precisely we do not assume that  $G$  admits a proper parabolic subgroup.

One important difference is that we only require that the ring  $R$  is henselian. We consider Zariski triviality on  $C$  with respect to henselian (or Nisnevich) topology on the base while Drinfeld-Simpson deal with Zariski triviality on  $C$  with respect to the étale topology on the base. We stated the semisimple case but the result extends in the reductive case by combining with the case of tori, see Theorem 7.1. We denote by  $F$  the function field of  $C$ ; in the case of a DVR, the main result leads to new cases of a local-global principle for  $G_F$ -torsors (Corollary 7.8). More precisely if  $C$  is smooth over  $R$  with geometrically connected fibers and  $G$  is a semisimple group over  $C$  (whose fundamental group is étale), then a torsor under  $G$  over  $F$  which is trivial at all completions of  $F$  at discrete valuations of  $F$  is trivial.

Let us review the contents of the paper. The toral case is quite different from the semisimple one since it works in higher dimensions; it is treated in section 2 by means of the proper base change theorem. The section 3 deals with generation by one-parameter subgroups, namely the Kneser-Tits problem. Section 4 extends Sorger's construction of the moduli stack of  $G$ -bundles [48] and discusses in detail its tangent bundle. The next section 5 recollects facts on patching for  $G$ -torsors and provides the main technical statement, namely the parametrization of the deformations of a given torsor in the henselian case in the presence of isotropy (Proposition 5.3); this refines Heinloth's uniformization [32]). Section 6 explains why this intermediate statement is enough for establishing the fact that deformations of a given torsor (in the henselian case) are locally trivial for the Zariski topology. One important point is that we can get rid of the isotropy assumptions. Finally section 7 provides a general theorem for reductive groups. We include at the end a short appendix 8 gathering facts on smoothness for morphisms of algebraic stacks.

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**Conventions and Notations.** We use mainly the terminology and notations of Grothendieck-Dieudonné [15, §9.4 and 9.6] which agree with that of Demazure-Grothendieck used in [45, Exp. I.4]

(a) Let  $S$  be a scheme and let  $\mathcal{E}$  be a quasi-coherent sheaf over  $S$ . For each morphism  $f : T \rightarrow S$ , we denote by  $\mathcal{E}_T = f^*(\mathcal{E})$  the inverse image of  $\mathcal{E}$  by the morphism  $f$ . We

denote by  $\mathbf{V}(\mathcal{E})$  the affine  $S$ -scheme defined by  $\mathbf{V}(\mathcal{E}) = \mathrm{Spec}(\mathrm{Sym}^\bullet(\mathcal{E}))$ ; it is affine over  $S$  and represents the  $S$ -functor  $Y \mapsto \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{E}_Y, \mathcal{O}_Y)$  [15, 9.4.9].

(b) We assume now that  $\mathcal{E}$  is locally free and denote by  $\mathcal{E}^\vee$  its dual. In this case the affine  $S$ -scheme  $\mathbf{V}(\mathcal{E})$  is of finite presentation (ibid, 9.4.11); also the  $S$ -functor  $Y \mapsto H^0(Y, \mathcal{E}_Y) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{E}_Y)$  is representable by the affine  $S$ -scheme  $\mathbf{V}(\mathcal{E}^\vee)$  which is also denoted by  $\mathbf{W}(\mathcal{E})$  [45, I.4.6].

It applies to the locally free coherent sheaf  $\mathcal{E}nd(\mathcal{E}) = \mathcal{E}^\vee \otimes_{\mathcal{O}_S} \mathcal{E}$  over  $S$  so that we can consider the affine  $S$ -scheme  $\mathbf{V}(\mathcal{E}nd(\mathcal{E}))$  which is an  $S$ -functor in associative commutative and unital algebras [15, 9.6.2]. Now we consider the  $S$ -functor  $Y \mapsto \mathrm{Aut}_{\mathcal{O}_Y}(\mathcal{E}_Y)$ . It is representable by an open  $S$ -subscheme of  $\mathbf{V}(\mathcal{E}nd(\mathcal{E}))$  which is denoted by  $\mathrm{GL}(\mathcal{E})$  (*loc. cit.*, 9.6.4).

(c) We denote by  $\mathbb{Z}[\epsilon] = \mathbb{Z}[x]/x^2$  the ring of dual integers and by  $S[\epsilon] = S \times_{\mathbb{Z}} \mathbb{Z}[\epsilon]$ . If  $G/S$  is an  $S$ -group space (i.e. an algebraic space in groups, called group algebraic space over  $S$  in [49, Tag 043H]) we denote by  $\mathrm{Lie}(G)$  the  $S$ -functor defined by  $\mathrm{Lie}(G)(T) = \ker(G(T[\epsilon]) \rightarrow G(T))$ . This  $S$ -functor is a functor in Lie  $\mathcal{O}_S$ -algebras, see [45, II.4.1] or [13, II.4.4]. More facts are collected in Appendix 8.2.

(d) If  $G/S$  is an affine smooth  $S$ -group scheme, we denote by  $\mathrm{Tors}_G(S)$  the groupoid of (right)  $G$ -torsors over  $S$  and by  $H^1(S, G)$  the set of isomorphism classes of  $G$ -torsors (locally trivial for the étale topology), we have a classifying map  $\mathrm{Tors}_G(S) \rightarrow H^1(S, G)$ ,  $E \mapsto [E]$ .

## 2. THE CASE OF TORI

In this section we prove variants of Theorem 1.1 for nice tori over proper schemes over local henselian noetherian rings (not necessarily relative curves). The main statement is Theorem 2.4 which is used in Theorem 7.2 on reductive group schemes.

**Lemma 2.1.** *Let  $X$  be a scheme and let  $T$  be an  $X$ -torus. Assume that  $T$  is split by a finite étale cover of degree  $d$ . Then  $dH^1(X, T) \subseteq H_{\mathrm{Zar}}^1(X, T)$ .*

*Proof.* Let  $f : Y \rightarrow X$  be a finite étale cover of degree  $d$  which splits  $T$ , that is  $T_Y \cong \mathbb{G}_{m, Y}^r$ . According to [11, 0.4], we have a norm map  $f_* : H^1(Y, T) \rightarrow H^1(X, T)$  such that the composite  $H^1(X, T) \xrightarrow{f^*} H^1(Y, T) \xrightarrow{f_*} H^1(X, T)$  is the multiplication by  $d$ . We claim that  $f_*(H^1(Y, T)) \subseteq H_{\mathrm{Zar}}^1(X, T)$ . Let  $x \in X$ . Then  $V_x = \mathrm{Spec}(\mathcal{O}_{X, x}) \times_X Y$  is a semilocal scheme so that  $\mathrm{Pic}(V_x) = 0$  [7, Chap. 2, Sect. 5, No. 3, Proposition 5]. Since  $T$  is split over  $V_x$ ,  $H^1(V_x, T) = 0$ . Since the construction of the norm commutes with base change, we get that  $\left(f_*(H^1(Y, T))\right)_{\mathcal{O}_{X, x}} = 0$ . In particular we have  $dH^1(X, T) \subseteq H_{\mathrm{Zar}}^1(X, T)$ .  $\square$

**Proposition 2.2.** *Let  $R$  be a henselian local ring of residue field  $\kappa$ . We denote by  $p$  the characteristic exponent of  $\kappa$  and let  $l$  be a prime number distinct from  $p$ . Let  $X$  be a proper  $R$ -scheme and let  $T$  be an  $X$ -torus.*

- (1) *For each  $i \geq 0$ ,  $\ker(H^i(X, T) \rightarrow H^i(X_\kappa, T))$  is  $l$ -divisible.*
- (2) *The kernel  $\ker(H^0(X, T) \rightarrow H^0(X_\kappa, T))$  is uniquely  $l$ -divisible.*
- (3) *We assume that  $T$  is locally isotrivial, that is there exists an open cover  $(U_i)_{i=1, \dots, n}$  of  $X$  and finite étale covers  $f_i : V_i \rightarrow U_i$  such that  $T_{V_i}$  is split for  $i = 1, \dots, n$ . Then there exists  $r \geq 0$  such that  $p^r \ker(H^1(X, T) \rightarrow H^1(X_\kappa, T)) \subseteq H_{Zar}^1(X, T)$ .*

*Proof.* (1) We consider the exact sequence of étale  $X$ -sheaves  $1 \rightarrow {}_lT \rightarrow T \xrightarrow{\times l} T \rightarrow 1$  which generalizes the Kummer sequence. It gives rise to the following commutative diagram

$$\begin{array}{ccccccc} H^i(X, {}_lT) & \longrightarrow & H^i(X, T) & \xrightarrow{\times l} & H^i(X, T) & \longrightarrow & H^{i+1}(X, {}_lT) \\ \downarrow \beta_1 & & \downarrow & & \downarrow & & \downarrow \beta_2 \\ H^i(X_\kappa, {}_lT) & \longrightarrow & H^i(X_\kappa, T) & \xrightarrow{\times l} & H^i(X_\kappa, T) & \longrightarrow & H^{i+1}(X_\kappa, {}_lT). \end{array}$$

The proper base change theorem [46, XII.5.5.(iii)] shows that  $\beta_1, \beta_2$  are isomorphisms. By diagram chase, we conclude that  $\ker(H^i(X, T) \rightarrow H^i(X_\kappa, T))$  is  $l$ -divisible.

(2) If  $i = 0$ , we can complete the left-hand side of the diagram with 0. By diagram chase it follows that  $\ker(H^0(X, T) \rightarrow H^0(X_\kappa, T))$  is uniquely  $l$ -divisible.

(3) Let  $(U_i)_{i=1, \dots, n}$  be an open cover of  $X$  and finite étale covers  $f_i : V_i \rightarrow U_i$  such that  $T_{V_i}$  is split for  $i = 1, \dots, n$ . Let  $d$  be the l.c.m. of the degrees of the  $f_i$ 's. We write  $d = p^r e$  with  $(e, p) = 1$ . Assertion (1) shows that  $\ker(H^1(X, T) \rightarrow H^1(X_\kappa, T))$  is  $e$ -divisible so that  $\ker(H^1(X, T) \rightarrow H^1(X_\kappa, T)) \subseteq eH^1(X, T)$ . Lemma 2.1 shows that  $dH^1(X, T) \subseteq H_{Zar}^1(X, T)$  which permits to conclude that we have the inclusion  $p^r \ker(H^1(X, T) \rightarrow H^1(X_\kappa, T)) \subseteq H_{Zar}^1(X, T)$ .  $\square$

**Remark 2.3.** Proposition 2.2.(1) fails completely for  $l = p$  if the residue field  $\kappa$  is of characteristic  $p > 0$  and already for  $\mathbb{G}_m$ . For example we take  $R = \mathbb{F}_p[[t]]$  (or  $\mathbb{Z}_p$ ). Then  $\ker(R^\times \rightarrow \mathbb{F}_p^\times)$  admits  $\mathbb{F}_p$  as quotient so is not  $p$ -divisible. For the  $H^1$ , we consider a smooth elliptic curve  $E$  over  $k$  having a  $k$ -point 0. We have  $\text{Pic}(E_{k[[t]]}) \xrightarrow{\sim} \text{Pic}(E_{k((t))}) = E(k((t))) \oplus \mathbb{Z} = E(k[[t]]) \oplus \mathbb{Z}$ . It follows that  $\ker(\text{Pic}(E_{k[[t]]}) \rightarrow \text{Pic}(E)) \xrightarrow{\sim} \ker(E(k[[t]]) \rightarrow E(k))$  so that admits a quotient isomorphic to  $\mathbb{F}_p = \text{Lie}(E)(\mathbb{F}_p)$ . Therefore  $\ker(\text{Pic}(E_{k[[t]]}) \rightarrow \text{Pic}(E))$  is not  $p$ -divisible.

**Theorem 2.4.** *Let  $R$  be a local henselian noetherian ring with residue field  $k$  and  $p = \text{char}(k)$ . Let  $X \rightarrow \text{Spec}(R)$  be a proper scheme and  $T$  an  $X$ -torus. Suppose  $T$  quasi-splits after a finite étale extension  $X'/X$  of degree prime to  $p$ , that is  $T \times_X X' \cong R_{X''/X'}(\mathbb{G}_m)$  where  $X'' \rightarrow X'$  is a finite étale cover. Then  $\ker(H^1(X, T) \rightarrow H^1(X_\kappa, T)) \subseteq H_{Zar}^1(X, T)$ .*

*Proof.* The theorem 90 of Hilbert-Grothendieck shows that  $H_{Zar}^1(X', T) = H^1(X', T_{X'})$ . By corestriction-restriction it follows that  $[X' : X]H^1(X, T) \subseteq H_{Zar}^1(X, T)$ . In particular we have

$$[X' : X] \ker(H^1(X, T) \rightarrow H^1(X_\kappa, T)) \subseteq H_{Zar}^1(X, T).$$

Since  $[X' : X]$  is prime to  $p$ , Proposition 2.2 yields that  $\ker(H^1(X, T) \rightarrow H^1(X_\kappa, T)) \subseteq H_{Zar}^1(X, T)$ .  $\square$

**Corollary 2.5.** *Suppose  $\text{char}(k) = 0$ . Then  $\ker(H^1(X, T) \rightarrow H^1(X_\kappa, T)) \subseteq H_{Zar}^1(X, T)$ .*

### 3. INFINITESIMAL KNESER-TITS PROBLEM

For a semisimple group scheme  $G$  defined over a semilocal ring  $R$ , we are interested in the quotient of  $G(R)$  by the subgroup generated by the unipotent radicals of parabolic subgroups of  $G$ .

