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LOCAL TRIVIALITY FOR G -TORSORS

P. GILLE, R. PARIMALA, AND V. SURESH

ABSTRACT. Let $C \rightarrow \text{Spec}(R)$ be a relative proper flat curve over an henselian base. Let G be a reductive C -group scheme. Under mild technical assumptions, we show that a G -torsor over C which is trivial on the closed fiber of C is locally trivial for the Zariski topology.

Keywords: Reductive group scheme, torsor, deformation.

MSC 2000: 14D23, 14F20.

1. INTRODUCTION

Let X be a smooth connected projective curve defined over an algebraically closed field k . Let $X^{af} = X \setminus \{\infty\}$ be the affine complement of a rational point. Given a local k -algebra R , Beauville and Laszlo showed that a vector bundle over $X \times_k R$ of trivial determinant is trivial over $X^{af} \times_k R$ [B-L1, 3.5]. A related result is that of Drinfeld and Simpson [D-S, th. 3]. Given a semisimple simply connected k -group G and a strictly henselian k -algebra R , they showed that a G -torsor over $X \times_k R$ is trivial over $X^{af} \times_k R$, in particular it is locally trivial. It has been generalized recently by Belkale-Fakhruddin [B-F1, B-F2] to a wider setting.

The purpose of this note is to investigate the intermediate case of a henselian k -algebra R and some variations. This means to consider Zariski triviality with respect to henselian (or Nisnevich) topology when Beauville-Laszlo (resp. Drinfeld-Simpson) deal with Zariski topology (resp. étale topology).

A significant example of our results is as follows where R is a noetherian henselian local ring with infinite residue field κ . Let G be a semisimple simply connected R -group. Let E be a G -torsor over $X \times_k R$ whose restriction on $X \times_R \kappa$ is trivial; then it is locally trivial for the Zariski topology (see Cor. 6.7).

This involves deformation theory using the algebraic stack of G -bundles and based on nice elements of loop groups defined by one parameter subgroups.

Let us review the contents of the paper. The toral case is quite different of the semisimple one since it works in higher dimensions; it is treated in section 2 by means of the proper base change theorem. The section 3 deals with generation by one

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parameter subgroups, namely the Kneser-Tits problem. Section 4 extends Sorger's construction of the moduli stack of G -bundles [So] and discusses in details its tangent bundle. The next section 5 recollects facts on patching for G -torsors and provides the main technical statement namely the parametrization of the deformations of a given torsor in the henselian case in presence of isotropy (Proposition 5.4). Section 6 explain why this intermediate statement is enough for establishing that deformations of a given torsor (in the henselian case) are locally trivial for the Zariski topology. One important point is that we can get rid of isotropy assumptions. Finally section 7 provides a general statement, also we included at the end a short appendix 8 gathering facts on smoothness for morphisms of algebraic stacks.

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Conventions and Notations. We use mainly the terminology and notations of Grothendieck-Dieudonné [EGAI, §9.4 and 9.6] which agrees with that of Demazure-Grothendieck used in [SGA3, Exp. I.4]

(a) Let S be a scheme and let \mathcal{E} be a quasi-coherent sheaf over S . For each morphism $f : T \rightarrow S$, we denote by $\mathcal{E}_{(T)} = f^*(\mathcal{E})$ the inverse image of \mathcal{E} by the morphism f . We denote by $\mathbf{V}(\mathcal{E})$ the affine S -scheme defined by $\mathbf{V}(\mathcal{E}) = \mathrm{Spec}(\mathrm{Sym}^\bullet(\mathcal{E}))$; it is affine over S and represents the S -functor $Y \mapsto \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{E}_{(Y)}, \mathcal{O}_Y)$ [EGAI, 9.4.9].

(b) We assume now that \mathcal{E} is locally free and denote by \mathcal{E}^\vee its dual. In this case the affine S -scheme $\mathbf{V}(\mathcal{E})$ is of finite presentation (ibid, 9.4.11); also the S -functor $Y \mapsto H^0(Y, \mathcal{E}_{(Y)}) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{E}_{(Y)})$ is representable by the affine S -scheme $\mathbf{V}(\mathcal{E}^\vee)$ which is also denoted by $\mathbf{W}(\mathcal{E})$ [SGA3, I.4.6].

It applies to the locally free coherent sheaf $\mathcal{E}nd(\mathcal{E}) = \mathcal{E}^\vee \otimes_{\mathcal{O}_S} \mathcal{E}$ over S so that we can consider the affine S -scheme $\mathbf{V}(\mathcal{E}nd(\mathcal{E}))$ which is an S -functor in associative commutative and unital algebras [EGAI, 9.6.2]. Now we consider the S -functor $Y \mapsto \mathrm{Aut}_{\mathcal{O}_Y}(\mathcal{E}_{(Y)})$. It is representable by an open S -subscheme of $\mathbf{V}(\mathcal{E}nd(\mathcal{E}))$ which is denoted by $\mathrm{GL}(\mathcal{E})$ (*loc. cit.*, 9.6.4).

(c) We denote by $\mathbb{Z}[\epsilon] = \mathbb{Z}[x]/x^2$ the ring of dual integers and by $S[\epsilon] = S \times_{\mathbb{Z}} \mathbb{Z}[\epsilon]$. If G/S is S -group space (i.e. an algebraic space in groups, called group algebraic space over S in [St, Tag043H]) we denote by $\mathrm{Lie}(G)$ the S -functor defined by $\mathrm{Lie}(G)(T) = \ker(G(T[\epsilon]) \rightarrow G(T))$. This S -functor is a functor in Lie \mathcal{O}_S -algebras, see [SGA3, II.4.1] or [D-G, II.4.4]. More facts are collected in Appendix 8.2.

(d) If G/S is a affine smooth S -group scheme, we denote by $\mathrm{Tors}_G(S)$ the groupoid of G -torsors over S and by $H^1(S, G)$ the set of isomorphism classes of G -torsors (locally trivial for the étale topology), we have a classifying map $\mathrm{Tors}_G(S) \rightarrow H^1(S, G)$, $E \mapsto [E]$.

2. THE CASE OF TORI

Lemma 2.1. *Let X be a scheme and let T be a X -torus. Assume that T is split by a finite étale cover of degree d . Then $dH^1(X, T) \subseteq H_{Zar}^1(X, T)$.*

Proof. Let $f : Y \rightarrow X$ be a finite étale cover of degree d which splits T , that is $T_Y \cong \mathbb{G}_{m, Y}^r$. According to [C-T-S, 0.4], we have a norm map $f_* : H^1(Y, T) \rightarrow H^1(X, T)$ such that the composite $H^1(X, T) \xrightarrow{f^*} H^1(Y, T) \xrightarrow{f_*} H^1(X, T)$ is the multiplication by d . We claim that $f_*(H^1(Y, T)) \subseteq H_{Zar}^1(X, T)$. Let $x \in X$. Then $V_x = \text{Spec}(\mathcal{O}_{X, x}) \times_X Y$ is a semi-local scheme so that $\text{Pic}(V_x) = 0$ [B:AC, Chap. 2, Sect. 5, No. 3, Proposition 5] so that $H^1(V_x, T) = 0$. Since the construction of the norm commutes with base change, we get that $\left(f_*(H^1(Y, T))\right)_{\mathcal{O}_{X, x}} = 0$. In particular we have $dH^1(X, T) \subseteq H_{Zar}^1(X, T)$. \square

Proposition 2.2. *Let R be an henselian local ring of residue field κ . We denote by p the characteristic exponent of κ . Let X be a proper R -scheme and let T be a X -torus.*

(1) *Let l be a prime number distinct of p . For each $i \geq 0$, the kernel $\ker(H^i(X, T) \rightarrow H^i(X_\kappa, T))$ is l -divisible.*

(2) *The kernel $\ker(H^0(X, T) \rightarrow H^0(X_\kappa, T))$ is uniquely l -divisible.*

(3) *We assume that T is locally isotrivial. There exists $r \geq 0$ such that $p^r \ker(H^1(X, T) \rightarrow H^1(X_\kappa, T)) \subseteq H_{Zar}^1(X, T)$.*

Proof. (1) We consider the exact sequence of étale X -sheaves $1 \rightarrow {}_lT \rightarrow T \xrightarrow{\times l} T \rightarrow 1$ which generalizes the Kummer sequence. It gives rise to the following commutative diagram

$$\begin{array}{ccccccc} H^i(C, {}_lT) & \longrightarrow & H^i(C, T) & \xrightarrow{\times l} & H^i(C, T) & \longrightarrow & H^{i+1}(C, {}_lT) \\ \downarrow \beta_1 & & \downarrow & & \downarrow & & \downarrow \beta_2 \\ H^i(C_\kappa, {}_lT) & \longrightarrow & H^i(C_\kappa, T) & \xrightarrow{\times l} & H^i(C_\kappa, T) & \longrightarrow & H^{i+2}(C_\kappa, {}_lT). \end{array}$$

The proper base change theorem [SGA4, XII.5.5.(iii)] shows that β_1, β_2 are isomorphisms. By diagram chase, we conclude that $\ker(H^i(C, T) \rightarrow H^i(C_\kappa, T))$ is l -divisible.

(2) If $i = 0$, we can complete the left handside of the diagram with 0. By diagram chase it follows that $\ker(H^0(C, T) \rightarrow H^0(C_\kappa, T))$ is uniquely l -divisible.

(3) There exists an open cover $(U_i)_{i=1, \dots, n}$ of X and finite étale covers $f_i : V_i \rightarrow U_i$ such that T_{V_i} is split for $i = 1, \dots, n$. Let d be the g.c.m of the degrees of the f_i 's. We write $d = p^r e$ with $(e, p) = 1$. Assertion (1) shows that $\ker(H^1(X, T) \rightarrow H^1(X_\kappa, T))$ is e -divisible so that $\ker(H^1(X, T) \rightarrow H^1(X_\kappa, T)) \subseteq eH^1(X, T)$. Lemma 2.1 shows that $dH^1(X, T) \subseteq H_{Zar}^1(X, T)$ which permits to conclude that $p^r \ker(H^1(X, T) \rightarrow H^1(X_\kappa, T)) \subseteq H_{Zar}^1(X, T)$. \square

Remark 2.3. By inspection of the proof, we see that we can take $r = 0$ in (3) if T (quasi)splits after a Galois extension X'/X of degree prime to p .

