HAL open science

Spectral Inequalities for the Schrödinger operator

Gilles Lebeau, Iván Moyano

▶ To cite this version:

Gilles Lebeau, Iván Moyano. Spectral Inequalities for the Schrödinger operator. 2019. hal-01977648

HAL Id: hal-01977648
https://hal.science/hal-01977648

Preprint submitted on 10 Jan 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

SPECTRAL INEQUALITIES FOR THE SCHRÖDINGER OPERATOR

GILLES LEBEAU AND IVÁN MOYANO

Abstract. In this paper we deal with the so-called “spectral inequalities”, which yield a sharp quantification of the unique continuation for the spectral family associated with the Schrödinger operator in $\mathbb{R}^d$

$$H_{g,V} = \Delta_g + V(x),$$

where Δ_g is the Laplace-Beltrami operator with respect to an analytic metric g, which is a perturbation of the Euclidean metric, and $V(x)$ a real valued analytic potential vanishing at infinity.

Contents

1. Introduction and main result

Let g be a Riemannian metric on $\mathbb{R}^d$, Δ_g the usual Laplacian in the metric g, and $V = V(x)$ a real potential function, not necessarily short range, such as the typical examples of long-range interactions in scattering theory (cf. [9, vol.IV Ch.XXX]).

In this paper we prove spectral inequalities for the Schrödinger operator

$$H_{g,V} := -\Delta_g + V(x), \qquad \text{in } \mathbb{R}^d, \quad d \geq 1.$$

Our approach relies on interpolation inequalities, in the spirit of the works [13, 10, 14], but adapted to the unbounded case. We will use spectral projectors, holomorphic extension arguments, and suitable interpolation estimates for holomorphic functions.

In the case $V = 0$, we recover some classical quantifications of the uncertainty principle dues to Zigmund [23, pp.202-208], Logvenenko and Svereda [15] and Kovrojkine [7, 8], among other references (see Section 2.1.2 for further details).

1.1. **Geometric conditions for the observability sets.** Given $R > 0$ and $x \in \mathbb{R}^d$, we denote by $B^g_R(x)$ the ball of radius R, with respect to the metric g centered in x. When $x = 0$, we simply write B^g_R and if moreover $g = Id$, one can simply write B_R as usual. We shall work with Lebesgue measurable sets $\omega \subset \mathbb{R}^d$ satisfying the condition

$$\exists R, \delta > 0 \quad \text{such that} \quad \frac{mes(\omega \cap B^g_R(x))}{mes(B^g_R(x))} \geq \delta, \quad \forall x \in \mathbb{R}^d. \tag{1.1}$$

Since in the present work the metric g will be asymptotically the flat metric, (1.1) can be replaced by the same condition with the Euclidean metric:

$$\exists R, \delta > 0, \quad \text{such that} \quad \inf_{x\in\mathbb{R}^d} mes\,\{t \in \omega,\, |x - t| < R\} \geq \delta. \tag{1.2}$$

1.2. **Main result: spectral inequality for the Schrödinger operator on $\mathbb{R}^d$.** Let $g = g_{ij}(x)$ be a Riemannian metric in $\mathbb{R}^d$ and consider the Laplace-Beltrami operator associated to g, i.e.,

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \operatorname{div}\left(\sqrt{\det g}\, g^{-1}\nabla u\right), \tag{1.3}$$

where $g^{-1}(x) = (g^{ij})(x)$ denotes as usual the inverse metric of g. Given $V = V(x)$ a real-valued potential function, one defines the associated Schrödinger operator

$$H_{g,V} := -\Delta_g + V(x), \qquad \text{on} \quad D(H_{g,V}), \tag{1.4}$$

where

$$D(H_{g,V}) = \left\{ u \in L^2(\mathbb{R}^d; \sqrt{\det g}\,\mathrm{d}x);\ H_{g,V}(u) \in L^2(\mathbb{R}^d; \sqrt{\det g}\,\mathrm{d}x) \right\}\ .$$

We will assume that the metric g and the real-valued potential V satisfy the following hypothesis:

$$\text{the metric } g \text{ is analytic, of the form } g = Id + \tilde{g} \text{ with } \lim_{|x|\to\infty} \tilde{g}(x) = 0, \tag{1.5}$$

$$\text{the potential } V \text{ is analytic, real valued, with } \lim_{|x|\to\infty} V(x) = 0, \tag{1.6}$$

$$\begin{aligned} &\exists a > 0 \text{ such that } g \text{ and } V \text{ extend holomorphically in the} \\ &\text{complex open set defined by } U_a := \left\{ z \in \mathbb{C}^d;\ |\operatorname{Im}(z)| < a \right\}, \end{aligned} \tag{1.7}$$

$$\begin{aligned} &\exists \epsilon \in (0, 1) \text{ such that } \tilde{g} \text{ and } V \text{ satisfy for } |\alpha| \leq 2 \\ &|\partial^\alpha \tilde{g}(z)| + |\partial^\alpha V(z)| \leq C_\alpha (1 + |z|)^{-\epsilon - |\alpha|}, \quad \forall z \in U_a. \end{aligned} \tag{1.8}$$

Under these hypothesis one can check (see Proposition 4.46) that the Schrödinger operator $H_{g,V}$ is an unbounded self-adjoint operator. In section 4 we will recall some basic facts on the functional calculus of self-adjoint operators. In particular, the spectral projectors $\Pi_\mu(g,V)$ are defined in (4.55). In this article we will prove that the family of specral projectors $\Pi_\mu(g,V)$ enjoy a spectral inequality, i.e., an observability inequality on a set $\omega \subset \mathbb{R}^d$ for low frequencies as long as the observability set satisfies (1.1).

Given $\mu \in \mathbb{R}$, let us introduce

$$\mu_{\pm}^{\frac{1}{2}} := \sqrt{\mu}, \quad \mu \geq 0, \qquad \mu_{\pm}^{\frac{1}{2}} := \pm i\sqrt{|\mu|}, \quad \mu < 0. \tag{1.9}$$

Our main result is the following:

THEOREM 1.1. *Let $\omega \subset \mathbb{R}$ be a measurable set satisfying the geometric condition (1.2) and let (g,V) satisfy hypothesis (1.5)–(1.8). Then, there exist constants $A = A(\omega,g,V) > 0$ and $C = C(\omega,g,V) > 0$ such that for all $\mu \in \mathbb{R}$ and all $f \in L^2(\mathbb{R}^d)$, one has*

$$\|\Pi_\mu(g,V)f\|_{L^2(\mathbb{R}^d;\sqrt{\det g}\,\mathrm{d}x)} \leq A\ |e^{C\mu_{\pm}^{\frac{1}{2}}}|\ \|\Pi_\mu(g,V)f\|_{L^2(\omega;\sqrt{\det g}\,\mathrm{d}x)}. \tag{1.10}$$

1.3. **A special case: spectral inequality for the Laplacian operator in $\mathbb{R}^d$.** When $g = Id$ and $V = 0$ as $H_0 = -\Delta_x$ is the usual flat Laplacian, it is well-known that that $\sigma(Id,0) = [0,\infty)$ is a purely absolutely continuous spectrum. Furthermore, the spectral projectors are explicitly determined through the Fourier transform, i.e.,

$$\Pi(Id,0)_\mu f = \frac{1}{(2\pi)^d}\int_{|\xi|<\mu} \hat{f}(\xi)e^{ix\cdot\xi}\,\mathrm{d}x,$$

recalling that the classical Fourier transform is defined by

$$\hat{g}(\xi) := \int_{\mathbb{R}^d} g(x)e^{-ix\cdot\xi}\,\mathrm{d}x, \quad \forall \xi \in \mathbb{R}^d, \quad \forall g \in L^2(\mathbb{R}^d).$$

As a result, one can recast (1.10) as the following familiar spectral inequality.

THEOREM 1.2. *Let $\omega \subset \mathbb{R}$ be a Lebesgue measurable set satisfying the geometric condition (1.2). Then, there exist constants $A = A(\omega) > 0, C = C(\omega) > 0$ such that for all $\mu \in \mathbb{R}_+$ and all $f \in L^2(\mathbb{R}^d)$, one has*

$$\|f\|_{L^2(\mathbb{R})} \leq A(\omega)e^{C(\omega)\mu}\|f\|_{L^2(\omega)}, \quad \textit{whenever} \quad \operatorname{supp}\hat{f} \subset \overline{B_\mu}. \tag{1.11}$$

2. Context of our results and previous works

2.1. Spectral inequalities, Logvenenko-Sereda inequalities and the uncertainty principle.

2.1.1. *Spectral Inequalities.* Given a compact Riemannian manifold M equipped with a metric g the spectral inequalities for $V = 0$ have been introduced by the first author and D. Jerison in [10] and also in [14], among other works (see [11] and the references therein).

