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Jacqueline Fleckinger, J.-P. Gossez, François de Thélin. Nodal structure of the solution of a cooperative elliptic system near an eigenvalue.. A tribute to Djairo Guedes de Figueiredo on the occasion of his 80th birthday. Cham: Birkhäuser/Springer (/hbk; 978-3-319-19902-3/ebook) 2015, 978-3-319-19902, 2015, Progress in Nonlinear Differential Equations and Their Applications, Springer, ISBN 978-3-319-19901-6. hal-01978076

HAL Id: hal-01978076

<https://hal.science/hal-01978076>

Submitted on 21 Jan 2019

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Nodal structure of the solution of a cooperative elliptic system near an eigenvalue

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We consider (in its variational formulation) the following $n \times n$ system defined on a smooth bounded domain $\Omega \subset \mathbb{R}^N$ with homogeneous Dirichlet boundary conditions:

$$(S_F) \quad -\Delta U = AU + \mu U + F \text{ in } \Omega, \quad U|_{\partial\Omega} = 0.$$

Here F is given with components $f_i \in L^p$, $p > N$, $1 \leq i \leq n$; U , with components u_i , $1 \leq i \leq n$ is the unknown. μ is a real parameter. We are interested by the change of sign of the solutions to (S_F) as μ goes over some critical values.

¹FGT280314.tex

1 Preliminaries

Matrix A :

A is a $n \times n$ cooperative matrix, which means that $a_{ij} \geq 0$ for $i \neq j$. Sometimes we assume it is a *strictly cooperative* matrix that is $a_{ij} > 0$ for $i \neq j$.

Definition 1 A square matrix B is a non singular M -matrix if it is of the form $\sigma I - C$ with $C \geq 0$ and $\sigma > \rho(C)$ the spectral radius of C .

Eigenvalues (ξ_k) of Matrix A , $k \in \{1, \dots, \hat{k}\}$:

Let us denote by ξ_k the real eigenvalues of A (written in a non increasing order) and by X_k the associated eigenvectors:

$$AX_k = \xi_k X_k. \tag{1}$$

$$1 \leq k \leq \hat{k} \leq n.$$

Eigenvalues (λ_s) of the Dirichlet Laplacian:

As usual we denote by $0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_s \leq \dots$ the eigenvalues of the Dirichlet Laplacian defined on Ω and by ϕ_s the orthonormal basis in $L^2(\Omega)$ of associated eigenfunctions; moreover, we choose ϕ_1 , associated to λ_1 , positive.

Eigenvalues $(\mu_{s,k})$ of System (S_F) :

We say that μ is an eigenvalue of (S_F) if there exists a non zero U satisfying

$$(S_0) : \quad -\Delta U = AU + \mu U \text{ in } \Omega, \quad U|_{\partial\Omega} = 0.$$

Proposition 1 With our above notations, the real eigenvalues of (S_F) are the numbers

$$\mu_{s,k} := \lambda_s - \xi_k, \quad \forall s \in \mathbb{N}^*, 1 \leq k \leq \hat{k}, \tag{2}$$

and the associated eigenvectors are $U = X_k \phi_s$.

2 Main Results

From now on we assume:

Hypothesis 1 We assume $\mu \neq \mu_{s,k}$ for any s and k .

Theorem 1 (Existence Result): *Let A be a cooperative matrix. We assume Hypothesis 1. Then, for $F \in (L^2(\Omega))^n$, System (S_F) has a unique solution U with components u_j in $H^2(\Omega) \cap H_0^1(\Omega)$.*

Consider System (S_F) involving Matrix A and fix some $s \geq 1$ with associated eigenpair (λ_s, ϕ_s) . For having nodal properties on the solution, we assume Hypotheses 2 and 3:

Hypothesis 2 *We assume $\mu_{s,k} = \mu_{s',k'} \Rightarrow \lambda_s = \lambda_{s'}$ and $\xi_k = \xi_{k'}$, or equivalently*

$$\xi_k \neq \xi_{k'} \text{ or } \lambda_s \neq \lambda_{s'} \Rightarrow \mu_{s,k} \neq \mu_{s',k'}.$$

Hypothesis 3 *We assume that ϕ_s has q nodal domains $\Omega^1, \dots, \Omega^I, \dots, \Omega^q$ and that these nodal domains enjoy the following two properties:*

(P₁) each Ω^I satisfies at each $x \in \partial\Omega^I$ the interior ball condition,

(P₂) for σ sufficiently small, say $0 < \sigma < \sigma_0$, each Ω_σ^I is arcwise connected, where $\Omega_\sigma^I := \{x \in \Omega^I : d(x, \partial\Omega^I) > \sigma\}$.