**3.1. Strictly proper parabolic subgroups.** Let  $G$  be a reductive group over an algebraically closed field  $k$ . We have a decomposition of its adjoint group

$$G_{ad} \xrightarrow{\sim} G_1 \times \cdots \times G_{r_s},$$

where the  $G_i$ 's are adjoint simple  $k$ -groups. Let  $Q$  be a parabolic subgroup of  $G$ . Then  $Q/C(G)$  is a parabolic subgroup of  $G_{ad}$  and decomposes as a product of parabolic subgroups  $\prod_i Q_i$ . We say that  $Q$  is a *strictly proper* parabolic subgroup of  $G$  if  $Q_i \subsetneq G_i$  for  $i = 1, \dots, r_s$ .

Let  $S$  be a scheme and let  $G$  be a reductive  $S$ -group scheme. Let  $P$  be a parabolic subgroup scheme of  $G$ . We say that  $P$  is *strictly proper* if for each point  $s \in S$ ,  $P_{\kappa(s)}$  is a strictly proper parabolic subgroup of  $G_{\kappa(s)}$ .

This definition is stable under base change and is local for the fppf topology.

**3.2. Last term of Demazure's filtration.** We continue with the  $S$ -parabolic subgroup  $P$  scheme of  $G$  assumed strictly proper. We consider Demazure's filtration of the unipotent radical  $U = \text{rad}_u(P)$

$$U = U_0 \supset U_1 \cdots \supset U_n \supsetneq 0$$

by vector subgroup schemes which are characteristic in  $P$  [45, XXVI.2.1]. The last  $U_n$  is central in  $U$ . If the Cartan-Killing type of  $G$  is constant and connected and if  $P$  is of constant type, the last term  $U_n$  is the right object for our purpose. This construction does not behave well under products and we need to refine it; it is enough to deal with the adjoint case since the unipotent radical of  $P$  and  $P/C(G)$  are isomorphic.

**Lemma 3.1.** *Assume that  $G$  is adjoint. The group scheme  $U = \text{rad}_u(P)$  admits a unique closed  $S$ -subgroup scheme  $U_{last}$  satisfying the following property:*

For each  $S$ -scheme  $Y$  such that  $G_Y, P_Y$  are of constant type and such that  $G \times_S Y \cong^{\phi} G_1 \times_Y G_2 \cdots \times_Y G_c$  where  $G_i$  is an adjoint semisimple  $Y$ -scheme whose absolute Cartan-Killing type is connected, then  $\phi(U_{last, Y}) = U_{1, last} \times_Y \cdots \times_Y U_{c, last}$  where  $U_{i, last}$  is the last term of Demazure's filtration of  $P_Y \cap G_i$  for  $i = 1, \dots, c$ .

Furthermore,  $U_{last}$  is a vector  $S$ -group scheme, is central in  $U$  and is  $\text{Aut}(G, P)$ -equivariant.

*Proof.* Without loss of generality, we can assume that  $G$  and  $P$  are of constant type. Since the required properties are local for the étale topology on  $S$ , it is convenient to reason by a descent argument.

We assume first that  $G = G_1 \times_S G_2 \cdots \times_S G_c$  where  $G_i$  is an adjoint semisimple  $S$ -scheme whose absolute Cartan-Killing type is connected. We have  $P = P_1 \times_S P_2 \cdots \times_S P_c$  and put  $U_{last} = U_{1, last} \times_S \cdots \times_S U_{c, last}$ . This is a vector  $S$ -group scheme, central in  $U = \text{rad}_u(P)$  and we claim that it is  $\text{Aut}(G, P)$ -equivariant.

Let  $\phi \in \text{Aut}(G, P)(S)$ . Up to localization, we may assume that there exist a permutation  $\sigma$  in  $c$  letters and  $S$ -isomorphisms  $\phi_i : G_{\sigma(i)} \xrightarrow{\sim} G_i$  ( $i = 1, \dots, c$ ) such that  $\phi(g_1, \dots, g_c) = (\phi_1(g_{\sigma(1)}), \dots, \phi_r(g_{\sigma(r)}))$ . Since each  $U_{i, last}$  is characteristic in  $P_i$ , it follows that  $\phi(U_{last}) = U_{last}$ . This shows that  $U_{last}$  satisfies the requirement. This is a vector  $S$ -group scheme which is central in  $U = \text{rad}_u(P)$  by construction.

*General case.* Locally for the étale topology over  $S$ ,  $(G, P)$  is isomorphic to  $(G_0, P_0)$  where  $G_0$  is an adjoint Chevalley  $S$ -group scheme and  $P_0$  is a standard parabolic subgroup of  $G_0$  [45, XXII.2.3 and XXVI.3.3]. We denote by  $U_0$  its unipotent radical and by  $U_{0, last} \subset U_0$  the  $S$ -subgroup defined in the first case. By faithfully flat descent  $U_{0, last}$  descends to  $U_{last} \subset P$  and satisfies the required property.  $\square$

**3.3. Subgroups attached to parabolic subgroups.** Let  $R$  be a ring and let  $G$  be a reductive  $R$ -group scheme. Let  $P$  be an  $R$ -parabolic subgroup of  $G$ ;  $P$  admits a Levi subgroup  $L$  [45, XXVI.2.3]. We consider the corresponding opposite  $R$ -parabolic subgroup  $P^-$  to  $P$ . We denote by  $E_P(R)$  the subgroup of  $G(R)$  which is generated by  $\text{rad}_u(P)(R)$  and  $\text{rad}_u(P^-)(R)$ ; it does not depend of the choice of  $L$  since Levi subgroups of  $P$  are  $\text{rad}_u(P)(R)$ -conjugated.

**Remark 3.2.** If  $R$  is a field, and  $P$  is a strictly proper parabolic  $R$ -subgroup, then  $E_P(R)$  does not depend on the choice of  $P$ . In this case, the group  $E_P(R) = G^{+, P}(R)$  is denoted by  $G^+(R)$  [6, prop. 6.2].

**3.4. Generation of the Lie algebra: the field case.** Let  $F$  be a field. If  $F$  is finite, we use the notation  $F_r$  for the unique extension of  $F$  of degree  $r$ .

**Lemma 3.3.** *Let  $G$  be a simply connected semisimple algebraic  $F$ -group with Lie algebra  $\mathfrak{g}$ . Let  $P$  be a strictly proper parabolic  $F$ -subgroup. Let  $U_{last}$  be the  $F$ -subgroup of  $\text{rad}_u(P)$  constructed in Lemma 3.1.*

- (1) If  $G$  is split, then  $G(F) \cdot \text{Lie}(U_{last})$  generates the  $F$ -vector space  $\mathfrak{g}$ .
- (2)  $\mathfrak{g}$  is the unique  $G$ -submodule of  $\mathfrak{g}$  containing  $\text{Lie}(U_{last})$ .
- (3) If  $F$  is infinite, then  $E_P(F) \cdot \text{Lie}(U_{last})$  generates the  $F$ -vector space  $\mathfrak{g}$ .
- (4) If  $F$  is finite, we have  $E_P(F_r) = G(F_r)$  for each  $r \geq 1$  and the quantity

$$r_F(G, P) = \text{Inf} \left\{ r \geq 1 \mid \text{the } F_r\text{-vector space } \mathfrak{g} \otimes_F F_r \text{ is generated by } E_P(F_r) \cdot \text{Lie}(U_{last})(F_r) \right\}$$

is  $< \infty$ .

*Proof.* (1) We can assume that  $G$  is almost simple. Let  $B$  be a Borel subgroup of  $P$  and let  $T$  be a maximal  $F$ -torus of  $B$ . Let  $\alpha$  be the maximal root of  $\Phi(G, T)$ , we have  $U_\alpha \subset U_{last}$  by construction. Since  $\alpha$  is a long root, the Lie subalgebra  $\mathfrak{g}_\alpha = \text{Lie}(U_\alpha)$  is called a long root subalgebra. Since roots of maximal length are conjugated under the Weyl group and since maximal split tori of  $G$  are  $G(F)$ -conjugated, it follows that all long root subalgebras are  $G(F)$ -conjugated. According to [51, lemma 1.1] (based on [34]), if  $G$  is not of rank one, the long root subalgebras generate the  $F$ -vector space  $\mathfrak{g}$ . Thus  $G(F) \cdot \text{Lie}(U_{last})$  generates the  $F$ -vector space  $\mathfrak{g}$ .

It remains then to deal with the case of  $\text{SL}_2$  and its standard Borel subgroup  $P = B$ . We observe that  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$  and  $\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$  are long root elements and form a  $F$ -basis of  $\mathfrak{sl}_2$ .

- (2) We can assume that  $F$  is algebraically closed so that the statement readily follows from (1).
- (3) This follows from the fact that  $E_P(F)$  is Zariski dense in  $G$  [6, cor. 6.9].
- (4) Since  $F$  is finite,  $G$  is quasi-split and we have  $E_P(F_r) = G(F_r)$  for each  $r \geq 1$  according to [50, 1.1.2]. Then (1) shows that  $G(F_\infty) \cdot \text{Lie}(U_{last})(F_\infty)$  generates the  $F_\infty$ -vector space  $\mathfrak{g} \otimes_F F_\infty$ . Then there exists  $r \geq 1$  such that  $G(F_r) \cdot \text{Lie}(U_{last})(F_r)$  generates the  $F_\infty$ -vector space  $\mathfrak{g} \otimes_F F_\infty$  and a fortiori the  $F_r$ -vector space  $\mathfrak{g} \otimes_F F_r$ .  $\square$

**Remark 3.4.** Under the hypothesis of Lemma 3.3, we assume furthermore that  $F$  is finite field.

- (a) We have  $r_{F_u}(G, P) = \text{Sup}(1, r_F(G, P) - u)$  for each  $u \geq 1$ .
- (b) If  $G$  is split, Lemma 3.3.(1) is rephrased by the formula  $r_F(G, P) = 1$ .

**3.5. Generation of the Lie algebra: Case of a semilocal ring.** The next statement is a variation of a result of Borel-Tits on the Whitehead groups over local fields [6, prop. 6.14].

**Lemma 3.5.** *Let  $R$  be a semilocal ring. Let  $G$  be a semisimple  $R$ -group scheme equipped with a strictly proper  $R$ -parabolic subgroup  $P$  of  $G$ . We assume that the fundamental group of  $G$  is étale. Let  $U_{last}$  be the  $R$ -subgroup  $P$  defined in Lemma 3.1. Let  $s_1, \dots, s_t$  be the closed points of  $\text{Spec}(R)$ . For each  $j$  such that  $\kappa(s_j)$  is finite we assume that  $G_{\kappa(s_j)}$  that  $r(G_{\kappa(s_j)}, P_{\kappa(s_j)}) = 1$ .*

(1) *There exist  $g_1, \dots, g_m \in E_P(R)$  such that the product map*

$$h : (U_{last})^m \rightarrow G, \quad (u_1, \dots, u_m) \mapsto {}^{g_1}u_1 \dots {}^{g_m}u_m$$

*is smooth at  $(1, \dots, 1)_{s_j}$  for  $j = 1, \dots, n$ .*

(2) *The map  $dh : \text{Lie}(U_{last})(R)^m \rightarrow \text{Lie}(G)(R)$  is onto.*

*Proof.* We denote by  $\mathfrak{m}_1, \dots, \mathfrak{m}_t$  the maximal ideals of  $R$ , by  $\kappa_j = \kappa(s_j) = R/\mathfrak{m}_j$  for  $j = 1, \dots, t$ .

(1) The hypothesis on the fundamental group of  $G$  implies that  $G^{sc} \rightarrow G$  is étale and then reduces to the simply connected case. Lemma 3.3.(1) implies that  $E_P(\kappa_j) \cdot \text{Lie}(U_{last})(\kappa_j)$  generates the  $\kappa_j$ -vector space  $\text{Lie}(G)(\kappa_j)$  if  $\kappa_j$  is infinite and this holds as well in the finite case granting to our assumption  $r(G_{\kappa_j}, P_{\kappa_j}) = 1$ .

**Claim 3.6.** *The map  $E_P(R) \rightarrow \prod_j E_P(\kappa(s_j))$  is onto.*

Let  $P^-$  be an opposite parabolic subgroup scheme of  $P$  and let  $U^-$  be its unipotent radical. Since  $E_P(R)$  (resp. each  $E_P(\kappa(s_j))$ ) is generated (by definition) by  $U(R)$  and  $U^-(R)$ , it is enough to show the surjectivity of  $U(R) \rightarrow \prod_j U(\kappa(s_j))$ . According to [45, XXVI.2.5], there exists a finitely generated locally free  $R$ -module  $\mathcal{E}$  such that  $U$  is isomorphic to  $\mathbf{W}(\mathcal{E})$  as  $R$ -scheme. Since  $\mathbf{W}(\mathcal{E})(R) = \mathcal{E}$  maps onto  $\prod_j \mathbf{W}(\mathcal{E})(\kappa(s_j)) = \prod_j \mathcal{E} \otimes_R \kappa(s_j)$ , the Claim is established.