In the non-compact case when $M = \mathbb{R}^d$ with the usual Euclidean metric, the methods of [13] have been extended in [12] in such a way that inequality (1.11) holds whenever $\omega \subset \mathbb{R}^d$ is an open set satisfying the geometric condition

$$\exists R > 0, \delta > 0 \text{ s.t. } \forall y \in \mathbb{R}^d,\ \exists y' \in \mathbb{R}^d \text{ with } B(y',R) \subset \omega \text{ and } d(y,y') < \delta. \tag{2.12}$$

This condition is thus proven to be sufficient for the null-controllability of the heat equation to hold, as well as for some hypoellipitic equations, like the Kolmogorov equation, arising in kinetic theory [16]. This condition is however far from being necessary (see Section 2.2 for details).

2.1.2. *Quantification of the uncertainity principle and the Logvenenko-Sereda inequality.* The classical uncertainty principle in harmonic analysis accounts for the fact that a function cannot be localised both in space and in the frequency variable (cf. [17, Prop. 10.2, p. 270]) :

$$\exists C>0, \quad \left(\int_{\mathbb{R}} x^2|f(x)|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} \xi^2|\hat{f}(\xi)|^2d\xi\right)^{\frac{1}{2}} \geq C\|f\|^2_{L^2(\mathbb{R})}, \quad \forall f\in L^2(\mathbb{R}).$$

A version of this inequality due to Amrein and Berthier (cf. [1]) guarantees that

$$\int_{\mathbb{R}\setminus\omega}|f(x)|^2dx+\int_{\mathbb{R}\setminus\Sigma}|\hat{f}(\xi)|^2d\xi \geq C\|f\|^2_{L^2(\mathbb{R})}, \quad mes(\omega)+mes(\Sigma)<\infty.$$

In the aim of giving more quantitative versions of the uncertainty principle in $\mathbb{R}^d$, one can find very significant literature (see [8] and the references therein for further details). In particular, Logvinenko and Sereda proved in [15] that the condition

$$E\subset\mathbb{R} \text{ measurable s.t. } \exists\gamma>0,\ a>0 \text{ s.t. } \frac{mes(E\cap I)}{mes(I)}\geq\gamma \tag{2.13}$$

whenever I is an interval of length a is sufficient to ensure that

$$\forall b\geq 0,\ \exists C=C(\gamma,a,b)>0 \text{ s.t.} \int_E |f(x)|^2\,\mathrm{d}x \geq C\|f\|^2_{L^2(\mathbb{R})}, \text{ if } \mathrm{supp}(\hat{f})\subset(-b,b). \tag{2.14}$$

On the other hand, the authors do not get a sharp estimate of C with respect to the parameters a,b,γ. This was achieved by Kovrijkine [7], where the author proves that

$$\exists K>0 \text{ such that } C(\gamma,a,b)=\left(\frac{\gamma}{K}\right)^{K(ab+1)},$$

with $\gamma<K$. Moreover, the Logvenenko-Sereda inequality (2.14) also holds in any $L^p(\mathbb{R})$ space with $p\in[1,\infty]$. This is possible by combining the Bernstein's inequality, a suitable Remez-type inequality and some previous results by A. Zigmund in lacunary series [23, pp. 202-208]. The same results were obtained by Nazarov in [19]. We refer to [6] and the references therein for more details on this subject.

Thus our Theorem 1.2 (when $V=0$ and $g=Id$) recovers the Logvinenko-Sereda inequality in $\mathbb{R}^d$.

2.2. **Controllability of the parabolic equations in the whole space.** Some recent work has been concerned with the problem of characterising the sets having *"good observability properties"* in the whole space in the context of the controllability of the heat equation and some other parabolic problems. More precisely, given an open set $\omega\subset\mathbb{R}^d$, for $d\geq 1$, let us consider the heat equation

$$(\partial_t-\Delta_x)\,u=\chi_\omega g, \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^d, \tag{2.15}$$

where χ_ω is the characteristic function of $\omega\subset\mathbb{R}^d$ and g is a forcing term, that we call a control, supported in $(0,T)\times\omega$. The small-time null-controllability of (2.15) in an L^2 setting is equivalent to the following property

$$\forall T>0,\ \forall u_0\in L^2(\mathbb{R}^d), \quad \exists g\in L^2((0,T)\times\omega) \quad \text{s.t} \quad u|_{t=T}=0, \tag{2.16}$$

where u is the solution of (2.15) with $u|_{t=0} = u_0$. According to the classical HUM method, the null controllability of (2.15) is equivalent to the observability for the adjoint system

$$(\partial_t - \Delta_x)\,\psi = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \tag{2.17}$$

which is equivalent to the following observability inequality:

$$\exists C_{obs} > 0 \quad \text{s.t. } \forall \psi \in L^2(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} |\psi|_{t=T}|^2 \,\mathrm{d}x \le \int_0^T \int_\omega |\psi(t,x)|^2 \,\mathrm{d}t\,\mathrm{d}x, \tag{2.18}$$

where ψ solves (2.17) with $\psi|_{t=0} = \psi_0$.

In the recent works [21, 5, 18], the authors use the Logvinenko-Sereda inequality to show that (2.18) holds if and only if $\omega \subset \mathbb{R}^d$ is a measurable set satisfying (1.1) for some $R, \delta > 0$.

2.3. **Outline of the paper.** In Section 3 we give a proof of Proposition 3.1 which is a basic interpolation estimate for holomorphic functions defined in a tubular neighborhood of $\mathbb{R}^d$. Although one can obtain this type of result like in [7] using the Remez inequality for polynomials as a starting point, we choose instead to present a proof which uses a Carleman estimate for functions of one complex variable (see Lemma 3.2).
In section 4 we recall some facts on the spectral theory of the Schrödinger operator and we introduce the Poisson kernel.
In section 5, we prove estimates on the holomorphic extension of solutions to the Poisson equation.
Finally, in section 6, we show that Theorem 1.1 is an easy consequence of the previous holomorphic extension estimates.

3. Carleman estimates and interpolation inequalities

Recall that for $a > 0$ we denote by U_a the tubular neighborhood of $\mathbb{R}^d$ in $\mathbb{C}^d$,

$$U_a = \{z \in \mathbb{C}^d, |Im(z)| < a\}.$$

Let $\mathcal{H}_a$ be the Banach space of holomorphic functions $f(x+iy)$ in U_a such that

$$\|f\|_{\mathcal{H}_a} := \sup_{|y|<a} \|f(.+iy)\|_{L^2(\mathbb{R}^d)} < \infty.$$

By the Paley-Wiener Theorem, $\mathcal{H}_a$ is the space of Fourier Transforms of functions $g \in L^2(\mathbb{R}^d)$ such that $\sup_{|\eta|<a} \|e^{\eta\cdot y} g(y)\|_{L^2(\mathbb{R}^d)} < \infty$.

The goal of this section is to prove the following result

PROPOSITION 3.1. *Let $\omega \subset \mathbb{R}^d$ satisfying the density condition (1.2). Then, there exist constants $C = C(a,\omega) > 0$ and $\nu = \nu(a,\omega) \in]0,1[$ such that*

$$\int_{\mathbb{R}^d} |f|^2 \,\mathrm{d}x \le C \left(\int_\omega |f|^2 \,\mathrm{d}x \right)^{\nu} \left(\int_{U_a} |f|^2 |\,\mathrm{d}z| \right)^{1-\nu}, \tag{3.19}$$

for any $f \in \mathcal{H}_a$.

Observe that Theorem 1.2 is an easy consequence of Proposition 3.1. In fact, if $f \in L^2(\mathbb{R}^d)$ is such that its Fourier transform $\hat{f}(\xi)$ is supported in the ball $|\xi| \le \mu$, by the Fourier inversion formula one has

$$f(x)=(2\pi)^{-d}\int_{|\xi|\le\mu} e^{ix\xi}\hat f(\xi)d\xi,\qquad \forall x\in\mathbb{R}^d.$$

Therefore f is the restriction to $\mathbb{R}^d$ of the holomorphic function $f(z)$ defined on $\mathbb{C}^d$ by $f(z)=(2\pi)^{-d}\int_{|\xi|\le\mu} e^{iz\xi}\hat f(\xi)d\xi$, and one has by Plancherel theorem

$$\|f(.+iy)\|^2_{L^2(\mathbb{R}^d)}=(2\pi)^{-d}\int_{|\xi|\le\mu}|e^{-y\xi}\hat f(\xi)|^2d\xi\le e^{2\mu|y|}\|f\|^2_{L^2(\mathbb{R}^d)}.$$

Therefore, from (3.19) we get

$$\|f\|^2_{L^2(\mathbb{R}^d)}\le C\left(\int_\omega |f|^2\,\mathrm{d}x\right)^\nu\left(c_d a^d e^{2\mu a}\|f\|^2_{L^2(\mathbb{R}^d)}\right)^{1-\nu},$$

where c_d is the volume of the unit sphere in $\mathbb{R}^d$, and this implies (1.11).