Remark 1 *Obviously if $s = 1$ then $q = 1$. In that case, (P_2) plays no role.*

Notations (r and \hat{r}): *Denote by r the largest integer l such that $\lambda_l < \lambda_s$ and by \hat{r} the smallest l such that $\lambda_l > \lambda_s$. Obviously if $s = 1$, r does not exist and $\hat{r} = 2$.*

We also introduce $\sigma_1 > 0$ such that

$$\begin{cases} d(\bar{\Omega}^I, \bar{\Omega}^J) \geq 4\sigma_1 \text{ if } \bar{\Omega}^I \cap \bar{\Omega}^J = \emptyset, \\ d(\bar{\Omega}^I, \partial\Omega) \geq 4\sigma_1 \text{ if } \bar{\Omega}^I \cap \partial\Omega = \emptyset. \end{cases} \quad (3)$$

Given $F \in (L^p(\Omega))^n$ with $p > N$, we suppose that F can be written as

(P₃) $F = \sum_{l=1}^r Y^l \phi_l + Z \phi_s + \hat{F}$, with Y^l and Z in \mathbb{R}^n and where the components of \hat{F} are orthogonal to the eigenspaces associated to $\lambda_1, \dots, \lambda_r, \lambda_s$.

Note that (P_3) is not a restriction on F when λ_s is simple.

We also fix some k such that $1 \leq k \leq \hat{k}$ where \hat{k} is defined in (1).

We will study the situation where μ stays near $\mu_{s,k}$ in the sense that $\mu_{r,k} < \mu < \mu_{s,k}$ or $\mu_{s,k} < \mu < \mu_{\hat{r},k}$ and we write $\mu = \mu_{s,k} + \eta$, $\eta \in \mathbb{R}$.

Notations ($B_i, M_i, n_k, m_k, \epsilon_k$): For this given k , denote by B_1, \dots, B_n the column vectors of matrix $B := \xi_k I - A$ and for $i = 1, \dots, n$,

$$M_i := \det(B_1, \dots, B_{i-1}, Z, B_{i+1}, \dots, B_n) \quad (4)$$

Denote by n_k the number of eigenvalues ξ_j of A such that $\xi_j > \xi_k$ and by m_k the multiplicity of ξ_k . Let ϵ_k be the sign of $(-1)^{n_k}(-\eta)^{m_k}$.

Let $U \in (H_0^1(\Omega))^n$ be the (unique) solution of (S_F) ; $U \in (W^{2,p}(\Omega))^n \subset (C^1(\bar{\Omega}))^n$.

Theorem 2 (Nodal Properties): *Let A be a $n \times n$ cooperative matrix and let s and k be as above. Assume Hypotheses 1, 2, 3 and let F satisfy (P_3) and suppose that there exists i_0 such that*

$$M_{i_0} > 0. \quad (5)$$

Take $\sigma > 0$ with $\sigma < \sigma_0$ and σ_1 . Then there exists $\delta = \delta(F, \sigma, i_0) > 0$ such that

(i) if $\mu_{s,k} - \delta < \mu < \mu_{s,k}$, then component u_{i_0} of U has exactly q nodal domains $\mathcal{O}_{i_0}^1, \dots, \mathcal{O}_{i_0}^I, \dots, \mathcal{O}_{i_0}^q$, such that

(i₁) $\Omega_\sigma^I \subset \mathcal{O}_{i_0}^I \subset \tilde{\Omega}_\sigma^I$ for $1 \leq I \leq q$, where $\tilde{\Omega}_\sigma^I := \{x \in \Omega : d(x, \Omega^I) < \sigma\}$,

(i₂) $(-1)^{n_k} u_{i_0}(x) \phi_s(x) > 0 \forall x \in \mathcal{O}_{i_0}^I \cap \Omega^I$ and any $1 \leq I \leq q$,

(i₃) if $\tilde{\mathcal{O}}_{i_0}^I \cap \tilde{\mathcal{O}}_{i_0}^J \neq \emptyset$ with $I \neq J$, then $u_{i_0}(x) u_{i_0}(y) < 0 \forall x \in \mathcal{O}_{i_0}^I, y \in \mathcal{O}_{i_0}^J$,

(ii) if $\mu_{s,k} < \mu < \mu_{s,k} + \delta$, then the same conclusion as in (i) above holds, with the only change that in (i₂) one now has $\epsilon_k u_{i_0}(x) \phi_s(x) > 0$ for all $x \in \mathcal{O}_{i_0}^I \cap \Omega^I$.

Remark 2 *We thus see that, for μ close to $\mu_{s,k}$, u_{i_0} looks like $\pm \phi_s$ in the sense that it has the same number of nodal domains, that each $\mathcal{O}_{i_0}^I$ appears as a small perturbation of the corresponding Ω^I (cf. (i₁)), with the same or opposite sign for u_{i_0} and ϕ_s on the intersection (cf. (i₂)). Moreover the sets $\mathcal{O}_{i_0}^I$ enjoy the property that a change of sign occurs when going from one $\mathcal{O}_{i_0}^I$ to an neighbouring one (cf. (i₃)); this latter property should be looked at as a regularity property (c.f. (3.7) in [7]).*

Remark 3 Hypothesis 3 is trivially satisfied for $N = 1$. In that case, we have a system of ODE and for any $s \geq 1$, λ_s is simple and $q = s$; the nodal domains are intervals. We extend here to a system previous results valid for one equation (Theorems 2.1 and 2.2 in [7]).

