



The Laplace transform $(\det s)^{-p} \exp \operatorname{tr}(s^{-1} w)$ and the existence of non-central Wishart distributions

Gérard Letac, Héli Ene Massam

► To cite this version:

Gérard Letac, Héli Ene Massam. The Laplace transform $(\det s)^{-p} \exp \operatorname{tr}(s^{-1} w)$ and the existence of non-central Wishart distributions. Journal of Multivariate Analysis, 2017. <hal-01977683>

HAL Id: hal-01977683

<https://hal.science/hal-01977683v1>

Submitted on 10 Jan 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

The Laplace transform $(\det s)^{-p} \exp \operatorname{tr}(s^{-1}w)$ and the existence of non-central Wishart distributions

G rard Letac^a, H l ne Massam^b

^a*Institut de math matiques de Toulouse, Universit  Paul-Sabatier, 31062, Toulouse, France*

^b*Department of Mathematics and Statistics, York University, Toronto, Ontario, Canada M3J 1P3*

Abstract

The problem considered in this paper is to find when the non-central Wishart distribution, defined on the cone $\overline{\mathcal{P}}_d$ of positive semidefinite matrices of order d and with a real-valued shape parameter p , does exist. This can be reduced to the study of the measures $m(n, k, d)$ defined on $\overline{\mathcal{P}}_d$ and with Laplace transform $(\det s)^{-n/2} \exp \operatorname{tr}(s^{-1}w)$, where n is an integer and $w = \operatorname{diag}(0, \dots, 0, 1, \dots, 1)$ has order d and rank k . Our two main results are the following: we compute $m(d-1, d, d)$ and we show that neither $m(d-2, d, d)$ nor $m(d-2, d-1, d)$ exists. These facts solve the problems of the existence and computation of these non-central Wishart distributions.

Keywords: Euclidean Jordan algebras, non-central Wishart, random matrices, zonal polynomials.

1. Introduction

The non-central Wishart distribution is traditionally defined as the distribution of the random symmetric real matrix $X = Y_1 Y_1^\top + \dots + Y_n Y_n^\top$, where $Y_1, \dots, Y_n \in \mathbb{R}^d$ are independent Gaussian column vectors with the same non-singular covariance matrix Σ and respective means m_1, \dots, m_n which are not necessarily equal. For s in the open cone \mathcal{P}_d of positive definite symmetric matrices of order d and $w = m_1 m_1^\top + \dots + m_n m_n^\top$ in the closed cone of positive semidefinite matrices $\overline{\mathcal{P}}_d$, one can readily derive the Laplace transform

$$\mathbb{E}(e^{-\operatorname{tr}(sX)}) = \frac{1}{\det(I_d + 2\Sigma s)^{n/2}} e^{-\operatorname{tr}\{2s(I_d + 2\Sigma s)^{-1}w\}}. \quad (1)$$

It is important to note that in this formula, the rank k of w is such that $k \leq \min(n, d)$.

In the same way that the familiar chi-square distribution with n degrees of freedom can be extended to the gamma distribution with a continuous shape parameter p by replacing the half integer $n/2$ by $p \in (0, \infty)$, it is tempting to extend the values that the power of $\det(I_d + 2\Sigma s)$ can take in (1). The question is then: given $\Sigma \in \mathcal{P}_d$ and $w \in \overline{\mathcal{P}}_d$, for which values of $p > 0$ does there exist a probability distribution on $\overline{\mathcal{P}}_d$ for X such that, for all $s \in \mathcal{P}_d$, we have

$$\mathbb{E}(e^{-\operatorname{tr}(sX)}) = \frac{1}{\det(I_d + 2\Sigma s)^p} e^{-\operatorname{tr}\{2s(I_d + 2\Sigma s)^{-1}w\}}. \quad (2)$$

The hypothetical distribution for X satisfying (2) can be called a non-central Wishart distribution with parameters $(2p, w, \Sigma)$, or $\mathcal{NCW}(2p, w, \Sigma)$ for short.

The first question is to determine what are the possible values of p . For $w = 0$, i.e., for the extension of the ordinary Wishart distributions, the problem is solved by Gindikin's theorem [5], which says that $\mathcal{NCW}(2p, 0, \Sigma)$ exists if and only if

$$p \in \Lambda_d = \left\{ \frac{1}{2}, \frac{2}{2}, \dots, \frac{d-2}{2} \right\} \cup [(d-1)/2, \infty). \quad (3)$$

Email addresses: gerard.letac@math.univ-toulouse.fr (G rard Letac), massamh@yorku.ca (H l ne Massam)

In Proposition 8 below, we give a proof of Gindikin's Theorem, which we borrow from [15] because it is the most adapted to the zonal polynomial techniques used in this paper. The Wishart distributions with $w = 0$ and when $2p$ not necessarily an integer and satisfying $2p > d - 1$ have densities proportional to $e^{-\text{tr}(\Sigma^{-1}x)}(\det x)^{p-(d+1)/2}$ concentrated on \mathcal{P}_d . A simple argument (see Proposition 1 below) shows that if $\mathcal{NCW}(2p, w, \Sigma)$ exists, then for all $\epsilon \geq 0$ the distribution $\mathcal{NCW}(2p, \epsilon w, \Sigma)$ also exists. By a limiting argument, this observation implies that if $\mathcal{NCW}(2p, w, \Sigma)$ exists, then $\mathcal{NCW}(2p, 0, \Sigma)$ exists. Hence $p \in \Lambda_d$ and the difficulty concerning the possible values of p is overcome.

The second question is a more difficult one. In the ordinary non-central Wishart (1), we have seen that the rank k of w must be such that $k \leq n = 2p$. Do we have a similar constraint for the $\mathcal{NCW}(2p, w, \Sigma)$ distribution? In other words, does the $\mathcal{NCW}(2p, w, \Sigma)$ defined by (2) fail to exist if $k > 2p$? We have a counterexample when $(d - 1)/2 < p < d/2$ and $k = d$. Indeed, similarly to the case $w = 0$, it has been proved (see, e.g., [9]) that, for any w , the distribution $\mathcal{NCW}(2p, w, \Sigma)$ exists and has a density if $(d - 1)/2 < p$.

The main objective of the present paper is to answer this question. We will see that:

- (a) If $p \in \{1/2, 2/2, \dots, (d - 2)/2\}$, then $\mathcal{NCW}(2p, w, \Sigma)$ exists if and only if the rank k of w is such that $k \leq 2p$.
- (b) If $p \geq (d - 1)/2$, then $\mathcal{NCW}(2p, w, \Sigma)$ exists for any w .

As just noted above, Statement (b) has already been proved for $p > (d - 1)/2$ and the density of the $\mathcal{NCW}(2p, w, \Sigma)$ distribution is known. For $p = (d - 1)/2$, Statement (b) is established by observing that

$$\mathcal{NCW}(d - 1, w, \Sigma) = \lim_{p \downarrow (d-1)/2} \mathcal{NCW}(2p, w, \Sigma).$$

2. Reduction of the problem: The measures $m(2p, k, d)$

Let k be an integer such that $0 \leq k \leq d$. We consider the diagonal matrix $I(k, d)$ with its first $d - k$ diagonal terms equal to 0 and the last k equal to 1, viz.

$$I(k, d) = \begin{bmatrix} 0_{d-k} & 0 \\ 0 & I_k \end{bmatrix}.$$

For $p \in \Lambda_d$ we define the positive measure $m(2p, k, d)$ on $\overline{\mathcal{P}_d}$ such that, for all $s \in \mathcal{P}_d$, we have

$$\int_{\overline{\mathcal{P}_d}} e^{-\text{tr}(sx)} m(2p, k, d)(dx) = \frac{1}{(\det s)^p} e^{\text{tr}\{s^{-1}I(k, d)\}}. \quad (4)$$

Note that $m(2p, k, d)$ may or may not exist. For example, formula (23) below shows that the density of $m(1, 1, 1)$ on $(0, \infty)$ is $\cosh(2\sqrt{x})/\sqrt{\pi x}$. More generally with $p > 0$, we have

$$m(0, 0, 1) = \delta_0, \quad m(0, 1, 1) = \delta_0 + \left\{ \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!(n-1)!} \right\} \mathbf{1}_{(0, \infty)}(x) dx,$$

$$m(p, 0, 1) = \frac{x^{p-1}}{\Gamma(p)} \mathbf{1}_{(0, \infty)}(x) dx, \quad m(p, 1, 1) = \left\{ \sum_{n=0}^{\infty} \frac{x^{n+p-1}}{n!\Gamma(n+p)} \right\} \mathbf{1}_{(0, \infty)}(x) dx,$$

where δ_0 denotes Kronecker's delta.

For $2p > d - 1$, formula (53) below gives $m(2p, d, d)$. If k and n are integers such that $0 \leq k \leq n \leq d$, formula (7) gives $m(n, k, d)$. Finally $m(1, 2, 2)$ is computed in Section 3. The measure $m(d - 1, d, d)$ is computed in Section 4.3 and details about $m(2, 3, 3)$ are given in Section 4.4. We will show below that these examples are the only cases of existence. For instance the function $s \mapsto \exp \text{tr}(s^{-1})$ on \mathcal{P}_d is not the Laplace transform of a positive measure if $d \geq 2$.

The following proposition links this unbounded measure $m(2p, k, d)$ defined by (4) with our initial existence problem. Proposition 1 is important for the solution to the problem of existence or non existence of the non-central Wishart with continuous shape parameter p : it focuses the problem at its core by ignoring the normalization constant and the parameter Σ , and by reducing the parameter w to its most important characteristic, namely its rank k .

Proposition 1. Let $\Sigma \in \mathcal{P}_d$, $w \in \overline{\mathcal{P}_d}$ and $p \in \Lambda_d$. Suppose that $\text{rank } w = k$. Then $\mathcal{NCW}(2p, w, \Sigma)$ as defined by (2) exists if and only if $m(2p, k, d)$ exists. Furthermore, if $m(2p, k, d)$ exists, then p belongs to the Gindikin set (3).

Proof. Assume that $m(2p, k, d)$ exists and let us show that $\mathcal{NCW}(2p, w, \Sigma)$ exists. The proof is based on the following principle. Let μ be a positive measure on a finite-dimensional real linear space E such that its Laplace transform $L_\mu(s) = \int_E e^{-\langle s, x \rangle} \mu(dx)$ is finite on some convex subset $D(\mu)$ of the dual space E^\top with a non-empty interior. Let a be a linear automorphism of E^\top and let $b \in E^\top$ such that $L_\mu\{a(b)\} < \infty$. Then there exists a probability $P(a, b)$ on E with Laplace transform $L_{P(a, b)}(s) = L_\mu\{a(s + b)\}/L_\mu\{a(b)\}$. This probability $P(a, b)$ is obtained in two steps: first take the image $\nu(dy)$ of $\mu(dx)$ by the map $x \mapsto a^\top(x) = y$, where a^\top is the adjoint of a . Its Laplace transform is $L_\nu(s) = L_\mu\{a(s)\}$. The second step constructs $P(a, b)(dy)$ as the probability $e^{-\langle b, y \rangle} \nu(dy)/L_\mu\{a(b)\}$: it is a member of the exponential family generated by ν .