There are  $g_1, \dots, g_m \in E_P(R)$  such that  $\text{Lie}(G)(\kappa(s_j))$  is generated by the  ${}^{g_i}\text{Lie}(U_{last})(\kappa(s_j))$  for  $j = 1, \dots, t$ . The differential of the product map

$$h : (U_{last})^m \rightarrow G, \quad (u_1, \dots, u_m) \mapsto {}^{g_1}u_1 \dots {}^{g_m}u_m$$

is  $\text{Lie}(U_{last})^m \rightarrow \text{Lie}(G)$ ,  $(x_1, \dots, x_m) \mapsto {}^{g_1}x_1 \dots {}^{g_m}x_m$ . It is onto by construction; we conclude that  $h$  is smooth at  $(1, \dots, 1)_{s_j}$  for  $j = 1, \dots, t$ .

(2) Then  $J = \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_t$  is the Jacobson radical of  $R$  and  $R/J \cong R/\mathfrak{m}_1 \times \dots \times R/\mathfrak{m}_t$ . Statement (1) shows that the map  $dh : \text{Lie}(U_{last})(R)^m \rightarrow \text{Lie}(G)(R)$  is surjective modulo  $\mathfrak{m}_i$  for  $i = 1, \dots, t$  so is surjective modulo  $J$ . Since  $\text{Lie}(G)(R)$  is finitely generated, Nakayama's lemma [36, II.4.2.3] enables us to conclude that  $dh$  is onto.  $\square$

## 4. MODULI STACK OF $G$ -TORSORS

**4.1. Setting.** Let  $S$  be a noetherian separated base scheme.

Let  $f : X \rightarrow S$  be a proper flat (finitely presented) scheme satisfying the resolution property fppf locally over  $S$ , i.e. every quasi-coherent  $\mathcal{O}_X$ -module of finite type is the

quotient of a finite locally free  $\mathcal{O}_X$ -module fppf locally over  $S$  [49, Tag 0F86]. This property is satisfied if  $X$  is projective over  $S$  and also if  $X$  is a divisorial scheme [47, II.2.24]. This applies in particular to the case  $X$  noetherian regular (*ibid*, II.2.7.1.1).

**Lemma 4.1.** *Let  $Y \rightarrow X$  be an affine scheme of finite type. Then the  $S$ -functor  $\prod_{X/S} Y$  is representable by an affine  $S$ -scheme of finite presentation.*

*Proof.* In the projective case, this is [30, §1.4]. According to the faithfully flat descent theorem, this assumption is local for the flat topology. This permits to assume that  $f : X \rightarrow S$  satisfies the resolution property. This matters in the following basic statement [17, 7.7.8, II.7.9], see also [22, §5.3, Th. 5.8] and its proof: if  $\mathcal{F}$  is flat coherent sheaf over  $X$ , the fppf  $S$ -sheaf  $T \mapsto H^0(X \times_S T, \mathcal{F}_{X \times_S T})$  is representable by a linear  $S$ -scheme.

According to [16, 1.7.15], there exists a closed immersion  $Y \rightarrow V(\mathcal{F})$  where  $\mathcal{F}$  is a quasi-coherent  $\mathcal{O}_X$ -module of finite type. The resolution property allows us to assume that  $\mathcal{F}$  is a quotient of a finite locally free  $\mathcal{O}_S$ -module  $\mathcal{E}$ . Since  $V(\mathcal{F}) \rightarrow V(\mathcal{E})$  is a closed immersion [15, 9.4.11.(v)], we get a closed embedding  $Y \rightarrow V(\mathcal{E})$ . For each  $S$ -scheme  $T$ , we have  $\left(\prod_{X/S} V(\mathcal{E})\right)(T) = V(\mathcal{E})(X \times_S T) = H^0(X \times_S T, \mathcal{E}_{X \times_S T}^\vee)$ . Since  $\mathcal{E}^\vee$  is a coherent flat module over  $X$ , the  $S$ -functor  $\prod_{X/S} V(\mathcal{E})$  is representable by an affine  $S$ -scheme. According to [8, §7.6, prop. 2.(2)],  $\prod_{X/S} Y$  is representable by a closed  $S$ -subscheme of  $\prod_{X/S} V(\mathcal{E})$ .

The  $S$ -scheme  $\prod_{X/S} Y$  is locally of finite presentation since its functor of points commutes with colimits; the affine  $S$ -scheme  $\prod_{X/S} Y$  is then of finite presentation.  $\square$

Let  $G$  be a smooth affine group scheme over  $X$ . Let  $BG = [X/G]$  the algebraic  $X$ -stack of  $G$ -bundles on  $X$  [43, 8.1.14]; it is smooth [49, Tag 0DLS]. For each  $X$ -scheme  $Y$ ,  $B_G(Y)$  is the groupoid of  $G_Y$ -torsors. We denote by  $\text{Bun}_G = \prod_{X/S} B_G$  the  $S$ -stack of  $G$ -bundles, i.e.  $\text{Bun}_G(T) = B_G(X \times_S T)$  for each  $S$ -scheme  $T$ .

**Lemma 4.2.** *Let  $P, Q$  be two  $G$ -torsors over  $X$ . Then the fppf  $S$ -sheaf*

$$T \mapsto \text{Isom}_{G_{X \times_S T}}(P_{X \times_S T}, Q_{X \times_S T})$$

*is representable by an  $S$ -scheme  $\text{Isom}^b(P, Q)$  which is affine of finite presentation.*

*Proof.* Since the  $G$ -torsors  $P$  and  $Q$  are locally isomorphic over  $X$  to  $G_X$  with respect to the étale topology, the faithfully flat descent theorem shows that the  $X$ -functor

$$D \mapsto \text{Isom}_{G \times_X D}(P \times_X D, Q \times_X D)$$

is representable by an affine smooth  $X$ -scheme. We denote it by  $\mathbf{Isom}_G(P, Q)$ . The  $S$ -functor

$$T \mapsto \text{Isom}_{G_{X_T}}(P_{X_T}, Q_{X_T})$$

is nothing but the scalar restriction  $\prod_{X/S} \mathbf{Isom}_G(P, Q)$ . According to Lemma 4.1, this  $S$ -functor is representable by an affine  $S$ -scheme of finite presentation.  $\square$

**Proposition 4.3.** *The  $S$ -stack  $\mathbf{Bun}_G$  is algebraic locally of finite type with affine diagonal.*

*Proof.* (1) This is an application of the general result by Hall-Rydh on Weil restriction of algebraic stacks [27, Th. 1.2]. More precisely we deal with  $BG \xrightarrow{g} X \xrightarrow{f}$  and we claim that

(\*)  $f \circ g$  is locally of finite presentation and has affine diagonal.

Since  $BG$  is smooth over  $X$ ,  $g$  is locally of finite presentation. Locally of finite presentation is stable under composition so  $f \circ g$  is locally of finite presentation. The affineness of the diagonal follows from Lemma 4.2. In particular,  $f \circ g$  has affine stabilizers (i.e. the diagonal of  $BG \rightarrow S$  has affine fibers). The quoted result shows that the  $S$ -stack  $\mathbf{Bun}_G$  is algebraic locally of finite type and with affine diagonal.  $\square$

We assume from now on that  $X = C$  is a relative curve. According to [49, Tag 0DMK],  $C$  is locally  $H$ -projective (that is embeds in a projective space) for the étale topology so satisfies the resolution property étale locally (and a fortiori fppf locally) over  $S$ .

**Proposition 4.4.** *The  $S$ -stack  $\mathbf{Bun}_G$  is a smooth algebraic stack locally of finite type with affine diagonal.*

*Proof.* The smoothness remains to be established. We use the criterion of formal smoothness [33, 2.6] (or [49, Tag 0DNV]). We are given an  $S$ -algebra  $R$  which is local Artinian with maximal ideal  $\mathfrak{m}$  such that  $\mathfrak{m}^2 = 0$  and a  $G$ -torsor  $P_0$  over  $C_0 = C \times_S R/\mathfrak{m}$ . We put  $G_0 = G_{C_0}$  and denote by  $H_0 = P_0 \wedge^{G_0} G_0$  the twisted group scheme over  $C_0$  with respect to the action of  $G_0$  on itself by inner automorphisms. According to [35, th. 8.5.9], the obstruction to lift  $P_0$  in a  $G$ -torsor over  $C \times_S R$  is a class of  $H^2(C_0, \mathcal{L}ie(G_0) \otimes_{\mathcal{O}_{C_0}} \mathfrak{m})$ . But  $R/\mathfrak{m} = \kappa$  is a field and  $C_0$  is of dimension 1 so that this group vanishes according to Grothendieck's vanishing theorem [31, III.2.7]. The formal smoothness criterion is satisfied so that the algebraic stack  $\mathbf{Bun}_G$  is smooth.  $\square$

**4.2. The tangent stack.** We consider now the tangent stack  $T(\mathbf{Bun}_G/S)$  [37, §17] which is algebraic (*loc. cit.*, 17.16). By definition, for each  $S$ -scheme  $Y$ , we have  $T(\mathbf{Bun}_G/S)(Y) = \mathbf{Bun}_G(Y[\epsilon])$  where  $Y[\epsilon] = Y \times_{\mathbb{Z}} \mathbb{Z}[\epsilon]$ . It comes with two 1-morphisms

$$\tau : T(\mathbf{Bun}_G/S) \rightarrow \mathbf{Bun}_G$$

and  $\sigma : \mathbf{Bun}_G \rightarrow T(\mathbf{Bun}_G/S)$ .

**Remark 4.5.** We can consider the smooth-étale site on  $\text{Bun}_G$  and the quasi-coherent sheaf  $\Omega_{\text{Bun}_G/S}^1$ ; its associated generalized vector bundle  $\mathbb{V}(\Omega_{\text{Bun}_G/S}^1)$  is an algebraic  $S$ -stack. There is a canonical 1-isomorphism between  $T(\text{Bun}_G/S)$  and  $\mathbb{V}(\Omega_{\text{Bun}_G/S}^1)$  (*loc. cit.*, 17.15). We shall not use that fact in the paper.

Our goal is the understanding of the fiber product of  $S$ -stacks

$$T_b\text{Bun}_G = T(\text{Bun}_G/S) \times_{\text{Bun}_G} S$$

where  $b : S \rightarrow \text{Bun}_G$  corresponds to the trivial  $G$ -bundle  $G$  over  $C$ . According to [37, 2.2.2], for each  $S$ -algebra  $B$ , the fiber category  $T_b\text{Bun}_G(B)$  has for objects the couples  $(F, f)$  where  $F$  is a  $G_{C_B}[\epsilon]$ -torsor and  $f : G_{C_B} \xrightarrow{\sim} F \times_{C_B[\epsilon]} C_B$  is a trivialization of  $G_{C_B}$ -torsors; an arrow  $(F_1, f_1) \rightarrow (F_2, f_2)$  is a couple  $(H, h)$  where  $H : F_1 \xrightarrow{\sim} F_2$  is an isomorphism of  $G_{C_B[\epsilon]}$ -torsors and  $h \in G(C_B)$  with the compatibility  $(H \times_{C_B[\epsilon]} C_B) \circ f_1 = f_2 \circ h$ .

**4.3. Relation with the Lie algebra.** We consider the Weil restriction  $G' = \prod_{C[\epsilon]/C} G_{C[\epsilon]}$ , this is an affine smooth  $C$ -group scheme [12, A.5.2]. It comes with a  $C$ -group homomorphism  $j : G \rightarrow G'$  and with a  $C[\epsilon]$ -group homomorphism  $q : G' \times_C C[\epsilon] \rightarrow G \times_S S[\epsilon]$ .

We consider the functor  $\Phi$  between the categories of  $G'$ -torsors over  $C$  and that of  $G$ -torsors over  $C[\epsilon]$  defined by the assignment defined by the assignment  $E' \mapsto q_*(E' \times_C C[\epsilon])$ . According to [25, VII.1.3.2], the restriction map  $H^1(V[\epsilon], G_{V[\epsilon]}) \rightarrow H^1(V, G)$  is bijective for each affine  $S$ -scheme  $V$ ; it follows that each  $G$ -torsor over  $C[\epsilon]$  is trivialized by an étale cover of  $C$  extended to  $C[\epsilon]$ . According to [45, XXIV.8.2] (see also [25, III.3.1.1]), it follows that we can define the functor  $\Psi$  by the assignment  $F' \mapsto \prod_{C[\epsilon]/C} (F')$ . The functors  $\Phi$  and  $\Psi$  are inverse of each others so that the groupoids  $\text{Tors}_{G'}(C)$  and  $\text{Tors}_{G \times_C C[\epsilon]}(C[\epsilon])$  are isomorphic.

We come now to Lie algebras considerations. By definition of the Lie algebra, the  $C$ -group  $G'$  fits in a split exact sequence of  $C$ -group schemes

$$0 \rightarrow \mathbf{W}(\mathcal{L}ie(G)) \xrightarrow{e} G' \xrightarrow{\pi} G \rightarrow 1$$

where  $\mathcal{L}ie(G) = \omega_{G/S}^\vee$  (see §8.2 and Remark 8.4.(a)).

