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► **To cite this version:**

Tuomas Hytönen, Stefanie Petermichl, Alexander Volberg. THE SHARP SQUARE FUNCTION ESTIMATE WITH MATRIX WEIGHT. *Discrete Analysis*, 2019. hal-01977600

HAL Id: hal-01977600

<https://hal.science/hal-01977600>

Submitted on 10 Jan 2019

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THE SHARP SQUARE FUNCTION ESTIMATE WITH MATRIX WEIGHT

TUOMAS HYTÖNEN, STEFANIE PETERMICHL, AND ALEXANDER VOLBERG

ABSTRACT. We prove the matrix A_2 conjecture for the dyadic square function, that is, an estimate of the form

$$\|S_W\|_{L^2_{\mathbb{C}^d}(W) \rightarrow L^2_{\mathbb{R}}} \lesssim [W]_{A_2},$$

where the focus is on the sharp linear dependence on the matrix A_2 constant. Moreover, we give a mixed estimate in terms of A_2 and A_∞ constants. Key is a sparse domination of a process inspired by the integrated form of the matrix-weighted square function.

1. INTRODUCTION

The theory of weights has drawn much attention in recent years. In the scalar-valued setting, we say that a non-negative locally integrable function is a dyadic A_2 weight iff

$$[w]_{A_2} = \sup_I \langle w \rangle_I \langle w^{-1} \rangle_I < \infty,$$

where the supremum runs over dyadic intervals and $\langle \cdot \rangle_I$ returns the average of a function over the interval I . It is classical that this is the necessary and sufficient condition for dyadic square function, maximal function, Hilbert transform and Calderón–Zygmund operators to be bounded on

$$L^2_{\mathbb{R}}(w) = \left\{ f : \|f\|_{L^2_{\mathbb{R}}(w)}^2 = \int |f|^2 w < \infty \right\}.$$

Partly inspired by applications to PDE, but also interesting in their own right, the precise dependence on the A_2 characteristic for the different operators has been under intensive investigation – these questions became known as A_2 conjectures. Most such improved estimates came at the cost and benefit of an array of fantastic new ideas and techniques in harmonic analysis. The first such example was the sharp weighted estimate of the dyadic square function by Hukovic–Treil–Volberg [7] followed by the martingale multiplier by Wittwer [27], the Beurling operator by Petermichl–Volberg [23], the Hilbert and Riesz transforms by Petermichl [20] [22] and all Calderón–Zygmund operators by Hytönen [11]. Somewhat later it has been discovered that many of these estimates can be slightly improved by replacing a

This research was conducted during the authors' NSF-supported participation in the Spring 2017 Harmonic Analysis Program at the Mathematical Sciences Research Institute in Berkeley, California. In addition, T.H. was supported by the Finnish Centre of Excellence in Analysis and Dynamics Research, S.P. by the ERC project CHRiSHarMa DLV-862402, and A.V. by the NSF grant DMS-160065.

half power of these norm estimates by a half power of the smaller A_∞ norm, the best estimate C in the inequality

$$\langle M_I w \rangle_I \leq C \langle w \rangle_I,$$

where M_I is the dyadic maximal function localized to I , see [9][12][16]. The field has since seen beautiful new proofs of these optimal results [17][14][10][15][25] and many extensions far beyond Calderón–Zygmund theory [8][2][5][18][1].

Inspired by applications to multivariate stationary stochastic processes, a theory of matrix weights was developed by Treil and Volberg [24], where the necessary and sufficient condition for boundedness of the Hilbert transform was found, the matrix A_2 characteristic. Aside from an early, excellent estimate for a natural maximal function in this setting by Christ–Goldberg [4][13], the optimal norm estimates for singular operators seemed out of reach. Some improvement was achieved in Bickel–Petermichl–Wick [3] with a new estimate for Hilbert and martingale transforms of $[W]_{A_2}^{3/2} \log(1 + [W]_{A_2})$. Recently, by Nazarov–Petermichl–Treil–Volberg [19] the logarithmic term was dropped and it was shown that for all Calderón–Zygmund operators there holds an estimate of the order $[W]_{A_2}^{3/2}$.