Remark 4 For example, if $N = 2$ and Ω is the unit ball, Hypothesis 3 is satisfied for $s = 1, 6, 15, \dots$ (See [7]). In that case, for $s = 2$, Hypothesis 3 is not satisfied but we conjecture that the result of Theorem 2 remains valid with the additional assumption $F \in (W^{1,p}(\Omega))^n$ (See [8] for $n = 1$). For $s = 4$, Hypothesis 3 is not satisfied and Theorem 2 is not valid (See [9] for $n = 1$).

Corollary 1 Let A be a cooperative matrix and $F \in (L^p(\Omega))^n$ with $p > N$. We assume Hypotheses 1 and 2 and that $M_{i_0} > 0$, for some i_0 . Then, there exists $\delta(F) > 0$ such that

- (i) For $\mu_{1,k} - \delta < \mu < \mu_{1,k}$, $(-1)^{n_k} u_{i_0} > 0$ in Ω .
- (ii) For $\mu_{1,k} < \mu < \mu_{1,k} + \delta$, $\epsilon_k u_{i_0} < 0$ in Ω .

Exemple 1 For $n = 2$ and A strictly cooperative , a simple calculus shows that A has exactly 2 distinct eigenvalues which are real:

$\xi_2 < \xi_1$ and therefore $n_1 = 0$, $n_2 = 1$, and $m_1 = m_2 = 1$, so that $\epsilon_1 = \text{sign}(-\eta)$ and $\epsilon_2 = -\text{sign}(-\eta)$. Assuming $M_1 > 0$ and $M_2 > 0$, we have

for $\mu_{1,1} - \delta < \mu < \mu_{1,1}$, $U \gg 0$, that is $u_1 > 0$ and $u_2 > 0$. (Maximum Principle).

for $\mu_{1,1} < \mu < \mu_{1,1} + \delta$, $U \ll 0$, that is $u_1 < 0$ and $u_2 < 0$. (Antimaximum Principle).

for $\mu_{1,2} - \delta < \mu < \mu_{1,2}$, $U \ll 0$, that is $u_1 < 0$ and $u_2 < 0$.

for $\mu_{1,2} < \mu < \mu_{1,2} + \delta$, $U \gg 0$, that is $u_1 > 0$ and $u_2 > 0$.

More details concerning this example are written at the end of this paper (Annex I).

3 Proofs

Proof of Proposition 1: Consider $U = X_k \phi_s$, one has

$$-\Delta U = -\Delta(X_k \phi_s) = X_k \lambda_s \phi_s \quad \text{and} \quad AU = A(X_k \phi_s) = (AX_k) \phi_s = \xi_k X_k \phi_s,$$

which shows that $\lambda_s - \xi_k$ is an eigenvalue of (S_F) .
 Conversely, assume μ is an eigenvalue of (S_F) . Hence

$$-\Delta u_i = \sum_j a_{ij} u_j + \mu u_i,$$

where at least one of the u_i say $u_j \neq 0$.
 Multiply by any ϕ_s ($s \in \mathbb{N}^*$) and integrate .

$$(\lambda_s - \mu)y_i = \sum_j a_{ij} y_j$$

where $y_i = \int u_i \phi_s$. Choosing s such that $y_j \neq 0$, we have for vector Y with components y_j

$$(\lambda_s - \mu)Y = AY$$

which means that $(\lambda_s - \mu)$ is an eigenvalue ξ_k of A . •

A derived system: Multiplying (S_F) by ϕ_s as above we obtain that the vector X with components $x_i = \int u_i \phi_s$ satisfies the following system

$$(\mathfrak{S}_{s,Z}) \quad ((\lambda_s - \mu)I - A)X = Z$$

where Z is the vector with components $z_i = \int f_i \phi_s$. Our proofs are based on the following lemma:

Lemma 1 : *Let A be a cooperative matrix, then there is some $\hat{\sigma}$ such that*

$$(\hat{\sigma}) \quad \text{for } \sigma \geq \hat{\sigma}, (\sigma I - A) \text{ is a non singular M - matrix.}$$

Moreover we have

$$\hat{\sigma} > \xi_1 \geq \xi_2 \geq \dots \geq \xi_k. \tag{6}$$

Proof of Lemma 1: Set $B = \tau I + A$ where $\tau > 0$ is such that $b_{ij} \geq 0$ for any i, j and take $\hat{\sigma} > \rho(B) - \tau$. It follows that for any $\sigma \geq \hat{\sigma}$, $(\sigma I - A) = (\tau + \sigma)I - B$ with $\tau + \sigma > \rho(B)$ so that $(\sigma I - A)$ is a non-singular M -matrix. Moreover, by Property D16 in [1] (see Annex II), all real eigenvalues of $(\hat{\sigma}I - A)$ are positive; that means

$$0 < \hat{\sigma} - \xi_1 \leq \hat{\sigma} - \xi_2 \leq \dots \leq \hat{\sigma} - \xi_k,$$

and the second result follows. •

We also use the following decomposition of spaces:

Definition 2 : The spaces E and H

For a given s_0 , we decompose the space $(L^2(\Omega))^n$ into 2 orthogonal spaces, one with finite dimension and the other one H is the set of $V \in (L^2(\Omega))^n$ with components v_i orthogonal to ϕ_s , $s \in \{1, \dots, s_0 - 1\}$, $i \in \{1, \dots, n\}$. Analogously we decompose the space $(H_0^1(\Omega))^n$ into 2 orthogonal spaces, one with finite dimension and the other one E is the set of $V \in (H_0^1(\Omega))^n$ with components v_i orthogonal to ϕ_s for $s \in \{1, \dots, s_0 - 1\}$, $i \in \{1, \dots, n\}$.