According to [25, III.3.2.1] we have an equivalence of groupoids between  $\text{Tors}_{\mathbf{W}(\mathcal{L}ie(G))}(C)$  and that of couples  $(E', \eta)$  where  $E'$  is a  $G'$ -torsor over  $C$  and  $\eta : G \xrightarrow{\sim} \pi_* E'$  is a trivialization. Taking into account the previous isomorphism of categories, we get then an equivalence of groupoids between  $\text{Tors}_{\mathbf{W}(\mathcal{L}ie(G))}(C)$  and that of couples  $(F, \xi)$  where  $F$  is a  $G$ -torsor over  $C[\epsilon]$  and  $\xi : G_C \rightarrow F \times_C C[\epsilon]$  is a trivialization; the morphisms are clear.

We come back now to the previous section involving a  $S$ -algebra  $R$  and the morphism  $b : \text{Spec}(R) \rightarrow \text{Bun}_G$  associated to the trivial  $G$ -torsor. By comparison it follows that the fiber category  $T_b\text{Bun}_G(R)$  is equivalent to  $\text{Tors}_{\mathbf{W}(\mathcal{L}ie(G))}(C)$ .

## 5. UNIFORMIZATION AND LOCAL TRIVIALITY

This section presents in a slightly more general manner than classical material on uniformization of  $G$ -bundles [1, 32, 33, 48].

**5.1. Loop groups.** We continue with the framework of the previous section and assume from now on that  $S = \mathrm{Spec}(R)$  is affine noetherian. We deal with a (proper flat) relative curve  $p : C \rightarrow \mathrm{Spec}(R)$ ; it satisfies étale locally the resolution property since it is locally  $H$ -projective [49, Tag 0E6F].

Let  $D$  be a finite flat  $S$ -scheme with a closed embedding  $s : D \rightarrow C$  such that

- (i) the complement  $C \setminus D$  is affine and will be denoted by  $V^D$ ;
- (ii)  $s$  factorizes through an affine  $R$ -subscheme  $V$  of  $C$ .

Note that (i) is satisfied if  $D$  is an effective Cartier divisor which is ample. Let  $V = \mathrm{Spec}(A)$ ,  $V^D = \mathrm{Spec}(A_D)$  and  $V \cap V^D = \mathrm{Spec}(A_{\sharp})$ ; this intersection is affine because the morphism  $C \rightarrow S$  is separated [49, Tag 01KP]. We denote by  $I \subset A$  the ideal defining  $D$ . We consider the completed ring  $\widehat{A} := \widehat{A}_I = \varprojlim_n A/I^n$ . We need some basic facts from commutative algebra (see [7, III.4.3, th. 3 and prop. 8] for (a) and (b)).

- (a)  $\widehat{A}$  is noetherian and flat over  $A$ .
- (b) The assignment  $\mathfrak{m} \mapsto \mathfrak{m}\widehat{A}$  provides a correspondence between the maximal ideals of  $A$  containing  $I$  and the maximal ideals of  $\widehat{A}$ ;
- (c) If  $R$  is semilocal so is  $\widehat{A}$ .

If  $R$  is local, the finite  $R$ -algebra  $A/I$  is semilocal so we get (c) from (b).

We recall that  $G$  is a smooth affine group scheme over  $C$ . We consider the following  $R$ -functors defined for each  $R$ -algebra  $B$  by:

- (1)  $L^+G(B) = G\left(\widehat{(A \otimes_R B)}_{I \otimes_R B}\right)$ ;
- (2)  $LG(B) = G\left(\widehat{(A \otimes_R B)}_{I \otimes_B \otimes_A A_{\sharp}}\right)$ .

**Example 5.1.** (a) The simplest example of our situation is  $C = \mathbb{P}_R^1 = V_{\infty} \cup V = \mathrm{Spec}(R[t]) \cup \mathrm{Spec}(R[t^{-1}])$  and for  $D$  the point 0 of  $C$ . In this case, we have  $A = R[t]$ ,  $I = tA$ ,  $A_{\sharp} = R[t, t^{-1}]$  and  $\widehat{A} = R[[t]]$ . For each  $R$ -algebra  $B$ , we have  $\widehat{(A \otimes_R B)}_{I \otimes_B} = B[[t]]$  and  $\widehat{(A \otimes_R B)}_{I \otimes_R B} \otimes_A A_{\sharp} = B[[t]] \otimes_{R[t]} R[t, t^{-1}] = B[[t]][\frac{1}{t}]$ . The standard notation for the last ring is  $B((t))$ .

**5.2. Patching.** For simplicity we assume that  $S = \text{Spec}(R)$  where  $R$  is a noetherian ring. If we are given an  $R$ -algebra  $B$  (not necessarily noetherian), we need to deal with the rings  $(\widehat{A \otimes_R B})_{I \otimes B}$  and  $(\widehat{A \otimes_R B})_{I \otimes B} \otimes_A A_\#$ . As pointed out by Bhatt [5, §1.3], the Beauville-Laszlo theorem [2] states that one can patch compatible quasi-coherent sheaves on  $\text{Spec}((\widehat{A \otimes_R B})_{I \otimes B})$  and  $V^D \times_R B$  to a quasi-coherent sheaf on  $C_B$ , provided the sheaves being patched are flat along  $\text{Spec}(B/IB)$ . In particular the square of functors

$$\begin{array}{ccc} \mathcal{C}(C_B) & \longrightarrow & \mathcal{C}\left((\widehat{A \otimes_R B})_{I \otimes B}\right) \\ \downarrow & & \downarrow \\ \mathcal{C}(V^D \times_R B) & \longrightarrow & \mathcal{C}\left((\widehat{A \otimes_R B})_{I \otimes B} \otimes_A A_\#\right) \end{array}$$

is cartesian where  $\mathcal{C}(X)$  stands for the category of flat quasi-coherent sheaves over the scheme  $X$  (resp. the category of flat affine schemes over  $X$ ). Note that if the ring  $B$  is noetherian, Ferrand-Raynaud's patching [20] (see also [40]) does the job.

**Proposition 5.2.** (1) *The square of functors*

$$\begin{array}{ccc} \text{Tors}_G(C_B) & \longrightarrow & \text{Tors}_G\left((\widehat{A \otimes_R B})_{I \otimes B}\right) \\ \downarrow & & \downarrow \\ \text{Tors}_G(V^D \times_R B) & \longrightarrow & \text{Tors}_G\left((\widehat{A \otimes_R B})_{I \otimes B} \otimes_A A_\#\right) \end{array}$$

*is cartesian.*

(2) *The  $S$ -functor  $LG$  is isomorphic to the functor associating to each  $R$ -algebra  $B$  the  $G$ -torsors over  $C_B$  together with trivializations on  $V^D \times_R B$  and on  $\text{Spec}((\widehat{A \otimes_R B})_{I \otimes B})$ .*

*Proof.* (1) Since  $G$  is affine and flat over  $C$ , it is a formal corollary of the patching statement.

(2) Let  $\mathcal{C}(B)$  be the the category of  $G$ -torsors over  $C_B$  together with trivializations on  $V^D \times_R B$  and on  $\text{Spec}((\widehat{A \otimes_R B})_{I \otimes B})$ . An object of  $\mathcal{C}(B)$  is a triple  $(E, f_1, f_2)$  where  $E$  is a  $G_{C_B}$ -torsor,  $f_1 : G_{V^D \times_R B} \xrightarrow{\sim} E_{V^D \times_R B}$  and  $f_2 : G_{(\widehat{A \otimes_R B})_{I \otimes B}} \xrightarrow{\sim} E_{(\widehat{A \otimes_R B})_{I \otimes B}}$  are trivializations. An element  $g \in LG(B) = G((\widehat{A \otimes_R B})_{I \otimes B} \otimes_A A_\#)$  gives rise to the right translation

$$(G_{V^D \times_R B})_{(\widehat{A \otimes_R B})_{I \otimes B} \otimes_A A_\#} \xrightarrow{\sim} (G_{(\widehat{A \otimes_R B})_{I \otimes B}})_{(\widehat{A \otimes_R B})_{I \otimes B} \otimes_A A_\#}.$$

It defines a  $G_C$ -torsor  $E_g$  with trivializations  $f_1$  and  $f_2$  on  $V^D \times_R B$  and on  $\text{Spec}((\widehat{A \otimes_R B})_{I \otimes B})$ . We get then a morphism  $\Phi : LG(B) \rightarrow \mathcal{C}(B)$ .

Conversely let  $c = (T, f_1, f_2)$  be an object of  $\mathcal{C}(B)$ . Then the map  $f_1^{-1}f_2 : G_{(\widehat{A \otimes_R B})_{I \otimes_B \otimes_A A_\sharp}} \rightarrow G_{(\widehat{A \otimes_R B})_{I \otimes_B \otimes_A A_\sharp}}$  is an isomorphism of  $G$ -torsors hence is the right translation by an element  $g = \Psi(c) \in LG(B)$ . The functors  $\Phi$  and  $\Psi$  provide the desired isomorphism of functors.  $\square$

Continuing with the  $R$ -algebra  $B$ , we have a factorization

$$\begin{array}{ccc} LG(B) & \xrightarrow{p} & \text{Bun}_G(B) \\ \downarrow & & \downarrow \text{class map} \\ c_G(B) := G(V^D \times_R B) \setminus LG(B) / L^+G(B) & \xrightarrow{p} & H^1(C_B, G). \end{array}$$

The map  $p$  is called the uniformization map. Proposition 5.2.(2) implies that the bottom map induces a bijection

$$(*) \quad c_G(B) \xrightarrow{\sim} \ker \left( H^1(C_B, G) \rightarrow H^1(V^D \times_R B, G) \times H^1((\widehat{A \otimes_R B})_{I \otimes_B \otimes_A A_\sharp}, G) \right).$$

**5.3. Link with the tangent space.** Our goal is to differentiate the mapping  $p : LG \rightarrow \text{Bun}_G$ . Let  $B$  be an  $R$ -algebra and consider the map

$$p[\epsilon] : LG(B[\epsilon]) \rightarrow \text{Bun}_G(B[\epsilon]).$$

We have  $(\widehat{A \otimes_R B}[\epsilon])_{I \otimes_B \otimes_A A_\sharp} = ((\widehat{A \otimes_R B})_{I \otimes_B \otimes_A A_\sharp})[\epsilon]$  so that  $LG(B[\epsilon]) = G(((\widehat{A \otimes_R B})_{I \otimes_B \otimes_A A_\sharp})[\epsilon])$ .

We consider the commutative diagram of categories<sup>1</sup>

$$\begin{array}{ccccc} 0 \longrightarrow \text{Lie}(G) \left( ((\widehat{A \otimes_R B})_{I \otimes_B \otimes_A A_\sharp}) \right) & \xrightarrow{e} & G \left( ((\widehat{A \otimes_R B})_{I \otimes_B \otimes_A A_\sharp})[\epsilon] \right) & \longrightarrow & G \left( ((\widehat{A \otimes_R B})_{I \otimes_B \otimes_A A_\sharp}) \right) \\ & & \downarrow \wr & & \downarrow \wr \\ & & LG(B[\epsilon]) & \xrightarrow{\quad} & LG(B) \\ & & \downarrow & & \downarrow \\ & & \text{Bun}_G(B[\epsilon]) & \xrightarrow{\quad} & \text{Bun}_G(B) \end{array}$$

where the first line is the exact sequence defining the Lie algebra. By considering the fiber at the trivial  $G$ -torsor  $b \in \text{Bun}_G(B)$ , we get then a functor

$$p \circ e : \text{Lie}(G) \left( (\widehat{A \otimes_R B})_{I \otimes_B \otimes_A A_\sharp} \right) \rightarrow T_b \text{Bun}_G(B).$$

Since  $\text{Lie}(G) \left( (\widehat{A \otimes_R B})_{I \otimes_B \otimes_A A_\sharp} \right) = \mathbf{W}(\mathcal{L}ie(G)) \left( (\widehat{A \otimes_R B})_{I \otimes_B \otimes_A A_\sharp} \right)$ , we have an  $R$ -functor

$$dp : \mathbf{LW}(\mathcal{L}ie(G)) \rightarrow T_b \text{Bun}_G.$$

We use now the equivalence of categories between  $T_b \text{Bun}_G(B)$  and  $\text{Tors}_{\mathbf{W}(\mathcal{L}ie(G))}(C_B)$  (cf. 4.3) and get the following compatibility with the classifying maps

<sup>1</sup>With the convention that a set defines a groupoid [49, Tag 001A].

$$\begin{array}{ccc}
 L\mathbf{W}(\mathcal{L}ie(G))(B) & \longrightarrow & \mathbf{W}(\mathcal{L}ie(G))(V^D \times_R B) \setminus L\mathbf{W}(\mathcal{L}ie(G))(B) / L^+\mathbf{W}(\mathcal{L}ie(G))(B) \\
 \downarrow dp & & \downarrow \\
 T_b\text{Bun}_G(B) & \xrightarrow{\text{class map}} & H^1(C_B, \mathbf{W}(\mathcal{L}ie(G))).
 \end{array}$$

We observe that the  $\mathbf{W}(\mathcal{L}ie(G))$ -torsors over affine schemes are trivial so that the top right map is an isomorphism according to the fact  $(*)$  above. Also  $H^1(C_B, \mathbf{W}(\mathcal{L}ie(G)))$  identifies with the coherent cohomology of the  $\mathcal{O}_S$ -module  $\mathcal{L}ie(G)$  [39, prop. III.3.7].