Let us apply this to the case where E is the Euclidean space of real symmetric matrices of order d with scalar product $\langle x, y \rangle = \text{tr}(xy)$ and where μ is $m(2p, k, d)$. Here $D(\mu) = \mathcal{P}_d$. We take $b = (2\Sigma)^{-1}$ and a to be the linear transformation $s \mapsto a(s) = qsq^\top$, where q is an invertible matrix of order d such that

$$2(2\Sigma)^{-1}w(2\Sigma)^{-1} = q^{-1}I(k, d)(q^\top)^{-1}. \quad (5)$$

We have $a^\top(x) = q^\top xq$. The distribution $P(a, b)$ is the non-central Wishart $\mathcal{NCW}(2p, w, \Sigma)$ distribution since

$$\frac{L_\mu\{a(s + b)\}}{L_\mu\{a(b)\}} = \frac{1}{\det(I_d + 2\Sigma s)^p} e^{-\text{tr}(2s(I_d + 2\Sigma s)^{-1}w)}. \quad (6)$$

Using (5), we show (6) as follows:

$$\begin{aligned} \text{tr}[(q^\top)^{-1}\{s + (2\Sigma)^{-1}\}^{-1}q^{-1} - (q^\top)^{-1}(2\Sigma)q^{-1}]I(k, d) \\ = \text{tr}[\{s + (2\Sigma)^{-1}\}^{-1} - 2\Sigma]q^{-1}I(k, d)(q^\top)^{-1}] = -\text{tr}(2s\{I_d + 2\Sigma s\}^{-1}w). \end{aligned}$$

The only thing left to prove is the existence of q satisfying (5). To see this, since the matrix $2(2\Sigma)^{-1}w(2\Sigma)^{-1}$ of $\overline{\mathcal{P}_d}$ has rank k , we write $2(2\Sigma)^{-1}w(2\Sigma)^{-1} = u\Delta u^\top$, where $\Delta = \text{diag}(0, \dots, 0, \lambda_1^2, \dots, \lambda_k^2)$ with $\lambda_i > 0$ and where u is an orthogonal matrix of order d . Taking $q = \text{diag}(1, \dots, 1, \lambda_1^{-1}, \dots, \lambda_k^{-1})u^\top$ provides a solution of (5).

The proof of the converse follows similar lines.

Suppose now that $m(2p, k, d)$ exists and let $w \in \overline{\mathcal{P}_d}$ of rank k . Then, from the first statement of the proposition, $\mathcal{NCW}(2p, \epsilon w, \Sigma)$ exists for all $\epsilon > 0$. The characteristic functions of $\mathcal{NCW}(2p, \epsilon w, \Sigma)$ and of $\mathcal{NCW}(2p, 0, \Sigma)$ are easily deduced from (2) and (1). Using these characteristic functions and Paul Lévy's continuity theorem, we deduce that $\mathcal{NCW}(2p, \epsilon w, \Sigma)$ converges weakly to $\mathcal{NCW}(2p, 0, \Sigma)$ when $\epsilon \rightarrow 0$. Therefore we can claim that $\mathcal{NCW}(2p, 0, \Sigma)$ also exists. From Gindikin's theorem, p must be in Λ_d defined by (3). \square

Example 1. When $0 \leq k \leq n \leq d$ we can use the above principle for constructing $\mathcal{NCW}[n, 2I(k, d), I_d]$ from $m(n, k, d)$. We take $q = I_d$ and $b = I_d/2$. Since a is the identity, we have

$$m(n, k, d)(dx) = 2^{dn/2} e^{2k} e^{\text{tr } x/2} \mathcal{NCW}[n, 2I(k, d), I_d](dx). \quad (7)$$

The next three propositions reformulate known facts in the language of the measures $m(2p, k, d)$.

Proposition 2. Let n and k be integers such that $0 \leq n, k \leq d$. The measure $m(n, k, d)$ exists for $0 \leq k \leq n \leq d$. Furthermore, the measure $m(d-1, d, d)$ exists.

Remark 1. The proof of the existence of $m(d-1, d, d)$ given below is easy, but its explicit computation is done in Section 4.

Proof. Formula (7) provides an explicit form of $m(n, k, d)$. For $2p > d-1$ the probability $\mathcal{NCW}(2p, I_d, I_d)$ exists as proved in Proposition 2 of [9]. This implies that

$$\lim_{p \downarrow (d-1)/2} \mathcal{NCW}(2p, I_d, I_d) = \mathcal{NCW}(d-1, I_d, I_d)$$

exists by considering the characteristic function of $\mathcal{NCW}(2p, I_d, I_d)$ deduced from (2) and Lévy's continuity theorem. From Proposition 1, we have the result. \square

Proposition 3. *Suppose $d \geq 3$. If $m(d-2, d-1, d)$ does not exist then, for any pair of integers (n, k) such that $0 \leq n < k < d$, $m(n, k, d)$ does not exist either. If $m(d-2, d, d)$ does not exist then, for any integer n such that $0 \leq n \leq d-2$, $m(n, d, d)$ does not exist either.*

Proof. Suppose that $m(n, k, d)$ exists for some pair (n, k) such that $0 \leq n < k < d$. We define $m'(dx)$ as the measure on $\overline{\mathcal{P}_d}$ with Laplace transform

$$\int_{\overline{\mathcal{P}_d}} e^{-\text{tr}(sx)} m'(dx) = \frac{1}{(\det s)^{(d-n-2)/2}} e^{\text{tr}[s^{-1}\{I(d-1, d) - I(k, d)\}]}$$

Since the rank of $I(d-1, d) - I(k, d)$ is equal to $d-1-k$ and is less than or equal to $d-n-2$ then, by Propositions 1–2, m' exists. Now we write the convolution $m(n, k, d) * m' = m(d-2, d-1, d)$, which contradicts the non-existence of $m(d-2, d-1, d)$. Similarly, suppose that $m(d-2, d, d)$ does not exist and that there exists n such that $0 \leq n \leq d-2$ and such that $m(n, d, d)$ exists. Then $m(n, d, d) * m(d-2-n, 0, d) = m(d-2, d, d)$ also leads to a contradiction. \square

An important result which is a consequence of [12] is the following.

Proposition 4. *If $d \geq 3$, the measure $m(d-2, d, d)$ does not exist.*

Apart from Proposition 9 below, our main result is as follows.

Proposition 5. *If $d \geq 3$, the measure $m(d-2, d-1, d)$ does not exist.*

The proof will be given in Section 6. In the remainder of the paper we develop the tools that lead us to this proof. They will also enable us to give another proof of Proposition 4. Let us emphasize the fact that Propositions 1–5 lead to a necessary and sufficient condition of existence of the distribution $\mathcal{NCW}(2p, w, \Sigma)$. It is worthwhile to make the following synthesis of Propositions 1–5 about the existence of $\mathcal{NCW}(2p, w, \Sigma)$ in the following proposition.

Proposition 6. *Let $\Sigma \in \mathcal{P}_d$, $w \in \overline{\mathcal{P}_d}$ with rank $k \in \{0, \dots, d\}$ and $p > 0$. Then the non-central Wishart distribution $\mathcal{NCW}(2p, w, \Sigma)$ exists if and only either $2p \geq d-1$ or $2p = n \in \{0, 1, \dots, d-2\}$ and $0 \leq k \leq n$. In particular, for $d = 2$, the probability $\mathcal{NCW}(2p, w, \Sigma)$ exists if and only if $2p \geq 1$. For an arbitrary d , $\mathcal{NCW}(0, w, \Sigma)$ exists if and only if $w = 0$.*

Proof. From Proposition 1, the existence of $\mathcal{NCW}(2p, w, \Sigma)$ is equivalent to the existence of $m(2p, k, d)$ when rank $w = k$.

\Rightarrow : We have seen in Proposition 1 that p must be in Λ_d as defined in (3). If furthermore $2p = n \in \{0, \dots, d-2\}$ let us show that $0 \leq k \leq n$. Suppose the contrary $0 \leq n < k$ holds. A reformulation of the first part of Proposition 3 is the following: if there exists (n, k) such that $0 \leq n < k < d$ and such that $m(n, k, d)$ exists then $m(d-2, d-1, d)$ exists. This contradicts the statement of Proposition 5. Thus the ‘if’ part of Proposition 6 is proved.

\Leftarrow : If $2p > d-1$ Proposition 2.2 of [9] proves the existence of $m(2p, k, d)$ without constraints on k . Passing to the limit when $2p = d-1$ shows the existence of $m(d-1, k, d)$ also for any k . If $2p = n \leq d-2$ and if $0 \leq k \leq n$, Proposition 2 shows that $m(n, k, d)$ exists. \square

3. Computation of $m(1, 2, 2)$

In this section, we compute $m(1, 2, 2)$, which exists, as we know from Proposition 2. We will use only elementary tools. We parameterize the cone \mathcal{P}_2 by the cone of revolution

$$C = \left\{ (x, y, z) \in \mathbb{R}^3 : x \geq \sqrt{y^2 + z^2} \right\}$$

using the mapping φ from C to $\overline{\mathcal{P}_2}$ defined by

$$(x, y, z) \mapsto \varphi(x, y, z) = \begin{bmatrix} x+y & z \\ z & x-y \end{bmatrix}. \quad (8)$$

Note that $\text{tr}\{\varphi(a, b, c)\varphi(x, y, z)\} = 2ax + 2by + 2cz$.