In [3] a notable improvement to the dyadic square function estimates of [24][26] was given, namely

$$\|S_W\|_{L^2_{\mathbb{C}^d}(W) \rightarrow L^2_{\mathbb{R}}} \lesssim [W]_{A_2} \log(1 + [W]_{A_2}).$$

The above estimate only features an extra logarithmic term as compared to the sharp, linear estimate in the scalar case [7]. Despite the advance on the matrix weighted Carleson lemma in [6], a natural tool for square function estimates in the scalar setting, the logarithmic term could not be removed. One of the many difficulties arising, stem from the non–commutativity and it seems that most convex functions are not matrix convex. In this paper, we remove the logarithmic term by other means and give the first sharp estimate of a singular operator in the matrix weighted setting:

$$\|S_W\|_{L^2_{\mathbb{C}^d}(W) \rightarrow L^2_{\mathbb{R}}} \lesssim [W]_{A_2}.$$

This estimate is known to be optimal among all upper bounds of the form $\phi([W]_{A_2})$ even in the scalar setting. Allowing for a more general dependence on the weight W , we even show that

$$\|S_W\|_{L^2_{\mathbb{C}^d}(W) \rightarrow L^2_{\mathbb{R}}} \lesssim [W]_{A_2}^{1/2} [W^{-1}]_{A_\infty}^{1/2}$$

using the matrix A_∞ characteristic. This coincides with the mixed A_2 – A_∞ bound in the scalar case obtained in [16].

2. NOTATION

Let \mathcal{D} be the collection of dyadic subintervals of $J = [0, 1]$. We call a $d \times d$ matrix–valued function W a weight, if W is entry–wise locally integrable and if $W(x)$ is positive semidefinite almost everywhere. One defines $L^2_{\mathbb{C}^d}(W)$ to be the space of vector functions with

$$\|f\|_{L^2_{\mathbb{C}^d}(W)}^2 = \int_J \|W^{1/2}(x)f(x)\|_{\mathbb{C}^d}^2 dx = \int_J \langle W(x)f(x), f(x) \rangle_{\mathbb{C}^d} dx < \infty.$$

The dyadic matrix Muckenhoupt A_2 condition is

$$[W]_{A_2} = \sup_{I \in \mathcal{D}} \|\langle W \rangle_I^{1/2} \langle W^{-1} \rangle_I^{1/2}\|^2 < \infty,$$

where we mean the operator norm. The dyadic matrix A_∞ condition is

$$[W]_{A_\infty} = \sup_{e \in \mathbb{C}^d} [W_e]_{A_\infty}$$

where $W_e(x) = \langle W(x)e, e \rangle_{\mathbb{C}^d}$. Let us recall the fact that $[W]_{A_\infty} \lesssim [W]_{A_2}$; see [19], Section 4.

Let $h_I = |I|^{-1/2}(\chi_{I_+} - \chi_{I_-})$ be the $L^2_{\mathbb{R}}$ normalized Haar function and let with $\sigma_I = \pm 1$

$$T_\sigma f(x) = \sum_{I \in \mathcal{D}} \sigma_I (f, h_I) h_I(x),$$

defined both on scalar-valued as well as vector-valued functions f . Recall that the dyadic square function for real-valued, mean zero functions f is defined as

$$Sf(x)^2 = \sum_I |(f, h_I)|^2 \frac{\chi_I(x)}{|I|}.$$

Its classical vector analog becomes the scalar-valued function

$$Sf(x)^2 = \sum_I \|(f, h_I)\|_{\mathbb{C}^d}^2 \frac{\chi_I(x)}{|I|}.$$

When working with matrix weights, it is customary to include the weight in the definition of the (scalar-valued) operator, such as done for example by Christ-Goldberg [4] for the maximal function

$$M_W f(x) = \sup_{I: x \in I} \frac{1}{|I|} \int_I \|W^{1/2}(x)W^{-1/2}(y)f(y)\|_{\mathbb{C}^d} dy.$$

Recall that $S_W : L^2_{\mathbb{C}^d}(W) \rightarrow L^2_{\mathbb{R}}$ is defined by (see [21])

$$S_W f(x)^2 = \mathbb{E}(\|W(x)^{1/2}T_\sigma f(x)\|_{\mathbb{C}^d}^2),$$

where \mathbb{E} is the expectation over independent uniformly distributed random signs $\sigma_I = \pm 1$. One calculates

$$\begin{aligned} \|S_W f\|_{L^2_{\mathbb{R}}}^2 &= \int_J \mathbb{E} \left(\sum_{I, I'} \sigma_I \sigma_{I'} h_I(x) h_{I'}(x) \langle W(x)(f, h_I), (f, h_{I'}) \rangle_{\mathbb{C}^d} \right) dx \\ &= \int_J \sum_{I, I'} \mathbb{E}(\sigma_I \sigma_{I'}) h_I(x) h_{I'}(x) \langle W(x)(f, h_I), (f, h_{I'}) \rangle_{\mathbb{C}^d} dx \\ &= \sum_I \langle \langle W \rangle_I (f, h_I), (f, h_I) \rangle_{\mathbb{C}^d}. \end{aligned}$$

The study of these sums was introduced by Volberg in [26]. Indeed, in the scalar setting, this square function S_w is bounded into the unweighted L^2 if and only if the classical dyadic square function is bounded into the weighted $L^2(w)$:

$$\|S_w f\|_{L^2_{\mathbb{R}}}^2 = \sum_I \langle w \rangle_I |(f, h_I)|^2 = \|Sf\|_{L^2_{\mathbb{R}}(w)}^2.$$

3. RESULTS

Here is our main theorem.