3. INFINITESIMAL KNESER-TITS PROBLEM

Let R be a commutative ring and let G be a reductive R -group scheme. Let P be a strictly proper R -parabolic subgroup of G (it means each projection of P on a semisimple quotient of G is proper) and assume that P admits Levi subgroups. Let P^- be an opposite R -parabolic subgroup to P . We denote by $E_P(R)$ the subgroup of $G(R)$ which is generated by $\text{rad}(P)(R)$ and $\text{rad}(P^-)(R)$. It does not depend on the choice of P^- [P-S, §1]. We denote by $G^{+,P}(R)$ the normal R -subgroup of $G(R)$ generated by $\text{rad}(P)(R)$ and $\text{rad}(P^-)(R)$ or equivalently by $E_P(R)$. The quotient group $W_P(R, G) = G(R)/G^{+,P}(R)$ is called the Whitehead group of G/R with respect to P .

Remark 3.1. If R is a field, $E_P(R)$ does not depend of the choice of P and the group $E_P(R) = G^{+,P}(R)$ is denoted by $G^+(R)$ [B-T, prop. 6.2]. If for each $s \in \text{Spec}(R)$ each semisimple quotient of G_s is of relative rank ≥ 2 , then $E_P(R)$ does not depend of the choice of P and $G^{+,P}(R)$ is denoted by $G^+(R)$.

The next statements 3.2, 3.5, 3.6 are variations of a result of Borel-Tits on the Whitehead groups over local fields [B-T, prop. 6.14].

Lemma 3.2. *Let $U = \text{rad}(P)$ and denote by $U' \subset U$ the last R -subgroup scheme of Demazure's filtration [SGA3, XXVI.2.1]. Let s_1, \dots, s_n be points of $\text{Spec}(R)$ whose residue fields are infinite or finite of characteristic $\neq 2$. We assume that G is semisimple and let $f : G^{\text{sc}} \rightarrow G$ be the universal cover. Assume that $\ker(f)$ is smooth.*

(1) *There exist $g_1, \dots, g_m \in G^{+,P}(R)$ such that the product map*

$$h : U'^m \rightarrow G, \quad (u_1, \dots, u_m) \mapsto g_1 u_1 \dots g_m u_m$$

is smooth at $(1, \dots, 1)_{s_j}$ for $j = 1, \dots, n$.

(2) *If R is semilocal, the map $dh : \text{Lie}(U')(R)^m \rightarrow \text{Lie}(G)$ is onto.*

This requires a variation on a statement of Riehm on proper subalgebras of Chevalley Lie algebras [R, §2, Lemma].

Lemma 3.3. *Let G be a semisimple group defined over a field F .*

(1) *Suppose that the characteristic exponent of F is odd and that G is split and almost simple. Let L be a Lie subalgebra of $\text{Lie}(G)$ which is $G(F)$ -stable and contains a long root element (i.e. associated to some long root). Then $L = \text{Lie}(G)$.*

(2) *Suppose that G is simply connected. Let P be a strictly proper parabolic subgroup of G and put $U = \text{rad}(P)$. We denote by $U' \subset U$ the last F -subgroup of Demazure's filtration [SGA3, XXVI.1.2], which is a vector F -group scheme.*

Then $\text{Lie}(G)$ is the unique Lie F -subalgebra of $\text{Lie}(G)$ containing $\text{Lie}(U')$ and which is stable under the adjoint action of G .

Proof. (1) Since roots of maximal length are conjugated under the Weyl group and since maximal split tori of G are $G(F)$ -conjugated, it follows that all long root elements are $G(F)$ -conjugated. It follows that L contains all long root elements. According to [C-S-U-W, prop. 3.3], we conclude that $L = \text{Lie}(G)$.

(2) Without loss of generality we can assume that F is algebraically closed and that G is almost simple. Let B be a Borel subgroup of P and let T be a maximal F -torus of B . Let U_{max} be the root subgroup attached to the maximal root of $\Phi(G, T)$ for the order defined by B . Then $U_{max} \subset U'$. Now let L be a Lie subalgebra of $\text{Lie}(G)$ which contains $\text{Lie}(U')$ and is stable under the adjoint action of G . If $\text{char}(\mathbf{k}) \neq 2$ statement (1) shows that $L = \text{Lie}(G)$ since L contains $\text{Lie}(U_{max})$.

It remains to discuss the characteristic two case. The argument of (1) shows that L contains all long root elements. The proper G -submodules of $\text{Lie}(G)$ are listed in [C-G-P, lemma 7.1.2]. Assume that $L \subsetneq \mathfrak{g} = \text{Lie}(G)$. Let \mathfrak{t} be a Cartan subalgebra of $\mathfrak{g} = \text{Lie}(G)$, up to conjugacy, we have that $L \subset \mathfrak{t} \oplus \mathfrak{g}_<$ where $\mathfrak{g}_<$ is generated by eigenspaces attached to short roots. Then $\mathfrak{g}_> \cap L = 0$ which contradicts the fact that L contains all long root elements. \square

We proceed now to the proof of Lemma 3.2.

Proof. Without loss of generality, we can assume that G and P are of constant type. The hypothesis implies that $G^{sc} \rightarrow G$ is étale, so boils down to the simply connected case. Let s_j be one of the point.

Case $\kappa(s_j)$ -infinite. According to Lemma 3.3.(2), $\text{Lie}(G)(\kappa(s_j))$ is the only subspace of $\text{Lie}(G)(\kappa(s_j))$ containing $\text{Lie}(U')(\kappa(s_j))$ and stable under the adjoint action of $G_{\kappa(s_j)}$. Since $G^+(\kappa(s_j))$ is Zariski dense in $G_{\kappa(s_j)}$, it follows that there exists $g_{1,j}, \dots, g_{m_j,j} \in G^+(\kappa(s_j))$ such that $\text{Lie}(G)(\kappa(s_j))$ is generated by the $g_{i,j} \text{Lie}(U')(\kappa(s_j))$.

Case $\kappa(s_j)$ finite. According to Lemma 3.3.(1), $\text{Lie}(G)(\kappa(s_j))$ is the only subspace of $\text{Lie}(G)(\kappa(s_j))$ containing $\text{Lie}(U')(\kappa(s_j))$ and stable under $G(\kappa(s_j))$. The group $G_{\kappa(s_j)}$ is quasi-split so that $G^+(\kappa(s_j)) = G(\kappa(s_j))$ [T, §1.1.2]; it follows that there exists $g_{1,j}, \dots, g_{m_j,j} \in G^+(\kappa(s_j))$ such that $\text{Lie}(G)(\kappa(s_j))$ is generated by the $g_{i,j} \text{Lie}(U')(\kappa(s_j))$.

We gather now both cases.

Claim 3.4. *The map $G^{+,P}(R) \rightarrow \prod_j G^{+,P}(\kappa(s_j))$ is onto.*

Let P^- be an opposite parabolic subgroup scheme of P and let U^- be its unipotent radical. Since $G^{+,P}(R)$ (resp. each $G^{+,P}(\kappa(s_j))$) is generated by $U(R)$ and $U^-(R)$, it is enough to show the surjectivity of $U(R) \rightarrow \prod_j U(\kappa(s_j))$. According to [SGA3, XXVI.2.5], there exists a f.g. locally free R -module \mathcal{E} such that U is isomorphic to $\mathbf{W}(\mathcal{E})$ as R -scheme. Since $\mathbf{W}(\mathcal{E})(R) = \mathcal{E}$ maps onto $\prod_j \mathbf{W}(\mathcal{E})(\kappa(s_j)) = \prod_j \mathcal{E} \otimes_R \kappa(s_j)$, the Claim is established.

There are $g_1, \dots, g_m \in G(R)$ such that $\text{Lie}(G)(\kappa(s_j))$ is generated by the $g_i \text{Lie}(U')(\kappa(s_j))$ for $j = 1, \dots, n$. The differential of the product map

$$h : U'^m \rightarrow G, \quad (u_1, \dots, u_m) \mapsto {}^{g_1}u_1 \dots {}^{g_m}u_m$$

is $\text{Lie}(U')^m \rightarrow \text{Lie}(G)$, $(x_1, \dots, x_m) \mapsto {}^{g_1}x_1 \dots {}^{g_m}x_m$. It is onto by construction and we conclude that h is smooth at $(1, \dots, 1)_{s_j}$ for $j = 1, \dots, n$.

(2) We assume that R is semilocal with maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_t$. Then $J = \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_t$ is the Jacobson radical of R and $R/J \cong R/\mathfrak{m}_1 \times \dots \times R/\mathfrak{m}_t$. Statement (1) shows that the map $dh : \text{Lie}(U')(R)^m \rightarrow \text{Lie}(G)(R)$ is surjective modulo \mathfrak{m}_i for $i = 1, \dots, t$ so is surjective modulo J . Since $\text{Lie}(G)(R)$ is finitely generated, Nakayama's lemma [Ks, II.4.2.3] enables us to conclude that dh is onto. \square

The two next statements will not be used in the paper but are applications of Lemma 3.2 to the rigidity of Whitehead groups. Let I be an ideal of R satisfying $I^2 = 0$ and consider the exact sequence ([D-G, proof of II.5.2.8] or Lemma 8.3.(2))

$$1 \rightarrow \text{Lie}(G)(R) \otimes_R I \xrightarrow{e^?} G(R) \rightarrow G(R/I) \rightarrow 1.$$

Lemma 3.5. *The sequence*

$$\text{Lie}(G)(R) \otimes_R I \rightarrow G(R)/G^{+,P}(R) \rightarrow G(R/I)/G^{+,P}(R/I) \rightarrow 1$$

is exact.