We will prove Proposition 3.1 in several steps. First we prove suitable Carleman estimates for holomorphic functions of one complex variable. In particular, we prove the interpolation estimate given in Lemma 3.3 below. We then deduce the multidimensional interpolation inequality given in Proposition 3.7. Finally, we get the proof of Proposition 3.1 by a simple covering argument.

3.1. **Carleman estimates for $\overline{\partial}$ in $\mathbb{C}$.** In this section, we prove basic estimates for holomorphic functions of one complex variable. We denote by $d\lambda$ the Lebesgue measure on $\mathbb{C}\simeq\mathbb{R}^2$.

Let $X\subset\mathbb{C}$ be an open bounded connected domain with regular boundary. We start with the following classical Carleman inequality.

LEMMA 3.2. *Let $\varphi(x)$ be a continuous function on X such that $\triangle\varphi=\nu$ is a Borel measure on X. For all $f\in C_0^\infty(X)$ and all $h>0$, the following inequality holds true*

$$4h^2\int_X e^{2\frac{\varphi(x)}{h}}|\overline{\partial}f(x)|^2d\lambda\ge h\int_X e^{2\frac{\varphi(x)}{h}}|f(x)|^2d\nu. \tag{3.20}$$

Proof. Let $f\in C_0^\infty(X)$ be given and let $Q\subset X$ be a compact set such that the support of f is contained in Q. We first assume that φ is smooth in a neighborhood of Q. Let $P:=\frac{2h}{i}\overline{\partial}$. We define the conjugate operator

$$P_\varphi:=e^{\frac{\varphi}{h}}Pe^{-\frac{\varphi}{h}}=A+iB,\quad A=\frac{h}{i}\partial_x-\varphi'_y=A^*,\ B=\frac{h}{i}\partial_y+\varphi'_x=B^*.$$

Set $g=e^{\frac{\varphi}{h}}f\in C_0^\infty(Q)$. One has $P_\varphi g=e^{\frac{\varphi}{h}}Pf$ and by integration by part we get

$$\begin{aligned}\|P_\varphi g\|^2_{L^2(X)}&=\|Ag\|^2_{L^2(X)}+\|Bg\|^2_{L^2(X)}+i(Bg,Ag)_{L^2(X)}-i(Ag,Bg)_{L^2(X)}\\&=\|Ag\|^2_{L^2(X)}+\|Bg\|^2_{L^2(X)}+i([A,B]g,g)_{L^2(X)}\end{aligned}\tag{3.21}$$

Since $[A,B]=AB-BA=-ih\triangle\varphi$, we get

$$4h^2\int_X e^{2\frac{\varphi(x)}{h}}|\overline{\partial}f(x)|^2d\lambda=\|P_\varphi g\|^2_{L^2(X)}\ge h\int_X e^{2\frac{\varphi(x)}{h}}|f(x)|^2\triangle\varphi(x)d\lambda,\tag{3.22}$$

and thus (3.20) holds true. Let now $\varphi(x)$ be a continuous function on X such that $\triangle\varphi = \nu$ is a Borel measure on X. Let χ_ε be a smooth approximation of the identity. Then $\varphi_\varepsilon = \varphi * \chi_\varepsilon$ and $\nu_\varepsilon = \nu * \chi_\varepsilon$ are well defined in a neighborhood V of Q for ε small and one has $\triangle\varphi_\varepsilon d\lambda = \nu_\varepsilon$ on V. The inequality (3.22) holds true for φ_ε. Since φ_ε converge uniformly to φ on Q, it remains to verify

$$\lim_{\varepsilon\to 0}\int_X e^{2\frac{\varphi_\varepsilon(x)}{h}}|f(x)|^2\triangle\varphi_\varepsilon(x)d\lambda = \int_X e^{2\frac{\varphi(x)}{h}}|f(x)|^2 d\nu \ .$$

Since the total variation on Q of the measures ν_ε is bounded, i.e $\sup_\varepsilon \int_Q d|\nu_\varepsilon| < \infty$, one has

$$\lim_{\varepsilon\to 0}\int_X |e^{2\frac{\varphi_\varepsilon(x)}{h}} - e^{2\frac{\varphi(x)}{h}}||f(x)|^2 d\nu_\varepsilon = 0 \ .$$

Then, the result follows from the convergence of the measures ν_ε to ν, i.e $\lim_{\varepsilon\to 0}\int_X g(x)d\nu_\varepsilon = \int_X g d\nu$ for any continuous function g with support in Q. The proof of Lemma 3.2 is complete.

□

Let $K \subset X$ a compact subset of X, and μ a positive Borel measure with support in K such that $\mu(K) = 1$. We will assume that μ satisfies the following hypothesis:

(3.23)
The potential function $W(x) = -\frac{1}{2\pi}\int \log|x-y|d\mu(y),$ solution in $\mathcal{D}'(\mathbb{R}^2)$ of the equation $-\triangle W = \mu,$ is continuous on $\mathbb{C} \simeq \mathbb{R}^2$.

For $y \in X$, we denote by $G(x, y)$ the Green function of the Dirichlet problem

$$\begin{cases} -\Delta_x G(x,y) = \delta_{x=y} \quad \text{in} \quad X, \\ G(.,y)|_{\partial X} = 0. \end{cases} \tag{3.24}$$

Recall that one has $G(x, y) > 0$ for all $(x, y) \in X \times X$ with $x \neq y$, and that there exist constants $0 < c_1 < c_2$ such that for all $x \in X$ with $\mathrm{dist}(x, \partial X)$ small one has

$$c_1 dist(x,\partial X) \le \inf_{y\in K} G(x,y), \quad \sup_{y\in K} G(x,y) \le c_2 dist(x,\partial X) \tag{3.25}$$

Moreover, $G(x, y)$ is analytic in $x \in X \setminus \{y\}$, and more precisely one has

$$\begin{cases} G(x,y) = -\frac{1}{2\pi}\int \log|x-y| + H(x,y) \\ -\Delta_x H(x,y) = 0, \\ H(x,y)|_{x\in\partial X} = \frac{1}{2\pi}\int \log|x-y|. \end{cases} \quad \text{in} \quad X, \tag{3.26}$$

Let $\Phi_\mu(x) = \int G(x, y)d\mu(y)$. Then Φ_μ satisfies

$$\begin{cases} -\Delta\Phi_\mu = \mu, \quad \text{in} \quad X, \\ \Phi_\mu|_{\partial X} = 0. \end{cases} \tag{3.27}$$

By assumption (3.23), the function $\Phi_\mu(x)$ is continuous on X. Moreover, Φ_μ is smooth on $\overline{X} \setminus K$, one has $\Phi_\mu(x) > 0$ for all $x \in X$, and

$$\left\{\begin{array}{l}\Phi_\mu(x) = W(x) + h(x) \\ -\Delta h = 0, \text{ in } X, \quad h|_{\partial X} = -W|_{\partial X}.\end{array}\right. \tag{3.28}$$