Proposition 7. Consider the positive measure μ on C such that, for $a > \sqrt{b^2 + c^2}$, we have

$$\frac{1}{\sqrt{a^2 - b^2 - c^2}} e^{\frac{2a}{a^2 - b^2 - c^2}} = \int_C e^{-2ax - 2by - 2cz} \mu(dx, dy, dz),$$

i.e., such that the image of μ by φ is $m(1, 2, 2)$. Then $\mu(dx, dy, dz) = r(dx, dy, dz) + f(x, y, z)\mathbf{1}_C(x, y, z)dxdydz$, where the singular part r is the image of the measure $g(2\sqrt{y^2 + z^2})dydz$ on \mathbb{R}^2 by the map $(y, z) \mapsto (x, y, z) = (\sqrt{y^2 + z^2}, y, z)$ with $g(t) = \{2 \cosh(2\sqrt{t})\}/(\pi t)$ and where, for $(x, y, z) \in C$,

$$f(x, y, z) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(x^2 - y^2 - z^2)^k}{k!(k+1)!} \sum_{m=0}^{\infty} \frac{1}{\Gamma(m+2k+5/2)} \frac{(2x)^m}{m!}.$$

Proof. Using the notation $D = \partial/\partial x$, the Faà di Bruno differentiation formula states that, if $f(t)$ and $g(x)$ are functions with enough derivatives, then

$$D^n f\{g(x)\} = \sum \frac{n!}{k_1! \cdots k_n!} (D^k f)\{g(x)\} \left\{ \frac{Dg(x)}{1!} \right\}^{k_1} \cdots \left\{ \frac{D^n g(x)}{n!} \right\}^{k_n}, \quad (9)$$

where $k = k_1 + \cdots + k_n$ and where the sum is taken on all integers $k_j \geq 0$ such that $k_1 + 2k_2 + \cdots + nk_n = n$. For a reference, see, e.g., [16]. We apply (9) to g defined by $x \mapsto x^2 - y^2 - z^2$ for fixed y, z and to $f(t) = t^n$. Noting that $D^3 g = 0$, we obtain

$$\frac{\partial^n}{\partial x^n} (x^2 - y^2 - z^2)^n = n!^2 \sum_{k_2=0}^{\lfloor n/2 \rfloor} \frac{1}{k_2!} \times \frac{(x^2 - y^2 - z^2)^{k_2}}{k_2!} \times \frac{(2x)^{n-2k_2}}{(n-2k_2)!}. \quad (10)$$

For simplification in the sequel we write $E = e^{-2ax - 2by - 2cz}$ and $F = e^{-2a\sqrt{y^2 + z^2} - 2by - 2cz}$. We now recall (see Formula 3.24 in [11]) that when $p > 1/2$ we have for $a > \sqrt{b^2 + c^2}$

$$\frac{1}{(a^2 - b^2 - c^2)^p} = \frac{2}{\sqrt{\pi}} \times \frac{1}{\Gamma(p)\Gamma(p-1/2)} \int_C (x^2 - y^2 - z^2)^{p-3/2} E dx dy dz. \quad (11)$$

Define

$$I_k(n) = (2a)^k \int_C (x^2 - y^2 - z^2)^n E dx dy dz = \frac{\sqrt{\pi}}{2} n! \Gamma(n+3/2) \times \frac{(2a)^k}{(a^2 - b^2 - c^2)^{n+3/2}},$$

where we apply (11) for $p = n + 3/2$. The idea of the proof is to write the Laplace transform of μ as follows:

$$\begin{aligned} \frac{e^{\frac{2a}{a^2 - b^2 - c^2}}}{\sqrt{a^2 - b^2 - c^2}} &= \frac{1}{\sqrt{a^2 - b^2 - c^2}} + \sum_{n=0}^{\infty} \frac{(2a)^{n+1}}{(n+1)!} \frac{1}{(a^2 - b^2 - c^2)^{n+3/2}} \\ &= \frac{1}{\sqrt{a^2 - b^2 - c^2}} + \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{(n+1)!n!\Gamma(n+3/2)} I_{n+1}(n). \end{aligned} \quad (12)$$

A first step is to observe that for $k \in \{0, \dots, n\}$, we have

$$I_k(n) = \int_C \frac{\partial^k}{\partial x^k} (x^2 - y^2 - z^2)^n E dx dy dz. \quad (13)$$

Let us prove it by induction on k . It is true for $k = 0$. Suppose that it is true for $k < n$ and let us show that (13) is true for $k + 1$. Observe that for fixed (y, z) , the root $\sqrt{y^2 + z^2}$ of the polynomial $x \mapsto (x^2 - y^2 - z^2)^n$ has order n and this implies that $\partial^k (x^2 - y^2 - z^2)^n / \partial x^k$ is zero for $x = \sqrt{y^2 + z^2}$. Using this remark and integrating by parts with $V(x) = e^{-2ax}$ and $U(x) = \partial^k (x^2 - y^2 - z^2)^n / \partial x^k$, we compute the following integral:

$$\int_{\sqrt{y^2+z^2}}^{\infty} 2ae^{-2ax} \frac{\partial^k}{\partial x^k} (x^2 - y^2 - z^2)^n dx = \int_{\sqrt{y^2+z^2}}^{\infty} e^{-2ax} \frac{\partial^{k+1}}{\partial x^{k+1}} (x^2 - y^2 - z^2)^n dx. \quad (14)$$

With (14) we are in position to prove (13). We have

$$\begin{aligned} I_{k+1}(n) &= 2a \int_C \frac{\partial^k}{\partial x^k} (x^2 - y^2 - z^2)^n E dx dy dz = \int_{\mathbb{R}^2} e^{-2by-2cz} \left[\int_{\sqrt{y^2+z^2}}^{\infty} 2ae^{-2ax} \frac{\partial^k}{\partial x^k} (x^2 - y^2 - z^2)^n dx \right] dy dz \\ &= \int_{\mathbb{R}^2} e^{-2by-2cz} \left[\int_{\sqrt{y^2+z^2}}^{\infty} e^{-2ax} \frac{\partial^{k+1}}{\partial x^{k+1}} (x^2 - y^2 - z^2)^n dx \right] dy dz = \int_C \frac{\partial^{k+1}}{\partial x^{k+1}} (x^2 - y^2 - z^2)^n E dx dy dz, \end{aligned}$$

which proves (13). We will need (13) only for $k = n$.

The second step is to express $I_{n+1}(n)$ as the Laplace transform of a positive measure. We compute $I_n(n)$ as expressed (13) by using again an integration by parts. The new fact for $k = n$ is that the integrated part will not disappear and will provide a term for the singular measure s given in the statement of the theorem. This calculation of the integrated part will use (10). Taking $V(x) = -e^{-2ax}$ and $U(x) = \partial^n (x^2 - y^2 - z^2)^n / \partial x^n$, we write

$$I_{n+1}(n) = 2aI_n(n) = \int_{\mathbb{R}^2} e^{-2by-2cz} \left[\int_{\sqrt{y^2+z^2}}^{\infty} 2ae^{-2ax} \frac{\partial^n}{\partial x^n} (x^2 - y^2 - z^2)^n dx \right] dy dz = A_n + S_n \quad (15)$$

with

$$A_n = \int_C \frac{\partial^{n+1}}{\partial x^{n+1}} (x^2 - y^2 - z^2)^n E dx dy dz, \quad (16)$$

$$S_n = n! \int_{\mathbb{R}^2} e^{-2by-2cz} \left[-e^{-2ax} (2x)^n \right]_{\sqrt{y^2+z^2}}^{\infty} dy dz = n! \int_{\mathbb{R}^2} (2\sqrt{y^2+z^2})^n F dy dz, \quad (17)$$

where (17) comes from (10) by keeping only the term $k_2 = 0$. We will carry this value of $I_{n+1}(n) = A_n + S_n$ in (12). Doing this, we can guess that S_n will contribute to the singular part of $m(1, 2, 2)$. But the term $1/\sqrt{a^2 - b^2 - c^2}$ in (12) will also contribute to it.

More specifically, the third step of the proof is to represent the function on $C \setminus \partial C$ defined by $(a, b, c) \mapsto 1/\sqrt{a^2 - b^2 - c^2}$ as a Laplace transform. Exploiting the Gaussian integral below, we obtain

$$\frac{1}{\sqrt{a^2 - b^2 - c^2}} = \frac{2}{\pi} \int_{\mathbb{R}^2} e^{-2a(u^2+v^2)-2b(u^2-v^2)-4cuv} dudv = \frac{2}{\pi} \int_{\mathbb{R}^2} (2\sqrt{y^2+z^2})^{-1} F dy dz.$$

To obtain the second identity, observe that the map on $\{(u, v) : u > 0\}$ defined by $y = u^2 - v^2$, $z = 2uv$ is a bijection with \mathbb{R}^2 ; the same is true with $\{(u, v) : u < 0\}$. Furthermore $dydz = 4(u^2 + v^2)dudv = 4\sqrt{y^2 + z^2}dudv$ and therefore $dudv = dydz/(4\sqrt{y^2 + z^2})$. This leads to the conclusion.

Now comes the fourth and final step. We use $\Gamma(n + 1/2) = (2n)! \sqrt{\pi}/(4^n n!)$ and we consider the function

$$g(t) = \frac{2}{\pi t} + \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{(n+1)! \Gamma(n+3/2)} t^n = \frac{2}{\pi t} \cosh(2\sqrt{t}).$$

We then define the measure $r(dx, dy, dz)$ concentrated on the boundary

$$\partial C = \left\{ (x, y, z) : x = \sqrt{y^2 + z^2} \right\}$$

of the cone C to be the image of the measure on \mathbb{R}^2

$$g(2\sqrt{y^2 + z^2})dydz = \frac{1}{\pi\sqrt{y^2 + z^2}} \cosh\{2^{3/2}(y^2 + z^2)^{1/4}\}dydz$$

by the map $(y, z) \mapsto (x, y, z) = (\sqrt{y^2 + z^2}, y, z)$. This measure r will be the singular part of the image μ of $m(1, 2, 2)$ by the reciprocal of φ defined by (8):

$$\begin{aligned} \int_C E ds &= \int_{\mathbb{R}^2} g(2\sqrt{y^2 + z^2}) F dydz = \int_{\mathbb{R}^2} F \frac{dydz}{\pi\sqrt{y^2 + z^2}} + \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \int_{\mathbb{R}^2} \frac{2^n (\sqrt{y^2 + z^2})^n}{(n+1)! \Gamma(n+3/2)} F dydz \\ &= \frac{1}{\sqrt{a^2 - b^2 - c^2}} + \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{S_n}{(n+1)! n! \Gamma(n+3/2)}. \end{aligned} \quad (18)$$

Finally we focus on the absolutely continuous part of μ . We will need the following formula, similar to (10) and also obtained using (9):

$$\frac{\partial^n}{\partial x^n} (x^2 - y^2 - z^2)^{n-1} = n! (n-1)! \sum_{k_2=1}^{\lfloor n/2 \rfloor} \frac{1}{(k_2-1)!} \times \frac{(x^2 - y^2 - z^2)^{k_2-1}}{k_2!} \times \frac{(2x)^{n-2k_2}}{(n-2k_2)!}.$$