Theorem 1.

$$S_W : L_{\mathbb{C}^d}^2(W) \rightarrow L_{\mathbb{R}}^2$$

has operator norm bounded by a constant multiple of

$$[W]_{A_2}^{1/2} [W^{-1}]_{A_\infty}^{1/2} \lesssim [W]_{A_2}^1.$$

This estimate is sharp among all upper bounds of the form $\phi([W]_{A_2})$.

The previously known best estimate [3] was $\|S_W\|_{L_{\mathbb{C}^d}^2(W) \rightarrow L_{\mathbb{R}}^2} \lesssim [W]_{A_2} \log(1 + [W]_{A_2})$. With a different method we drop the logarithmic term and improve the single power A_2 constant to split into a $A_2 - A_\infty$ estimate.

A key to the proof is the following sparse domination result of independent interest. Recall that a collection of intervals $\mathcal{S} \subset \mathcal{D}$ is called sparse, if there are disjoint subsets $E(I) \subset I$ for every $I \in \mathcal{S}$ such that $|E(I)| \geq \frac{1}{2}|I|$.

Proposition 1. *Given $f \in L_{\mathbb{C}^d}^2(W)$, there exists a sparse collection $\mathcal{S} \subset \mathcal{D}$ such that*

$$\|S_W f\|_{L_{\mathbb{R}}^2}^2 \lesssim \sum_{I \in \mathcal{S}} \langle \|\langle W \rangle_I^{1/2} f\|_{\mathbb{C}^d} \rangle_I^2 |I| = \left\| \left(\sum_{I \in \mathcal{S}} \langle \|\langle W \rangle_I^{1/2} f\|_{\mathbb{C}^d} \rangle_I^2 \chi_I \right)^{1/2} \right\|_{L_{\mathbb{R}}^2}^2.$$

Proof of the Theorem assuming the Proposition. Sharpness follows from the scalar case. Let us attend to the upper estimate $\|S_W f\|_{L_{\mathbb{R}}^2} \lesssim [W]_{A_2}^{1/2} [W^{-1}]_{A_\infty}^{1/2} \|f\|_{L_{\mathbb{C}^d}^2(W)}$. Substituting $W^{-1/2} f$ in place of f and using the bound from the Proposition, we should show that

$$\left\| \left(\sum_{I \in \mathcal{S}} \langle \|\langle W \rangle_I^{1/2} W^{-1/2} f\|_{\mathbb{C}^d} \rangle_I^2 \chi_I \right)^{1/2} \right\|_{L_{\mathbb{R}}^2} \lesssim [W]_{A_2}^{1/2} [W^{-1}]_{A_\infty}^{1/2} \|f\|_{L_{\mathbb{C}^d}^2}.$$

But the left hand side is the $L_{\mathbb{R}}^2$ norm of the sparse square function $S_{3,W} f$ defined in [19], for which it was proved [19] that

$$\|S_{3,W}\|_{L_{\mathbb{C}^d}^2 \rightarrow L_{\mathbb{R}}^2} \lesssim [W]_{A_2}^{1/2} [W^{-1}]_{A_\infty}^{1/2}.$$

And this is exactly the estimate we needed. \square

Proof of the Proposition. Consider the collection of first stopping intervals $L \subset J$ determined by

$$\|\langle W \rangle_L^{1/2} \langle W \rangle_J^{-1/2}\| > C_1$$

or

$$\sum_{I: I \supseteq L} \|\langle W \rangle_J^{1/2}(f, h_I)\|_{\mathbb{C}^d}^2 \frac{1}{|I|} > C_2 \langle \|\langle W \rangle_J^{1/2} f\|_{\mathbb{C}^d} \rangle_J^2$$

but for all $L' \supsetneq L$

$$\|\langle W \rangle_{L'}^{1/2} \langle W \rangle_J^{-1/2}\| \leq C_1$$

and

$$\sum_{I: I \supseteq L'} \|\langle W \rangle_J^{1/2}(f, h_I)\|_{\mathbb{C}^d}^2 \frac{1}{|I|} \leq C_2 \langle \|\langle W \rangle_J^{1/2} f\|_{\mathbb{C}^d} \rangle_J^2.$$