We will denote by $C_\mu > 0$ the constant

$$C_\mu = \sup_{x\in X} \Phi_\mu(x) = \sup_{x\in K} \Phi_\mu(x)\ . \tag{3.29}$$

Observe that from (3.25), one has for all $x \in X$ with $\mathrm{dist}(x, \partial X)$ small

$$\Phi_\mu(x) \leq c_2 \mathrm{dist}(x, \partial X). \tag{3.30}$$

Let $Y \subset X$ an open subset of X with regular boundary such that $K \subset Y$ and $\overline{Y} \subset X$. We will denote by $c_Y > 0$ the constant

$$c_Y = \inf_{x\in Y, y\in K} G(x,y)\ . \tag{3.31}$$

From $\Phi_\mu(x) = \int G(x,y)d\mu(y)$ we get

$$c_Y \leq \inf_{x\in Y} \Phi_\mu(x)\ . \tag{3.32}$$

Let $\Psi_Y(x) = -\int_Y G(x,y)d\lambda(y)$ be the solution of

$$\triangle\Psi_Y = 1_{x\in Y}, \quad \Psi_Y|_{\partial X} = 0\ . \tag{3.33}$$

The function Ψ_Y is continuous on $\overline{X}$ and one has $\Psi_Y(x) \leq 0$ for all $x \in X$. We denote by C_Y the constant

$$C_Y = \sup_{x\in Y} |\Psi_Y(x)|\ . \tag{3.34}$$

Let $\rho > 0$ such that $2\rho C_Y \leq c_Y$ and define φ by the formula

$$\varphi(x) = \Phi_\mu(x) + \rho\Psi_Y(x)\ . \tag{3.35}$$

Then φ is continuous on X and one has $\triangle\varphi = -\mu + \rho 1_{x\in Y}$. Thus we can apply (3.20) and we get for all $f \in C_0^\infty(X)$

$$4h^2 \int_X e^{2\frac{\varphi(x)}{h}} |\overline{\partial} f(x)|^2 d\lambda + h\int_K e^{2\frac{\varphi(x)}{h}} |f(x)|^2 d\mu \geq h\rho \int_Y e^{2\frac{\varphi(x)}{h}} |f(x)|^2 d\lambda. \tag{3.36}$$

Let g be an holomorphic function in X such that $g \in L^2(X)$. We will apply (3.36) to ψg, with $\psi \in C_0^\infty(X)$ such that ψ is equal to 1 in a neighborhood of $\overline{Y}$, and $\nabla\psi$ is supported in $\mathrm{dist}(x, \partial X) \leq r$ with $r > 0$ small enough to have $4c_2 r \leq c_Y$. By our choice of the constants ρ and r, the following inequalities hold true:

$$\begin{aligned}\sup_{x\in K} \varphi(x) &\leq \sup_{x\in K} \Phi_\mu(x) = C_\mu, \\ \inf_{x\in Y} \varphi(x) &\geq \inf_{x\in Y} \Phi_\mu(x) - \rho \sup_{x\in Y} |\Psi_Y(x)| \geq c_Y/2, \\ \sup_{x\in \mathrm{support}(\nabla\psi)} \varphi(x) &\leq \sup_{dist(x,\partial X)<r} \Phi_\mu(x) \leq c_Y/4\ .\end{aligned} \tag{3.37}$$

Since $\overline{\partial}(\psi g) = g\overline{\partial}\psi$, we get that the following Lemma holds true, with $M = \|\overline{\partial}\psi\|_{L^\infty}$.

LEMMA 3.3. *For every holomorphic function $g \in L^2(X)$ and all $h > 0$, the following inequality holds true*

$$4h^2 M e^{\frac{c_Y}{2h}} \int_X |g(x)|^2 d\lambda + h e^{\frac{2C_\mu}{h}} \int_K |g(x)|^2 d\mu \geq h\rho e^{\frac{c_Y}{h}} \int_Y |g(x)|^2 d\lambda. \tag{3.38}$$

REMARK 3.4. *Observe that from the definition (3.29) of C_μ, one has obviously $C_\mu \geq c_Y$.*

PROPOSITION 3.5. *There exists a constant C, depending only on X, Y such that for all holomorphic function $g \in L^2(X)$, the following interpolation inequality holds true*

$$\int_Y |g(x)|^2 d\lambda \leq C\big(\int_K |g(x)|^2 d\mu\big)^\delta \big(\int_X |g(x)|^2 d\lambda\big)^{1-\delta}, \quad \delta = \frac{c_Y}{4C_\mu - c_Y}\ . \tag{3.39}$$

Proof. Set $\mathcal{X} = \int_X |g(x)|^2 d\lambda$, $\mathcal{O} = \int_K |g(x)|^2 d\mu$ and $\mathcal{Y} = \int_Y |g(x)|^2 d\lambda$. Inequality (3.38) reads

$$4hMe^{\frac{-c_Y}{2h}} \mathcal{X} + e^{\frac{2C_\mu - c_Y}{h}} \mathcal{O} \geq \rho \mathcal{Y}\ . \tag{3.40}$$

If $\mathcal{O} = 0$, by taking the limit $h \to 0$, we get $\mathcal{Y} = 0$, hence we may assume $\mathcal{O} > 0$. Let h_* be such that

$$\frac{\mathcal{X}}{\mathcal{O}} = F(h_*), \quad F(h) = \frac{1}{4hM} e^{\frac{4C_\mu - c_Y}{2h}}\ .$$

If $h_* \geq 1$, we use $F(h_*) \leq F(1)$, hence $\mathcal{X} \leq F(1)\mathcal{O}$, and we write

$$\mathcal{Y} \leq \mathcal{X} \leq F(1)^\delta \mathcal{O}^\delta \mathcal{X}^{1-\delta}.$$

Observe that $F(1)^\delta$ is independent of C_μ, hence depends only on X, Y.
If $h_* \leq 1$, we use $e^{\frac{4C_\mu - c_Y}{2h_*}} \leq 4M\frac{\mathcal{X}}{\mathcal{O}}$, and we write

$$4h_* M e^{\frac{-c_Y}{2h_*}} \mathcal{X} + e^{\frac{2C_\mu - c_Y}{h_*}} \mathcal{O} = 2e^{\frac{2C_\mu - c_Y}{h_*}} \mathcal{O} \leq (4M)^{1-\delta} \mathcal{O}^\delta \mathcal{X}^{1-\delta}.$$

The proof of Proposition 3.5 is complete. □

From Proposition 3.5 we deduce the following Lemma.

LEMMA 3.6. *Let $X \subset \mathbb{C}$ be a complex neighborhood of $[0,1]$. There exists constants C, c depending only on X such that the following holds true. If $E \subset [0,1]$ is a measurable set with positive measure $|E| > 0$, and g a holomorphic and bounded function on X, one has:*

$$\sup_{x\in[0,1]} |g(x)| \leq \frac{C}{|E|^{\delta/2}} \big(\int_E |g(x)|^2 dx\big)^{\delta/2} \big(\sup_{x\in X} |g(x)|\big)^{1-\delta}, \quad \delta = \frac{c}{1 + |\log |E||}\ . \tag{3.41}$$

Proof. We may assume X bounded with regular boundary. We apply Proposition 3.5 with $K = [0,1]$ and the measure μ defined by $\int g d\mu = |E|^{-1} \int_E g(x)dx$. By Cauchy integral formula, if $Y \subset\subset X$ is a complex neighborhood of $[0,1]$, there exists a constant C such that

$$\sup_{x\in[0,1]} |g(x)| \leq C\big(\int_Y |g(x)|^2 d\lambda\big)^{1/2}.$$

Thus, from (3.39) it just remains to verify the lower bound on δ. By formula (3.39), this is equivalent to get a upper bound on C_μ. From formula (3.28) we get

$$C_\mu \le C + \sup_{x\in[0,1]} \frac{1}{|E|}\int_E |\log|x-t||dt \le C(1+|\log|E||),$$

with C independent of E. The proof of Lemma 3.3 is complete. □

3.2. **Interpolation estimates in $\mathbb{R}^d$.** Let $R>0$ be given. Let $X\subset\mathbb{C}^d$ be a bounded complex neighborhood of the closed Euclidean ball $B_R=\{x\in\mathbb{R}^d,\ |x|\le R\}$. Let $E\subset B_R$ be a measurable set with positive Lebesgue measure, $|E|>0$. The goal of this section is to prove the following interpolation inequality.