The absolutely continuous part of μ is given by (16) and (12). Its density is

$$\begin{aligned} f(x, y, z) &= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{(n+1)! n! \Gamma(n+3/2)} \frac{\partial^{n+1}}{\partial x^{n+1}} (x^2 - y^2 - z^2)^n \\ &= \frac{2}{\sqrt{\pi}} \sum_{n=2}^{\infty} \frac{1}{(n-1)! n! \Gamma(n+1/2)} \frac{\partial^n}{\partial x^n} (x^2 - y^2 - z^2)^{n-1} \\ &= \frac{2}{\sqrt{\pi}} \sum_{n=2}^{\infty} \frac{1}{\Gamma(n+1/2)} \sum_{k_2=1}^{\lfloor n/2 \rfloor} \frac{1}{(k_2-1)!} \times \frac{(x^2 - y^2 - z^2)^{k_2-1}}{k_2!} \times \frac{(2x)^{n-2k_2}}{(n-2k_2)!} \\ &= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(x^2 - y^2 - z^2)^k}{k! (k+1)!} \sum_{m=0}^{\infty} \frac{1}{\Gamma(m+2k+5/2)} \frac{(2x)^m}{m!}. \end{aligned} \quad (19)$$

From (19), (12) and (16) the Laplace transform of f is

$$\int_C E f dx dy dz = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{A_n}{(n+1)! n! \Gamma(n+3/2)}. \quad (20)$$

Let us add (18) and (20) and use (12). What we get shows that the parameterization μ of $m(1, 2, 2)$ by φ is the sum of r and of the absolutely continuous part with density f . Formula (20) shows that f is as given in Proposition 7. \square

Remark 2. The following point is essential for understanding Section 4. Remark first that the image by φ of the measure $r(dx, dy, dz)$ is concentrated on the set $S_1 \subset \overline{\mathcal{P}}_2$ of matrices of rank 1. Any element of S_1 can be written as

$$u \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix} u^\top,$$

where u is an orthogonal matrix of $\mathbb{O}(2)$ and $\lambda_1 > 0$. We can compute the image of $r(dx, dy, dz)$ by the map

$$\begin{bmatrix} x+y & z \\ z & x-y \end{bmatrix} = \begin{bmatrix} \sqrt{y^2+z^2}+y & z \\ z & \sqrt{y^2+z^2}-y \end{bmatrix} \mapsto \lambda_1 = 2\sqrt{y^2+z^2}. \quad (21)$$

If $A_t = \{(x, y, z) : 2\sqrt{y^2+z^2} < t\}$, then using polar coordinates $y = \lambda_1 \cos \alpha$, $z = \lambda_1 \sin \alpha$ with Jacobian equal to $\lambda_1/2$, we have

$$r(A_t) = \int_{A_t} r(dx, dy, dz) = \int_{2\sqrt{y^2+z^2} < t} g(2\sqrt{y^2+z^2}) dy dz = \frac{\pi}{2} \int_0^t g(\lambda_1) \lambda_1 d\lambda_1.$$

Since $g(\lambda_1) = 2(\pi\lambda_1)^{-1} \cosh 2\sqrt{\lambda_1}$, the image of the measure r by the map (21) is

$$\cosh(2\sqrt{\lambda_1}) \mathbf{1}_{(0,\infty)}(\lambda_1) d\lambda_1. \quad (22)$$

Now an important observation is the following: consider the measure $m(1, 1, 1)(d\lambda)$ on $(0, \infty)$ whose Laplace transform is, for $s > 0$,

$$\frac{1}{\sqrt{s}} e^{1/s} = \sum_{n=0}^{\infty} \frac{1}{n! s^{n+1/2}} = \int_0^{\infty} e^{-s\lambda} \sum_{n=0}^{\infty} \frac{\lambda^{n-1/2}}{n! \Gamma(n+1/2)} d\lambda = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-s\lambda} \frac{1}{\sqrt{\lambda}} \cosh(2\sqrt{\lambda}) d\lambda.$$

This last line implies

$$m(1, 1, 1)(d\lambda) = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\lambda}} \cosh(2\sqrt{\lambda}) \mathbf{1}_{(0,\infty)}(\lambda) d\lambda \quad (23)$$

and one observes that $m(1, 1, 1)$ is quite close to (22). To summarize this remark, the singular part of $m(1, 2, 2)$ can be seen as the image of $\sqrt{\pi\lambda_1} m(1, 1, 1)(d\lambda_1) \otimes du$ by the map

$$(u, \lambda_1) \mapsto u \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix} u^\top$$

from $(0, \infty) \times \mathbb{O}(2)$, where du is the uniform probability on $\mathbb{O}(2)$. This is the key to the generalization of the computation from $m(1, 2, 2)$ to the computation of $m(d-1, d, d)$ for $d \geq 2$ done in the next section.

4. Computation of the measure $m(d-1, d, d)$

Before stating Proposition 9 which describes $m(d-1, d, d)$ we set some notations, we recall some facts about zonal functions and polynomials and we prove three lemmas. The Lebesgue measure dx on the space of real symmetric matrices of order d has the normalization associated to the Euclidean structure given by $\langle x, y \rangle = \text{tr}(xy)$. Note that Muirhead [13] has a different normalization. As mentioned on page ix of the Introduction of [13], zonal functions are the essential tool of non-central distribution theory. Our reference will be Faraut and Koranyi [4], abbreviated from now on as FK.

4.1. Zonal functions

Let \mathcal{E}_d be the set of sequences $\kappa = (m_1, \dots, m_d)$ of d integers such that $m_1 \geq \dots \geq m_d \geq 0$. For $\kappa \in \mathcal{E}_d$, we consider the two zonal polynomials

$$\Phi_\kappa^{(d)}(x) = \Phi_{m_1, \dots, m_d}^{(d)}(x), \quad C_\kappa^{(d)}(x) = C_\kappa^{(d)}(I_d) \Phi_\kappa^{(d)}(x),$$

where $C_\kappa^{(d)}(I_d)$ is defined below in (25). In FK p. 228, the Φ_κ are called spherical rather than zonal polynomials, and on p. 234 the notation Z_κ is used instead of our notation $C_\kappa^{(d)}$. We use the definitions given in FK, while [13] and [18] have other ways to introduce the zonal polynomials. To define $\Phi_\kappa^{(d)}$ we consider

$$\Delta_\kappa(x) = \Delta_1(x)^{m_1-m_2} \Delta_2(x)^{m_2-m_3} \dots \Delta_{d-1}(x)^{m_{d-1}-m_d} \Delta_d(x)^{m_d},$$

where for the real symmetric matrix $x = (x_{ij})_{1 \leq i, j \leq d}$, the function $\Delta_k(x) = \det(x_{ij})_{1 \leq i, j \leq k}$ is the principal determinant of x of order k . The function $\Phi_\kappa^{(d)}$ is then defined by

$$\Phi_\kappa^{(d)}(x) = \int_{\mathbb{O}(d)} \Delta_\kappa(uxu^\top) du, \quad (24)$$

where du is the Haar probability on the orthogonal group $\mathbb{O}(d)$. When $x \in \mathcal{P}_d$ definition (24) makes sense even when m_1, \dots, m_d are complex numbers. In that case $\Phi_{m_1, \dots, m_d}^{(d)}(x)$ is no longer a polynomial and is called a zonal function.

From (24) one obtains easily the following formula for any x and y in \mathcal{S}_d :

$$\int_{\mathbb{O}(d)} \Phi_\kappa^{(d)}(xuyu^\top) du = \Phi_\kappa^{(d)}(x)\Phi_\kappa^{(d)}(y).$$

To give the value of the constant $C_\kappa^{(d)}(I_d)$, we need the notations $\ell(\kappa) = \max\{j : m_j > 0\}$, $|\kappa| = m_1 + \dots + m_d$ and

$$\Gamma_d(z_1, \dots, z_d) = \prod_{j=1}^d \Gamma\left(z_j - \frac{j-1}{2}\right)$$

defined whenever $z_j - (j-1)/2 > 0$ for all $j \in \{1, \dots, d\}$. If p is a real number, we use the notational convention

$$\Gamma_d(z+p) = \Gamma_d(z_1+p, \dots, z_d+p).$$

If $\kappa \in \mathcal{E}_d$ and $p > (d-1)/2$, we define the Pochhammer symbol as $(p)_\kappa = \Gamma_d(\kappa+p)/\Gamma_d(p)$. Since $p \mapsto (p)_\kappa$ is a polynomial, we extend its definition to the whole line. If $\kappa \in \mathcal{E}_d$, the constant $C_\kappa^{(d)}(I_d)$ is

$$C_\kappa^{(d)}(I_d) = C_{m_1, \dots, m_d}^{(d)}(I_d) = 2^{2|\kappa|} |\kappa|! \left(\frac{d}{2}\right)_\kappa \frac{\prod_{1 \leq i < j \leq \ell(\kappa)} (2m_i - 2m_j - i + j)}{\prod_{i=1}^{\ell(\kappa)} \{2m_i + \ell(\kappa) - i\}!} \quad (25)$$

$$= \frac{|\kappa|!}{((d+1)/2)_\kappa} \times \prod_{1 \leq i \leq j \leq d} \frac{B[(j-i+1)/2, 1/2]}{B[m_i - m_j + (j-i+1)/2, 1/2]}. \quad (26)$$

Form (25) of the spherical polynomials is given in [13] p. 237 formula (38), where there is a reference to [2] for a proof. Form (26) can be proved from FK by combining Propositions XI. 4.2 (i), p. 230, and XI.4.4, p. 232, and the definition of $Z_\kappa(x) = C_\kappa^{(d)}(x)$ on the last line of p. 234. We never consider $C_\kappa^{(d)}(x)$ if $\kappa \notin \mathcal{E}_d$. The exact value of $C_\kappa^{(d)}(I_d)$ will be crucial in the proof of Proposition 9 when we need the following formula which is Formula (3), p. 259, of Muirhead [13]:

$$e^{\text{tr } x} = \sum_{\kappa \in \mathcal{E}_d} \frac{1}{|\kappa|!} C_\kappa^{(d)}(x). \quad (27)$$

Since we have introduced the zonal polynomials, we use them here to recall the result of [15] leading to the elegant proof by Peddada and Richards of Gindikin's Theorem.