**5.4. Heinloth's section.** This statement is a variation over a local henselian noetherian base of a result due to Heinloth when the residue field is algebraically closed [32, cor. 8].

**Proposition 5.3.** *Assume that  $S = \text{Spec}(R)$  with  $R$  local noetherian henselian with residue field  $\kappa$ . We assume that  $G$  is semisimple and that its fundamental group is smooth over  $C$ . We assume that  $G_{D_\kappa}$  admits a strictly proper parabolic  $D_\kappa$ -subgroup  $Q$  such that  $n_{\kappa(s)}(G_{\kappa(s)}, Q_{\kappa(s)}) = 1$  for each point  $s \in D_\kappa$  with finite residue field.*

(1) *There exists a map  $F : \mathbf{A}_R^n \rightarrow LG$  such that the composite*

$$f : \mathbf{A}_R^n \xrightarrow{F} LG \xrightarrow{p} \text{Bun}_G$$

*is a map of stacks, maps  $0_R$  to the trivial  $G$ -torsor  $b$  and such that*

$$df_{0_R} : R^n \rightarrow T_b\text{Bun}_G(R)$$

*is essentially surjective. Furthermore there exists a neighborhood  $\mathcal{N}$  of  $0_\kappa$  in  $\mathbf{A}_R^n$  such that  $f|_{\mathcal{N}}$  is smooth.*

(2) *Let  $E$  be a  $G$ -bundle over  $C$  such that  $E \times_C C_\kappa$  is trivial. Then  $E$  is trivial on  $V^D$ .*

*Proof.* (1) The proof goes by a differential argument. The  $R$ -module  $H^1(C, \text{Lie}(G))$  is finitely generated over  $R$  [31, III.5.2] and we lift a generating family of  $H^1(C, \text{Lie}(G))$  to a family of elements  $Y_1, \dots, Y_r$  of  $\text{Lie}(G) \left( \widehat{A} \otimes_A A_{\frac{1}{4}} \right)$ . We have noticed that  $\widehat{A}$  is a semilocal noetherian ring (Ex. 5.1.(b)). We want now to apply Lemma 3.5 to  $G_{\widehat{A}}$  with respect to the closed points of  $\text{Spec}(\widehat{A})$ . Let  $Z$  be the  $C$ -scheme of strictly parabolic subgroups of  $G$  [45, XXVI.3]. It is a smooth proper  $C$ -scheme. It follows that its Weil restriction  $\prod_{D/R}(Z_D)$  is smooth so that Hensel's lemma shows that  $Q$  lifts to a strictly proper parabolic  $D$ -subgroup scheme  $P_D$  of  $G_D$ . Similarly  $Z(\widehat{A}) \rightarrow Z(A/I) = Z(D)$  is onto so that  $P_D$  lifts a strictly proper parabolic  $\widehat{A}$ -subgroup scheme  $P$  of  $G_{\widehat{A}}$ . We put  $U = \text{rad}_u(P)$ , it is a smooth affine  $\widehat{A}$ -group scheme and we denote by  $U_{\text{last}}$  its  $R$ -subgroup scheme constructed in §3.3.

Lemma 3.5 provides elements  $g_1, \dots, g_m \in E_P(\widehat{A})$  such that the product map

$$h : (U_{\text{last}})^m \rightarrow G, \quad (u_1, \dots, u_m) \mapsto g^1 u_1 \dots g^m u_m$$

induces a surjective differential

$$dh : \mathrm{Lie}(U_{last})^m(\widehat{A}) \rightarrow \mathrm{Lie}(G)(\widehat{A}), \quad (X_1, \dots, X_m) \mapsto {}^{g^1}X_1 + \dots + {}^{g^m}X_m.$$

In other words we have

$$\mathrm{Lie}(G)(\widehat{A}) = {}^{g^1}\mathrm{Lie}(U_{last})(\widehat{A}) + {}^{g^2}\mathrm{Lie}(U_{last})(\widehat{A}) + \dots + {}^{g^m}\mathrm{Lie}(U_{last})(\widehat{A})$$

so that (using the identity of Lemma 8.3.(2))

$$\begin{aligned} & \mathrm{Lie}(G)(\widehat{A} \otimes_A A_{\sharp}) = \mathrm{Lie}(G)(\widehat{A}) \otimes_A A_{\sharp} \\ = & {}^{g^1}\mathrm{Lie}(U_{last})(\widehat{A}) \otimes_A A_{\sharp} + {}^{g^2}\mathrm{Lie}(U_{last})(\widehat{A}) \otimes_A A_{\sharp} + \dots + {}^{g^m}\mathrm{Lie}(U_{last})(\widehat{A}) \otimes_A A_{\sharp}. \end{aligned}$$

We can write

$$(**) \quad Y_i = \sum_{j=1, \dots, m} c_{i,j} {}^{g^j}Z_{i,j}$$

where  $Z_{i,j} \in \mathrm{Lie}(U_{last})(\widehat{A})$  and  $c_{i,j} \in \widehat{A} \otimes_A A_{\sharp}$  for each  $j$ .

Since  $U_{last}$  is an  $\widehat{A}$ -vector group scheme, there is a canonical identification  $\exp : \mathbf{W}(\mathcal{L}ie(U_{last})) \xrightarrow{\sim} U_{last}$ ,  $X \mapsto \exp(X)$ . We consider the polynomial ring  $B = R[t_{i,j}]$  where  $i = 1, \dots, r, j = 1, \dots, m$ . We consider the map of  $R$ -functors  $F : \mathbf{A}_R^{rm} \rightarrow LG$  defined by the element

$$\prod_{i=1, \dots, r; j=1, \dots, m} {}^{g^i}\exp\left(t_{i,j} c_{i,j} Z_{i,j}\right) \in G\left(\widehat{(A \otimes_R B)}_{I \otimes_R B} \otimes_A A_{\sharp}\right) = LG(B)$$

where we can take for example the lexicographic order. It induces an  $R$ -map  $f : \mathbf{A}_R^{rm} \rightarrow \mathrm{Bun}_G$  of stacks mapping  $0_R$  to the trivial  $G$ -bundle. Taking into account the last compatibility of §5.3, its differential at  $0_R$

$$df : R^{rm} \rightarrow T_b\mathrm{Bun}_G(R)$$

factorizes through  $L\mathbf{W}(\mathcal{L}ie(G))(B)$ . More precisely we have a commutative diagram

$$\begin{array}{ccccc} R^{rm} & \xrightarrow{h} & L\mathbf{W}(\mathcal{L}ie(G))(R) & \longrightarrow & H^1(C, \mathrm{Lie}(G)) \\ & & \searrow dp & & \uparrow \\ & & & & \text{class map} \\ & & & & \uparrow \\ T_b\mathrm{Bun}_G(R) & \xrightarrow{df_0} & & & \end{array}$$

where  $h$  maps the basis element  $e_{i,j} \in R^{rm}$  to  $c_{i,j} {}^{g^j}Z_{i,j}$  in  $L\mathbf{W}(\mathcal{L}ie(G))(R)$ . We take into account the identity (\*\*). By  $R$ -linearity, the image of  $R^{rm} \rightarrow H^1(C, \mathrm{Lie}(G))$  contains all  $Y_i$ 's. Since the  $Y_i$ 's generate the  $R$ -module  $H^1(C, \mathrm{Lie}(G))$ , we conclude that  $df_0$  is essentially surjective.

The formation of  $R^1p_*\mathcal{L}ie(G)$  commutes with base change, we have an isomorphism  $H^1(C, \mathcal{L}ie(G)) \otimes_R \kappa \xrightarrow{\sim} H^1(C_0, \mathcal{L}ie(G))$  so that  $df_{0,k} : k^{rm} \rightarrow H^1(C_0, \mathrm{Lie}(G))$  is onto as well.

It follows that  $f$  is smooth locally at  $0_{\kappa}$  according to the Jacobian smoothness criterion 8.1 stated in the appendix. Thus there is  $\mathcal{N}$  as claimed in the statement.

(2) We see  $E$  as an object of  $\text{Bun}_G(R)$  and consider the fiber product

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{f} & \text{Bun}_G \\ \uparrow & & \uparrow E \\ Y & \xrightarrow{\pi} & \text{Spec}(R). \end{array}$$

Then  $Y$  is an  $R$ -algebraic space [43, 8.16] which is smooth over  $R$ . Let  $Q$  be the  $G$ -torsor over  $C_{\mathcal{N}}$  defined by  $f : \mathcal{N} \rightarrow \text{Bun}_G$ . The algebraic space  $Y$  is representable by the  $\mathcal{N}$ -scheme  $\text{Isom}^b(Q, E_{\mathcal{N}})$  defined in Lemma 4.2. Hensel's Lemma shows that  $Y(R) \neq \emptyset$ . It follows that there exists  $u \in \mathcal{N}(R)$  which maps to  $E$ . Since the map  $\mathcal{N} \rightarrow \text{Bun}_G$  factorizes through  $LG$ , we conclude that the  $G$ -torsor  $E$  is trivial on  $V^D$ .  $\square$

## 6. PROOF OF THE MAIN RESULT

We need the following consequence of Poonen's result [44] and its refinement by Moret-Bailly [41].

**Proposition 6.1.** *Let  $F$  be a field. Let  $Y$  be an irreducible  $F$ -scheme of finite type of positive dimension. We denote by  $Y_0^{\text{sep}}$  the set of separable closed points of  $Y$ . Let  $X$  be an  $F$ -scheme of finite type and let  $f : X \rightarrow Y$  be a smooth surjective  $F$ -morphism. Then*

$$\Sigma_f(Y) = \left\{ y \in Y_0 \mid X_y(F(y)) \neq \emptyset \right\}$$

*is Zariski dense in  $Y$ . If furthermore  $Y$  is  $F$ -smooth, then the set*

$$\Sigma_f^{\text{sep}}(Y) = \left\{ y \in Y_0^{\text{sep}} \mid X_y(F(y)) \neq \emptyset \right\}$$

*is Zariski dense in  $Y$ .*

*Proof.* Since shrinking is allowed, this reduces to show that the set  $\Sigma_f(Y)$  is non-empty which is [44, remark p. 225]. The second fact uses the refinement of [41, page 472].  $\square$

**Lemma 6.2.** *Let  $F$  be field. Let  $X$  be an irreducible  $F$ -scheme of finite type and of dimension  $\geq 1$ . Let  $X_0^{\text{sep}}$  be the set of separable closed points of  $X$ . Let  $H$  be a semisimple  $X$ -group scheme. Then the set*

$$X(s) = \left\{ x \in X_0 \mid H_{F(x)} \text{ is split} \right\}$$

*is Zariski dense in  $X$ . If furthermore,  $X$  is smooth, then the set*

$$X^{\text{sep}}(s) = \left\{ x \in X_0^{\text{sep}} \mid H_{F(x)} \text{ is split} \right\}$$

*is Zariski dense in  $X$ .*

*Proof.* We consider the  $X$ -scheme  $Y = \text{Isom}(H_0, H)$  (where  $H_0$  is the Chevalley form of  $H$ ) which is affine smooth over  $X$  [45, XXIV.1.9]. Proposition 6.1 applied to  $Y \rightarrow X$  yields that  $X(s)$  is dense in  $X$ .  $\square$

We can proceed to the proof of Theorem 1.1.

*Proof.* We are given two  $G$ -torsors over  $C$  such that  $E \times_C C_\kappa$  is isomorphic to  $E' \times_C C_\kappa$ . Up to consider the twisted  $R$ -group scheme  ${}^{E'}G$ , we can assume that  $E' = G$  without loss of generality. Let  $\Theta$  be the set of irreducible components of  $C_\kappa$  and denote by  $C_\kappa^\theta$  the component attached to  $\theta \in \Theta$ .

*Case (I).* Since each  $C^{sm} \cap C_\kappa^\theta$  is smooth and nonempty, Lemma 6.2 provides two fully distinct families of closed separable points  $(c_1^\theta)_{\theta \in \Theta}$  and  $(c_2^\theta)_{\theta \in \Theta}$  of  $C^{sm} \cap C_\kappa^\theta$  such that  $G_{c_i^\theta}$  is a split semisimple  $\kappa(c_i^\theta)$ -group for  $i = 1, 2$  and each  $\theta \in \Theta$ . Let  $Q_i^\theta$  be a  $\kappa(c_i^\theta)$ -Borel subgroup of  $G_{c_i^\theta}$  for each  $\theta$  and  $i = 1, 2$ . Taking into account Lemma 3.3.(1), we have  $n_{\kappa(s)}(G_{\kappa(c_i^\theta)}, Q_{\kappa(c_i^\theta)}) = 1$  if  $\kappa$  is finite (that is, there are enough long root elements). Since  $R$  is henselian there exists finite étale extensions  $R_i^\theta$  of  $R$  which lifts  $\kappa(c_i^\theta)/\kappa$  and  $R_i^\theta$  is henselian as well [49, Tag 04GH]. Since  $C^{sm}$  is smooth over  $R$ , Hensel's lemma applies to  $C^{sm} \times_R R_i^\theta$  shows that each  $c_i^\theta$  lifts in a closed  $R$ -subscheme  $D_i^\theta \rightarrow C$  which is finite étale over  $R$ . We put  $D_i = \bigsqcup_{\theta \in \Theta} D_i^\theta$  for  $i = 1, 2$ .