PROPOSITION 3.7. *There exists constants $C>0,\delta\in]0,1]$, depending only on X, R and $|E|$, such that for all holomorphic function $g\in L^2(X)$, the following interpolation inequality holds true*

$$\int_{B_R}|g(x)|^2dx \le C\Big(\int_E|g(x)|^2dx\Big)^\delta\Big(\int_X|g(z)|^2|dz|\Big)^{1-\delta}. \tag{3.42}$$

Proof. The proof of Proposition 3.7 is an usual consequence of Lemma 3.6. We recall it for the reader's convenience. We may assume $R=1$. If $\tilde X\subset\subset X$ is a complex neighborhood of $B=B_1$, by Cauchy integral formula one has

$$\sup_{x\in\tilde X}|g(x)|\le C\Big(\int_X|g(z)|^2|dz|\Big)^{1/2}.$$

Therefore, replacing X by $\tilde X$, we just have to prove

$$\sup_{x\in B}|g(x)|\le C\Big(\int_E|g(x)|^2dx\Big)^{\delta/2}\Big(\sup_{x\in X}|g(x)|\Big)^{1-\delta}. \tag{3.43}$$

Let $x_0\in B$ such that $|g(x_0)|=\sup_{x\in B}|g(x)|$. Let $\rho\in]0,1[$ such that $|B_\rho|\le|E|/2$. Set $\tilde E=E\cap\{\rho<|x|<1\}$. One has $|\tilde E|\ge|E|/2$. Let S^{d-1} be the unit sphere in $\mathbb{R}^d$ and let c_d be its volume. For $\omega\in S^{d-1}$, define $r_\omega\in[0,2]$ as the largest value of $r\ge0$ such that $x_0+r\omega\in B_1$ and set

$$E_\omega=\{r\in]\rho,r_\omega],\ \text{such that}\ \ x_0+r\omega\in E\}.$$

We denote by $|E_\omega|\in[0,2]$ the Lebesgue measure of $E_\omega\subset[0,2]$. One has

$$|\tilde E|=\int_{S^{d-1}}\Big(\int_{r\in E_\omega}r^{d-1}dr\Big)d\sigma(\omega)\le 2^{d-1}\int_{S^{d-1}}|E_\omega|d\sigma(\omega). \tag{3.44}$$

Set

$$V=\{\omega\in S^{d-1}\ \text{such that}\ |E_\omega|\ge\frac{|\tilde E|}{2^dc_d}\}.$$

By (3.44) one has $|\tilde E|\le|\tilde E|/2+2^dVol(V)$, hence $Vol(V)\ge|\tilde E|/2^{d+1}$.
Observe that there exists $a>0$ such that X is a complex neighborhood of B_{1+a}. Therefore, for each $\omega\in V$, the function of one complex variable z,

$$g_\omega(z)=g\Big(x_0+\frac{z}{r_\omega+a}\omega\Big)$$

is defined in a complex neighborhood Z of the interval $[0,1]$ independent of ω and one has $x_0+\frac{z}{r_\omega+a}\omega\in X$ for $z\in Z$. Therefore, we can apply Lemma 3.3, and we get for all $\omega \in V$,

$$|g(x_0)|^{2/\delta} = |g_\omega(0)|^{2/\delta} \leq C \left(\int_{E_\omega} |g(x_0 + r\omega)|^2 dr \right) \big(\sup_{x \in X} |g(x)| \big)^{\frac{2(1-\delta)}{\delta}} ,$$

with C, δ depending only on X and $|E|$ since with have the lower bound $|E_\omega| \geq \frac{|E|}{c_d 2^{d+1}}$. By integration in $\omega \in V$, using $E_\omega \subset]\rho, 2[$, we get

$$Vol(V)|g(x_0)|^{2/\delta} \leq \frac{C}{\rho^{d-1}} \int_{\omega \in V} \int_{E_\omega} |r^{d-1} g(x_0 + r\omega)|^2 dr d\sigma(\omega) \quad \big(\sup_{x \in X} |g(x)| \big)^{\frac{2(1-\delta)}{\delta}} .$$

Since

$$\int_{\omega \in V} \int_{E_\omega} |r^{d-1} g(x_0 + r\omega)|^2 dr d\sigma(\omega) \leq \int_E |g|^2 dx ,$$

we get that (3.43) holds true. The proof of Proposition 3.7 is complete. □

3.3. **Proof of Proposition 3.1.** In this section we prove Proposition 3.1. Let $R, \delta > 0$ given by the assumption (1.2). For $k \in \mathbb{Z}^d$, let $B_R(k) = k + B_R$ the closed ball of radius R centered at k. Increasing R if necessary, we may assume that the family $(B_R(k))_{k \in \mathbb{Z}^d}$ is a covering of $\mathbb{R}^d$ and that we have

$$\int_{\mathbb{R}^d} |f|^2 dx \leq \sum_{k \in \mathbb{Z}^d} \int_{B_R(k)} |f|^2 dx .$$

Let $X \subset U_{a/2}$ be a complex neighborhood of $B_R(0)$ and for any $k \in \mathbb{Z}^d$ set $\omega_k = \omega \cap B_R(k)$. By assumption, one has $|\omega_k| \geq \delta$ for all k. By Proposition 3.7, there exists constants $C, \nu > 0$ independent of $k \in \mathbb{Z}^d$ such that

$$\int_{B_R(k)} |f(x)|^2 dx \leq C \big(\int_{\omega_k} |f(x)|^2 dx \big)^\nu \big(\int_{X+k} |f(z)|^2 |dz| \big)^{1-\nu} . \tag{3.45}$$

Set

$$c_k = \int_{B_R(k)} |f(x)|^2 dx, \quad a_k = \int_{\omega_k} |f(x)|^2 dx, \quad b_k = \int_{X+k} |f(z)|^2 |dz|$$

By Hölder's inequality with $1/p = \nu, 1/q = 1 - \nu$, we get

$$\sum_k c_k \leq C \sum_k a_k^\nu b_k^{1-\nu} \leq C (\sum_k a_k)^\nu (\sum_k b_k)^{1-\nu} .$$

It remains to observe that one has

$$\sum_k a_k \leq C \int_\omega |f(x)|^2 dx, \quad \sum_k b_k \leq C \int_{U_a} |f(z)|^2 |dz| .$$

The proof of Proposition 3.1 is complete.

4. Spectral analysis

4.1. **Description of the spectrum.** The goal of this section is to give a description of the spectrum of the operator $H_{g,V}$ defined by (1.4). To do this, we apply the long-range scattering theory developed in [9, Chap. 30], which yields the following result.

PROPOSITION 4.1. *The operator $H_{g,V}$ defined by (1.4) is self-adjoint in the space $L^2(\mathbb{R}^d; \sqrt{\det g}\,\mathrm{d}x)$. The spectrum of $H_{g,V}$ is of the form*

$$\Sigma(H_{g,V}) = \Lambda \cup \{0\} \cup H_{ac}, \tag{4.46}$$

where $H_{ac} = (0, +\infty)$ is the absolutely continuous spectrum, and Λ is the set of non zero eigenvalues. Moreover, there exists $E_0 > 0$ such that $\Lambda \subset [-E_0, 0)$ and any eigenvalue $\lambda \in \Lambda$ is isolated with finite multiplicity.

REMARK 4.2. *The set Λ may be empty, or finite, or countable, and 0 is its only possible accumulation point.*

Proof. According to [9, Sec. 30.2], let us split the operator $H_{g,V}$ in the following manner

$$H_{g,V} = -\Delta + \tilde{V}(x, \partial), \tag{4.47}$$

where Δ is the flat Laplacian in $\mathbb{R}^d$ and

$$\tilde{V}(x, \partial) = V(x) - \operatorname{div}\left[(g^{-1} - Id)\nabla\right] - \frac{g^{-1}}{\sqrt{\det g}} \nabla(\sqrt{\det g}) \cdot \nabla.$$

Observe that we may further write

$$\begin{aligned} \tilde{V}(x, \partial) &= \partial_i\left[(\delta_{ij} - g^{ij})\partial_j\right] - \frac{g^{-1}}{\sqrt{\det g}} \nabla(\sqrt{\det g}) \cdot \nabla + V(x) \\ &= V^L(x, \partial) + V^S(x, \partial), \end{aligned} \tag{4.48}$$

with

$$V^L(x, \partial) := (\delta_{ij} - g^{ij})\partial^2_{i,j} - \partial_i(g^{ij})\partial_j + V(x), \tag{4.49}$$

$$V^S(x, \partial) := -\frac{1}{\sqrt{\det g}} g^{ij} \partial_j(\sqrt{\det g})\partial_i, \tag{4.50}$$

where we have used the expression in components and Einstein's convention. We have to verify that the operator $\tilde{V} = V^S + V^L$ is a 1-admissible perturbation of the flat Laplacian , i.e., the short-range $V^S(x, \partial) = \sum_{i=1}^d V_i^S(x)\partial_i$ satisfies

$$V_i^S(x) \le C_S(1 + |x|)^{-1-\epsilon}, \tag{4.51}$$

and the long-range part $V^L(x, \partial) = \sum_{|\alpha| \le 2} V_\alpha^L(x)\partial^\alpha$ satisfies

$$\exists \epsilon > 0, \forall |\alpha| \le 2, \quad |\partial^\beta V_\alpha^L(x)| \le C_{\alpha,\beta}(1 + |x|)^{-|\beta|-\epsilon}, |\beta| = 0, 1, \tag{4.52}$$

The estimates (4.51) and (4.52) follows from (1.8) and the fact that one can write $g^{-1} = Id + \hat{g}$ and $\det g = 1 + f$ where $\hat{g}$ and f satisfy (1.8). Thus the operator $H_{g,V}$ is a 1-admissible perturbation of the flat Laplacian. As a consequence of [9, Theorem 30.2.10, p.295] we get that the eigenvalues of the operator $H_{g,V}$ in $\mathbb{R} \setminus \{0\}$ are isolated with finite multiplicity. Furthermore, applying [20, Theorem 1, p. 530] ensures that $H_{g,V}$ does not have eigenvalues in $\mathbb{R}_+$. Finally, since for $f \in D(H_{V,g})$ one has

$$\langle H_{g,V} f, f \rangle_{L^2(\mathbb{R}^d; \sqrt{\det g}\,\mathrm{d}x)} \ge \int_{\mathbb{R}^d} V(x)|f|^2 \sqrt{\det g}\,\mathrm{d}x \ge \inf_{x \in \mathbb{R}^d} V(x) \|f\|^2_{L^2(\mathbb{R}^d; \sqrt{g}\,\mathrm{d}x)},$$

one get $\Sigma(H_{g,V}) \subset [-E_0, 0)$ for some $E_0 > 0$.