Proposition 8. *Let $p > 0$ and let W be a random variable of $\overline{\mathcal{P}}_d$ such that $E(e^{\text{tr } sW}) = \{\det(I_d - s)\}^{-p}$ when $I_d - s$ is positive definite. Then, for all $\kappa \in \mathcal{E}_d$, we have $E\{\Phi_\kappa^{(d)}(W)\} = (p)_\kappa$ and W exists if and only if p is in Λ_d as defined by (3).*

Proof. Consider the scalar product for polynomials on \mathcal{S}_d defined in FK, p. 220, as

$$\langle p, q \rangle = p\left(\frac{\partial}{\partial x}\right)q(x)\Big|_{x=0}.$$

Then (see FK, p. 234) the spherical polynomials are an orthogonal family. In particular

$$\begin{aligned} E\{\Phi_\kappa^{(d)}(W)\} &= \Phi_\kappa^{(d)}\left(\frac{\partial}{\partial s}\right)E(e^{\text{tr } sW})\Big|_{s=0} = \Phi_\kappa^{(d)}\left(\frac{\partial}{\partial s}\right)\{\det(I_d - s)\}^{-p}\Big|_{s=0} \\ &= \Phi_\kappa^{(d)}\left(\frac{\partial}{\partial s}\right) \sum_{\kappa' \in \mathcal{E}_d} \frac{(p)_{\kappa'}}{|\kappa'|!} C_{\kappa'}^{(d)}(s)\Big|_{s=0} \end{aligned} \quad (28)$$

$$= (p)_\kappa \|\Phi_\kappa^{(d)}\|^2 C_\kappa^{(d)}(I_d) = (p)_\kappa \quad (29)$$

where (28) comes from (27), (29) comes from the orthogonality of the spherical functions, and the norm of $\Phi_\kappa^{(d)}$ is given in FK, p. 234. Since $\Phi_\kappa^{(d)}(W) \geq 0$ we have $(p)_\kappa \geq 0$ for all $\kappa \in \mathcal{E}_d$ and this implies that $p \in \Lambda_d$. \square

It is interesting to compute the zonal polynomials for $d = 2$. This is given as Exercise 5, p. 237, in FK. Define the Legendre polynomials $(P_k)_{k=0}^\infty$ by their generating function

$$\sum_{k=0}^{\infty} P_k(x) z^k = \frac{1}{\sqrt{1 - 2zx + z^2}}.$$

Further let

$$x = \begin{bmatrix} a+b & c \\ c & a-b \end{bmatrix}$$

in \mathcal{P}_2 . Then for $(m_1, m_2) \in \mathcal{E}_2$ we have

$$\Phi_{m_1, m_2}(x) = (a^2 - b^2 - c^2)^{(m_1+m_2)/2} P_{m_1-m_2} \left(\frac{a}{\sqrt{a^2 - b^2 - c^2}} \right). \quad (30)$$

We now detail the proof of (30). The Legendre polynomial P_k satisfies

$$P_k(\cosh t) = \frac{1}{\pi} \int_0^\pi (\cosh t + \cos u \sinh t)^k du. \quad (31)$$

To check this, call Q_k the right-hand side of (31). The computation of $\sum_{k=0}^\infty Q_k z^k$ gives $(1 - 2zx + z^2)^{-1/2}$ for $|z|$ small enough. This proves (31). Let

$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

and observe that $\mathbb{SO}(2) = \{R(\theta) : \theta \in \mathbb{R}\}$ and $\mathbb{O}(2) \setminus \mathbb{SO}(2) = J\mathbb{SO}(2)$. Writing

$$\begin{pmatrix} B \\ C \end{pmatrix} = R(2\theta) \begin{pmatrix} b \\ c \end{pmatrix},$$

a small calculation yields

$$R(\theta) \begin{bmatrix} a+b & c \\ c & a-b \end{bmatrix} R(-\theta) = \begin{bmatrix} a+B & C \\ C & a-B \end{bmatrix}, \quad (32)$$

$$JR(\theta) \begin{bmatrix} a+b & c \\ c & a-b \end{bmatrix} R(-\theta)J = \begin{bmatrix} a+B & -C \\ -C & a-B \end{bmatrix}. \quad (33)$$

The two formulas (32) and (33) enable us to compute the zonal polynomial

$$\begin{aligned} \Phi_{m_1, m_2}(x) &= \int_{\mathbb{O}(2)} \Delta_{m_1, m_2}(uxu^\top) du = \frac{1}{2} \int_{\mathbb{SO}(2)} \Delta_{m_1, m_2}(uxu^\top) du + \frac{1}{2} \int_{\mathbb{O}(2) \setminus \mathbb{SO}(2)} \Delta_{m_1, m_2}(uxu^\top) du \\ &= \frac{1}{4\pi} \int_0^{2\pi} [\Delta_{m_1, m_2}\{R(\theta)xR(-\theta)\} + \Delta_{m_1, m_2}\{JR(\theta)xR(-\theta)J\}] d\theta \\ &= (a^2 - b^2 - c^2)^{m_2} \frac{1}{2\pi} \int_0^{2\pi} (a+B)^{m_1-m_2} d\theta \\ &= (a^2 - b^2 - c^2)^{m_2} \frac{1}{\pi} \int_0^\pi (a + \sqrt{b^2 + c^2} \cos \theta)^{m_1-m_2} d\theta \\ &= (a^2 - b^2 - c^2)^{\frac{m_1+m_2}{2}} \frac{1}{\pi} \int_0^\pi \left(\frac{a}{\sqrt{a^2 - b^2 - c^2}} + \frac{\sqrt{b^2 + c^2}}{\sqrt{a^2 - b^2 - c^2}} \cos \theta \right)^{m_1-m_2} d\theta. \end{aligned}$$

Using form (31) of the Legendre polynomial yields (30). Using (26) we also obtain

$$C_{m_1, m_2}^{(2)}(I_2) = \frac{(m_1 + m_2)!}{(m_1 - m_2)! m_2!} \times \frac{1}{(3/2 + m_1 - m_2)_{m_2}}. \quad (34)$$

4.2. Three properties of zonal functions

Lemma 1. Let $x = \begin{bmatrix} x_1 & x_{12} \\ x_{21} & x_2 \end{bmatrix} \in \mathcal{P}_d$ and $[x]_1 = x_1 \in \mathcal{P}_{d-1}$. Then for all complex numbers m_1, \dots, m_d we have

$$\Phi_{m_1, \dots, m_d}^{(d)}(x) = (\det x)^{m_d} \int_{\mathbb{O}(d)} \Phi_{m_1, \dots, m_{d-1}}^{(d-1)}([uxu^\top]_1) du.$$

Proof. Consider $v = \begin{bmatrix} v_1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{O}(d)$, where $v_1 \in \mathbb{O}(d-1)$. Observe that for any $y \in \mathcal{P}_d$ we have

$$[vyv^\top]_1 = v_1[y]_1 v_1^\top. \quad (35)$$

We write

$$\Phi_{m_1, \dots, m_d}^{(d)}(x) = \int_{\mathbb{O}(d)} \Delta_{m_1, \dots, m_{d-1}, m_d}(uxu^\top) du \quad (36)$$

$$= (\det x)^{m_d} \int_{\mathbb{O}(d)} \Delta_{m_1, \dots, m_{d-1}}([uxu^\top]_1) du \quad (37)$$

$$= (\det x)^{m_d} \int_{\mathbb{O}(d)} \Delta_{m_1, \dots, m_{d-1}}([vuxu^\top v^\top]_1) du \quad (38)$$

$$= (\det x)^{m_d} \int_{\mathbb{O}(d)} \Delta_{m_1, \dots, m_{d-1}}(v_1[uxu^\top v^\top]_1 v_1^\top) du \quad (39)$$

$$= (\det x)^{m_d} \int_{\mathbb{O}(d)} \left[\int_{\mathbb{O}(d-1)} \Delta_{m_1, \dots, m_{d-1}}\{v_1[uxu^\top v^\top]_1 v_1^\top\} dv_1 \right] du \quad (40)$$

$$= (\det x)^{m_d} \int_{\mathbb{O}(d)} \Phi_{m_1, \dots, m_{d-1}}^{(d-1)}([uxu^\top]_1) du. \quad (41)$$

In the equations above, (36) comes from the definition of $\Phi_{\kappa}^{(d)}(x)$, (37) separates the roles of $[uxu^\top]_1$ and $\det(uxu^\top) = \det x$ in the definition of $\Delta_{\kappa}(uxu^\top)$, (38) uses the fact that du is the Haar probability, (39) follows from (35) applied to $y = uxu^\top$, (40) uses the fact that the Haar measure dv_1 of $\mathbb{O}(d-1)$ has mass 1 and (41) comes from the definition of $\Phi_{m_1, \dots, m_{d-1}}^{(d-1)}(x)$. \square

Lemma 2. If $x \in \mathcal{P}_d$, then $\Phi_{m_1, \dots, m_d}^{(d)}(x^{-1}) = \Phi_{-m_d, \dots, -m_1}^{(d)}(x)$.

Proof. Define $p \in \mathbb{O}(d)$ by

$$p = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

and define $\Delta_{m_1, \dots, m_{d-1}, m_d}^\top(x) = \Delta_{m_1, \dots, m_{d-1}, m_d}(p x p^\top)$. We can write $\Phi_{\kappa}^{(d)}(x^{-1})$ as

$$\int_{\mathbb{O}(d)} \Delta_{m_1, \dots, m_{d-1}, m_d}(ux^{-1}u^\top) du = \int_{\mathbb{O}(d)} \Delta_{-m_d, \dots, -m_1}^\top(uxu^\top) du \quad (42)$$

$$= \int_{\mathbb{O}(d)} \Delta_{-m_d, \dots, -m_1}(uxu^\top) du \quad (43)$$

In this list (42) comes FK, Proposition VII.1.5 (ii), p. 127, which states that $\Delta_{m_1, \dots, m_{d-1}, m_d}(x^{-1}) = \Delta_{-m_d, \dots, -m_1}^\top(x)$, and (43) comes from the invariance of the Haar probability du on $\mathbb{O}(d)$ by $u \mapsto pu$. \square

Lemma 3. If $x \in \mathcal{P}_d$, then $\Phi_{m_1, \dots, m_d}^{(d)}(x)(\det x)^p = \Phi_{m_1+p, \dots, m_d+p}^{(d)}(x)$.

Proof. By definition,

$$\Phi_{\kappa}^{(d)}(x)(\det x)^p = \int_{\mathbb{O}(d)} \Delta_{m_1, \dots, m_d}(uxu^\top)(\det uxu^\top)^p du = \int_{\mathbb{O}(d)} \Delta_{m_1+p, \dots, m_d+p}(uxu^\top) du = \Phi_{m_1+p, \dots, m_d+p}^{(d)}(x),$$

which completes the argument. \square

4.3. The derivation of $m(d-1, d, d)$

Proposition 9. Define the singular measure $r(dt)$ on $\overline{\mathcal{P}_d}$ and concentrated on the set S_{d-1} of symmetric matrices of rank $d-1$ as the image of the product measure

$$\frac{(\pi \det x)^{1/2}}{\Gamma(d/2)} m(d-1, d-1, d-1)(dx) \otimes du$$

by the map from $\mathcal{P}_{d-1} \times \mathbb{O}(d)$ to $\overline{\mathcal{P}_d}$ defined by

$$(x, u) \mapsto t = u \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} u^\top = u \tilde{x} u^\top.$$

Define

$$f_d(t) = (\det t)^{-1} \left[\sum_{\kappa \in \mathcal{E}'_d} \frac{C_{\kappa}^{(d)}(t)}{|\kappa|! \Gamma_d\{\kappa + (d-1)/2\}} \right], \quad (44)$$

where $\mathcal{E}'_d = \{\kappa \in \mathcal{E}_d ; m_d > 0\}$. Then $m(d-1, d, d)(dt) = r(dt) + f_d(t)I_{\mathcal{P}_d}(t)dt$.