Proof. We are given an element $g \in G(R)$ whose reduction in $G(R/I)$ belongs to $G^{+,P}(R/I)$. Since the maps $G(R) \rightarrow G(R/I)$ and $\text{rad}(P^\pm)(R) \rightarrow \text{rad}(P^\pm)(R/I)$ are onto, it follows that $G^{+,P}(R) \rightarrow G^{+,P}(R/I)$ is onto. Therefore we have $g \in e^{(\text{Lie}(G)(R) \otimes_R I)} G^{+,P}(R)$. This shows the exactness. \square

Proposition 3.6. *We assume that R is semilocal with infinite residue fields. Assume that G is semisimple and let $f : G^{sc} \rightarrow G$ be the universal cover. Assume that $\ker(f)$ is smooth. The map $G(R)/G^{+,P}(R) \rightarrow G(R/I)/G^{+,P}(R/I)$ is an isomorphism.*

Proof. Let s_1, \dots, s_n be the closed points of $\text{Spec}(R)$. We have to show the inclusion $e^{(\text{Lie}(G)(R) \otimes_R I)} \subset G^{+,P}(R)$. Lemma 3.2 provides elements $g_1, \dots, g_m \in G(R)$ such that the product map $h : U'^m \rightarrow G$, $(u_1, \dots, u_m) \mapsto {}^{g_1}u_1 \dots {}^{g_m}u_m$ is smooth at $(1, \dots, 1)_{s_j}$ for $j = 1, \dots, n$. Nakayama lemma shows that $dh : \text{Lie}(U')(R)^m \rightarrow \text{Lie}(G)$ is onto. By construction we have $e^{(h(\text{Lie}(U')(R)^m \otimes_R I))} \subseteq G^{+,P}(R)$. Thus $e^{(\text{Lie}(G)(R) \otimes_R I)} \subseteq G^{+,P}(R)$ as desired. \square

4. MODULI STACK OF G -TORSORS

4.1. **Setting.** Let S be a noetherian separated base scheme. Let $p : C \rightarrow S$ be a projective relative curve (that is all geometric fibers are algebraic curves). We assume that C is integral and that the map $\mathcal{O}_S \rightarrow p_*(\mathcal{O}_C)$ is universally an isomorphism. This implies that p is cohomologically flat in dimension zero. We recall that p is cohomologically flat in degree 1 for each S -flat coherent sheaf \mathcal{F} over C [I, 8.3.11.1]; it means that the formation of $R^1p_*\mathcal{F}$ commutes with base change.

Let G be a smooth affine group scheme over C . We assume that G admits a linear representation $i : G \rightarrow \mathrm{GL}(\mathcal{E})$ where \mathcal{E} is a locally free coherent sheaf of constant rank such that

- (I) i is a closed immersion;
- (II) the fppf sheaf $\mathrm{GL}(\mathcal{E})/G$ is representable by an affine C -scheme.

Note that the smoothness of $\mathrm{GL}(\mathcal{E})$ implies that $\mathrm{GL}(\mathcal{E})/G$ is smooth [SGA3, VI_B.9.2]. Also this assumption is satisfied for example if G is a semisimple group scheme [Ma, prop. 3.2].

Lemma 4.1. *Let Y be a S -scheme and let P, Q be two G -torsors over C_Y . Then the fppf Y -sheaf*

$$Y' \mapsto \mathrm{Isom}_{G_{C_{Y'}}}(P_{C_{Y'}}, Q_{C_{Y'}})$$

is representable by a Y -scheme $\mathrm{Isom}^b(P, Q)$ which is affine of finite presentation.

Proof. We consider the fppf sheaf in groups $ad(P) = \underline{\mathrm{Aut}}_G(P)$ over C_Y [Gd, III.1.4.8]. Since G is smooth affine over C , faithfully flat descent for affine schemes applies and shows that $ad(P)$ is represented by a smooth affine C_Y -group scheme. We see P as a $(ad(P), G)$ -bitorsor (*ibid*, §III.1.5) so defines the $(G, ad(P))$ -sheaf bitorsor P° over C_Y (called the opposite torsor of G) which is representable by a smooth affine C_Y -scheme. According to [Gd, III.1.6.4], we have an isomorphism of fppf sheaves $h : Q \wedge^G P^\circ \xrightarrow{\sim} \underline{\mathrm{Isom}}_G(P, Q)$. By the same descent argument, $Q \wedge^G P^\circ$ is representable by a smooth affine C_Y -scheme, say Z . We consider the global section Y -functor

$\prod_{C_Y/Y} (Z/Y)$ defined by

$$\left(\prod_{C_Y/Y} (Z/Y) \right) (Y') = Z(C_Y \times_{Y'} Y) = Z(C_{Y'}) = \mathrm{Isom}_{G_{C_{Y'}}}(P_{C_{Y'}}, Q_{C_{Y'}})$$

for each Y -scheme Y' . According to Grothendieck [FGA, TDTE, §C.2], $\prod_{C_Y/Y} (Z/Y)$

is representable. We denote by $\mathrm{Isom}^b(P, Q)$ the relevant Y -scheme. It is actually affine over Y by the argument of [Hd, §1.4]. Since the global section functor is locally of finite presentation, we conclude that $\mathrm{Isom}^b(P, Q)$ is of finite presentation. \square

For each S -scheme Y , we denote by $\mathrm{Bun}_G(Y) = \mathrm{Tors}_G(C_Y)$ the groupoid of $C_Y = C \times_S Y$ -torsors under G_{C_Y} . This defines a S -stack. The following extends partially

[He1, prop. 1]. It is based on the crucial case of GL_n and use of a dévissage method due to Sorger [So, §3.6].

Proposition 4.2. *The S -stack Bun_G is a smooth (quasi-separated) algebraic stack of finite type.*

Proof. We shall firstly establish that the stack Bun_G is algebraic. In the \mathbf{GL}_n -case, the stack $\mathrm{Bun}_{\mathrm{GL}_n}$ is that of locally free coherent modules of rank n denoted by $\mathcal{F}ib_{C/S,n}$. It is algebraic of finite type [L-M-B, th. 4.6.21].

The next case is that of $\mathrm{GL}(\mathcal{E})$ for \mathcal{E} locally free of rank n . We denote by $P = \underline{\mathrm{Isom}}(\mathcal{O}_C^n, \mathcal{E})$ the associated GL_n -torsor over C . Since $\mathrm{GL}(\mathcal{E}) = P \wedge^{\mathrm{GL}_n} \mathrm{GL}_n$ is the twist of GL_n by P , twisting by P provides an isomorphism of stacks $\mathrm{Bun}_{\mathrm{GL}_n} \rightarrow \mathrm{Bun}_G$, so that Bun_G is an algebraic stack of finite type according to the preceding case. For the general case, the representation $i : G \rightarrow \mathrm{GL}(\mathcal{E})$ gives rise to a 1-morphism of S -stacks $\varphi : \mathrm{Bun}_G \rightarrow \mathrm{Bun}_{\mathrm{GL}(\mathcal{E})}$.

Claim 4.3. *The morphism φ is representable.*

We are given a S -scheme U and a morphism $\eta : U \rightarrow \mathrm{Bun}_{\mathrm{GL}(\mathcal{E})}$, that is a $\mathrm{GL}(\mathcal{E})$ -bundle F_η over C_U . We consider the S -stack $B_\eta = U \times_{\mathrm{Bun}_{\mathrm{GL}(\mathcal{E})}} \mathrm{Bun}_G$. On the other hand, we consider the fppf sheaf $M_\eta = F_\eta \wedge (\mathrm{GL}(\mathcal{E})/G)$; it is representable by an affine U -scheme of finite presentation by faithfully flat descent. We denote by $q : F_\eta \rightarrow M_\eta$ the quotient map, this is a G -torsor.

Let V be a U -scheme. Then $B_\eta(V)$ is the groupoid of pairs (P, α) where P is a G -torsor over V and $\alpha : P \wedge^G \mathrm{GL}(\mathcal{E}) \xrightarrow{\sim} F_{\eta,V}$ is an isomorphism of $\mathrm{GL}(\mathcal{E})$ -torsors.

Each point $m \in M_\eta(C_V)$ defines the G -torsor $P_m = q^{-1}(m)$ and a trivialization $\alpha_m : P_m \wedge^G \mathrm{GL}(\mathcal{E}) \xrightarrow{\sim} F_{\eta,V}$ given by $(p, f) \mapsto i_m(p)f$ (where $i_m : P_m \rightarrow F_{\eta,V}$ is the obvious embedding). We get then a morphism of U -stacks $j : \prod_{C_U/U} M_\eta \rightarrow B_\eta$. The point is that the U -functor $\prod_{C_U/U} M_\eta$ is representable by a V -scheme [FGA, TDTE, §C.2] which is affine of finite presentation. Since the groupoid $\prod_{C_U/U} (M_\eta)$ is equivalent to $B_\eta(V)$ for each V [Gd, III.2.1], we have shown that φ is representable.

According to [St, 86.15.4, Tag 05UM], the Claim implies that Bun_G is an algebraic stack. To show the smoothness, we can use the criterion of formal smoothness [He2, 2.6] (or [St, §98.8, Tag 0D9V]). We are given a S -ring R which is local Artinian with maximal ideal \mathfrak{m} such that $\mathfrak{m}^2 = 0$ and a G -torsor P_0 over $C_0 = C \times_S R/\mathfrak{m}$. We put $G_0 = G_{C_0}$ and denote by $H_0 = P_0 \wedge^{G_0} G_0$ the twisted group scheme over C_0 . According to [I, th. 2.6], the obstruction to lift P_0 in a G -torsor over $C \times_S R$ is a class of $H^2(C_0, \mathcal{L}ie(H_0) \otimes_{\mathcal{O}_{C_0}} \mathfrak{m})$. But $R/\mathfrak{m} = \kappa$ is a field and C_0 is of dimension 1 so that this group vanishes according to Grothendieck's vanishing theorem [Ha, III.2.7]. The formal smoothness criterion is satisfied so that the algebraic stack Bun_G is smooth.

It remains to show that Bun_G is of finite type. We take an atlas $\eta : U \rightarrow \mathrm{Bun}_{\mathrm{GL}(\mathcal{E})}$ where U is smooth of finite type (and quasi-separated). Since $B_\eta \rightarrow \mathrm{Bun}_G$ is an atlas and $B_\eta \rightarrow U$ is affine of finite presentation, B_η is of finite type over S (and quasi-separated). We conclude that Bun_G is of finite type. \square

4.2. The tangent stack. We consider now the tangent stack $T(\text{Bun}_G/S)$ [L-M-B, §17] which is algebraic (*loc. cit.*, 17.16). By definition, for each S -scheme Y , we have $T(\text{Bun}_G/S)(Y) = \text{Bun}_G(Y[\epsilon])$ where $Y[\epsilon] = Y \times_{\mathbb{Z}} \mathbb{Z}[\epsilon]$. It comes with two 1-morphisms

$$\tau : T(\text{Bun}_G/S) \rightarrow \text{Bun}_G$$

and $\sigma : \text{Bun}_G \rightarrow T(\text{Bun}_G/S)$.