Proof of Theorem 1: We fix μ satisfying Hypothesis 1. Uniqueness is obvious; we only consider existence.

We choose s_0 such that we have simultaneously

$$\lambda_{s_0-1} < \lambda_{s_0}, \quad (7)$$

and

$$\lambda_{s_0} - \mu > \hat{\sigma}, \quad (8)$$

where $\hat{\sigma}$ is defined in Lemma 1. This is clearly possible since $\lambda_s \rightarrow +\infty$ as $s \rightarrow +\infty$.

The consequence of (8) is that $((\lambda_l - \mu)I - A)$ is a non singular M -matrix and therefore is invertible for $l \geq s_0$.

Claim: For any $G \in H$, System (S_G) has a unique solution $V \in E$.

Proof of the claim: For proving this we proceed by approximation in finite dimensional spaces.

Let E^r (resp. H^r) be the subspace of vectors in E (resp. H) with components generated by $\phi_{s_0}, \phi_{s_0+1}, \dots, \phi_{s_0+r}$.

We consider a sequence $G^r \in H^r$, $r \in \mathbb{N}^*$, such that $G^r \rightarrow G$ in $(L^2(\Omega))^n$ as $r \rightarrow \infty$.

We first seek a solution $V^r \in E^r$ of System (S_{G^r}) :

$$(S_{G^r}) \quad -\Delta V^r = AV^r + \mu V^r + G^r \text{ in } \Omega, \quad V^r|_{\partial\Omega} = 0.$$

Multiplying System (S_{G^r}) by ϕ_l we get for each $l \in \{s_0, s_0 + 1, \dots, s_0 + r\}$:

$$(\mathfrak{S}_{l, Y_l^r}) \quad ((\lambda_l - \mu)I - A) X = Y_l^r,$$

where $Y_l^r = \int G^r \phi_l$. Since, again by Hypothesis 1, the matrix $(\lambda_l - \mu)I - A$ is invertible, it has a unique solution $X = X_l^r$.

This gives a solution $V^r \in E^r$ of System (S_{G^r}) with components $(v_i^r)_{1 \leq i \leq n}$.

Multiplying (S_{G^r}) by V^r , and using the variational characterization of λ_{s_0} , we obtain, since $a_{ij} \geq 0$ for $i \neq j$, for $1 \leq i \leq n$:

$$\begin{aligned} \lambda_{s_0} \int (v_i^r)^2 &\leq \int |\nabla v_i^r|^2 = \sum_{j=1}^n a_{ij} \int v_i^r v_j^r + \mu \int (v_i^r)^2 + \int g_i^r v_i^r \\ &\leq \sum_{j \neq i} a_{ij} \left(\int (v_i^r)^2 \right)^{1/2} \left(\int (v_j^r)^2 \right)^{1/2} + (a_{ii} + \mu) \int (v_i^r)^2 + \left(\int (v_i^r)^2 \right)^{1/2} \left(\int (g_i^r)^2 \right)^{1/2}. \end{aligned}$$

Denoting by \tilde{X}^r [resp. \tilde{Z}^r] the vector with components $\|v_i^r\|_{L^2}$ [resp. $\|g_i^r\|_{L^2}$] we derive

$$((\lambda_{s_0} - \mu)I - A) \tilde{X}^r \leq \tilde{Z}^r.$$

By (8), $((\lambda_{s_0} - \mu)I - A)$ is a non singular M -matrix. Then, it follows from property (N39) in [1] (or see Annex II) that

$$0 \leq \tilde{X}^r \leq ((\lambda_{s_0} - \mu)I - A)^{-1} \tilde{Z}^r.$$

Setting for any matrix $B = (b_{ij})$, $\|B\| = (\sum_{i,j} b_{ij}^2)^{1/2}$, we obtain

$$\|\tilde{X}^r\| \leq \|((\lambda_{s_0} - \mu)I - A)^{-1} \tilde{Z}^r\| \leq \|((\lambda_{s_0} - \mu)I - A)^{-1}\| \|\tilde{Z}^r\|.$$

Hence

$$\|V^r\|_{(L^2)^n} \leq C_1 \|G^r\|_{(L^2)^n} \leq C_2 \|G\|_{(L^2)^n},$$

By regularity properties of the Laplacian,

$$\|V^r\|_{(H^2)^n} \leq C_3 \|G\|_{(L^2)^n}. \quad (9)$$

Finally a compactness argument proves now that V^r converges to the solution $V \in E$ of (S_G) and **the claim is proved.**

End of the proof of Theorem 1: Let $F \in (L^2(\Omega))^n$; we write

$$F = \hat{F} + G$$

where $G \in H$ and $\hat{F} \in (L^2(\Omega))^n$ where $\hat{F} = \sum_{l=1}^{s_0-1} \hat{F}^l \phi_l$.