Proof. The function $f_d(t)$ is an analytic function around $t = 0$ because, from the definition (24) of $\Phi_{\kappa}^{(d)}$, the polynomial $C_{m_1, \dots, m_d}^{(d)}(t)$ is divisible by $(\det t)^{m_d}$. Therefore $(\det t)^{-1} C_{\kappa}^{(d)}(t)$ is a polynomial when $\kappa \in \mathcal{E}'_d$. Recall the following (see FK, Lemma XI.2.3, p. 226, or Muirhead [13] Theorem 7.2.7, p. 248):

$$\int_{\mathcal{P}_d} e^{-\text{tr}(sx)} (\det x)^{p-(d+1)/2} \frac{\Phi_{\kappa}^{(d)}(x)}{\Gamma_d(\kappa + p)} dx = \Phi_{\kappa}^{(d)}(s^{-1})(\det s)^{-p}. \quad (45)$$

Note that the choice of the suitable Lebesgue measure dx is crucial in (45). Formula (45) holds true for $p + m_d > (d-1)/2$. For $m_d = 0$, this was proven in the references [13] and FK. When $m_d > 0$ we observe that

$$\Phi_{\kappa}^{(d)}(x) = \Phi_{m_1, \dots, m_d}^{(d)}(x) = (\det x)^{m_d} \Phi_{m_1-m_d, \dots, m_{d-1}-m_d, 0}^{(d)}(x) = (\det x)^{m_d} \Phi_{\kappa-m_d}^{(d)}(x). \quad (46)$$

As a consequence

$$\int_{\mathcal{P}_d} e^{-\text{tr}(sx)} (\det x)^{p-(d+1)/2} \frac{\Phi_{\kappa}^{(d)}(x)}{\Gamma_d(\kappa + p)} dx = \int_{\mathcal{P}_d} e^{-\text{tr}(sx)} (\det x)^{p+m_d-(d+1)/2} \frac{\Phi_{\kappa-m_d}^{(d)}(x)}{\Gamma_d(\kappa + p)} dx \quad (47)$$

$$= \Phi_{\kappa-m_d}^{(d)}(s^{-1})(\det s)^{-p-m_d} \quad (48)$$

$$= \Phi_{\kappa}^{(d)}(s^{-1})(\det s)^{-p}, \quad (49)$$

where (47) and (49) come from (46), and (48) from (45), where p is replaced by $p + m_d$.

From (27) we know that, for $2p \geq d-1$, the Laplace transform of $m(2p, d, d)$ is

$$\int_{\mathcal{P}_d} e^{-\text{tr}(sx)} m(2p, d, d)(dx) = (\det s)^{-p} \sum_{\kappa \in \mathcal{E}_d} \frac{C_{\kappa}^{(d)}(s^{-1})}{|\kappa|!}. \quad (50)$$

Observe that the Laplace transform of $f_d(t)\mathbf{1}_{\mathcal{P}_d}(t)dt$ as defined by (44) is easily deduced from (45) and is equal to

$$\int_{\mathcal{P}_d} e^{-\text{tr}(st)} f_d(t) dt = (\det s)^{-(d-1)/2} \left\{ \sum_{\kappa \in \mathcal{E}'_d} \frac{C_\kappa^{(d)}(s^{-1})}{|\kappa|!} \right\}. \quad (51)$$

In (50), take $2p = d - 1$. Using the Laplace transform (51), we now want to prove that the Laplace transform of $r(dt)$ is

$$\int_{\mathcal{P}_d} e^{-\text{tr}(st)} r(dt) = (\det s)^{-(d-1)/2} \left\{ \sum_{\kappa \in \mathcal{E}_d \setminus \mathcal{E}'_d} \frac{C_\kappa^{(d)}(s^{-1})}{|\kappa|!} \right\} = (\det s)^{-(d-1)/2} \left\{ \sum_{\kappa \in \mathcal{E}_{d-1}} \frac{C_{(\kappa,0)}^{(d)}(s^{-1})}{|\kappa|!} \right\}. \quad (52)$$

To prove (52) we start from the definition of $r(dt)$. Observe first that, for $2p > d - 1$, (45) and (50) imply

$$m(2p, d, d)(dx) = (\det x)^{p-(d+1)/2} \left\{ \sum_{\kappa \in \mathcal{E}_d} \frac{C_\kappa^{(d)}(x)}{|\kappa|! \Gamma_d(\kappa + p)} \right\} \mathbf{1}_{\mathcal{P}_d}(x) dx. \quad (53)$$

In particular, in (53) let us replace d by $d - 1$ and let $2p = d - 1$. We obtain

$$(\det x)^{1/2} m(d-1, d-1, d-1)(dx) = \left[\sum_{\kappa \in \mathcal{E}_{d-1}} \frac{C_\kappa^{(d-1)}(x)}{|\kappa|! \Gamma_{d-1}(\kappa + (d-1)/2)} \right] \mathbf{1}_{\mathcal{P}_{d-1}}(x) dx. \quad (54)$$

We can now write

$$\begin{aligned} \int_{\mathcal{P}_d} e^{-\text{tr}(st)} r(dt) &= \frac{\pi^{1/2}}{\Gamma(d/2)} \int_{\mathbb{O}(d)} \left[\int_{\mathcal{P}_{d-1}} e^{-\text{tr}(su\tilde{x}u^\top)} \sum_{\kappa \in \mathcal{E}_{d-1}} \frac{C_\kappa^{(d-1)}(x)}{|\kappa|! \Gamma_{d-1}(\kappa + (d-1)/2)} dx \right] du \\ &= \frac{\pi^{1/2}}{\Gamma(d/2)} \sum_{\kappa \in \mathcal{E}_{d-1}} \frac{C_\kappa^{(d-1)}(I_{d-1})}{|\kappa|!} \int_{\mathbb{O}(d)} \left[\int_{\mathcal{P}_{d-1}} e^{-\text{tr}([u^\top su]_1 x)} \frac{\Phi_\kappa^{(d-1)}(x)}{\Gamma_{d-1}(\kappa + (d-1)/2)} dx \right] du. \end{aligned} \quad (55)$$

Equality (55) comes from (54) and the definition of r . Now we compute the last double integral as follows

$$\int_{\mathbb{O}(d)} \left[\int_{\mathcal{P}_{d-1}} e^{-\text{tr}([u^\top su]_1 x)} \frac{\Phi_\kappa^{(d-1)}(x)}{\Gamma_{d-1}(\kappa + (d-1)/2)} dx \right] du = \int_{\mathbb{O}(d)} (\det[usu^\top]_1^{-1})^{d/2} \Phi_\kappa^{(d-1)}([usu^\top]_1^{-1}) du \quad (56)$$

$$= \int_{\mathbb{O}(d)} \Phi_{\kappa+d/2}^{(d-1)}([usu^\top]_1^{-1}) du \quad (57)$$

$$= \int_{\mathbb{O}(d)} \Phi_{-m_{d-1}-d/2, \dots, -m_1-d/2}^{(d-1)}([usu^\top]_1) du \quad (58)$$

$$= \Phi_{-m_{d-1}-d/2, \dots, -m_1-d/2, 0}^{(d)}(s) \quad (59)$$

$$= \Phi_{0, m_1+d/2, \dots, m_{d-1}+d/2}^{(d)}(s^{-1}) \quad (60)$$

$$= \Phi_{-d/2, m_1, \dots, m_{d-1}}^{(d)}(s^{-1}) (\det s^{-1})^{d/2} \quad (61)$$

$$= \Phi_{m_1, \dots, m_{d-1}, 0}^{(d)}(s^{-1}) (\det s^{-1})^{(d-1)/2}. \quad (62)$$

In the equations above, equality (56) follows from (45) by replacing (d, p) by $(d-1, d/2)$. Equalities (57) and (61) come from Lemma 3. In the identities following (58) we have replaced κ by (m_1, \dots, m_{d-1}) for clarity. Formulas (58) and (60) come from Lemma 2, and (59) comes from Lemma 1. The proof of (62) is more involved and is a consequence of formula (iii) in Theorem XIV 3.1 of FK, where we replace (d, r, λ, μ) respectively by $1, d$ and

$$\begin{aligned} \lambda &= \left(m_1 + \frac{d-1}{4}, m_2 + \frac{d-3}{4}, \dots, m_{d-1} - \frac{d-3}{4}, -\frac{d-1}{4} \right), \\ \mu &= \left(-\frac{d-1}{4}, m_1 + \frac{d-1}{4}, m_2 + \frac{d-3}{4}, \dots, m_{d-1} - \frac{d-3}{4} \right). \end{aligned}$$

The fact that μ is a permutation of λ and the reference above imply (62). Now we observe that

$$\frac{\pi^{1/2}}{\Gamma(d/2)} C_\kappa(I_{d-1}) = C_{\kappa,0}(I_d), \quad (63)$$

implied by formula (26). Finally we gather (55), (62) and (63) to obtain

$$\int_{\overline{\mathcal{P}}_d} e^{-\text{tr}(st)} r(dt) = (\det s)^{-(d-1)/2} \sum_{\kappa \in \mathcal{E}_{d-1}} \frac{C_{(\kappa,0)}(s^{-1})}{|\kappa|!},$$

which proves (52) and Proposition 9 itself. \square

4.4. Example: $m(2, 3, 3)$

To illustrate Proposition 9 we consider the function $f_3(t)$ defined on \mathcal{P}_3 by (44). More specifically we have

$$f_3(t) = \frac{1}{\det t} \sum_{m_1 \geq m_2 \geq m_3 > 0} \frac{C_{m_1, m_2, m_3}^{(3)}(t)}{(m_1 + m_2 + m_3)! m_1! \Gamma(m_2 + 1/2) (m_3 - 1)!}.$$

We also consider the measure $m(2, 2, 2)(dt)$ on \mathcal{P}_2 parameterized by $(a, b, c) \mapsto t = \varphi(a, b, c)$ as in (8). From (54), we have

$$\begin{aligned} m(2, 2, 2)(da, db, dc) &= \frac{1}{(\det t)^{1/2}} \sum_{m_1 \geq m_2 \geq 0} \frac{C_{m_1, m_2}^{(2)}(t)}{(m_1 + m_2)! m_1! \Gamma(m_2 + 1/2)} \\ &= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a^2 - b^2 - c^2)^{n+(k-1)/2} 2^{2n}}{(3/2 + k)_n (n+k)! k! (2n)!} P_k \left(\frac{a}{\sqrt{a^2 - b^2 - c^2}} \right). \end{aligned}$$

This last formula is obtained by using the calculations done in Section 4.2 for $\Phi_{m_1, m_2}^{(2)}$ in (30), for $C_{m_1, m_2}^{(2)}(I_2)$ in (34) and the change of indexes $(m_1 - m_2, m_1) = (k, n)$. Finally the singular measure $r(dt)$ concentrated on the set of matrices of rank 2 in the cone $\overline{\mathcal{P}}_3$ of positive semidefinite matrices of order 3 is constructed as follows. One considers the product of $m(2, 2, 2)(da, db, dc)$ by the uniform probability measure du on the orthogonal group $\mathbb{O}(3)$. The measure $r(dt)$ is the image of this product measure by the following map:

$$(a, b, c, u) \mapsto t = u \begin{bmatrix} a+b & c & 0 \\ c & a-b & 0 \\ 0 & 0 & 0 \end{bmatrix} u^\top.$$

Proposition 9 says that the measure $m(2, 3, 3)(dt)$ on the set $\overline{\mathcal{P}}_3$ of semipositive definite matrices of order 3 defined by the Laplace transform $(\det s)^{-1} \exp \text{trace } s^{-1}$ is the sum $r(dt) + f_3(t) \mathbf{1}_{\mathcal{P}_3}(t) dt$.