Remark 4.4. On the other hand, we can consider the smooth-étale site on Bun_G and the quasi-coherent sheaf $\Omega_{\text{Bun}_G/S}^1$; its associated generalized vector bundle $\mathbb{V}(\Omega_{\text{Bun}_G/S}^1)$ is an algebraic S -stack. There is a canonical 1-isomorphism between $T(\text{Bun}_G/S)$ and $\mathbb{V}(\Omega_{\text{Bun}_G/S}^1)$ (*loc. cit.*, 17.15). We shall not use that fact in the paper.

Our goal is the understanding of the fiber product of S -stacks

$$\mathcal{T}_b = T(\text{Bun}_G/S) \times_{\text{Bun}_G} S$$

where $b : S \rightarrow \text{Bun}_G$ corresponds to the trivial G -bundle G over C . According to [L-M-B, 2.2.2], for each S -ring B , the fiber category $\mathcal{T}_b(B)$ has for objects the couples (F, f) where F is a $G_{C_B}[\epsilon]$ -torsor and $f : G_{C_B} \xrightarrow{\sim} F \times_{C_B[\epsilon]} C_B$ is a trivialization of G_{C_B} -torsors; an arrow $(F_1, f_1) \rightarrow (F_2, f_2)$ is a couple (H, h) where $H : F_1 \xrightarrow{\sim} F_2$ is an isomorphism of $G_{C_B[\epsilon]}$ -torsors and $h \in G(C_B)$ with the compatibility $(H \times_{C_B[\epsilon]} C_B) \circ f_1 = f_2 \circ h$.

4.3. Relation with the Lie algebra. We consider the Weil restriction $G' = \prod_{C[\epsilon]/C} G$, this is an affine smooth C -group scheme [C-G-P, A.5.2]. It comes with a C -group homomorphism $j : G \rightarrow G'$ and with a $C[\epsilon]$ -group homomorphism $q : G' \times_C C[\epsilon] \rightarrow G \times_S S[\epsilon]$.

We consider the functor Φ between the categories of G' -torsors over C and that of G -torsors over $C[\epsilon]$ defined by the assignment $E' \mapsto q_*(E' \times_C C[\epsilon])$.

For each affine open subset V of C , the map $H^1(V, G') \rightarrow H^1(V[\epsilon], G)$ is bijective [Gd, VII.1.3.2] so that each G -torsor over $C[\epsilon]$ is trivialized by an étale cover of C extended to $C[\epsilon]$. According to [SGA3, XXIV.8.2] (see also [Gd, III.3.1.1]), it follows that we can define the functor Ψ by the assignment $F' \mapsto \prod_{C[\epsilon]/C} (F')$. The functors Φ

and Ψ are inverse of each others so that the groupoids $\text{Tors}_{G'}(C)$ and $\text{Tors}_{G \times_C C[\epsilon]}(C[\epsilon])$ are isomorphic. We come now to Lie algebra considerations. By definition of the Lie algebra, the C -group G' fits in an exact sequence of C -group schemes

$$0 \rightarrow \mathbf{W}(\mathcal{L}ie(G)) \xrightarrow{e^?} G' \xrightarrow{\pi} G \rightarrow 1$$

where $\mathcal{L}ie(G) = \omega_{G/S}^\vee$ (see §8.2 and Remark 8.4.(a)). According to [Gd, III.3.2.1] we have an equivalence of groupoids between $\text{Tors}_{\mathbf{W}(\mathcal{L}ie(G))}(C)$ and that of couples (E', η) where E' is a G' -torsor over C and $\eta : G \xrightarrow{\sim} \pi_* E'$ is a trivialization. Taking into account the previous isomorphism of categories, we get then an equivalence of

groupoids between $\text{Tors}_{\mathbf{W}(\mathcal{L}ie(G))}(C)$ and that of couples (F, ξ) where F is a G -torsor over $C[\epsilon]$ and $\xi : G_C \rightarrow F \times_C C[\epsilon]$ is a trivialization; the morphisms are clear.

We come back now to the previous section involving a S -ring R and the morphism $b : \text{Spec}(R) \rightarrow \text{Bun}_G$ associated to the trivial G -torsor. By comparison it follows that the fiber category $\mathcal{T}_b(R)$ is equivalent to $\text{Tors}_{\mathbf{W}(\mathcal{L}ie(G))}(C)$.

5. UNIFORMIZATION AND LOCAL TRIVIALITY

5.1. Loop groups. We continue with the framework of the previous section and assume from now on for simplicity that $S = \text{Spec}(R)$ is affine noetherian. Examples of our setting are listed below.

Examples 5.1. (a) If $S = \text{Spec}(R)$ with R noetherian local, this condition is equivalent to that the curve C is R -flat and the geometric fibers of C are reduced and connected [EGAIII, 7.8.6].

(b) We assume in the sequel that R is a DVR with fraction field K and residue field κ .

(b1) If C_K is furthermore geometrically integral, the map $\mathcal{O}_S \rightarrow p_*(\mathcal{O}_C)$ is an isomorphism [L, 8.3.6].

(b2) We assume furthermore that the scheme C is normal, satisfies $\mathcal{O}_S \xrightarrow{\sim} p_*(\mathcal{O}_C)$ and that the g.c.d. of the geometric multiplicities of the irreducible components C_κ is prime to the characteristic exponent of κ . Then C/R is cohomologically flat [Ra, Introduction].

(b3) We assume that furthermore C is normal, that C_K is geometrically integral, the g.c.d of the geometric multiplicities of the irreducible components C_κ is prime to the characteristic exponent of κ . By combining (b1) and (b2), we get that $\mathcal{O}_S \xrightarrow{\sim} p_*(\mathcal{O}_C)$ and that C/R is cohomologically flat.

We are given a finite S -scheme D with a closed embedding $s : D \rightarrow C$ such that

- (i) D is an effective Cartier divisor which is ample;
- (ii) s factorizes through an affine R -subscheme V of C .

Note that (i) implies that the open complement $V' = C \setminus D$ is affine over S . Let $V = \text{Spec}(A)$, $V' = \text{Spec}(A')$ and $V \cap V' = \text{Spec}(A_\#)$; this intersection is affine because the morphism $C \rightarrow S$ is a separated [St, 25.21.7.(1), Tag 01KP]. We denote by $I \subset A$ the ideal defining D . We consider the completed ring $\widehat{A} := \widehat{A}_I = \varprojlim_n A/I^n$. We need some basic facts from commutative algebra (see [B:AC, III.4.3, th. 3 and prop. 8] for (a) and (b)).

- (a) \widehat{A} is noetherian and flat over A .
- (b) The assignment $\mathfrak{m} \mapsto \mathfrak{m}\widehat{A}$ provides a correspondence between the maximal ideals of A containing I and the maximal ideals of \widehat{A} ;
- (c) If R is semilocal so is \widehat{A} .

If R is local, the finite R -algebra A/I is semilocal so we get (c) from (b).

We recall that G is a smooth affine group scheme over C , it admits a linear representation $i : G \rightarrow \mathrm{GL}(\mathcal{E})$ where \mathcal{E} is a locally free coherent sheaf of constant rank such that i is a closed immersion and such that the fppf sheaf $\mathrm{GL}(\mathcal{E})/G$ is representable by an affine C -scheme. We consider the following R -functors defined for each R -algebra B by:

- (1) $L^+G(B) = G\left(\widehat{(A \otimes_R B)}_{I \otimes_R B}\right)$;
- (2) $LG(B) = G\left(\widehat{(A \otimes_R B)}_{I \otimes B} \otimes_A A_\sharp\right)$.

Example 5.2. (a) The simplest example of our situation is $C = \mathbb{P}_R^1 = V' \cup V = \mathrm{Spec}(R[t]) \cup \mathrm{Spec}(R[t^{-1}])$ and for D the point 0 of C . In this case, we have $A = R[t]$, $I = tA$, $A_\sharp = R[t, t^{-1}]$ and $\widehat{A} = R[[t]]$. For each R -algebra B , we have $\widehat{(A \otimes_R B)}_{I \otimes B} = B[[t]]$ and $\widehat{(A \otimes_R B)}_{I \otimes_R B} \otimes_A A_\sharp = B[[t]] \otimes_{R[t]} R[t, t^{-1}] = B[[t]][\frac{1}{t}]$. The standard notation for the last ring is $B((t))$.

5.2. Patching. For simplicity we assume that $S = \mathrm{Spec}(R)$ where R is a noetherian ring. If we are given a R -ring B (not necessarily noetherian), we need to deal with the rings $\widehat{(A \otimes_R B)}_{I \otimes B}$ and $\widehat{(A \otimes_R B)}_{I \otimes B} \otimes_A A_\sharp$. As pointed out by Bhatt [Bh, §1.3], the Beauville-Laszlo theorem [B-L2] states that one can patch compatible quasi-coherent sheaves on $\mathrm{Spec}(\widehat{(A \otimes_R B)}_{I \otimes B})$ and V'_B to a quasi-coherent sheaf on C_B , provided the sheaves being patched are flat along $\mathrm{Spec}(B/IB)$. In particular the square of functors

$$\begin{array}{ccc} \mathcal{C}(C_B) & \longrightarrow & \mathcal{C}\left(\widehat{(A \otimes_R B)}_{I \otimes B}\right) \\ \downarrow & & \downarrow \\ \mathcal{C}(V'_B) & \longrightarrow & \mathcal{C}\left(\widehat{(A \otimes_R B)}_{I \otimes B} \otimes_A A_\sharp\right) \end{array}$$

is cocartesian where $\mathcal{C}(X)$ stands for the category of flat quasi-coherent sheaves over the scheme X (resp. the category of flat affine schemes over X). Note that if the ring B is noetherian, Ferrand-Raynaud's patching [F-R] (see also [M-B]) does the job.

Proposition 5.3. (1) *The square of functors*

$$\begin{array}{ccc} \mathrm{Tors}_G(C_B) & \longrightarrow & \mathrm{Tors}_G\left(\widehat{(A \otimes_R B)}_{I \otimes B}\right) \\ \downarrow & & \downarrow \\ \mathrm{Tors}_G(V'_B) & \longrightarrow & \mathrm{Tors}_G\left(\widehat{(A \otimes_R B)}_{I \otimes B} \otimes_A A_\sharp\right) \end{array}$$

is cocartesian.