Proceeding as above, we find \hat{X}^l solving System $(\mathfrak{S}_{l, \hat{F}^l})$ since, by Hypothesis 1, $(\lambda_l - \mu)I - A$ is invertible, for $1 \leq l \leq s_0 - 1$. Hence System $(S_{\hat{F}})$ has a unique solution $\hat{U} = \sum_{l=1}^{s_0-1} \hat{X}^l \phi_l$.

Therefore $U = \hat{U} + V$, where V is given by the Claim above, is the unique solution of (S_F) . •

Technical lemmas:

We consider $\hat{\sigma}$ defined in Lemma 1 and let K be some positive constant; we choose t such that

$$\lambda_t > \max\{\lambda_{t-1}, \hat{\sigma} + K\} \quad (10)$$

Lemma 2

i) There is some $\epsilon > 0$ such that, for any $l \in \mathbb{N}$ and any $\mu \in \mathbb{R}$,

$$\text{Det}((\lambda_l - \mu)I - A) = (\lambda_l - \mu - \xi_1) \dots (\lambda_l - \mu - \xi_{\hat{k}}) Q(\lambda_l - \mu), \quad (11)$$

where $Q(\lambda_l - \mu) \geq \epsilon > 0$

ii) For any $\mu < K$ and t satisfying (10), $(\lambda_t - \mu)I - A$ is a non singular M -matrix and we have

$$\text{Det}((\lambda_t - \mu)I - A) \geq (\hat{\sigma} - \xi_1)^{\hat{k}} \epsilon > 0, \quad (12)$$

Proof of Lemma 2:

i) This determinant is the characteristic polynomial of A and its computation shows that the quotient polynomial is such that $Q(\lambda_l - \mu) \geq \epsilon > 0$ since it involves only the non real eigenvalues of A .

ii) By Lemma 1, $(\lambda_t - \mu)I - A$ is a non singular M -matrix since $\lambda_t - \mu > \hat{\sigma} + K - \mu > \hat{\sigma}$ and by (6), for $1 \leq k \leq \hat{k}$,

$$\lambda_t - \mu - \xi_k > \hat{\sigma} + K - \mu - \xi_k > \hat{\sigma} - \xi_1 > 0.$$

Therefore, for any $\mu < K$, (12) follows and the lemma is proved. •

The spaces: As above we use Definition 2 and introduce the spaces E and H for $s_0 = t$.

Let $F^\perp \in H$ and $W \in (H_0^1(\Omega))^n$ be the solution of the following system

$$(S_{F^\perp}) \quad -\Delta W = AW + \mu W + F^\perp \text{ in } \Omega, \quad W|_{\partial\Omega} = 0,$$

where F^\perp is given with components $f_i^\perp \in L^p$, $p > N$, $1 \leq i \leq n$.

Lemma 3 *We assume Hypothesis 1, and choose t satisfying Equation (10). Then all components w_j of W are orthogonal to $\phi_1, \dots, \phi_{t-1}$ that is $W \in E$ and set $U^\perp = W$.*

Moreover, for $\mu < K$, there exists a constant C , independent of μ such that

$$\|U^\perp\|_{(L^2)^n} \leq C \|F^\perp\|_{(L^2)^n} \text{ and } \|U^\perp\|_{(C^1(\bar{\Omega}))^n} \leq C \|F^\perp\|_{(L^2)^n}. \quad (13)$$

The main point here is the fact that for these μ , $(\lambda_t - \mu)I - A$ is a non singular M-matrix (by Lemma 1).

Proof of Lemma 3:

$$-\Delta w_i = \Sigma a_{ij} w_j + \mu w_i + f_i^\perp.$$

Multiplying by ϕ_l for $1 \leq l \leq s - 1$, we obtain

$$(\lambda_l - \mu) \int w_i \phi_l = \Sigma a_{ij} \int w_j \phi_l$$

Denote by Z the vector with components $\int w_i \phi_l$, we obtain

$$((\lambda_l - \mu)I - A) Z = 0.$$

By Hypothesis 1, this implies $Z = 0$ and therefore $W \in E$.