5. Convolution lemmas in the cone $\overline{\mathcal{P}}_d$

Let \mathcal{G}_k the set of linear subspaces G of dimension k of a Euclidean space E of dimension d . Let us endow \mathcal{G}_k with the uniform distribution, i.e., the unique probability on \mathcal{G}_k such that $G \sim uG$ for all $u \in \mathbb{O}(d)$. Lemma 4 below describes an intuitively obvious fact. For the sake of completeness, we offer a proof, although other ones may already exist in the literature.

Lemma 4. *Let F be a fixed linear subspace of dimension n of the Euclidean space E of dimension d . If the random linear subspace G of E has the uniform distribution on \mathcal{G}_k and if $k \leq d - n$, then $\Pr(G \cap F \neq \{0\}) = 0$.*

Proof. It is enough to prove the lemma for $E = \mathbb{R}^d$, $F = \{0\} \times \mathbb{R}^{d-n}$ and $k = d - n$. Let Z_1, \dots, Z_n be iid random variables in \mathbb{R}^d following the standard Gaussian distribution $\mathcal{N}(0, I_d)$. Let G be the random linear subspace of E

generated by the vectors Z_1, \dots, Z_n . Since for all $u \in \mathbb{O}(d)$ we have $(uZ_1, \dots, uZ_n) \sim (Z_1, \dots, Z_n)$, clearly $G \sim uG$ and G follows the uniform distribution. Introduce the matrix

$$M = [Z_1, \dots, Z_n] = (Z_{ij})_{1 \leq i \leq d, 1 \leq j \leq n},$$

whose columns are the vectors Z_1, \dots, Z_n . Then $x_1 Z_1 + \dots + x_n Z_n = MX$, where $X = (x_1, \dots, x_n)^\top$. Now $G \cap F \neq \{0\}$ implies that there exists a non-zero X such that the first n elements of MX are zero. In other terms, considering the square matrix M_1 of order n defined by $M_1 = (Z_{ij})_{1 \leq i, j \leq n}$, we have that $G \cap F \neq \{0\}$ implies that there exists a non-zero X such that $M_1 X = 0$. This happens if and only if $\det M_1 = 0$. Since the n^2 entries of the matrix M_1 are independent $N(0, 1)$ variables, the event $\det M_1 = 0$ has probability zero and the lemma is proved. \square

In the sequel we will denote by S_b the set of $x \in \overline{\mathcal{P}_d}$ with $b = \text{rank } x = 0, \dots, d$. Of course $S_d = \mathcal{P}_d$.

Lemma 5. *Let Y be a random variable in S_b and assume that $uYu^\top \sim Y$ for all u in the orthogonal group $\mathbb{O}(d)$. Let $x_0 \in S_a$. Then $x_0 + Y$ is concentrated on S_{a+b} if $a + b \leq d$ or on $S_d = \mathcal{P}_d$ if $a + b \geq d$. Furthermore if $x_0 \in S_c$, if $x_0 + Y$ is concentrated on S_{a+b} and if $a + b < d$, then $c = a$.*

Remark 3. If $a + b = d$ and if $x_0 + Y$ is concentrated on $S_{a+b} = S_d$, x_0 could be on any S_c with $a \leq c \leq k$.

Proof. Apply Lemma 4 to $F = x_0 \mathbb{R}^d$ and to $G = Y \mathbb{R}^d$. Then almost surely we have $\dim(F + G) = a + b$ if $a + b \leq d$. Furthermore we always have $\text{rank}(x_0 + Y) \leq a + b$.

To see that $\text{rank}(x_0 + Y) = a + b$ almost surely, let us suppose that $(x_0 + Y) \mathbb{R}^d \neq E = F + G$. Let x'_0 and Y' be the restrictions of the endomorphisms x_0 and Y to the linear space E . Since x_0 and Y are symmetric, this implies that $x'_0 E = F$ and $Y' E = G$. Since $(x'_0 + Y') E \neq E$, there exists $v \in E \setminus \{0\}$ which is orthogonal to $(x'_0 + Y') E$ and thus $(x'_0 + Y') v = 0$. Since $x'_0 v \in F$ and $Y' v \in G$ and since $F \cap G = \{0\}$, this implies that $x'_0 v = Y' v = 0$, and v is in $\text{Ker}(x'_0) \cap \text{Ker}(Y')$. But since we have almost surely $F \oplus G = E$ (a direct sum, not necessarily an orthogonal one), we have also almost surely $\text{Ker}(x'_0) \oplus \text{Ker}(Y') = E$, which implies $\text{Ker}(x'_0) \cap \text{Ker}(Y') = \{0\}$. Thus almost surely $v = 0$, which is a contradiction. Finally, $(x_0 + Y) \mathbb{R}^d = E = F + G$ and $\text{rank}(x_0 + Y) = a + b$. If $a + b > d$ then F contains a subspace of dimension $d - b$ and $\dim(F + G) = d$.

To conclude, suppose now that $x_0 \in S_c$, and that $x_0 + Y$ is concentrated on S_{a+b} with $a + b < k$. If $c + b < k$ then, by the first part of the lemma, $x_0 + Y$ is concentrated on S_{c+b} . But $S_{a+b} = S_{c+b}$ implies $c = a$. If $c + b \geq k$ then, by the first part of the lemma again, $x_0 + Y$ is concentrated on S_d . This is impossible since $S_{a+b} \neq S_d$. \square

Lemma 6. *Let μ and ν be positive measures on $\overline{\mathcal{P}_d}$ such that ν is concentrated on S_b and ν is invariant by the transformations $x \mapsto uxu^\top$, $u \in \mathbb{O}(d)$. Let $a \in \{0, \dots, d-2\}$ be such that $a + b < d$. Then $\mu * \nu$ is concentrated on S_{a+b} if and only if μ is concentrated on S_a . Furthermore $\mu * \nu$ is concentrated on $S_d = \mathcal{P}_d$ if μ is concentrated on S_{d-b} .*

Proof. \Rightarrow : For $y_0 \in S_b$ consider the distribution $K_{y_0}(dy)$ on S_b of the random variable Uy_0U^\top , where U is uniformly distributed on the orthogonal group $\mathbb{O}(k)$. Let D_b the set of diagonal elements y_0 of S_b of the form $y_0 = \text{diag}(\lambda_1, \dots, \lambda_b, 0, \dots, 0)$ such that $\lambda_1 \geq \dots \geq \lambda_b > 0$. Then there exists a unique positive measure ν_0 on D_b such that the following disintegration holds:

$$\nu(dy) = \int_{D_b} \nu_0(dy_0) K_{y_0}(dy).$$

It follows that

$$(\mu * \nu)(dx) = \int_{S_b} \nu(dy) \mu(dx - y) = \int_{S_b} \mu(dx - y) \int_{D_b} \nu_0(dy_0) K_{y_0}(dy) = \int_{D_b} \nu_0(dy_0) \int_{S_b} \mu(dx - y) K_{y_0}(dy).$$

Therefore the measure $\mu * K_{y_0}$ is concentrated on S_{a+b} for ν_0 almost all $y_0 \in D_b$. From Lemma 5, this implies that μ is concentrated on S_a .

\Leftarrow : If μ is concentrated on S_a with $a + b \leq d$, it is an easy consequence of Lemma 5 that $\mu * \nu$ is concentrated on S_{a+b} . \square

Lemma 7. Let $a, b \in \{1, \dots, d-1\}$ such that $a+b < d$. If $m(a, a+b, d)$ exists, it is concentrated on S_a .

Proof. From the Laplace transforms of the measures we know that $m(a, a+b, d) * m(b, 0, d) = m(a+b, a+b, d)$. From Proposition 2.2, we know that $m(a+b, a+b, d)$ is concentrated on S_{a+b} since this is the case for the singular non-central Wishart $\mathcal{N}(n, I(n, d), I_d)$ with $n = a+b$. Since the Laplace transform of $m(b, 0, d)$ is $(\det s)^{-b/2}$, we know that $m(b, 0, d)$ is invariant by the transformations $x \mapsto uxu^\top$ for any $u \in \mathbb{O}(d)$. By Lemma 6 we deduce that $m(a, a+b, d)$ is concentrated on S_a if it exists. \square

6. $m(d-2, d-1, d)$ and $m(d-2, d, d)$ do not exist for $d \geq 3$

In this section we prove Propositions 4 and 5.