(2) The S -functor LG represents the functor associating to each R -ring B the G -torsors over C_B together with trivializations on V'_B and on $\text{Spec}((\widehat{A \otimes_R B})_{I \otimes B})$.

Proof. (1) Since G is affine and flat over C , it is a formal corollary of the patching statement.

(2) Let $\mathcal{C}(B)$ be the the category of G -torsors over C_B together with trivializations on V'_B and on $\text{Spec}((\widehat{A \otimes_R B})_{I \otimes B})$. An object of \mathcal{C} is a triple (E, f_1, f_2) where E is a G_{C_B} -torsor, $f_1 : G_{V'_B} \xrightarrow{\sim} E_{V'_B}$ and $f_2 : G_{(\widehat{A \otimes_R B})_{I \otimes B}} \xrightarrow{\sim} E_{(\widehat{A \otimes_R B})_{I \otimes B}}$ are trivializations. An element $g \in LG(B) = G((\widehat{A \otimes_R B})_{I \otimes B} \otimes_A A_\sharp)$ gives rise to the right translation

$$(G_{V'_B})_{(\widehat{A \otimes_R B})_{I \otimes B} \otimes_A A_\sharp} \xrightarrow{\sim} (G_{(\widehat{A \otimes_R B})_{I \otimes B}})_{(\widehat{A \otimes_R B})_{I \otimes B} \otimes_A A_\sharp}.$$

It defines a G_C -torsor E_g with trivializations f_1 and f_2 on V'_B and on $\text{Spec}((\widehat{A \otimes_R B})_{I \otimes B})$. We get then a morphism $\Phi : LG(B) \rightarrow \mathcal{C}(B)$.

Conversely let $c = (T, f_1, f_2)$ be an object of $\mathcal{C}(B)$. Then the map $f_1^{-1}f_2 : G_{(\widehat{A \otimes_R B})_{I \otimes B} \otimes_A A_\sharp} \rightarrow G_{(\widehat{A \otimes_R B})_{I \otimes B} \otimes_A A_\sharp}$ is an isomorphism of G -torsors hence is the right translation by an element $g = \Psi(c) \in G((\widehat{A \otimes_R B})_{I \otimes B} \otimes_A A_\sharp)$. The functors Φ and Ψ provide the desired equivalence of categories. \square

Continuing with the R -ring B , we have a factorization

$$\begin{array}{ccc} LG(B) & \xrightarrow{p} & \text{Bun}_G(R) \\ \downarrow & & \downarrow \text{class map} \\ c_G(B) := G(V'_B) \setminus LG(B) / L^+G(B) & \xrightarrow{p} & H^1(C_B, G). \end{array}$$

The map p is called the uniformization map. Proposition 5.3.(2) implies that the bottom map induces a bijection

$$(*) \quad c_G(B) \xrightarrow{\sim} \ker \left(H^1(C_B, G) \rightarrow H^1(V_B, G) \times H^1((\widehat{A \otimes_R B})_{I \otimes R}, G) \right).$$

5.3. Link with the tangent space. Our goal is to differentiate the mapping $p : LG \rightarrow \text{Bun}_G$. Let B be a R -algebra and consider the map

$$p : LG(B[\epsilon]) \rightarrow \text{Bun}_G(B[\epsilon]).$$

We have $(\widehat{A \otimes_R B}[\epsilon])_{I \otimes R B[\epsilon]} = ((\widehat{A \otimes_R B})_{I \otimes B})[\epsilon]$ so that $LG(B[\epsilon]) = G((\widehat{A \otimes_R B})_{I \otimes B} \otimes_A A_\sharp)[\epsilon]$. We consider the commutative diagram of categories

$$\begin{array}{ccccc}
 0 \longrightarrow \mathrm{Lie}(G) \left(((\widehat{A \otimes_R B})_{I \otimes B} \otimes_A A_{\sharp}) \right) & \longrightarrow & G \left(((\widehat{A \otimes_R B})_{I \otimes B} \otimes_A A_{\sharp})[\epsilon] \right) & \longrightarrow & G \left(((\widehat{A \otimes_R B})_{I \otimes B} \otimes_A A_{\sharp}) \right) \\
 & & \downarrow & & \downarrow \\
 & & LG(B[\epsilon]) & \longrightarrow & LG(B) \\
 & & \downarrow & & \downarrow \\
 & & \mathrm{Bun}_G(B[\epsilon]) & \longrightarrow & \mathrm{Bun}_G(B)
 \end{array}$$

where the first line is the exact sequence defining the the Lie algebra. By considering the fiber at the trivial G -torsor $b \in \mathrm{Bun}_G(B)$, we get then a functor

$$p \circ e : \mathrm{Lie}(G) \left((\widehat{A \otimes_R B})_{I \otimes B} \right) \rightarrow \mathcal{T}_b(B).$$

Since $\mathrm{Lie}(G) \left((\widehat{A \otimes_R B})_{I \otimes B} \right) = \mathbf{W}(\mathcal{L}ie(G)) \left((\widehat{A \otimes_R B})_{I \otimes B} \right)$ we have a R -functor

$$dp : L\mathbf{W}(\mathcal{L}ie(G)) \rightarrow \mathcal{T}_b.$$

We use now the equivalence of categories between $\mathcal{T}_b(B)$ and $\mathrm{Tors}_{\mathbf{W}(\mathcal{L}ie(G))}(C_B)$ and get the following compatibility with the classifying maps

$$\begin{array}{ccc}
 L\mathbf{W}(\mathcal{L}ie(G))(B) \longrightarrow \mathbf{W}(\mathcal{L}ie(G))(V'_B) \setminus L\mathbf{W}(\mathcal{L}ie(G))(B) / L^+\mathbf{W}(\mathcal{L}ie(G))(B) & \longrightarrow & H^1(C_B, \mathbf{W}(\mathcal{L}ie(G))) \\
 \downarrow dp & \nearrow \text{class map} & \\
 \mathcal{T}_b(B) & &
 \end{array}$$

We observe that the $\mathbf{W}(\mathcal{L}ie(G))$ -torsors over affine schemes are trivial so that the top right map is an isomorphism according to the fact (*) above. Also $H^1(C_B, \mathbf{W}(\mathcal{L}ie(G))) = H^1(C_B, \mathbf{W}(\mathcal{L}ie(G)))$ identifies with the coherent cohomology of the \mathcal{O}_S -module $\mathcal{L}ie(G)$ [Mn, prop. III.3.7].

5.4. Heinloth's section. This statement is a variation over a local henselian noetherian base of a result due to Heinloth when the residue field is algebraically closed [He1, cor. 8].

Proposition 5.4. *Assume that $S = \mathrm{Spec}(R)$ with R local noetherian henselian with residue field κ which is infinite (resp. finite of characteristic $\neq 2$). We assume that G is semisimple and that its fundamental group is smooth over R . We assume that G_{D_κ} admits a strictly proper parabolic D_κ -subgroup (resp. is split).*

(1) *There exists a map $F : \mathbf{A}_R^n \rightarrow LG$ such that the composite*

$$f : \mathbf{A}_R^n \xrightarrow{F} LG \xrightarrow{p} \mathrm{Bun}_G$$

is a map of stacks, maps 0_R to the trivial G -torsor b and such that

$$df_{0_R} : R^n \rightarrow T_{\mathrm{Bun}_G, b}(R)$$

is essentially surjective. Furthermore there exists a neighborhood \mathcal{N} of 0_κ in \mathbf{A}_R^n such that $f|_{\mathcal{N}}$ is smooth.

(2) Let E be a G -bundle over C such that $E \times_C C_\kappa$ is trivial. Then E is trivial on V' .

Proof. (1) We proceed first to the case of infinite residue field. The proof goes by a differential argument. The R -module $H^1(C, \mathrm{Lie}(G))$ is finitely generated over R [Ha, III.5.2] and we lift a generating family of $H^1(C, \mathrm{Lie}(G))$ to a family of elements Y_1, \dots, Y_r of $\mathrm{Lie}(G)(\widehat{A} \otimes_A A_\#)$. We have noticed that \widehat{A} is a semilocal noetherian ring (Ex. 5.2.(b)). We want now to apply Lemma 3.2 to $G_{\widehat{A}}$ with respect to the closed points of $\mathrm{Spec}(\widehat{A})$. Since the scheme of parabolic groups of G is smooth [SGA3, XXVI.3.5], the Hensel lemma shows that G_D admits a strictly proper parabolic D -subgroup scheme. The same argument shows that $G_{\widehat{A}}$ admits a strictly proper parabolic \widehat{A} -subgroup scheme P . We put $U = \mathrm{rad}(P)$, it is a smooth affine \widehat{A} -group scheme. We denote by U' its last R -subgroup scheme of Demazure's filtration [SGA3, XXVI.2.1]. Lemma 3.2 provides elements $g_1, \dots, g_m \in G^{+,P}(\widehat{A})$ such that the product map

$$h : U'^m \rightarrow G, \quad (u_1, \dots, u_m) \mapsto {}^{g_1}u_1 \dots {}^{g_m}u_m$$

induces a surjective differential

$$dh : \mathrm{Lie}(U')^m(\widehat{A}) \rightarrow \mathrm{Lie}(G)(\widehat{A}), \quad (X_1, \dots, X_m) \rightarrow {}^{g_1}X_1 + \dots + {}^{g_m}X_m.$$

In other words we have

$$\mathrm{Lie}(G)(\widehat{A}) = {}^{g_1}\mathrm{Lie}(U')(\widehat{A}) + {}^{g_2}\mathrm{Lie}(U')(\widehat{A}) + \dots + {}^{g_m}\mathrm{Lie}(U')(\widehat{A})$$

so that (using the identity of Lemma 8.3.(2))

$$\begin{aligned} \mathrm{Lie}(G)(\widehat{A} \otimes_A A_\#) &= \mathrm{Lie}(G)(\widehat{A}) \otimes_A A_\# \\ &= {}^{g_1}\mathrm{Lie}(U')(\widehat{A}) \otimes_A A_\# + {}^{g_2}\mathrm{Lie}(U')(\widehat{A}) \otimes_A A_\# + \dots + {}^{g_m}\mathrm{Lie}(U')(\widehat{A}) \otimes_A A_\#. \end{aligned}$$

We can write

$$(**) \quad Y_i = \sum_{j=1, \dots, m} c_{i,j} {}^{g_j}Z_{i,j}$$

where $Z_{i,j} \in \mathrm{Lie}(U')(\widehat{A})$ and $c_{i,j} \in \widehat{A} \otimes_A A_\#$ for each j .