Since  $C$  is projective over  $R$  and  $D_i$  is semilocal,  $D_i$  is a closed  $R$ -subscheme of an affine open  $R$ -subscheme of  $C$  and  $D_i$  is finite étale over  $R$ . For each point  $s \in \text{Spec}(R)$ ,  $D_{i,s}$  consists of smooth points of  $C_s$  so is an effective Cartier divisor, hence  $D_i$  is a relative Cartier divisor for  $C/R$  [49, end of Tag 062Y]. By construction,  $D_{i,\kappa}$  has positive degree on each irreducible component of  $C_\kappa$  so is ample [38, §7.5, prop. 5.5] so that  $D_i$  is ample [17, 4.7.1] and hence  $C \setminus D_i$  is affine for  $i = 1, 2$ .

We claim that the scheme  $C \setminus D_i$  is affine for  $i = 1, 2$ . The group  $G \times D_{i,\kappa}$  admits a Borel subgroup (resp. is split) for  $i = 1, 2$ . Now let  $E$  be a  $G$ -torsor over  $C$  such that  $E \times_C C_\kappa$  is trivial. Proposition 5.3.(2) shows that  $E|_{C \setminus D_i}$  is trivial for  $i = 1, 2$ . Since  $C = (C \setminus D_1) \cup (C \setminus D_2)$ , we conclude that the  $G$ -torsor  $E$  is locally trivial for the Zariski topology.

*Case (II).* In this case  $R$  is a henselian DVR. Lemma 6.2 provides two fully distinct families of closed points  $(c_1^\theta)_{\theta \in \Theta}$  and  $(c_2^\theta)_{\theta \in \Theta}$  of  $C_\kappa$  such that  $G_{c_i^\theta}$  is a split semisimple  $\kappa(c_i^\theta)$ -group for  $i = 1, 2$  and for each  $\theta \in \Theta$ .

We use now that there is closed  $R$ -embedding  $i : C \rightarrow \mathbb{P}_R^N$  [49, Tag 0E6F]. We note that  $C \rightarrow \text{Spec}(R)$  is a regular fibered surface in the sense of Liu's book [38]. According to [38, §8.3, lemma 3.35], there exists an effective Cartier (equivalently Weil) "horizontal" divisor  $D_i^\theta$  of  $C$  such that  $C_\kappa \cap \text{Supp}(D_i^\theta) = c_i^\theta$  for each  $\theta \in \Theta$  and  $i = 1, 2$  (note it is finite flat over  $S$ ). We consider the effective Cartier divisors  $D_i = \bigsqcup_{\theta \in \Theta} D_i^\theta$  for  $i = 1, 2$ ;  $D_i$  is finite flat over  $S$ . According to [49, Tag 056Q],  $D_i$  is a relative Cartier divisor on  $C$  and  $D_{i,\kappa}$  is an effective Cartier divisor of  $C_\kappa$ . Since  $G_{\kappa(c_i)}$  is split, we have that  $G \times D_{i,\kappa}$  is split for  $i = 1, 2$  by using the smoothness of

the scheme  $\text{Isom}(G_0, G)$  (where  $G_0$  is the Chevalley form of  $G$ ). Repeating verbatim the argument of Case (I) finishes the proof.  $\square$

**Remark 6.3.** In the proof of (1), an important step is the construction of the divisor  $D$  such that  $D$  is finite étale over  $R$  and  $G_D$  is split. Though our construction is quite different, a similar argument has been used by Panin and Fedorov in their proof of Grothendieck-Serre's conjecture [19, prop. 4.1].

## 7. EXTENSION TO REDUCTIVE GROUPS

We gather here our results in a single long statement.

**Theorem 7.1.** *Assume that  $R$  is local henselian noetherian of residue field  $\kappa$ . Let  $p$  be the characteristic exponent of  $\kappa$ . Let  $f : C \rightarrow \text{Spec}(R)$  be a relative curve of relative dimension 1 and denote by  $C^{sm}$  the smooth locus of  $f$ . We assume that one of the following holds:*

- (I)  $C_\kappa^{sm}$  is dense in  $C_\kappa$ ;
- (II)  $R$  is a DVR and  $C$  is integral regular.

Let  $G$  be a reductive  $C$ -group scheme and consider its presentation [45, XXII.6.2.3]

$$1 \rightarrow \mu \rightarrow G^{sc} \times_C \text{rad}(G) \rightarrow G \rightarrow 1,$$

where  $\text{rad}(G)$  is the radical  $C$ -torus of  $G$  and  $G^{sc}$  is the simply connected universal cover of  $DG$ . We assume that

- (i)  $\mu$  is étale over  $C$ ;
- (ii) the  $C$ -torus  $\text{rad}(G)$  is quasi-split by a finite étale extension  $\tilde{C}/C$  of degree prime to  $p$ .

Let  $E, E'$  be two  $G$ -torsors over  $C$  such that  $E \times_C C_\kappa$  is isomorphic to  $E' \times_C C_\kappa$ . Then  $E$  and  $E'$  are locally isomorphic for the Zariski topology.

*Proof.* Once again we can assume that  $E' = G$ . We consider the following commutative diagram

$$\begin{array}{ccccccc} H^1(C, \mu) & \longrightarrow & H^1(C, G^{sc}) \times H^1(C, T) & \longrightarrow & H^1(C, G) & \longrightarrow & H^2(C, \mu) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(C_\kappa, \mu) & \longrightarrow & H^1(C_\kappa, G^{sc}) \times H^1(C_\kappa, T) & \longrightarrow & H^1(C_\kappa, G) & \longrightarrow & H^2(C_\kappa, \mu), \end{array}$$

where the horizontal lines are exact sequences of pointed sets. On the other hand, the proper base change theorem for étale cohomology shows that the maps  $H^i(C, \mu) \rightarrow H^i(C_\kappa, \mu)$  are bijective for  $i = 1, 2$  [46, XII.5.5.(iii)]. By diagram chase, it follows that the map

$$\begin{aligned} & \ker(H^1(C, G^{sc}) \rightarrow H^1(C_\kappa, G^{sc})) \times \ker(H^1(C, T) \rightarrow H^1(C_\kappa, T_\kappa)) \\ & \rightarrow \ker(H^1(C, G) \rightarrow H^1(C_\kappa, G)) \end{aligned}$$

is onto. The first kernel (resp. the second one) consists of Zariski locally trivial according to Theorem 1.1 (resp. Proposition 2.2.(3) and Theorem 2.4). Thus the third kernel consists of Zariski locally trivial torsors.  $\square$

We have the following refinement of Theorem 7.1 which answers a question of Olivier Benoist.

**Theorem 7.2.** *Assume that  $R$  is local henselian noetherian of residue field  $\kappa$ . Let  $f : C \rightarrow \text{Spec}(R)$  be a smooth relative curve of relative dimension 1. Let  $G$  be a reductive  $C$ -group scheme which satisfies the same assumption as in Theorem 7.1.*

*Let  $E, E'$  be  $G$ -torsors over  $C$ . Then the following are equivalent:*

- (i) *The  $G$ -torsors  $E, E'$  are locally isomorphic for the Zariski topology;*
- (ii) *The  $G_{C_\kappa}$ -torsors  $E_{C_\kappa}, E'_{C_\kappa}$  are locally isomorphic for the Zariski topology.*

The proof of Theorem 7.2 involves the Iwasawa decomposition; since the reference [Br-T, cor. 7.3.2.(ii)] is for the semisimple case, we provide a short proof for the reductive case.

**Lemma 7.3.** *Let  $O$  be a henselian DVR of fraction field  $K$ . Let  $H$  be an  $O$ -reductive group scheme and  $P$  a minimal  $O$ -parabolic subgroup of  $H$ . Let  $P = U \rtimes L$  be a Levi decomposition (it exists according to [45, XXVI.2.3]) and let  $S$  be the maximal split central  $O$ -torus of  $L$ . Then we have an isomorphism*

$$S(K)/S(O) \xrightarrow{\sim} U(K) \backslash H(K) / H(O).$$

*Proof.* Let  $\mathfrak{X} \cong H/P$  be the  $O$ -scheme of parabolic subgroups of  $H$  with same type as  $P$  [45, §XXVI.3]. Since  $\mathfrak{X}$  is a projective  $O$ -scheme, we have  $\mathfrak{X}(O) = \mathfrak{X}(K)$ . Since  $H(O)$  (resp.  $H(K)$ ) acts simply transitively on  $\mathfrak{X}(O)$  (resp.  $\mathfrak{X}(K)$ ) according to [45, XXVI.2.5], we get that  $H(O)/P(O) = H(K)/P(K)$ . It follows that

$$(7.1) \quad H(K) = P(K)H(O) = U(K)L(K)H(O).$$

Since  $L/S$  is  $O$ -anisotropic, it is  $K$ -anisotropic [26, 3.4]. We have then  $(L/S)(O) = (L/S)(K)$  according to the Bruhat-Tits-Rousseau's theorem [26, 3.5]. Hilbert 90 theorem states that  $H^1(O, S) = H^1(K, S) = 0$  hence  $L(O)/S(O) = L(K)/S(K)$ . It follows that  $L(K) = S(K)L(O)$ ; by taking into account the identity (7.1) we obtain  $H(K) = U(K)S(K)H(O)$ . We have proven that the map

$$S(K)/S(O) \rightarrow U(K) \backslash H(K) / H(O)$$

is onto. For establishing the injectivity we are given  $s, s' \in S(K)$  such that  $s' = ush$  with  $u \in H(K), h \in H(O)$ . We consider the  $O$ -subgroup scheme  $M = U \rtimes S$  of  $P$ . Then  $h = u_0 s_0 \in M(O) = U(O) \rtimes S(O)$  so that  $s' = u s u_0 s_0$ . We conclude that  $s' = s s_0$ .  $\square$

We can now proceed to the proof of Theorem 7.2.

*Proof.* Once again we can assume that  $E' = G$ . The implication (i)  $\implies$  (ii) is obvious. Conversely we assume that  $E_{C_\kappa}$  is locally trivial for the Zariski topology. Then there exists a divisor  $D_\kappa = \{c_1, \dots, c_t\}$  of  $C_\kappa$  such that the  $G$ -torsors  $E_{V_\kappa^D}$  and  $E_{D_\kappa}$  are trivial. We denote by  $\kappa_i = \kappa(c_i)$  the residue field, it is finite separable over  $\kappa$ . Let  $R_i$  be the finite étale cover of  $R$  which lifts  $\kappa_i/\kappa$ ; Hensel's lemma applies to  $C \times_R R_i$  shows that each  $c_i$  lifts in a closed  $R$ -subscheme  $D_i \rightarrow C$  which is finite étale over  $R$ . We put  $D = D_1 \sqcup D_2 \cdots \sqcup D_t$ , choose  $V = \text{Spec}(A)$  containing  $D$ , as in §5.

**Claim 7.4.** *The  $G$ -torsor  $E_{C_\kappa}$  extends to a  $G$ -torsor  $F$  whose restrictions to  $V^D$  and  $D$  are trivial.*

We postpone the proof of the Claim. Assuming the Claim, the  $G$ -torsors  $E$  and  $F$  are isomorphic on  $C_\kappa$ . Theorem 7.1 shows that  $E$  and  $F$  are locally isomorphic for the Zariski topology. It follows that  $E_{V^D}$  is locally trivial. By varying the choices of  $D$ , we can find a cover of  $C$  by open subsets  $V'_1, \dots, V'_r$  such that  $E_{V'_i}$  is locally trivial for  $i = 1, \dots, r$ . Thus  $E$  is locally trivial for the Zariski topology. For proving the Claim, we use the uniformization bijections

$$\begin{array}{ccc} G(V^D) \backslash L_G(R) / L_G^+(R) & \xrightarrow{\sim} & \ker \left( H^1(C, G) \rightarrow H^1(V^D, G) \times H^1(\widehat{A}, G) \right) \\ \downarrow & & \downarrow \\ G(V_\kappa^D) \backslash L_G(\kappa) / L_G^+(\kappa) & \xrightarrow{\sim} & \ker \left( H^1(C_\kappa, G) \rightarrow H^1(V_\kappa^D, G) \times H^1(\widehat{A}_\kappa, G) \right). \end{array}$$

using the notations of §5. The  $G_{C_\kappa}$ -torsor  $E_\kappa$  arises then from an element  $g \in L_G(\kappa)$ . We shall show that the map  $L_G(R) \rightarrow L_G(\kappa) / L_G^+(\kappa)$  is onto, that is,

$$\phi : G(\widehat{A} \otimes_A A_\#) \rightarrow G(\widehat{A}_\kappa \otimes_A A_\#) / G(\widehat{A}_\kappa)$$

is onto (which implies Claim 7.4). According to [52, 24.19], we have a decomposition  $\widehat{A} = \widehat{A}_1 \times \cdots \times \widehat{A}_t$  in complete local rings where  $\widehat{A}_i$  has residue field  $\kappa_i$ . We have a compatible decomposition  $\widehat{A}_\kappa = \mathcal{A}_1 \times \cdots \times \mathcal{A}_t$  where  $\mathcal{A}_i$  is a complete DVR of residue field  $\kappa_i$ . Since the map  $\widehat{A} \rightarrow \widehat{A}_\kappa$  is onto (by the Mittag-Leffler's condition), it follows that each map  $\widehat{A}_i \rightarrow \mathcal{A}_i$  is onto. We are then reduced to show that each map

$$\phi_i : G(\widehat{A}_i \otimes_A A_\#) \rightarrow G(\mathcal{A}_i \otimes_A A_\#) / G(\mathcal{A}_i)$$

is onto. We fix an index  $i$ . Let  $\mathcal{F}_i$  be the fraction field of  $\mathcal{A}_i$ , we have  $\mathcal{A}_i \otimes_A A_\# = \mathcal{F}_i$ . The quotient  $G(\mathcal{F}_i) / G(\mathcal{A}_i)$  is described by the Iwasawa decomposition.