$\square$

4.2. **Spectral projectors and Poisson kernels.** According to the spectral theorem (cf. [4, Section 2.5]) applied to the self-adjoint operator $H_{g,V}$, there exist a measure $\mathrm{d}\nu$ on $\mathbb{R}\times\mathbb{N}$, supported in $\Sigma(g,V)\times\mathbb{N}$, and a unitary operator

$$U : L^2(\mathbb{R}^d; \sqrt{\det g}\,\mathrm{d}x) \longrightarrow L^2(\mathbb{R}\times\mathbb{N}; \mathrm{d}\nu)$$

such that

$$UH_{g,V}U^{-1}(h) = \sigma h(\sigma,n), \quad h \in L^2(\mathbb{R}\times\mathbb{N}; \mathrm{d}\nu). \tag{4.53}$$

If F is a bounded Borel measurable function on $\mathbb{R}$, the operator $F(H_{g,V})$ is defined by the formula

$$UF(H_{g,V})U^{-1} = F(\sigma). \tag{4.54}$$

In particular, for $\lambda\in\mathbb{R}$, the spectral projector $\Pi_\lambda(g,V)$, associated with the function $F(\sigma) = \mathbb{1}_{\sigma<\lambda}$, is defined by

$$\Pi_\lambda(g,V)(f) = U^{-1}\big(\mathbb{1}_{\sigma<\lambda}U(f)\big). \tag{4.55}$$

For $s\geq 0$, we define the Poisson operator $\mathbb{P}_{s,\pm}$, associated with the function $F(\sigma) = e^{-s\sigma_\pm^{1/2}}$, by the formula

$$\mathbb{P}_{s,\pm}(f) = U^{-1}\big(e^{-s\sigma_\pm^{1/2}}U(f)\big), \tag{4.56}$$

where the function $\sigma_\pm^{1/2}$ is defined in (1.9).

PROPOSITION 4.3. *Let $f\in L^2(\mathbb{R}^d; \sqrt{\det g}\,\mathrm{d}x)$ and set $u_\pm(s,x) = \mathbb{P}_{s,\pm}(f)(x)$. Then $u_\pm(s,x)$ satisfies the following elliptic boundary value problem*

$$\begin{cases} (-\partial_s^2 + H_{g,V})u = 0, & \textit{in } (0,\infty)\times\mathbb{R}^d, \\ \lim_{s\to 0^+} u(s,x) = f(x), & \textit{in } L^2(\mathbb{R}^d; \sqrt{\det g}\,\mathrm{d}x). \end{cases} \tag{4.57}$$

Proof. Setting

$$v(s,\sigma,n) = U(u(s,.)), \qquad g(\sigma,n) = Uf(x),$$

the boundary value problem (4.57) writes

$$\begin{cases} (-\partial_s^2 + \sigma)v(s,\sigma,n) = 0, & \text{in } (0,\infty)\times\mathbb{R}\times\mathbb{N}, \\ \lim_{s\to 0^+} v(s,\sigma,n) = g(\sigma,n), & \text{in } L^2(\mathbb{R}\times\mathbb{N}; \mathrm{d}\nu). \end{cases} \tag{4.58}$$

which is true since by construction one has

$$v(s,\sigma,n) = e^{-s\sigma_\pm^{\frac{1}{2}}}g(\sigma,n).$$

□

5. Analytic estimates for second order elliptic operators

5.1. **Holomorphic extensions estimates.** Throughout this section we use $N\in\mathbb{N}$ instead of d to denote the dimension of the space. Let $B_R = \{x\in\mathbb{R}^N, |x|<R\}$ and let $X\subset\mathbb{C}^N$ a complex neighborhood of the closed ball $\overline{B}_R$. Let $D_0>0, d_0>0$ be given. We denote by $\mathcal{Q} = \mathcal{Q}_{X,D_0,d_0}$ the family of second order differential operators $Q(x,\partial_x)$ of the form

$$Q(x,\partial_x) = \sum_{|\alpha|\leq 2} q_\alpha(x)\partial^\alpha \tag{5.59}$$

where the functions $q_\alpha(x)$ are holomorphic in X, and such that

$$\begin{aligned} &\sup_\alpha \|q_\alpha\|_{L^\infty(X)} \leq D_0, \\ &\sum_{|\alpha|=2} \mathrm{Re}(q_\alpha(x))\xi^\alpha \geq d_0|\xi|^2, \quad \forall x \in X, \quad \forall \xi \in \mathbb{R}^N \ . \end{aligned} \tag{5.60}$$

Let $R' \in]0, R[$. The goal of this section is to prove the two following propositions. In this subsection, we will denote by C_j various constants independent of $Q \in \mathcal{Q}$ and of a particular solution $u \in L^2(B_R)$ of the equation $Qu = 0$.

PROPOSITION 5.1. *There exists a constant C such that for any $Q \in \mathcal{Q}$ and $u \in L^2(B_R)$ such that $Qu = 0$, the following inequality holds true*

$$\|u\|_{L^\infty(B_{R'})} \leq C\|u\|_{L^2(B_R \setminus B_{R'})} \tag{5.61}$$

PROPOSITION 5.2. *There exists constants $C_j > 0$ such that for any $Q \in \mathcal{Q}$ and $u \in L^2(B_R)$ such that $Qu = 0$, the function u extends as a holomorphic function in the set*

$$Y = \{z \in \mathbb{C}^N, |Re(z)| < R', |Im(z)| < C_1(R' - |Re(z)|)\}$$

and the following inequality holds true

$$\sup_{z \in Y} |u(z)| \leq C_2\|u\|_{L^2(B_R)} \tag{5.62}$$

REMARK 5.3. *It will be essential in the proof of Proposition 5.5 below that the constants C_j can be chosen independent of $Q \in \mathcal{Q}$.*

Proof. The proof of Proposition 5.1 is classical. Let $\varphi \in C_0^\infty(B_R)$ equal to 1 in a neighborhood of $\overline{B}_{R'}$ and $\psi \in C_0^\infty(B_R)$ equal to 1 in a neighborhood of the support of φ. Let $s > N/2$. By classical pseudo-differential calculus, since Q is elliptic, there exist a pseudo-differential operator E of degree -2 such that

$$EQ = \varphi + \psi T\psi.$$

where T is a pseudo-differential operator of degree $-s$ such that

$$\|\psi T(\psi f)\|_{H^s} \leq C_2\|f\|_{L^2(B_R)}$$

Since the construction of E, T involves only a finite number of derivatives of the coefficients of Q, from (5.60), the constant C_2 is independent of $Q \in \mathcal{Q}$. From $Qu = 0$, we get $\varphi u = -\psi T\psi u$ and therefore

$$\|\varphi u\|_{H^s} = \|\psi T(\psi u)\|_{H^s} \leq C_2\|u\|_{L^2(B_R)} \ . \tag{5.63}$$

Let us now prove (5.61) by a contradiction argument. If (5.61) is untrue, one can find a sequence $Q_n \in \mathcal{Q}$ and a sequence $u_n \in L^2(B_R)$ such that $Q_n u_n = 0$, $\|u_n\|_{L^\infty(B_{R'})} = 1$ and $\|u_n\|_{L^2(B_R \setminus B_{R'})} \to 0$. The sequence u_n is bounded in $L^2(B_R)$ and from (5.63), φu_n is bounded in H^s. Thus we may assume that u_n weakly converge in L^2 to some $u \in L^2(B_R)$ and, since $s > N/2$, that u_n converge strongly to u in $L^\infty(B_{R'})$. Then we have $\|u\|_{L^\infty(B_{R'})} = 1$ and $u = 0$ on $B_R \setminus B_{R'}$. Let X' an open neighborhood of B_R such that $\overline{X'} \subset X$. We may also assume that Q_n converge in $\mathcal{Q}(X', D_0, d_0)$ to some $Q \in \mathcal{Q}(X', D_0, d_0)$. Then u satisfies $Qu = 0$ and since Q is elliptic with analytic coefficient and $u = 0$ on $B_R \setminus B_{R'}$ we get $u = 0$ on B_R, in contradiction with $\|u\|_{L^\infty(B_{R'})} = 1$.