Since U' is a \widehat{A} -vector group scheme, there is a canonical identification $\exp : \mathbf{W}(\mathrm{Lie}(U')) \xrightarrow{\sim} U'$, $X \mapsto \exp(X)$. We consider the polynomial ring $B = R[t_{i,j}]$ where $i = 1, \dots, r, j = 1, \dots, m$. We consider the map of R -functors $F : \mathbf{A}_R^{rm} \rightarrow LG$ defined by the element

$$\prod_{i=1, \dots, r, j=1, \dots, m} {}^{g_i}\exp\left(t_{i,j} c_{i,j} Z_{i,j}\right) \in G\left(\widehat{(A \times_R B)}_{I \otimes_R B} \otimes_A A_\#\right) = LG(B)$$

where we can take for example the lexicographic order. It induces a R -map $f : \mathbf{A}_R^{rm} \rightarrow \text{Bun}_G$ of stacks mapping 0_R to the trivial G -bundle. Taking into account the last compatibility of §5.3, its differential at 0_R

$$df : R^{rm} \rightarrow \mathcal{T}_b(R)$$

factorizes through $L\mathbf{W}(\mathcal{L}ie(G))(B)$. More precisely we have a commutative diagram

$$\begin{array}{ccccc} R^{rm} & \xrightarrow{h} & L\mathbf{W}(\mathcal{L}ie(G))(R) & \longrightarrow & H^1(C, \text{Lie}(G)) \\ \downarrow df_0 & \nearrow dp & & \nearrow \text{class map} & \\ \mathcal{T}_b(R) & & & & \end{array}$$

where h maps the basis element $e_{i,j} \in R^{mr}$ to $c_{i,j} g_j Z_{i,j}$ in $L\mathbf{W}(\mathcal{L}ie(G))(R)$. We take into account the identity (**). By R -linearity, the image of $R^{mr} \rightarrow H^1(C, \text{Lie}(G))$ contains all Y_i 's. Since the Y_i 's generate the R -module $H^1(C, \text{Lie}(G))$, we conclude that df_0 is essentially surjective.

The formation of $R^1 p_* \mathcal{L}ie(G)$ commutes with base change, we have an isomorphism $H^1(C, \mathcal{L}ie(G)) \otimes_R \kappa \xrightarrow{\sim} H^1(C_0, \mathcal{L}ie(G))$ so that $df_{0,k} : k^{mr} \rightarrow H^1(C_0, \text{Lie}(G))$ is onto as well.

It follows that f is smooth locally at 0_κ according to the Jacobian smoothness criterion 8.5 stated in the appendix. Thus there is \mathcal{N} as claimed in the statement.

In the finite residued field case, our assumption is stronger so that we can assume that G_D is split so that $G_{\hat{A}}$ admits a strictly Borel \hat{A} -subgroup scheme B . Then Lemma 3.2 still work and the proof is verbatim the same.

(2) We see E as an object of $\text{Bun}_G(R)$ and consider the fiber product

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{f} & \text{Bun}_G \\ \uparrow & & \uparrow E \\ Y & \xrightarrow{\pi} & \text{Spec}(R). \end{array}$$

Then Y is an R -algebraic space [O, 8.16] which is smooth over R . Since Bun_G is quasi-separated over $\text{Spec}(R)$, f is quasi-separated (use [St, 92.3, Tag 04XB]). By base change, it follows that π is quasi-separated. The Hensel Lemma 8.1 for algebraic spaces shows that $Y(R) \neq \emptyset$. It follows that there exists $u \in \mathcal{N}(R)$ which maps to E . Since the map $\mathcal{N} \rightarrow \text{Bun}_G$ factorizes through LG , we conclude that the G -torsor E is trivial on V' . \square

6. MAIN RESULT

Theorem 6.1. *Assume that $S = \text{Spec}(R)$ with R local henselian noetherian of residue field κ which is assumed infinite (resp. finite of characteristic $\neq 2$). Let $f : C \rightarrow \text{Spec}(R) = S$ be a flat projective curve and assume that one of the following holds:*

- (I) C is smooth with geometrically connected fibers;
- (II) R is a DVR and the map $\mathcal{O}_S \rightarrow f_*\mathcal{O}_C$ is universally an isomorphism.

We assume that

- (i) The fundamental group μ/C of G is étale;
- (iii) $G_{C_\kappa}^{\text{sc}}$ is generically quasi-split.

Then the following hold:

- (1) Let E be a G -torsor over C such that $E \times_C C_\kappa$ is trivial. Then E is locally trivial for the Zariski topology.
- (2) Let E, E' be two G -torsors over C such that ${}^E G_{C_\kappa}$ is generically quasi-split and such that $E \times_C C_\kappa$ is isomorphic to $E' \times_C C_\kappa$. Then E and E' are locally isomorphic for the Zariski topology.

Remarks 6.2. (a) Note that $\mathcal{O}_S \rightarrow f_*\mathcal{O}_C$ is universally an isomorphism in case (I) and that case (II) includes the case when C is normal, satisfies $\mathcal{O}_S \rightarrow f_*\mathcal{O}_X$ and the *g.c.d.* of the geometric multiplicities of the irreducible components of C_κ is prime to the characteristic exponent of k .

(b) Once again the technical assumptions (i) and (ii) are satisfied if κ is of characteristic zero or if the characteristic of κ is not a torsion prime for the type of G .

We need the following auxiliary lemma.

Lemma 6.3. *Let X be a smooth geometrically connected curve defined over a field F and X_0 the set of closed points of X . Let H be a reductive X -group scheme such that $H_{F_s(X)}$ is quasi-split (resp. split). Then the set*

$$X(qs) = \left\{ x \in X_0 \mid H_{F(x)} \text{ is quasi-split (resp. split) and } F(x) \text{ is separable over } F \right\}$$

is dense in X .

Proof. We consider first the particular case when X is an open subset of the affine line \mathbf{A}_F^1 . Our assumption on H in 6.3 provides a finite separable field extension L/F such that $H_{L(X)}$ is quasi-split. Since the X -scheme of Borel subgroups of H is projective, it follows that H_{X_L} admits a Borel subgroup. Hence for each point $x \in X(L)$, the L -group H_x is quasi-split. We consider the set

$$\Sigma(L) = \left\{ x \in X_0 \mid F(x) \cong L \right\}.$$

We have $\Sigma(L) \subset X(L)$ so that $\Sigma(L) \subset X(qs)$. If F is infinite, $\Sigma(L)$ is dense in X and we are done. If F is finite, we need to take the union of the $\Sigma(L')$ for L' running over the finite extensions of L . We conclude that $X(qs)$ is dense in X .

We come to the general case. Up to shrinking X , we may assume that there exists a finite étale morphism $f : X \rightarrow Y$ where Y is an open subset of the affine line \mathbf{A}_k^1 . We put $G = R_{X/Y}(H)$, it is a reductive Y -group scheme. We have $f^{-1}(Y(qs)) \subset X(qs)$. By the preceding case, $Y(qs)$ is dense in Y . We conclude that $X(qs)$ is dense in X .

The proof of the split variant is the same by taking a finite separable field extension L/F such that $H_{L(X)}$ is split. \square

Remark 6.4. According to Steinberg's theorem [Se1, III.2.2], the assumption $H_{F_s(X)}$ is split in 6.3 is satisfied if F is perfect. It is also satisfied if the characteristic of F is a good prime for $DH_{F(X)}$ [Se2, §4.4].

For the finite field case, we have the following variant.

Lemma 6.5. *Let F be a finite field. Let X be a smooth geometrically connected curve defined over a field F . Let H be a semisimple X -group scheme. Then the set*

$$X(qs) = \left\{ x \in X_0 \mid H_{F(x)} \text{ is split} \right\}$$

is dense in X .

Proof. Let L/F be a finite field extension such that $H_{L(X)}$ is quasi-split. Let $\pi : Y \rightarrow X$ be the minimal Galois cover which splits the quasi-split form of H [SGA3, XXIV.3.11]. We apply Tchebotarev's density theorem [Ro, 9.13.A] to the Galois cover $\psi : Y_L \rightarrow Y \rightarrow X$. Then there exist infinitely many points $x \in X_0$ such that $\psi^{-1}(x)$ consists of $\deg(\psi)$ -points with residue fields $k(x)$. For such a point, H_x is quasi-split and is an inner form whence is split. \square

We can proceed to the proof of Theorem 6.1.

Proof. (1) The infinite residue field case is the main one and the finite case is added in parenthesis (with resp.). Let Θ be set of irreducible components of C_κ and denote by C_κ^θ the component attached to $\theta \in \Theta$. Lemma 6.3 (resp. 6.5) provides in two fully distinct families of closed smooth points $(c_1^\theta)_{\theta \in \Theta}$ and $(c_2^\theta)_{\theta \in \Theta}$ of $C_\kappa \setminus D_\kappa$ such that $G_{c_i^\theta}$ is a quasi-split (resp. split) semisimple $\kappa(c_i^\theta)$ -group for $i = 1, 2$ and each $\theta \in \Theta$.

Case (I). Our assumption is that C is smooth over R . Hensel lemma shows that each c_i^θ lifts in a closed R -subscheme $D_i^\theta \rightarrow C$ which is finite étale over R . We put $D_i = \bigsqcup_{\theta \in \Theta} D_i^\theta$ for $i = 1, 2$. Since C is projective over R and D_i is semilocal, D_i is a closed R -subscheme of an affine open R -subscheme of C .

Claim 6.6. *The scheme $C \setminus D_i$ is affine for $i = 1, 2$.*

The point is that $D_{i,\kappa}$ is an ample divisor of the curve C_κ so that D_i is an ample divisor of C [EGAIII, 4.7.1]. We have that $G \times D_{i,\kappa}$ admits a Borel subgroup (resp. is split) for $i = 1, 2$.

Now let E be a G -torsor over C such that $E \times_C C_\kappa$ is trivial. Proposition 5.4.(2) shows that $E|_{C \setminus D_i}$ is trivial for $i = 1, 2$. Since $C = (C \setminus D_1) \cup (C \setminus D_2)$, we conclude that the G -torsor E is locally trivial for the Zariski topology.