Let  $Q_i$  be a minimal parabolic  $\kappa_i$ -subgroup of  $G_{\kappa_i}$ , it lifts to an  $\widehat{A}_i$ -parabolic  $P_i$  subgroup of  $G_{\widehat{A}_i}$ . We have  $P_i = U_i \rtimes L_i$  where  $U_i$  is the unipotent radical of  $P_i$  and  $L_i$  is a Levi subgroup. Let  $S_i \subset L_i$  be the maximal  $\widehat{A}_i$ -split central torus of [45, XXVI.7.8]; then  $S_{i, \kappa_i}$  is a maximal  $\kappa_i$ -torus of  $G_{\kappa_i}$  and  $S_{i, \mathcal{F}_i}$  is a maximal  $\mathcal{F}_i$ -torus of  $G_{\mathcal{F}_i}$ . We write the Iwasawa decomposition (Lemma 7.3)

$$S_i(\mathcal{F}_i) / S_i(\mathcal{A}_i) \xrightarrow{\sim} U_i(\mathcal{F}_i) \backslash G(\mathcal{F}_i) / G(\mathcal{A}_i).$$

In particular,  $U_i(\mathcal{F}_i) S_i(\mathcal{F}_i)$  maps onto  $G(\mathcal{F}_i)/G(\mathcal{A}_i)$ . We consider the commutative diagram

$$\begin{array}{ccc} U_i(\widehat{A}_i \otimes_A A_{\#}) S_i(\widehat{A}_i \otimes_A A_{\#}) & \longrightarrow & U_i(\mathcal{F}_i) S_i(\mathcal{F}_i) \\ \downarrow & & \downarrow \\ G(\widehat{A}_i \otimes_A A_{\#})/G(\widehat{A}_i) & \xrightarrow{\phi_i} & G(\mathcal{F}_i)/G(\mathcal{A}_i). \end{array}$$

The surjectivity of  $\phi_i$  follows of the next

**Claim 7.5.** *The maps  $U_i(\widehat{A}_i \otimes_A A_{\#}) \rightarrow U_i(\mathcal{F}_i)$  and  $S_i(\widehat{A}_i \otimes_A A_{\#}) \rightarrow S_i(\mathcal{F}_i)$  are onto.*

Since the map  $\widehat{A}_i \rightarrow \mathcal{A}_i$  is onto it follows that the map  $\widehat{A}_i \otimes_A A_{\#} \rightarrow \mathcal{A}_i \otimes_{A_{\kappa}} A_{\#} = \mathcal{F}_i$  is onto. According to [45, XXVI.1.12],  $U_i$  is isomorphic (as  $\widehat{A}_i$ -scheme) to a vector group scheme so that  $U_i(\widehat{A}_i \otimes_A A_{\#}) \rightarrow U_i(\mathcal{F}_i)$  is onto.

Since  $S_i$  is a split torus, it is enough to check the surjectivity of  $(\widehat{A}_i \otimes_A A_{\#})^{\times} \rightarrow \mathcal{F}_i^{\times}$ . Since  $\widehat{A}_i$  is a regular local ring, the divisor  $D_{\widehat{A}_i}$  is principal, i.e.  $D_{\widehat{A}_i} = \text{div}(\pi_i)$  with  $\pi_i \in \widehat{A}_i$ . It follows that  $\widehat{A}_i \otimes_A A_{\#} = \widehat{A}_i \left[ \frac{1}{\pi_i} \right]$  and  $\mathcal{F}_i = \mathcal{A}_i \left[ \frac{1}{\bar{\pi}_i} \right]$  where  $\bar{\pi}_i$  is the image of  $\pi_i$  in  $\mathcal{A}_i$ .

An element of  $\mathcal{F}_i^{\times}$  is of the shape  $\bar{\pi}_i^r (a_0 + a_1 \bar{\pi}_i + a_2 \bar{\pi}_i^2 + \dots)$  with  $a_0 \in \mathcal{A}_i^{\times}$ ,  $a_1, a_2, \dots \in \mathcal{A}_i$ . It is lifted to  $\left( \widehat{A}_i \left[ \frac{1}{\pi_i} \right] \right)^{\times}$  by the element  $\pi_i^r (\tilde{a}_0 + \tilde{a}_1 \pi_i + \tilde{a}_2 \pi_i^2 + \dots)$  where the  $\tilde{a}_i$ 's lift the  $a_i$ 's. Claim 7.5 is established and ends the proof.  $\square$

**Remarks 7.6.** (a) In case (II), Claim 7.4 holds provided  $D_{\kappa}$  is contained in the smooth locus of  $C_{\kappa}$ .

(b) In the case  $G$  is semisimple simply connected and  $k$  is algebraically closed it is well-known that  $G$ -torsors over  $C_{\kappa}$  are locally trivial for the Zariski topology. Theorem 7.2 provides then an alternative proof of Drinfeld-Simpson's theorem in this case.

**Corollary 7.7.** *Assume that  $R$  is a henselian DVR with finite residue field  $\kappa$ . Let  $f : C \rightarrow \text{Spec}(R)$  be a smooth proper curve. Let  $G$  be a semisimple simply connected  $C$ -group scheme. Then  $H_{\text{Zar}}^1(C, G) = H^1(C, G)$ .*

*Proof.* By a theorem of Harder [30], we have  $H^1(\kappa(C_{\kappa}^{\theta}), G) = 1$  for each connected component  $C_{\kappa}^{\theta}$  of  $C_{\kappa}$ . Nisnevich's theorem [42] (see also [26]) shows that  $H_{\text{Zar}}^1(C_{\kappa}, G) = H^1(C_{\kappa}, G)$ . Thus the corollary follows from Theorem 7.2.  $\square$

**Corollary 7.8.** *Assume that  $R$  is an henselian DVR with residue field  $\kappa$ . Let  $f : C \rightarrow \text{Spec}(R)$  be a smooth projective curve such that its generic fiber is connected. Let  $G$  be a reductive  $C$ -group scheme which satisfies the same assumption as in Theorem 7.1. Let  $F$  be the function field of  $C$ . Then, the local-global principle holds for  $G$ -torsors over  $F$  with respect to all discrete valuations arising from codimension one points of  $C$ .*

*Proof.* Let  $\xi \in H^1(F, G)$  which is trivial over  $F_v$  for all completions at discrete valuations arising from codimension one points of  $C$ . By glueing [24, cor. A.8], there is an element  $\zeta \in H^1(U, G)$  which maps to  $\xi$  over  $F$  where  $U \subset C$  contains all points of codimension 1. According to [10, th. 6.13], we have  $H^1(C, G) = H^1(U, G)$  so that we can assume that  $U = X$ . Since  $\xi$  is trivial over the completion  $F_w$  of  $F$  at the discrete valuation  $w$  associated to the special fiber of  $C$ , we claim that the specialisation  $\zeta_\kappa \in H^1(C_\kappa, G)$  of  $\zeta$  is generically trivial. This follows from the fact that  $H^1(O_w, G)$  injects in  $H^1(F_w, G)$  due to Bruhat-Tits (see [26, Th. 5.1]). According to Nisnevich's theorem [42] (see also [26]) this class is Zariski locally trivial on  $C_\kappa$ . Theorem 7.2 enables us to conclude that  $\eta$  is Zariski locally trivial and hence  $\xi$  is trivial.  $\square$

**Remarks 7.9.** (a) The only case where we knew that local-global principle for  $G$  simply connected group defined over  $C$  (for arbitrary residue fields) is when  $G_F$  is  $F$ -rational. In this case, Harbater, Hartmann and Krashen established their ‘‘patching local-global principle’’ [28, th. 3.7] which implies our local-global principle according to [9, Th. 4.2.(ii)].

(b) The special case when  $R$  is the ring of integers of a  $p$ -adic field was already known [9, Th. 4.8].

## 8. APPENDICES

The purpose of this appendix is to provide proofs to statements for algebraic spaces and stacks which are well-known among experts.

**8.1. Jacobian criterion for stacks.** Let  $S$  be a scheme and let  $\mathcal{X}, \mathcal{Y}$  be quasi-separated algebraic  $S$ -stacks of finite presentation. Let  $g : \mathcal{X} \rightarrow \mathcal{Y}$  be a 1-morphism over  $S$ . We have a 1-morphism  $Tg : T(\mathcal{X}) \rightarrow T(\mathcal{Y})$  of algebraic stacks [37, 17.14, 17.16].

Let  $s \in S$  and denote by  $K$  the residue field of  $s$ . Let  $x : \text{Spec}(K) \rightarrow \mathcal{X}$  be a 1-morphism mapping to  $s$ . We put  $T(\mathcal{X})_x = T(\mathcal{X}/S) \times_{\mathcal{X}} \text{Spec}(K)$  and denote by  $\text{Tan}_x(\mathcal{X})$  the category  $T(\mathcal{X})_x(K)$ . We denote by  $y = g \circ x : \text{Spec}(K) \rightarrow \mathcal{Y}$  and get the tangent morphism  $(Tg)_x : \text{Tan}_x(\mathcal{X}) \rightarrow \text{Tan}_y(\mathcal{Y})$ .

**Proposition 8.1.** *We assume that  $\mathcal{X}$  is smooth at  $x$  over  $S$ . Then the following assertions are equivalent:*

- (i) *The morphism  $g$  is smooth at  $x$ ;*
- (ii) *The tangent morphism  $(Tg)_x : \mathrm{Tan}_x(\mathcal{X}) \rightarrow \mathrm{Tan}_y(\mathcal{Y})$  is essentially surjective.*

*Furthermore, under those conditions,  $\mathcal{Y}$  is smooth at  $y$  over  $S$ .*

*Proof.* In the case of a morphism  $g : X \rightarrow Y$  of  $S$ -schemes locally of finite presentation such that  $g(x) = y$  and  $X$  is smooth at  $x$  over  $S$ , we have that  $K = \kappa(x) = \kappa(y)$  so that the statement is a special case of [18, 17.11.1]. We proceed now to the stack case.

(i)  $\implies$  (ii). Up to shrinking, we can assume that  $\mathcal{X}$  is smooth over  $S$  and that  $g$  is smooth. We denote by  $i : \mathrm{Spec}(K) \rightarrow \mathrm{Spec}(K[\epsilon])$ .

We are given an object of  $\mathrm{Tan}_y(\mathcal{Y})$ , that is a morphism couple  $(y', \eta)$  where  $y' : \mathrm{Spec}(K[\epsilon]) \rightarrow \mathcal{Y}$  together with a 2-morphism  $\eta : y' \circ i \rightarrow y$ . We remind that  $g$  is formally smooth [49, Tag 0DP0] that is, if it is formally smooth on objects as a 1-morphism in categories fibered in groupoids [49, Tag 0DNN]. A special case is for the following commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{x} & \mathfrak{X} \\ \downarrow i & \searrow \eta & \downarrow g \\ \mathrm{Spec}(K[\epsilon]) & \xrightarrow{y'} & \mathfrak{Y} \end{array},$$

where  $\eta : y' \circ i \rightarrow y = g \circ x = y$  is a 2-morphism witnessing the commutativity of the diagram; there exists a triple  $(x', \alpha, \beta)$  where :

- (i)  $x' : \mathrm{Spec}(K[\epsilon]) \rightarrow \mathfrak{X}$  is a morphism;
- (ii)  $\alpha : x' \circ i \rightarrow x$ ,  $\beta : y' \rightarrow g \circ x'$  are 2-arrows such that  $\eta = (id_g \star \alpha) \circ (\beta \star id_i)$ .

It follows that  $g(x', \alpha) = (g \circ x', g \circ \alpha) \in \mathrm{Tan}_y(\mathcal{Y})$  is isomorphic to  $(y', \eta)$ . This establishes the essential surjectivity of the tangent morphism.