In order to prove Proposition 5.2, we will use complex deformation arguments.

We first prove that for $C_1 > 0$ small enough u extends as a holomorphic function in Y . For $r \in [0, R']$ let us define the non negative function $\psi_r(t)$:

$$\psi_r(t) := \max(R' - \sqrt{r^2 + t^2}, 0), \qquad t \in \mathbb{R}.$$

The function ψ_r is Lipschitz with $|\psi'(r)| \le 1$, and $\operatorname{supp} \psi_r = \left\{|t| \le \sqrt{R'^2 - r^2}\right\}$. Observe that $\psi_r(t)$ is decreasing in r, $\psi_{R'}(t) = 0$, $\psi_0(t) = \max(R' - |t|, 0)$. Take $C_1 > 0$ small. For $r \in [0, R']$, let K_r be the compact set in $\mathbb{C}^N$

$$K_r := \left\{ z \in \mathbb{C}^N; |\operatorname{Im}(z)|^2 \le C_1 \psi_r^2(|\operatorname{Re}(z)|),\, |\operatorname{Re} z| \le \sqrt{R'^2 - r^2} \right\} \ .$$

The interior Ω_r of K_r, is given by

$$\Omega_r := \left\{ z \in \mathbb{C}^N; |\operatorname{Im}(z)|^2 < C_1 \psi_r^2(|\operatorname{Re}(z)|),\, |\operatorname{Re} z| < \sqrt{R'^2 - r^2} \right\} \ .$$

The open sets Ω_r are decreasing in r and one has $\Omega_{R'} = \emptyset$, $\Omega_0 = Y$.
Let $I = \{r \in [0, R'[$ such that u extends as a holomorphic function in $\Omega_r\}$.
Since $Qu = 0$, the function u is analytic near $\operatorname{Im}(z) = 0$. Thus for r close to R' one has $r \in I$. From $\cup_{r>\rho}\Omega_r = \Omega_\rho$, we get that I is of the form $[r_0, R'[$. In order to prove $r_0 = 0$, it is sufficient to prove that if $0 < r \in I$, u extends near any point $z_0 \in K_r \setminus \Omega_r$. If $\operatorname{Im}(z_0) = 0$, this is true since u is analytic in B_R. If $\operatorname{Im}(z_0) \neq 0$, one has $|\operatorname{Re}(z_0)| < \sqrt{R'^2 - r^2}$, $|\operatorname{Im}(z_0)|^2 = C_1 \psi_r^2(|\operatorname{Re} z_0|)$, and locally near z_0, Ω_r is defined by $f < 0$ with

$$f(z) = |\operatorname{Im}(z)|^2 - C_1 \psi_r^2(|\operatorname{Re}(z)|).$$

Let $\partial = \frac{1}{2}(\partial_x - i\partial_y)$. At $z_0 = a + ib$, one has $b^2 = C_1 \psi_r^2(|a|)$ and

$$\partial f(z_0) = \zeta_0 = \xi_0 + i\eta_0, \quad \xi_0 = C_1 \psi_r(|a|) \frac{a}{\sqrt{r^2 + a^2}}, \quad \eta_0 = -b \ .$$

This implies $|\xi_0| \le \sqrt{C_1} |\eta_0|$. Therefore, if $q(z, \zeta) = \sum_{|\alpha|=2} q_\alpha(z) \zeta^\alpha$ is the principal symbol of Q, one finds, using $|b| = \sqrt{C_1}\psi_r(|a|) \le \sqrt{C_1} R'$

$$\operatorname{Re} q(z_0, \zeta_0) = \operatorname{Re} q(a, \zeta_0) + O(|b||\zeta_0|^2) = -\operatorname{Re} q(a, b) + O(\sqrt{C_1} b^2) \ .$$

By the second line of (5.60) this implies $q(z_0, \zeta_0) \neq 0$ for C_1 small. Then the result follows from the Zerner Lemma that we recall for the reader's convenience.

LEMMA 5.4 (M.Zerner). *Let $Q(z, \partial) = \sum_{\alpha, |\alpha| \le m} q_\alpha(z) \partial_z^\alpha$ be a linear differential operator with holomorphic coefficients defined in a neighborhood U of 0 in $\mathbb{C}^N$ and let $q(z, \zeta) = \sum_{|\alpha|=m} q_\alpha(z) \zeta^\alpha$ be its principal symbol. Let $f \in \mathscr{C}^1(\mathbb{C}^N; \mathbb{R})$ be a real function such that $f(0) = 0$ and $\partial f(0) \neq 0$. Let u be a holomorphic function defined in $U \cap \{f < 0\}$, such that Qu extends holomorphically to U. Then, if $q(0, \partial f(0)) \neq 0$, u extends holomorphically near 0.*

Finally, let us verify that (5.62) holds true. Let $R' < R_1 < R_2 < R$. Let $\varphi \in C_0^\infty(B_{R_2})$, equal to 1 on B_{R_1}. Let $\delta > 0$ small and $\mathcal{D} = \{w \in \mathbb{R}^N,\ |w| \le \delta\}$. For $w \in \mathcal{D}$, we deform the real ball B_R into the countour Σ_w

$$\Sigma_w = \{z \in \mathbb{C}^d, \quad \exists x \in B_R, \quad z = x + iw\varphi(x)\} \ .$$

By the first part of the proof of Proposition 5.2, that we apply with some $R' \in]R_2, R[$, if δ is small enough, the function u extends holomorphically near any $z \in \Sigma_w, w \in \mathcal{D}$. Let $u_w(x) = u(x + iw\varphi(x))$. Then $u_w \in L^2(B_R)$ and one has

$Q_w(u_w) = 0$ where Q_w is the operator induced by Q on Σ_w.
One has $Q_w \in \mathcal{Q}(X', D_0', d_0')$ with (X', D_0', d_0') close to (X, D_0, d_0) if δ is small enough. Now, we apply Proposition 5.1 with $R' = R_2$. We get in particular

$$\sup_{w\in\mathcal{D}} \sup_{x\in B_{R_1}} |u(x+iw)| \leq C\|u\|_{L^2(B_R\setminus B_{R_2})}.$$

By taking C_1 small enough, this implies that (5.61) holds true.

□

5.2. **Estimates on solutions of the Poisson equation.** Let $f \in L^2(\mathbb{R}^d; \sqrt{\det g}\, \mathrm{d}x)$ and $u_\pm(s,x) = \mathbb{P}_{s,\pm}(f)(x)$ the solution of the Poisson equation (4.57). Let $s_0 > 0$. The goal of this subsection is to prove the following proposition

PROPOSITION 5.5. *There exists constants $b > 0, C > 0$ independent of f such that, $u_\pm(s_0, .)$ extends as an holomorphic function in the set $U_b = \{z \in \mathbb{C}^d, |Im(z)| < b\}$. Moreover $u(s_0, z) \in \mathcal{H}_b$ and one has*

$$\int_{z\in U_b} |u_\pm(s_0, z)|^2 |dz| \leq C\|f\|_{L^2}^2. \tag{5.64}$$

Proof. Recall that $u_\pm(s, x)$ is a solution of the elliptic equation

$$(-\partial_s^2 + H_{g,V})u_\pm = 0, \quad \text{in } (0,\infty)\times\mathbb{R}^d. \tag{5.65}$$