Case (II). In this case R is a henselian DVR. According to [L, §8.3, lemma 3.35], there exists an effective Cartier “horizontal” divisor D_i^θ of C (note it is finite over S) such that $C_\kappa \cap \text{Supp}(D_i^\theta) = C_i^\theta$ for each $\theta \in \Theta$ and $i = 1, 2$. We consider the Cartier effective divisors $D_i = \bigsqcup_{\theta \in \Theta} D_i^\theta$ for $i = 1, 2$, D_i is finite over S . The argument of case

(I) shows that $C \setminus D_i$ is affine and that D_i embeds in an affine R -subscheme of C . Since $G_{\kappa(c_i)}$ admits a Borel subgroup (resp. is split), we have that $G \times D_{i,\kappa}$ admits a Borel subgroup (resp. is split) for $i = 1, 2$ by using the smoothness of the scheme of Borel subgroups of G (resp. of the scheme $\text{Isom}(G_0, G)$ where G_0 is the Chevalley form of G). Repeating verbatim the argument of Case (I) finishes the proof.

(2) We apply (1) to the twisted R -group scheme ${}^E G$.

□

The case of a smooth curve defined over a field is of special interest.

Corollary 6.7. *Let X be a smooth projective algebraic curve defined over a field k . Let R be a local noetherian henselian k -ring whose residue field κ is infinite or finite of characteristic $\neq 2$. Let G be a semisimple R -group scheme whose fundamental group is étale over X .*

(1) *Let E be a G -torsor over X_R such that $E \times_X X_\kappa$ is trivial. Then E is locally trivial for the Zariski topology.*

(2) *Let E, E' be G -torsor over X_R such that $E \times_X X_\kappa$ is isomorphic to $E \times_{X_R} X_\kappa$. Then E and E' are locally isomorphic for the Zariski topology.*

7. EXTENSION TO REDUCTIVE GROUPS

We gather here our results in a single long statement.

Theorem 7.1. *Assume that $S = \text{Spec}(R)$ with R local henselian noetherian of residue field κ which is infinite or finite of characteristic $\neq 2$. Let p the characteristic exponent of κ . Let $f : C \rightarrow \text{Spec}(R) = S$ be a flat projective curve which is integral. Assume that one of the following holds:*

(I) *C is smooth with geometrically connected fibers;*

(II) *R is a DVR and $\mathcal{O}_S \rightarrow f_* \mathcal{O}_X$ is universally an isomorphism.*

Let G be a reductive C -group scheme and consider its presentation [SGA3, XXII.6.2.3]

$$1 \rightarrow \mu \rightarrow G^{\text{sc}} \times_C \text{rad}(G) \rightarrow G \rightarrow 1$$

where $\text{rad}(G)$ is the radical C -torus of G and G^{sc} is the simply connected universal cover of DG . We assume that

- (i) μ is étale over C ;
- (ii) the C -torus $\text{rad}(G)$ is split by a p' -Galois cover of the shape

$$X \xrightarrow{\pi} C \times_S \text{Spec}(R') \rightarrow C$$

where R'/R is a connected Galois cover and π is a finite Galois cover such that the map $\mathcal{O}_{S'} \rightarrow g_* \mathcal{O}_X$ is universally an isomorphism where $g = p_2 \circ \pi : X \rightarrow S' = \text{Spec}(R')$.

- (iii) $G_{C_{\kappa_s}}^{\text{sc}}$ is generically quasi-split.

Then the following hold:

(1) Let E be a G -torsor over C such that $E \times_C C_{\kappa}$ is trivial. Then E is locally trivial for the Zariski topology.

(2) Let E, E' be two G -torsors over C such that $({}^E G^{\text{sc}})_{C_{\kappa_s}}$ is generically quasi-split and such that $E \times_C C_{\kappa}$ is isomorphic to $E' \times_C C_{\kappa}$. Then E and E' are locally isomorphic for the Zariski topology.

Proof. We consider the following commutative diagram

$$\begin{array}{ccccccc} H^1(C, \mu) & \longrightarrow & H^1(C, G^{\text{sc}}) \times H^1(C, T) & \longrightarrow & H^1(C, G) & \longrightarrow & H^2(C, \mu) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(C_{\kappa}, \mu) & \longrightarrow & H^1(C_{\kappa}, G^{\text{sc}}) \times H^1(C_{\kappa}, T) & \longrightarrow & H^1(C_{\kappa}, G) & \longrightarrow & H^2(C_{\kappa}, \mu), \end{array}$$

where the horizontal lines are exact sequences of pointed sets. On the other hand, the proper base change theorem for étale cohomology shows that the maps $H^i(C_{\kappa}, \mu) \rightarrow H^i(C_{\kappa}, \mu)$ are bijective for $i = 1, 2$ [SGA4, XII.5.5.(iii)]. By diagram chase, it follows that the map

$$\begin{aligned} & \ker(H^1(C, G^{\text{sc}}) \rightarrow H^1(C_{\kappa}, G^{\text{sc}})) \times \ker(H^1(C, T) \rightarrow H^1(C_{\kappa}, T_{\kappa})) \\ & \rightarrow \ker(H^1(C, G) \rightarrow H^1(C_{\kappa}, G)) \end{aligned}$$

is onto. The first kernel (resp. the second one) consists of Zariski locally trivial according to Theorem 6.1 (resp. Proposition 2.2.(3) and Remark 2.3). Thus the third kernel consists of Zariski locally trivial. The assertion (2) follows by twisting. \square

8. APPENDIX: FACTS ON SMOOTHNESS

The purpose of this appendix is to provide proofs to statements which are well-known among experts.

8.1. Hensel lemma for algebraic spaces. Rydh proved several generalizations for étale morphisms of algebraic spaces from the case of schemes [Ry, app. A] including the Hensel lemma. Our purpose here is to prove the following variant of [EGAIV, 18.5.17].

Proposition 8.1. *Let R be a henselian local ring with residue field κ . Let $f : X \rightarrow Y$ be a smooth morphism of (quasi-separated) R -algebraic spaces. Let $y \in Y(R)$ and let $x_0 \in X(\kappa)$ with the image y_0 of y by $Y(R) \rightarrow Y(\kappa)$ is the image of x_0 by $X(\kappa) \rightarrow Y(\kappa)$. Then there exists $x \in X(R)$ mapping to y and x_0 .*

Proof. Up to pull-back everything by $y : \text{Spec}(R) \rightarrow Y$, we can assume that $Y = \text{Spec}(R)$. Now we use that there is an étale morphism $f : U \rightarrow X$ where U is an affine scheme such that $x_0 = f(u_0)$ for some $u_0 \in U(\kappa)$ [K, II.6.4]. By composition, the morphism $U \rightarrow Y = \text{Spec}(R)$ is smooth and the usual Hensel lemma applies [EGAIV, 18.5.17]. \square

8.2. Lie algebra of a S -group space. Let $f : X \rightarrow Y$ be a morphism of S -algebraic spaces. We consider the quasi-coherent sheaf $\Omega_{Y/X}^1$ on Y defined in [St, 68.1.2, Tag 04CT]. Let T be a S -scheme equipped with a closed subscheme T_0 defined by a quasi-coherent ideal \mathcal{I} such that $\mathcal{I}^2 = 0$. According to [O, 7.A page 167] for any commutative diagram of algebraic spaces

$$\begin{array}{ccc} T_0 & \xrightarrow{x_0} & X \\ \downarrow & \swarrow \text{dotted} & \downarrow f \\ T & \xrightarrow{y} & Y \end{array}$$

if there exists a dotted arrow filling in the diagram then the set of such dotted arrows form a torsor under $\text{Hom}_{\mathcal{O}_{T_0}}(x^*\Omega_{Y/X}^1, \mathcal{I})$. We extend to group spaces well-known statements on group schemes [SGA3, II.4.11.3].

Lemma 8.2. *Let G be a S -group space. We denote by $e_G : S \rightarrow G$ the unit point and put $\omega_{G/S} = e_G^*(\Omega_{G/S}^1)$.*

- (1) *There is a canonical isomorphism of S -functors $\text{Lie}(G) \xrightarrow{\sim} \mathbf{V}(\omega_{G/S})$ which is compatible with the \mathcal{O}_S -structure.*
- (2) *If $\omega_{G/S}$ is a locally free coherent sheaf, then $\text{Lie}(G) \xrightarrow{\sim} \mathbf{W}(\omega_{G/S}^\vee)$. In particular we have an isomorphism*

$$\text{Lie}(G)(R) \otimes_R R' \xrightarrow{\sim} \text{Lie}(G)(R')$$

for each morphism of S -rings $R \rightarrow R'$.

- (3) *Assume that G is smooth and quasi-separated over S . Then $\omega_{G/S}$ is a finite locally free coherent sheaf and (2) holds.*

Under the conditions of (2) or (3), we denote also by $\mathcal{L}ie(G) = \omega_{G/S}^\vee$ the locally free coherent sheaf.

Proof. (1) Let T_0 be a S -scheme and consider $T = T_0[\epsilon]$. We apply the above fact to the morphism $G \rightarrow S$ and the points $x_0 = e_{G_{T_0}}$ and $y : T \rightarrow S$ the structural morphism. It follows that $\ker(G(T) \rightarrow G(T_0))$ is a torsor under $\mathrm{Hom}_{\mathcal{O}_{T_0}}(e_{G_{T_0}}^* \Omega_{G/S}^1, \epsilon \mathcal{O}_{T_0}) \cong \mathrm{Hom}_{\mathcal{O}_{T_0}}(e_{G_{T_0}}^* \Omega_{G/S}^1, \mathcal{O}_{T_0}) = \mathrm{Hom}_{\mathcal{O}_{T_0}}(\omega_{G/S}^1 \otimes_{\mathcal{O}_S} \mathcal{O}_{T_0}, \mathcal{O}_{T_0})$. We have constructed an isomorphism of S -functors $\mathrm{Lie}(G) \xrightarrow{\sim} \mathbf{V}(\omega_{G/S})$ and the compatibility of \mathcal{O}_S -structures is a straightforward checking.

(2) If $\omega_{G/S}$ is a locally free coherent sheaf, then $\mathrm{Lie}(G) \xrightarrow{\sim} \mathbf{V}(\omega_{G/S}) \xrightarrow{\sim} \mathbf{W}(\omega_{G/S}^\vee)$. The next fact follows from [EGAIII, 12.2.3].