(ii)  $\implies$  (i). According to [37, Thm. 6.3], there exists a smooth 1-morphism  $\varphi : Y \rightarrow \mathcal{Y}$  and a point  $y_1 \in Y(K)$  mapping to  $y$  such that  $Y$  is an affine scheme. We note that  $K = \kappa(y_1)$ . We consider the fiber product  $\mathcal{X}' = \mathcal{X} \times_{\mathcal{Y}} Y$ , it is an algebraic stack and there exists a 1-morphism  $x' : \mathrm{Spec}(K) \rightarrow \mathcal{X}'$  lifting  $x$  and  $y_1$ . There exists a smooth 1-morphism  $\psi : X' \rightarrow \mathcal{X}'$  and a point  $x_1 \in X'(K)$  mapping to  $x$  such that  $X'$  is an affine scheme. By construction we have again that  $K = \kappa(x_1)$ . We have then the commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{\psi} & \mathcal{X}' & \xrightarrow{g'} & Y \\ & & \downarrow & \searrow & \downarrow \varphi \\ & & \mathcal{X} & \xrightarrow{g} & \mathcal{Y} \end{array}.$$

According to [37, Lem. 17.5.1], the square

$$\begin{array}{ccc} T(\mathcal{X}'/S) & \xrightarrow{Tg'} & T(Y/S) \\ \downarrow & & \downarrow T\varphi \\ T(\mathcal{X}/S) & \xrightarrow{Tg} & T(\mathcal{Y}/S) \end{array}$$

is 2–cartesian. It follows that the square

$$\begin{array}{ccc} \mathrm{Tan}_{x'}(\mathcal{X}') & \xrightarrow{(Tg')_{x'}} & \mathrm{Tan}_{y_1}(Y) \\ \downarrow (T\psi)_{x_1} & & \downarrow (T\varphi)_{y_1} \\ \mathrm{Tan}_x(\mathcal{X}) & \xrightarrow{(Tg)_x} & \mathrm{Tan}_y(\mathcal{Y}) \end{array}$$

is 2-cartesian. Our assumption is that the bottom morphism is essentially surjective, it follows that  $(Tg')_{x'} : \mathrm{Tan}_{x_1}(\mathcal{X}') \rightarrow \mathrm{Tan}_{y_1}(Y)$  is essentially surjective as well. Since  $\psi$  is smooth, the map  $(T\psi)_{x_1} : \mathrm{Tan}_{x'}(\mathcal{X}') \rightarrow \mathrm{Tan}_{x_1}(\mathcal{X}')$  is essentially surjective. By composition it follows that  $\mathrm{Tan}_{x_1}(\mathcal{X}') \rightarrow \mathrm{Tan}_{y_1}(Y)$  is essentially surjective. Since  $X'$  and  $Y$  are locally of finite presentation over  $S$ , the case of schemes yields that  $g' \circ \psi : X' \rightarrow Y$  is smooth at  $x'$ . By definition of smoothness for morphisms of stacks [43, §8.2], we conclude that  $g$  is smooth at  $x$ .

We assume (ii) and shall show that  $\mathcal{Y}$  is smooth at  $y$  over  $S$ . Using the diagrams of the proof, we have seen that the  $S$ –morphism  $X' \rightarrow Y$  of schemes is smooth at  $x'$ . Once again the classical Jacobian criterion [18, 17.11.1] applies and shows that  $Y$  is smooth at  $y_1$  over  $S$ . By definition of smoothness for stacks, we get that  $\mathcal{Y}$  is smooth at  $y$  over  $S$ .  $\square$

**8.2. Lie algebra of an  $S$ -group space.** Let  $S$  be a scheme. Let  $f : X \rightarrow Y$  be a morphism of  $S$ –algebraic spaces. We consider the quasi-coherent sheaf  $\Omega_{X/Y}^1$  on  $X$  defined in [49, Tag 04CT]. Let  $T$  be an  $S$ –scheme equipped with a closed subscheme  $T_0$  defined by a quasi-coherent ideal  $\mathcal{I}$  such that  $\mathcal{I}^2 = 0$ . According to [43, 7.A page 167] for any commutative diagram of algebraic spaces

$$\begin{array}{ccc} T_0 & \xrightarrow{x_0} & X \\ \downarrow & \searrow & \downarrow f \\ T & \xrightarrow{y} & Y \end{array}$$

if there exists a dotted arrow filling in the diagram then the set of such dotted arrows form a torsor under  $\mathrm{Hom}_{\mathcal{O}_{T_0}}(x_0^* \Omega_{X/Y}^1, \mathcal{I})$ . We extend to group spaces well-known statements on group schemes [45, II.4.11.3].

**Lemma 8.2.** *Let  $G$  be an  $S$ -group space. We denote by  $e_G : S \rightarrow G$  the unit point and put  $\omega_{G/S} = e_G^*(\Omega_{G/S}^1)$ .*

(1) *There is a canonical isomorphism of  $S$ -functors  $\mathrm{Lie}(G) \xrightarrow{\sim} \mathbf{V}(\omega_{G/S})$  which is compatible with the  $\mathcal{O}_S$ -structure.*

(2) If  $\omega_{G/S}$  is a locally free coherent sheaf, then  $\mathrm{Lie}(G) \xrightarrow{\sim} \mathbf{W}(\omega_{G/S}^\vee)$ . In particular we have an isomorphism

$$\mathrm{Lie}(G)(R) \otimes_R R' \xrightarrow{\sim} \mathrm{Lie}(G)(R')$$

for each morphism of  $S$ -algebras  $R \rightarrow R'$ .

(3) Assume that  $G$  is smooth and quasi-separated over  $S$ . Then  $\omega_{G/S}$  is a finite locally free coherent sheaf and (2) holds.

Under the conditions of (2) or (3), we denote also by  $\mathcal{L}ie(G) = \omega_{G/S}^\vee$  the locally free coherent sheaf.

*Proof.* (1) Let  $T_0$  be an  $S$ -scheme and consider  $T = T_0[\epsilon]$ . We apply the above fact to the morphism  $G \rightarrow S$  and the points  $x_0 = e_{G_{T_0}}$  and  $y : T \rightarrow S$  the structural morphism. It follows that  $\ker(G(T) \rightarrow G(T_0))$  is a torsor under  $\mathrm{Hom}_{\mathcal{O}_{T_0}}(e_{G_{T_0}}^* \Omega_{G/S}^1, \epsilon \mathcal{O}_{T_0}) \cong \mathrm{Hom}_{\mathcal{O}_{T_0}}(e_{G_{T_0}}^* \Omega_{G/S}^1, \mathcal{O}_{T_0}) = \mathrm{Hom}_{\mathcal{O}_{T_0}}(\omega_{G/S}^1 \otimes_{\mathcal{O}_S} \mathcal{O}_{T_0}, \mathcal{O}_{T_0})$ . We have constructed an isomorphism of  $S$ -functors  $\mathrm{Lie}(G) \xrightarrow{\sim} \mathbf{V}(\omega_{G/S})$  and the compatibility of  $\mathcal{O}_S$ -structures is a straightforward checking.

(2) If  $\omega_{G/S}$  is a locally free coherent sheaf, then  $\mathrm{Lie}(G) \xrightarrow{\sim} \mathbf{V}(\omega_{G/S}) \xrightarrow{\sim} \mathbf{W}(\omega_{G/S}^\vee)$ . The next fact follows from [17, 12.2.3].

(3) According to [49, Tag 0CK5],  $\Omega_{G/S}^1$  is a finite locally free coherent sheaf over  $G$ . It follows that  $\omega_{G/S}^1$  is a finite locally free coherent sheaf over  $S$ . □

**Lemma 8.3.** *Let  $G$  be a smooth  $S$ -group space and let  $T$  be an  $S$ -scheme equipped with a closed subscheme  $T_0$  defined by a quasi-coherent ideal  $\mathcal{I}$  such that  $\mathcal{I}^2 = 0$ . We denote by  $t_0 : T_0 \rightarrow S$  the structural morphism,  $G_0 = G \times_S T_0$  and assume that  $t_0$  is quasi-compact and quasi-separated.*

(1) *We have an exact sequence of fppf (resp. étale, Zariski) sheaves on  $S$*

$$0 \rightarrow \mathbf{W}\left((t_0)_*(\mathcal{L}ie(G_0) \otimes_{\mathcal{O}_{T_0}} \mathcal{I})\right) \rightarrow \prod_{T/S} G \rightarrow \prod_{T_0/S} G \rightarrow 1.$$

(2) *If  $T = \mathrm{Spec}(A)$  is affine and  $T_0 = \mathrm{Spec}(A/I)$ , we have an exact sequence*

$$0 \rightarrow \mathrm{Lie}(G)(A) \otimes_A I \rightarrow G(A) \rightarrow G(A/I) \rightarrow 1.$$

*Proof.* (1) We have

$$\mathrm{Hom}_{\mathcal{O}_{T_0}}(\omega_{G_0/T_0}, \mathcal{I}) = H^0(T_0, \mathcal{L}ie(G_0) \otimes_{\mathcal{O}_{T_0}} \mathcal{I}) = H^0\left(T, t_{0,*}(\mathcal{L}ie(G_0) \otimes_{\mathcal{O}_{T_0}} \mathcal{I})\right),$$

whence an exact sequence

$$0 \rightarrow H^0\left(T, t_{0,*}(\mathcal{L}ie(G_0) \otimes_{\mathcal{O}_{T_0}} \mathcal{I})\right) \rightarrow G(T) \rightarrow G(T_0).$$

Now let  $h : S' \rightarrow S$  be a flat morphism locally of finite presentation and denote by  $G' = G \times_S S'$ ,  $h_T : T' \rightarrow T$ , ... the relevant base change to  $S'$ . Since  $t_0$  is quasi-compact and quasi-separated, the flatness of  $h$  yields an isomorphism [49, Tag 02KH]

$$h^*(t_{0,*}(\mathcal{L}ie(G_0) \otimes_{\mathcal{O}_{T_0}} \mathcal{I})) \xrightarrow{\sim} t'_{0,*}(h_{T_0}^*(\mathcal{L}ie(G_0) \otimes_{\mathcal{O}_{T_0}} \mathcal{I})) = t'_{0,*}(\mathcal{L}ie(G'_0) \otimes_{\mathcal{O}_{T'_0}} \mathcal{I}').$$

The similar sequence for  $T'_0$  reads then

$$0 \rightarrow H^0(T', t_{0,*}(\mathcal{L}ie(G_0) \otimes_{\mathcal{O}_{T_0}} \mathcal{I})) \rightarrow G(T') \rightarrow G(T'_0).$$

We have then an exact sequence of fppf sheaves

$$0 \rightarrow \mathbf{W}\left((t_0)_*(\mathcal{L}ie(G_0) \otimes_{\mathcal{O}_{T_0}} \mathcal{I})\right) \rightarrow \prod_{T/S} G \rightarrow \prod_{T_0/S} G.$$

For  $T$  an affine scheme, the map  $G(T) \rightarrow G(T_0)$  is onto since the smooth  $S$ -group space  $G$  is formally smooth [49, Tag 04AM], that is, it satisfies the infinitesimal lifting criterion [49, Tag 049S, 060G].

It implies the exactness for the the Zariski, étale and fppf topologies.

(2) We can assume that  $S = T = \text{Spec}(A)$ . In this case, we have

$$H^0\left(S, (t_0)_*(\mathcal{L}ie(G_0) \otimes_{\mathcal{O}_{T_0}} \mathcal{I})\right) = H^0(T_0, \mathcal{L}ie(G_0) \otimes_{\mathcal{O}_{T_0}} \mathcal{I}) = \text{Lie}(G_0)(A/I) \otimes_{A/I} I.$$

We have  $\text{Lie}(G_0)(A/I) = \text{Lie}(G)(A) \otimes_A A/I$  in view of Lemma 8.3.(2) whence the identification  $\text{Lie}(G_0)(A/I) \otimes_{A/I} I \cong \text{Lie}(G)(A) \otimes_A I$ . Then (1) provides the exact sequence

$$0 \rightarrow \text{Lie}(G)(A) \otimes_A I \rightarrow G(A) \rightarrow G(A/I)$$

and the right map is onto since  $G$  is smooth. □

**Remarks 8.4.** (a) A special case of (1) is  $T = S[\epsilon]$  and  $T_0 = S$ . We get an exact sequence of fppf (resp. étale, Zariski) sheaves on  $S$

$$0 \rightarrow \mathbf{W}(\mathcal{L}ie(G)) \rightarrow \prod_{S[\epsilon]/S} G \rightarrow G \rightarrow 1.$$

(b) In the group scheme case, (2) is established in [13, proof of II.5.2.8].

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UMR 5208 INSTITUT CAMILLE JORDAN - UNIVERSITÉ CLAUDE BERNARD LYON 1 43 BOULEVARD DU 11 NOVEMBRE 1918 69622 VILLEURBANNE CEDEX - FRANCE  
*Email address:* gille@math.univ-lyon1.fr

DEPARTEMENT OF MATHEMATICS AND COMPUTER SCIENCE, MSC W401, 400 DOWMAN DR. EMORY UNIVERSITY ATLANTA, GA 30322 USA  
*Email address:* parimala.raman@emory.edu

DEPARTEMENT OF MATHEMATICS AND COMPUTER SCIENCE, MSC W401, 400 DOWMAN DR. EMORY UNIVERSITY ATLANTA, GA 30322 USA  
*Email address:* suresh@mathcs.emory.edu