We first choose $R \in]0, s_0/2]$. We denote here by $B_R \subset \mathbb{R}^{d+1}$ the ball

$$B_R = \{(\sigma, x) \in \mathbb{R}^{d+1}, \quad \sigma^2 + |x|^2 < R^2\}\ .$$

For $w \in \mathbb{R}^d$, we define the function $u_w(\sigma, x)$ by the formula

$$u_w(\sigma, x) = u_\pm(s_0 + \sigma, w + x)\ .$$

Then one has $u_w \in L^2(B_R)$, and u_w satisfies the equation

$$Q_w(u_w) = 0 \quad \text{on} \quad B_R$$

with $-Q_w = \tau_{(-s_0,-w)}(-\partial_s^2 + H_{g,V})\tau_{(s_0,w)}$ where $\tau_{(s_0,w)}$ is the translation by (s_0, w). By hypothesis (1.7) and (1.8) there exist (X, D_0, d_0) such that $Q_w \in \mathcal{Q}(X, D_0, d_0)$ for all w. Let $R' < R$. By Proposition 5.2, there exist $b > 0$ and $C > 0$ independent of $w \in \mathbb{R}^d$ such that

$$\sup_{|x|<R', |y|<b} |u_w(0, x+iy)|^2 \leq C\int_{B_R} |u_w(\sigma, x)|^2 d\sigma dx\ .$$

This implies

$$\sup_{|y|<b} \int_{B_{R'}} |u_\pm(s_0, w+x+iy)|^2 dx \leq C\int_{B_R} |u_\pm(s_0+\sigma, w+x)|^2 d\sigma dx. \tag{5.66}$$

Applying (5.66) at points $w_k = hk,\ k \in \mathbb{Z}^d$ with h small enough and adding all these inequalities, we get with a different constant C

$$\sup_{|y|<b} \int_{\mathbb{R}^d} |u_\pm(s_0, x+iy)|^2 dx \leq C\int_{\mathbb{R}^d}\int_{s\in[s_0-R', s_0+R']} |u_\pm(s,x)|^2 ds dx. \tag{5.67}$$

This proves $u_\pm(s_0, z) \in \mathcal{H}_b$ and (5.64) follows from $\|u_\pm(s,.)\|_{L^2(\mathbb{R}^d)} \leq \|f\|_{L^2(\mathbb{R}^d)}$ for all $s > 0$. □

6. Proof of Theorem 1.2

Let $\mu \in \mathbb{R}$ and let $f \in L^2(\mathbb{R}^d; \sqrt{\det g}\, \mathrm{d}x)$ be such that $f = \Pi_\mu(g, V)f$. Take $s_0 > 0$. Since the support of $U(f)$ is contained in $\sigma < \mu$, we can define the function

$$h = U^{-1}\big(e^{s_0 \sigma_\pm^{1/2}} U(f)\big)$$

which satisfies

$$\|h\|_{L^2} \leq |e^{s_0 \mu_\pm^{1/2}}| \|f\|_{L^2} \ .$$

The function $u_\pm(s, .) = \mathbb{P}_{s,\pm}(h)$ is solution of the Poisson equation with data h on $s = 0$ and one has by construction

$$f = u_\pm(s_0, .) \ . \tag{6.68}$$

By Proposition 5.5, there exists $b > 0$ such that $f \in \mathcal{H}_b$ and one has

$$\int_{z \in U_b} |f(z)|^2 |dz| \leq C\|h\|^2_{L^2} \leq C|e^{2s_0 \mu_\pm^{1/2}}| \|f\|^2_{L^2}. \tag{6.69}$$

By Proposition 3.1, we get that there exists $\nu > 0$ such that

$$\int_{\mathbb{R}^d} |f|^2 \, \mathrm{d}x \leq C \left(\int_\omega |f|^2 \, \mathrm{d}x \right)^\nu \left(\int_{U_b} |f|^2 |\, \mathrm{d}z| \right)^{1-\nu} , \tag{6.70}$$

From (6.69) and (6.70) we get

$$\int_{\mathbb{R}^d} |f|^2 \, \mathrm{d}x \leq C|e^{\frac{2(1-\nu)s_0 \mu_\pm^{1/2}}{\nu}}| \int_\omega |f|^2 \, \mathrm{d}x \ .$$

The proof of Theorem 1.2 is complete.

References

[1] W.O. Amrein and A.M. Berthier. On support properties of L^p functions and their Fourier transforms. *Journal of Functional Analysis*, vol. 24, 1977, pp 258-267.

[2] L. Boutet de Monvel (L.). Convergence dans le domaine complexe des sŕies de fonctions propres. *C.R.A.S. Paris, t.287, série A,* 1978, pp 855–856.

[3] N. Burq and R. Joly. Exponential decay for the damped wave equation in unbounded domains. *Commun. Contemp. Math.*, 18(6):1650012, 27, 2016

[4] E.B. Davies. *Spectral Theory and Differential Operators*, Cambridge studies in Advanced Mathematics, vol. 42, Cambridge University Press, Cambridge, 1995.

[5] M. Egidi and I. Veselić. Sharp geometric condition for null-controllability of the heat equation on $\mathbb{R}^d$ and consistent estimates on the control cost. *ArXiv. 1711.06088v1*, 2017.

[6] M. Egidi, I. Nakić, A. Seelmann, M. Täufer and I. Veselić. Null-controllability and control cost estimates for the heat equation on unbounded and large domains. arXiv:1810.11229, 2018.

[7] O. Kovrojkine. Some results related to the Logvinenko-Sereda inequality. *Proc. Amer. Math. Soc.*, vol. 129 (2001), 3037-3047.

[8] O. Kovrojkine. The Uncertainty Principle for relatively dense sets and lacunary spectra.

[9] L. Hörmander. *Linear Partial Differential Operators.* 4 vols., Berlin, Springer, 1983.

[10] D. Jerison and G. Lebeau. Nodal sets of sums of eigenfunctions. In *Harmonic analysis and partial differential equations (Chicago, IL, 1996)* Chicago Lectures in Mathematics, The University of Chicago Press, 223-239, 1999.

[11] J. Le Rousseau and G. Lebeau. On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations. *ESAIM Control Optim. Calc. Var.* vol. 18, pp. 712-747, 2012.

[12] J. Le Rousseau and I. Moyano. Null-controllability of the Kolmogorov equation in the whole phase space. *J. Differential Equations*, vol. 260. pp. 3193-3233. 2016.

[13] G. Lebeau and L. Robbiano. Contrôle exact de l'équation de la chaleur. *Comm. Partial Differential Equations,* vol. 20, pp. 335-356, 1995.

[14] G. Lebeau and E. Zuazua. Null-controllability of a system of linear thermoelasticity. *Arch. Rational Mech. Anal.,* vol. 141, pp. 297-329, 1998.

[15] V.N. Logvinenko and J.F. Sereda. Equivalent norms in spaces of entire functions fo exponential type. *Tero. Funk. Anal. i Prilozen.* Vyp, 20 (1974), 102-111.

[16] I. Moyano. *Contrôlabilité de quelques équations cinétiques collisionnelles et non collisionnelles : Fokker-Planck et Vlasov-Navier-Stokes.* Actes du Séminaire Laurent Schwartz (EDP et Applications). Nov. 2016.

[17] C. Muscalu and W. Schlag. *Classical and Multilinear Harmonic Analysis (volume 1)*, Cambridge studies in advanced mathematics, vol. 137, Cambridge University Press, 2013.

[18] I. Nakić, M. Täufer, M. Tautenhahn and I. Veselić. Scale-free unique continuation principle for spectral projectors, eigenvalue lifting and Wegner estimates for random Schrödinger operators. *Analysis & PDEs*, Vol. 11 (2018), No. 4, 1049–108.

[19] F.L. Nazarov. Local estimates of exponential polynomials and their application to inequalities of uncertainty principle type. *St. Petersburg Math. Journal* Vol 5 (1994), pp. 663-717.

[20] B. Simon. On positive eigenvalues of one-body Schrödinger operators. *Comm. Pure and Appl. Math.*, vol. XXII, 531-538 (1967).

[21] G. Wang, M. Wang, C. Zhang and Y. Zhang. Observable sets, observability, interpolation inequality and spectral inequality for the heat equation in $\mathbb{R}^n$. *ArXiv. 1711.04279v1*, 2017.

[22] M. Zerner. Domaine d'holomorphie des fonctions vérifiant une équation aux dérivées partielles. *C.R.A.S. Paris*, t.272, série A, (1971) pp 1646-1648.

[23] A. Zigmund. *Trigonometric Series*, Vols I and II. Cambridge University Press, New York, 1968.

Gilles Lebeau, Laboratoire J.A. Dieudonné, UMR CNRS 7351, Université de Nice Sophia-Antipolis, Parc Valrose, 06108 Nice Cedex 02, France

E-mail address: gilles.lebeau@unice.fr

Iván Moyano, DPMMS, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, CB3 0WA, United Kingdom

E-mail address: im449@dpmms.cam.ac.uk