(3) According to [St, 68.7.16, Tag 0CK5], $\Omega_{G/S}^1$ is a finite locally free coherent sheaf over G . It follows that $\omega_{G/S}^1$ is a finite locally free coherent sheaf over S . \square

Lemma 8.3. *Let G be a smooth S -group space and let T be a S -scheme equipped with a closed subscheme T_0 defined by a quasi-coherent ideal \mathcal{I} such that $\mathcal{I}^2 = 0$. We denote by $t_0 : T_0 \rightarrow S$ the structural morphism, $G_0 = G \times_S T_0$ and assume that t_0 is quasi-compact and quasi-separated.*

(1) *We have an exact sequence of fppf (resp. étale, Zariski) sheaves on S*

$$0 \rightarrow \mathbf{W}\left((t_0)_*(\mathcal{L}ie(G_0) \otimes_{\mathcal{O}_{T_0}} \mathcal{I})\right) \rightarrow \prod_{T/S} G \rightarrow \prod_{T_0/S} G \rightarrow 1.$$

(2) *If $T = \mathrm{Spec}(A)$ is affine and $T_0 = \mathrm{Spec}(A/I)$, we have an exact sequence*

$$0 \rightarrow \mathrm{Lie}(G)(A) \otimes_A I \rightarrow G(A) \rightarrow G(A/I) \rightarrow 1.$$

Remarks 8.4. (a) A special case of (1) is $T = S[\epsilon]$ and $T_0 = S$. We get an exact sequence of fppf (resp. étale, Zariski) sheaves on S

$$0 \rightarrow \mathbf{W}(\mathcal{L}ie(G)) \rightarrow \prod_{S[\epsilon]/S} G \rightarrow G \rightarrow 1.$$

(b) In the group scheme case, (2) is established in [D-G, proof of II.5.2.8].

Proof. (1) We have

$$\mathrm{Hom}_{\mathcal{O}_{T_0}}(\omega_{G_0/T_0}, \mathcal{I}) = H^0(T_0, \mathcal{L}ie(G_0) \otimes_{\mathcal{O}_{T_0}} \mathcal{I}) = H^0\left(T, t_{0,*}(\mathcal{L}ie(G_0) \otimes_{\mathcal{O}_{T_0}} \mathcal{I})\right),$$

whence an exact sequence

$$0 \rightarrow H^0\left(T, t_{0,*}(\mathcal{L}ie(G_0) \otimes_{\mathcal{O}_{T_0}} \mathcal{I})\right) \rightarrow G(T) \rightarrow G(T_0).$$

Now let $h : S' \rightarrow S$ be a flat morphism locally of finite presentation and denote by $G' = G \times_S S'$, $h_T : T' \rightarrow T, \dots$ the relevant base change to S' . Since t_0 is quasi-compact and quasi-separated, the flatness of h yields an isomorphism [St, 28.5.2, Tag

02KH]

$$h^*(t_{0,*}(\mathcal{L}ie(G_0) \otimes_{\mathcal{O}_{T_0}} \mathcal{I})) \xrightarrow{\sim} t'_{0,*} \left(h_{T_0}^* (\mathcal{L}ie(G_0) \otimes_{\mathcal{O}_{T_0}} \mathcal{I}) \right) = t'_{0,*} (\mathcal{L}ie(G'_0) \otimes_{\mathcal{O}_{T'_0}} \mathcal{I}').$$

The similar sequence for T'_0 reads then

$$0 \rightarrow H^0 \left(T', t_{0,*} (\mathcal{L}ie(G_0) \otimes_{\mathcal{O}_{T_0}} \mathcal{I}) \right) \rightarrow G(T') \rightarrow G(T'_0).$$

We have then an exact sequence of fppf sheaves

$$0 \rightarrow \mathbf{W} \left((t_0)_* (\mathcal{L}ie(G_0) \otimes_{\mathcal{O}_{T_0}} \mathcal{I}) \right) \rightarrow \prod_{T/S} G \rightarrow \prod_{T_0/S} G$$

For T an affine scheme, the map $G(T) \rightarrow G(T_0)$ is onto since the smooth S -group space G is formally smooth [St, 68.19.6, Tag 04AM]. It implies the exactness for the Zariski, étale and fppf topologies.

(2) We can assume that $S = T = \text{Spec}(A)$. In this case, we have

$$H^0 \left(S, (t_0)_* (\mathcal{L}ie(G_0) \otimes_{\mathcal{O}_{T_0}} \mathcal{I}) \right) = H^0(T_0, \mathcal{L}ie(G_0) \otimes_{\mathcal{O}_{T_0}} \mathcal{I}) = \text{Lie}(G_0)(A/I) \otimes_{A/I} I.$$

We have $\text{Lie}(G_0)(A/I) = \text{Lie}(G)(A) \otimes_A A/I$ in view of Lemma 8.2.(2) whence the identification $\text{Lie}(G_0)(A/I) \otimes_{A/I} I \cong \text{Lie}(G)(A) \otimes_A I$. Then (1) provides the exact sequence

$$0 \rightarrow \text{Lie}(G)(A) \otimes_A I \rightarrow G(A) \rightarrow G(A/I)$$

and the right map is onto since G is smooth. \square

8.3. Jacobian criterion for stacks. Let S be a scheme and let \mathcal{X}, \mathcal{Y} be quasi-separated algebraic S -stacks of finite presentation. Let $g : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism. We have a 1-morphism $Tg : T(\mathcal{X}) \rightarrow T(\mathcal{Y})$ of algebraic stacks [L-M-B, 17.14, 17.16].

Let $s \in S$ and denote by K the residue field of s . Let $x : \text{Spec}(K) \rightarrow \mathcal{X}$ be a 1-morphism mapping to s . We put $T(\mathcal{X})_x = T(\mathcal{X}/S) \times_{\mathcal{X}} \text{Spec}(K)$ and denote by $\text{Tan}_x(\mathcal{X})$ the category $T(\mathcal{X})_x(K)$. We denote by $y = h \circ x : \text{Spec}(K) \rightarrow \mathcal{Y}$ and get the tangent morphism $(Tg)_x : \text{Tan}_x(\mathcal{X}) \rightarrow \text{Tan}_y(\mathcal{Y})$.

Proposition 8.5. *We assume that \mathcal{X} is smooth at x over S . Then the following assertions are equivalent:*

- (i) *The morphism g is smooth at x ;*
- (ii) *The tangent morphism $(Tg)_x : \text{Tan}_x(\mathcal{X}) \rightarrow \text{Tan}_y(\mathcal{Y})$ is essentially surjective.*

Furthermore, under those conditions, \mathcal{Y} is smooth at y over S .

Proof. In the case of a morphism $g : X \rightarrow Y$ of S -schemes locally of finite presentation such that $g(x) = y$ and X is smooth at x over S , we have that $K = \kappa(x) = \kappa(y)$ so that the statement is a special case of [EGAIV, 17.11.1]. We proceed now to the stack case.

(i) \implies (ii). Up to shrinking, we can assume that \mathcal{X} is smooth over S and that g is smooth. Also g is formally smooth [St, 98.8.4, Tag 0DNN] that is satisfies the

relevant infinitesimal lifting criterion. It applies in particular to $K[\epsilon]$ whence the essential surjectivity of the tangent morphism.

(ii) \implies (i). According to [L-M-B, Thm. 6.3], there exists a smooth 1-morphism $\varphi : Y \rightarrow \mathcal{Y}$ and a point $y_1 \in Y(K)$ mapping to y such that Y is an affine scheme. We note that $K = \kappa(y_1)$. We consider the fiber product $\mathcal{X}' = \mathcal{X} \times_{\mathcal{Y}} Y$, it is an algebraic stack and there exists a 1-morphism $x' : \text{Spec}(K) \rightarrow \mathcal{X}'$ lifting x and y_1 . There exists a smooth 1-morphism $\psi : X' \rightarrow \mathcal{X}'$ and a point $x_1 \in X'(K)$ mapping to x such that X' is an affine scheme. By construction we have again that $K = \kappa(x_1)$. We have then the commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{\psi} & \mathcal{X}' & \xrightarrow{g'} & Y \\ & & \downarrow & & \downarrow \varphi \\ & & \mathcal{X} & \xrightarrow{g} & \mathcal{Y}. \end{array}$$

According to [L-M-B, Lem. 17.5.1], the square

$$\begin{array}{ccc} T(\mathcal{X}'/S) & \xrightarrow{Tg'} & T(Y/S) \\ \downarrow & & \downarrow T\varphi \\ T(\mathcal{X}/S) & \xrightarrow{Tg} & T(\mathcal{Y}/S) \end{array}$$

is 2-cartesian. It follows that the square

$$\begin{array}{ccc} \text{Tan}_{x'}(\mathcal{X}') & \xrightarrow{(Tg')_{x'}} & \text{Tan}_{y_1}(Y) \\ \downarrow (T\psi)_{x_1} & & \downarrow (T\varphi)_{y_1} \\ \text{Tan}_x(\mathcal{X}) & \xrightarrow{(Tg)_x} & \text{Tan}_y(\mathcal{Y}) \end{array}$$

is 2-cartesian. Our assumption is that the bottom morphism is essentially surjective, it follows that $(Tg')_{x'} : \text{Tan}_{x'}(\mathcal{X}') \rightarrow \text{Tan}_{y_1}(Y)$ is essentially surjective as well. Since ψ is smooth, the map $(T\psi)_{x_1} : \text{Tan}_{x'}(X') \rightarrow \text{Tan}_{x_1}(\mathcal{X}')$ is essentially surjective. By composition it follows that $\text{Tan}_{x_1}(X') \rightarrow \text{Tan}_{y_1}(Y)$ is essentially surjective. Since X' and Y are locally of finite presentation over S , the case of schemes yields that $g' \circ \psi : X' \rightarrow Y$ is smooth at x' . By definition of smoothness for morphisms of stacks [O, §8.2], we conclude that g is smooth at x .

We assume (ii) and shall show that \mathcal{Y} is smooth at y over S . Using the diagrams of the proof, we have seen that the S -morphism $X' \rightarrow Y$ of schemes is smooth at x' . Once again the classical Jacobian criterion [EGAIV, 17.11.1] applies and shows that Y is smooth at y_1 over S . By definition of smoothness for stacks, we get that \mathcal{Y} is smooth at y over S . \square

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