Proof of Proposition 4. Suppose that $m(d-2, d, d)$ exists. Then $m(d-2, d, d) * m(2, 0, d) = m(d, d, d)$. From (45) and (50), the measure $m(d, d, d)$ has a C^∞ density g . As a consequence

$$\int_{\mathcal{P}_d} e^{-\text{tr}(sx)} m(d-2, d, d)(dx) = \det s \int_{\mathcal{P}_d} e^{-\text{tr}(sx)} g(x) dx = (-1)^n \int_{\mathcal{P}_d} e^{-\text{tr}(sx)} \det(\partial/\partial x) g(x) dx.$$

This implies that $m(d-2, d, d)(dx) = (-1)^n \det(\partial/\partial x) g(x) dx$ has a density. However, since $m(d-2, d, d) * m(1, 0, d) = m(d-1, d, d)$ this would imply that $m(d-1, d, d)$ is absolutely continuous, which contradicts Proposition 9. \square

Proof of Proposition 5. Suppose that $m(d-2, d-1, d)$ exists. By Lemma 7, the measure $m(d-2, d-1, d)$ is concentrated on S_{d-2} . Therefore there exists a positive measure $m(dy) = m(dy_1, \dots, dy_{d-2})$ on $\mathbb{R}^{d(d-2)} = \mathbb{R}^d \times \dots \times \mathbb{R}^d$ such that for all $s \in \mathcal{P}_d$ we have

$$\int_{\mathbb{R}^{d(d-2)}} e^{-(y_1^\top s y_1 + \dots + y_{d-2}^\top s y_{d-2})} m(dy) = \frac{1}{\det s^{(d-2)/2}} e^{\text{tr}\{s^{-1}I(d-1, d)\}}.$$

We write the elements $y = (y_1, \dots, y_{d-2})$ more conveniently with the help of the transposed matrix $y^\top = (y_{i,j})$ with $d-2$ rows $y_1^\top, \dots, y_{d-2}^\top$ and d columns c_1, \dots, c_d

$$y^\top = \begin{bmatrix} y_1^\top \\ \vdots \\ y_{d-2}^\top \end{bmatrix} = \begin{bmatrix} y_{1,1} & \dots & y_{1,d} \\ \vdots & \vdots & \vdots \\ y_{d-2,1} & \dots & y_{d-2,d} \end{bmatrix} = [c_1, \dots, c_d].$$

With this notation introduce the Gram matrix $G(c) = G(c_1, \dots, c_d) = (\langle c_j, c_k \rangle)_{1 \leq j, k \leq d}$ and denote by $m(dc)$ what we denoted by $m(dy)$ before. We get

$$\int_{\mathbb{R}^{d(d-2)}} e^{-\text{tr}\{sG(c)\}} m(dc) = \frac{1}{\det s^{(d-2)/2}} e^{\text{tr}\{s^{-1}I(d-1, d)\}}. \quad (64)$$

Equality (64) means that $m(d-2, d-1, d)(dx)$ is the image of $m(dc)$ by $c \mapsto x = G(c)$.

Now in (64) we choose $s = \text{diag}(1, s_1)$ where s_1 is a symmetric positive definite matrix of order $d-1$. We also desintegrate $m(dc)$ by introducing a probability kernel $K(c_2, \dots, c_d; dc_1)$ and a positive measure $m_1(dc_2, \dots, dc_d)$ such that

$$e^{-\|c_1\|^2} m(dc_1, dc_2, \dots, dc_d) = m_1(dc_2, \dots, dc_d) K(c_2, \dots, c_d; dc_1).$$

With these notations we can write

$$\begin{aligned} \frac{1}{\det s_1^{(d-2)/2}} e^{\text{tr}(s_1^{-1})} &= \int_{\mathbb{R}^{d(d-2)}} e^{-\text{tr}\{sG(c)\}} m(dc) = \int_{\mathbb{R}^{d(d-2)}} e^{-\|c_1\|^2} e^{-\text{tr}\{s_1 G(c_2, \dots, c_d)\}} m(dc_1, dc_2, \dots, dc_d) \\ &= \int_{\mathbb{R}^{(d-1)(d-2)}} e^{-\text{tr}\{s_1 G(c_2, \dots, c_d)\}} \left\{ \int_{\mathbb{R}^{d-2}} K(c_2, \dots, c_d; dc_1) \right\} m_1(dc_2, \dots, dc_d) \\ &= \int_{\mathbb{R}^{(d-1)(d-2)}} e^{-\text{tr}\{s_1 G(c_2, \dots, c_d)\}} m_1(dc_2, \dots, dc_d) \end{aligned}$$

since K is a probability kernel. The last equality says that the image of $m_1(dc_2, \dots, dc_d)$ by the map $(c_2, \dots, c_d) \mapsto x = G(c_2, \dots, c_d)$ is nothing but $m(d-2, d-1, d-1)(dx)$. Denote $G_2 = G(c_2, \dots, c_d)$ for simplicity. Since c_2, \dots, c_d are vectors of a Euclidean space of dimension $d-2$, the rank of G_2 is less than or equal to $d-2$. To prove this elementary fact of linear algebra we use $G_2 \in \overline{\mathcal{P}_{d-1}}$. This implies that if $x = (x_2, \dots, x_d)^\top$, then $G_2 x = 0$ if and only if $x^\top G_2 x = 0$. Since $x^\top G_2 x = \|x_2 c_2 + \dots + x_d c_d\|^2$ the linear space of $x \in \mathbb{R}^{d-1}$ such that $x_2 c_2 + \dots + x_d c_d = 0$ has at least dimension 1, the kernel of the endomorphism of \mathbb{R}^{d-1} with matrix G_2 has at least dimension 1 and its image has at most dimension $d-2$. This contradicts Proposition 9 which says that $m(d-2, d-1, d-1)$ has an absolutely continuous part and therefore charges matrices with rank $d-1$. \square

7. Bibliographical comments and acknowledgments

The question of the existence of the non-central Wishart distribution was addressed in [9] where we claimed, in Proposition 2.3, that such a distribution exists if and only if p is in Λ_d without any restriction on $\Sigma \in \mathcal{P}_d$ or $w \in \mathcal{P}_d$. However the proof of the ‘if’ part was not given in [9] since we considered it obvious that if p was in the part $\{1/2, 2/2, \dots, (d-1)/2\}$ of Λ_d , then $\mathcal{NCW}(2p, w, \Sigma)$ did exist without restriction on the rank of w . This gap in [9] was kindly pointed out to us, in a private communication, by E. Mayerhofer who later showed in [12] that the statement was not only unproven, but false. More specifically, Mayerhofer [12] showed that if $\mathcal{NCW}(2p, w, \Sigma)$ exists, if $d \geq 3$ and if $n = 2p$ is in $\{1, 2, \dots, d-2\}$, then $\text{rank } w \leq n+1$. We reproved it in a different form in Proposition 2.4 of the present paper. Mayerhofer [12] used a stochastic process valued in the set of symmetric matrices in order to prove this statement. Finally, he conjectured in [12] that $\text{rank } w \leq n$ holds, and Proposition 2.5 shows that the conjecture is true.

Our present paper corrects the mistake in [9] by giving a necessary and sufficient condition for the existence of $\mathcal{NCW}(2p, w, \Sigma)$ through Proposition 2.6 and thus gives a proof of the aforementioned conjecture by Mayerhofer. We follow the lines of the arXiv paper [10], with a correction to the proof of Proposition 2.4 and a complement to Proposition 2.1. Recently, Graczyk, Małeck and Mayerhofer in [8] have also given a proof of the same result, using the same techniques as [12]. Clearly, the methods of the present paper are based on linear algebra and are of a different nature.

Because of the gap in the ‘if’ part of Proposition 2.3 of [9], it happened that the proof of the ‘only if’ part became also incomplete. This fact has been pointed out by Piotr Graczyk in a private communication. With the help of Mauro Piccioni, we have been able to design the simple proof of the second part of Proposition 2.1 of the present paper. We have taken the idea of Proposition 2.3 from [12]. Jacques Faraut helped us with Lemma 4.2.

The ‘only if’ part of Proposition 2.6 has a long history: for $w = 0$, an unconvincing proof appears in [14] and is commented upon in [1]. Later on, this result was conjectured by Eaton [3], who was unaware of [5]. Gindikin’s proof is described in FK Theorem VII 3.1, where it is explained why (3) is also called the Wallach set. Shanbhag in [17] gave an elementary and elegant proof, which is now the classical one. Peddada and Richards [15] gave the proof described in Proposition 4.1 of the present paper. They also proved that $p \in \Lambda$ when w has rank one.

The proper mathematical framework for this paper is that of Euclidean Jordan algebras rather than the linear spaces of real symmetric matrices. But working in that general framework might have obscured our statements without adding any insight: the extension of our results to Euclidean Jordan algebras is straightforward. However, past experience comparing the real and the complex case [6, 7] makes us aware that replacing real symmetric matrices by Hermitian ones could lead to more elegant formulas than in the real case. To conclude, our deepest thanks go to Eberhard Mayerhofer, Jacques Faraut, Piotr Graczyk, Mauro Piccioni and the two referees.

Acknowledgments. H. Massam gratefully acknowledges support from the Natural Sciences and Engineering Research Council of Canada through Discovery Grant No A8947.

References

- [1] M. Casalis, G. Letac, Characterization of the Jorgensen set in generalized linear models, TEST 3 (1994) 145–162.
- [2] A.G. Constantine, Some non-central distribution problems in multivariate analysis, Ann. Math. Statist. 34 (1963) 1270–1285.
- [3] M.L. Eaton, Multivariate Statistics: A Vector Space Approach, Wiley, New York, 1983.
- [4] J. Faraut, A. Korányi, Analysis on Symmetric Cones, Oxford University Press, 1994.
- [5] S. Gindikin, Invariant generalized functions in homogeneous spaces, J. Funct. Anal. Appl. 9 (1975) 50–52.
- [6] P. Graczyk, G. Letac, H. Massam, The complex Wishart distribution and the symmetric group, Ann. Statist. 31 (2003) 287–309.

- [7] P. Graczyk, G. Letac, H. Massam, The hyperoctahedral group, symmetric group representations and the moments of the real Wishart distribution, *J. Theoret. Probab.* 18 (2005) 1–42.
- [8] P. Graczyk, J. Małeck, E. Mayerhofer, Characterisation of the Wishart processes and Wishart distributions, *ArXiv*: 1607.00206v3, 2016.
- [9] G. Letac, H. Massam, The non-central Wishart as an exponential family and its moments, *J. Multivariate Anal.* 99 (2008) 1393–1417.
- [10] G. Letac, H. Massam, Existence and non-existence of the non-central Wishart distribution, *ArXiv*: 1108.2849, 2011.
- [11] G. Letac, J. Wesołowski, Laplace transforms which are negative powers of quadratic polynomials, *Trans. Amer. Math. Soc.* 360 (2008) 6475–6496.
- [12] E. Mayerhofer, On the existence of non-central Wishart distributions, *J. Multivariate Anal.* 114 (2013) 448–456.
- [13] R.J. Muirhead, *Aspects of Multivariate Analysis*, Wiley, New York, 1982.
- [14] I. Olkin, H. Rubin, A characterization of the Wishart distribution, *Ann. Math. Statist.* 33 (1962) 1272–1280.
- [15] S.D. Peddada, D.St.P. Richards, Proof of a conjecture of M.L. Eaton on the characteristic function of the Wishart distribution, *Ann. Probab.* 19 (1991) 869–874.
- [16] S. Roman, The formula of Faà di Bruno, *Amer. Math. Monthly* 87 (1980) 806–809.
- [17] D.N. Shanbhag, The Davidson–Kendall problem and related results on the structure of the Wishart distribution, *Austr. J. Statist.* 30A (1989) 272–280.
- [18] A. Takemura, *Zonal Polynomials*, Lecture Notes Monograph Series, Institute of Mathematical Statistics, Hayward, CA, 1984.