Multiplying by w_i , for $1 \leq l \leq s-1$, and using the variational characterization of λ_s , since $\int w_i \phi_l = 0$, we obtain, since $a_{ij} \geq 0$ for $i \neq j$,

$$\begin{aligned} \lambda_s \int (w_i)^2 &\leq \int |\nabla w_i|^2 = \Sigma a_{ij} \int w_j w_i + \mu \int (w_i)^2 + \int w_i f_i^\perp \\ &\leq \Sigma_{i \neq j} a_{ij} (\int (w_j)^2)^{1/2} (\int (w_i)^2)^{1/2} + (a_{ii} + \mu) \int (w_i)^2 + (\int (f_i^\perp)^2)^{1/2} (\int (w_i)^2)^{1/2}. \end{aligned}$$

We divide by $(\int w_i^2)^{1/2}$, and denote by X (resp. Z) the vector with components $\|w_i\|_{L^2}$ (resp. $\|f_i^\perp\|_{L^2}$); we derive

$$((\lambda_s - \mu)I - A) X \leq Z.$$

By Lemma 1 $((\lambda_s - \mu)I - A)$ is a non singular M -matrix, and we get from property (N39) in [1] (or see Annex II below) that

$$X \leq ((\lambda_s - \mu)I - A)^{-1} Z.$$

From (12), in Lemma 2, $\|((\lambda_s - \mu)I - A)^{-1}\|$ is bounded independently of μ , and therefore

$$\|X\| \leq \|((\lambda_s - \mu)I - A)^{-1} Z\| \leq \|((\lambda_s - \mu)I - A)^{-1}\| \|Z\| \leq C \|Z\|.$$

Since $W = U^\perp$ this implies

$$\|U^\perp\|_{(L^2)^n} = \|W\|_{(L^2)^n} \leq C\|F^\perp\|_{(L^2)^n}.$$

We have the first relation in (13) of Lemma 3.

Proceeding now as in the case of one equation, we deduce from Sobolev imbedding theorem, and from regularity properties of the Laplacian that

$$\|W\|_{C^1(\bar{\Omega})} = \|U^\perp\|_{C^1(\bar{\Omega})} \text{ is bounded independently of } \mu, \quad (14)$$

and the second result follows. •

Lemma 4 *We assume Hypothesis 2. For any $l \in \mathbb{N}$ there exists $\delta_l > 0$ such that for $\mu = \mu_{s,k} + \eta$ with $|\eta| \leq \delta_l$, the determinant of System \mathfrak{S}_{l,Z^l} is*

$$|Det((\lambda_l - \mu)I - A)| \geq \frac{1}{2^k} |\lambda_l - \lambda_s|^{n_k} \prod_{\xi_j \neq \xi_k} |\mu_{l,j} - \mu_{s,k}| \epsilon > 0,$$

where ϵ is defined in Lemma 2.

Proof of Lemma 4: Take $\mu = \mu_{s,k} + \eta$ with $|\eta| \leq \delta_l$ and

$$\delta_l = \frac{1}{2} \text{Min}\{|\lambda_l - \lambda_s|, \text{Min}_{\xi_j \neq \xi_k} \{|\mu_{l,j} - \mu_{s,k}|\}\}.$$

Note that, by Hypothesis 2, $\delta_l > 0$.

$$Det((\lambda_l - \mu)I - A) = (\lambda_l - \mu - \xi_1) \dots (\lambda_l - \mu - \xi_{\hat{k}}) Q(\xi_k - \eta).$$

For the n_k indices j such that $\xi_j = \xi_k$

$$\lambda_l - \mu - \xi_j = \lambda_l - \lambda_s - \eta$$

so that

$$|\lambda_l - \mu - \xi_j| \geq |\lambda_l - \lambda_s| - \delta_l \geq \frac{1}{2} |\lambda_l - \lambda_s|.$$

For $\xi_j \neq \xi_k$

$$\begin{aligned} |\lambda_l - \mu - \xi_j| &= |\mu_{l,j} - \mu_{s,k} - \eta| \\ &\geq |\mu_{l,j} - \mu_{s,k}| - |\delta_l| \geq \frac{1}{2} |\mu_{l,j} - \mu_{s,k}|, \end{aligned}$$

and the Lemma follows. •

Proof of Theorem 2 : From (P_3) in Hypothesis 3,

$$F = \sum_{l=1}^r Y^l \phi_l + Z \phi_s + \hat{F},$$

where \hat{F} is orthogonal to the eigenspaces associated to $\lambda_1, \dots, \lambda_r, \lambda_s$, the solution U of (S_F) is

$$U = \sum_{l=1}^r U^l + X \phi_s + \hat{U}$$

where $U^l = X^l \phi_l$ is solution of (S_{F^l}) , and \hat{U} satisfies $(S_{\hat{F}})$.

We compute first the components on ϕ_s :

Let $\mu = \mu_{s,k} + \eta < \mu_{\hat{r},k}$. Multiplying (S_F) by ϕ_s we derive that X solves $(\mathfrak{S}_{s,Z})$:

$$((\xi_k - \eta)I - A)X = ((\lambda_s - \mu)I - A)X = Z.$$

For $\eta \in \mathbb{R}$ we define

$B_1^\eta, \dots, B_n^\eta$ the column vectors of matrix $B^\eta := (\xi_k - \eta)I - A$.

By Cramer's formulae the i^{th} component of X is $x_i = M_i^\eta / D$ where

$$M_i^\eta = \det(B_1^\eta, \dots, B_{i-1}^\eta, Z, B_{i+1}^\eta, \dots, B_n^\eta)$$

and as in (11),

$$D = \det((\xi_k - \eta)I - A) = \prod_{j=1}^{j=k} (\xi_k - \eta - \xi_j) Q(\xi_k - \eta)$$

with $Q(\xi_k - \eta) \geq \epsilon > 0$ for all η . For $|\eta|$ sufficiently small, say $|\eta| \leq \delta'$,

$\xi_k - \eta - \xi_j$ has the sign of $\xi_k - \xi_j$ if $\xi_k \neq \xi_j$,

$\xi_k - \eta - \xi_j = -\eta$ if $\xi_j = \xi_k$.

Therefore, for $|\eta| < \delta'$, D has the sign of $(-1)^{n_k} (-\eta)^{m_k}$, and $\lim_{\eta \rightarrow 0} D_{\eta \rightarrow 0} = 0$.

Since $M_{i_0}^\eta > 0$, we obtain that $\lim_{\eta \rightarrow 0} \epsilon_k x_{i_0} = +\infty$ as $\eta \rightarrow 0$.

We establish now upperbounds for the remaining parts of the solution:

We choose $\mu < \mu_{\hat{r},k}$ and \hat{s} such that $\lambda_{\hat{s}} > \lambda_{\hat{s}-1}$ and $\lambda_{\hat{s}} > \hat{\sigma} + \mu_{\hat{r},k}$, where $\hat{\sigma}$ is defined in Lemma 1. Here $K = \mu_{\hat{r},k}$. It follows from Lemma 3 that $(\lambda_{\hat{s}} - \mu)I - A$ is a non singular M -matrix.

Now we use again the sets E and H defined in Definition 2 for $s_0 = \hat{s}$. For $F \in (L^p(\Omega))^n$, we decompose \hat{F} in (P_3) and write

$$F = \sum_{l=1}^r F^l + Z\phi_s + \sum_{l=\hat{r}}^{\hat{s}-1} F^l + F^\perp$$

where $F^\perp \in H$ and $F^l = Z^l\phi_l$; (if $\hat{s} = \hat{r}$, obviously $\sum_{l=\hat{r}}^{\hat{s}-1} = 0$)
The solution U of (S_F) is

$$U = \sum_{l=1}^r U^l + X\phi_s + \sum_{l=\hat{r}}^{\hat{s}-1} U^l + U^\perp,$$

where $U^\perp \in E$ satisfies (S_{F^\perp}) , the components x_i of X are computed before and U^l satisfies (S_{F^l}) for $l = 1, \dots, r$ and $l = \hat{r}, \dots, \hat{s} - 1$.

Since $\mu < \mu_{\hat{r},k}$, we can take $\mu = \mu_{s,k} + \eta$ with $|\eta| \leq \delta_s$ so that the Hypotheses of Lemma 3 are satisfied; it follows again (13) that is :

$\|U^\perp\|_{C^1(\overline{\Omega})}$ is bounded independently of μ .

For $l = 1, \dots, r$ and $l = \hat{r}, \dots, \hat{s} - 1$, $U^l = X^l\phi_l$ satisfies (S_{F^l}) ; so multiplying (S_{F^l}) by ϕ_l , we obtain that X^l satisfies (\mathfrak{S}_{l,Z^l}) .

Again, it follows from Cramer's formulae that X^l , and hence U^l , are bounded independently of μ .

Setting $\delta = \min\{\delta', \delta_1, \dots, \delta_r, \delta_{\hat{r}}, \dots, \delta_{\hat{s}}\}$ where δ' is defined above and $\mu = \mu_{s,k} + \eta$ with $|\eta| \leq \delta$ we derive that for all these l , $\|U^l\|_{C^1(\overline{\Omega})}$ are bounded independently of μ .

Combining the results concerning the U^l and U^\perp with the behavior of $Z\phi_s$ near $\mu_{s,k}$, we may write

$$u_{i_0} = \alpha(\mu)\phi_s + \tilde{u},$$

with $\lim_{\mu \rightarrow \mu_{s,k}} \alpha(\mu) = \infty$ and $\|\tilde{u}\|_{C^1(\overline{\Omega})}$ bounded independently of μ . Using for u_{i_0} the arguments developed in the proof of Theorem 3.1 in [7] we derive Theorem 2. •

4 Annex I: Example for a 2×2 system:

Sign of the solutions for μ near $\mu_{1,1}$ or $\mu_{1,2}$

We develop the example announced in Section 2. Consider the 2×2 system (S_F) involving Matrix A :

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and assume $a_{12} > 0$, $a_{21} > 0$. A has 2 distinct real eigenvalues $\xi_2 < \xi_1$. The eigenvalues are the roots of the characteristic polynomial P , and since $P(a_{11}) = P(a_{22}) = -a_{12}a_{21} < 0$, we have

$$\xi_2 < a_{11} < \xi_1 \text{ and } \xi_2 < a_{22} < \xi_1. \quad (15)$$

For a given Z with components z_1 and z_2 , we consider ($\mathfrak{S}_{1,Z}$):

$$((\lambda_1 - \mu)I - A)X = Z$$

that we write also

$$\begin{aligned} (\lambda_1 - \mu - a_{11})x_1 - a_{12}x_2 &= z_1 \\ -a_{21}z_1 + (\lambda_1 - \mu - a_{22})x_2 &= z_2. \end{aligned}$$

We use Cramer's formula:

$$x_1 = \frac{N_1}{D}, \quad x_2 = \frac{N_2}{D},$$

where

$$D = \det((\lambda_1 - \mu)I - A) = (\mu - \mu_{1,1})(\mu - \mu_{1,2}).$$

$$N_1 = z_1(\lambda_1 - \mu - a_{22}) + z_2a_{12}, \quad N_2 = z_2(\lambda_1 - \mu - a_{11}) + z_1a_{21}.$$

Obviously $D < 0$ for $\mu_{1,1} < \mu < \mu_{1,2}$ and > 0 otherwise.

Near an eigenvalue $\mu_{1,k}$, $k = 1, 2$: Write

$$\mu = \mu_{1,k} + \eta = \lambda_1 - \xi_k + \eta. \quad (16)$$

As $\mu \rightarrow \mu_{1,k}$, $D \rightarrow 0$ and

$$N_1 = z_1(\xi_k - a_{22}) + z_2a_{12} - \eta z_1 \rightarrow M_1 := z_1(\xi_k - a_{22}) + z_2a_{12},$$

$$N_2 = z_2(\xi_k - a_{11}) + z_1a_{21} - \eta z_2 \rightarrow M_2 := z_2(\xi_k - a_{11}) + z_1a_{21}.$$

Here M_1 and M_2 are defined in (4).

Near the first eigenvalue $\mu_{1,1}$, $k = 1$

As $\eta \rightarrow 0$, $D \rightarrow 0$ with the sign of $-\eta$. If $z_1 = \int f\phi_1 > 0$ and $z_2 = \int f_2\phi_1 > 0$, by 15, $M_1 > 0$ and $M_2 > 0$. For η small enough, N_1 and N_2 have the same sign than M_1 and M_2 , and are positive. Finally

$$\{\mu \nearrow \mu_{1,1} \Rightarrow x_1 \rightarrow +\infty, x_2 \rightarrow +\infty \text{ and } U \gg 0\}.$$

$$\{\mu \searrow \mu_{1,1} \Rightarrow x_1 \rightarrow -\infty, x_2 \rightarrow -\infty \text{ and } U \ll 0\}.$$

If $z_1 < 0$ and $z_2 < 0$, the signs are reversed.

Near the eigenvalue $\mu_{1,2}$, $k = 2$

As $\eta \rightarrow 0$, $D \rightarrow 0$ with the sign of $+\eta$. Assuming $M_1 > 0$, $M_2 > 0$, we obtain that, for η small enough, $N_1 > 0$, $N_2 > 0$. Finally

$$\{\mu \nearrow \mu_{1,2} \Rightarrow x_1 \rightarrow -\infty, x_2 \rightarrow -\infty \text{ and } U \ll 0\};$$

$$\{\mu \searrow \mu_{1,2} \Rightarrow x_1 \rightarrow +\infty, x_2 \rightarrow +\infty \text{ and } U \gg 0\}.$$

The sign of the components of the solutions change as μ goes over an eigenvalue $\mu_{1,k}$, $k = 1, 2$.

5 Annex II: Non singular M -matrices

A matrix A which can be expressed

$$sI - B, \quad s > \rho(B), \quad B \geq 0,$$

with B a matrix with all terms non negative ($B \geq 0$) and $\rho(B)$ its spectral radius is called a **non singular M -matrix**.

We list here some of the 50 equivalent properties of a non singular M -matrix shown in Bermann and Plemmons, ([1], p132 to 138).

(A1) All of the principal minors of A are positive.

(A5) A does not reverse the sign of any vector; that is if $X \neq 0$ with components X_i and $Y = AX$, Y with components Y_i , then for some subscript i : $X_i Y_i > 0$.

(D15) $A + tI$ is non singular for each $t \geq 0$.

(D16) Every real eigenvalue of A is positive.

(I27) A is "semipositive"; that is there exists $X \gg 0$ with $AX \gg 0$.

(I28) There exists $X > 0$ with $AX \gg 0$.

(N38) A is with positive inverse; that is A^{-1} exists and $A^{-1} \geq 0$.

(N39) A is monotone; that is

$$AX \geq 0 \Rightarrow X \geq 0 \text{ for all } X \in \mathbb{R}^n.$$

5.1 An example

$$B = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}.$$

B is a strictly cooperative matrix with non real eigenvalues: $P(\lambda) = (3 - \lambda)(\lambda^2 + 3\lambda + 3)$ gives the eigenvalues of B : $3, (-3 \pm (i\sqrt{3})/2$. Obviously $\rho(B) = 3$ and $sI - B$ is a non singular M -matrix for $s > 3$.

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