Consider \mathcal{S}^1 , the collection of all first stopping intervals. Our final sum splits after this step

$$\begin{aligned} & \sum_I \|\langle W \rangle_I^{1/2}(f, h_I)\|_{\mathbb{C}^d}^2 \\ &= \sum_{I:\forall L \in \mathcal{S}^1: I \not\subseteq L} \|\langle W \rangle_I^{1/2}(f, h_I)\|_{\mathbb{C}^d}^2 + \sum_{I:\exists L \in \mathcal{S}^1: I \subseteq L} \|\langle W \rangle_I^{1/2}(f, h_I)\|_{\mathbb{C}^d}^2. \end{aligned}$$

We estimate the first sum.

$$\begin{aligned} & \sum_{I:\forall L \in \mathcal{S}^1: I \not\subseteq L} \langle \langle W \rangle_I(f, h_I), (f, h_I) \rangle_{\mathbb{C}^d} \\ & \leq \sum_{I:\forall L \in \mathcal{S}^1: I \not\subseteq L} \|\langle W \rangle_I^{1/2} \langle W \rangle_J^{-1/2}\|^2 \langle \langle W \rangle_J(f, h_I), (f, h_I) \rangle_{\mathbb{C}^d} \\ & \leq C_1^2 \sum_{I:\forall L \in \mathcal{S}^1: I \not\subseteq L} \langle \langle W \rangle_J(f, h_I), (f, h_I) \rangle_{\mathbb{C}^d} \\ & = C_1^2 \int_J \sum_{I:\forall L \in \mathcal{S}^1: I \not\subseteq L} \|\langle W \rangle_J^{1/2}(f, h_I)\|_{\mathbb{C}^d}^2 \frac{\chi_I(x)}{|I|} dx \\ & \leq C_1^2 C_2 \langle \|\langle W \rangle_J^{1/2} f \|_{\mathbb{C}^d} \rangle_J^2 |J|. \end{aligned}$$

The first step is a triangle inequality for norms, the second step uses the first stopping condition and the last step uses the second stopping condition and a resulting pointwise estimate. By iteration we have the domination

$$\sum_I \langle \langle W \rangle_I(f, h_I), (f, h_I) \rangle_{\mathbb{C}^d} \leq C_1^2 C_2 \sum_{L \in \mathcal{S}} \langle \|\langle W \rangle_L^{1/2} f \|_{\mathbb{C}^d} \rangle_L^2 |L|,$$

precisely the integrated form of $S_{3,W}$, provided \mathcal{S} is sparse.

It remains to show the collection \mathcal{S} is sparse for large enough C_1 and C_2 . The collection stemming from

$$\sum_{I:L \subseteq I \subseteq J} \|\langle W \rangle_J^{1/2}(f, h_I)\|_{\mathbb{C}^d}^2 \frac{1}{|I|} > C_2 \langle \|\langle W \rangle_J^{1/2} f \|_{\mathbb{C}^d} \rangle_J^2$$

is sparse for large enough C_2 because of the (unweighted) weak type boundedness of $S : L_{\mathbb{C}^d}^1 \rightarrow L_{\mathbb{R}}^1$ (with a universal constant), where S is the standard square function

$$Sg(x) := \sum_{I \in \mathcal{D}} \|(g, h_I)\|_{\mathbb{C}^d}^2 \frac{\chi_I(x)}{|I|},$$

now applied to $g = \langle W \rangle_J^{1/2} f$; indeed, the above stopping condition means that $Sg(x) > \sqrt{C_2} \langle \|g\|_{\mathbb{C}^d} \rangle_J$ for all $x \in L$ and thus the union of these intervals L is contained in $\{Sg > \sqrt{C_2} \langle \|g\|_{\mathbb{C}^d} \rangle_J\}$, which has measure at most $C_2^{-1/2} \|S\|_{L_{\mathbb{C}^d}^1 \rightarrow L_{\mathbb{R}}^1}$.

The collection stemming from $\|\langle W \rangle_L^{1/2} \langle W \rangle_J^{-1/2}\| > C_1$ is sparse because

$$\begin{aligned}
C_1^2 \sum_L |L| &< \sum_L |L| \|\langle W \rangle_L^{1/2} \langle W \rangle_J^{-1/2}\|^2 \\
&\leq \sum_L |L| \|\langle W \rangle_L^{1/2} \langle W \rangle_J^{-1/2}\|_{S_2}^2 \\
&= \sum_L |L| \operatorname{tr}(\langle W \rangle_J^{-1/2} \langle W \rangle_L \langle W \rangle_J^{-1/2}) \\
&= \sum_L \int_L \operatorname{tr}(\langle W \rangle_J^{-1/2} W(x) \langle W \rangle_J^{-1/2}) dx \\
&\leq \int_J \operatorname{tr}(\langle W \rangle_J^{-1/2} W(x) \langle W \rangle_J^{-1/2}) dx \\
&= \operatorname{tr}(I_d) |J| = d |J|.
\end{aligned}$$

The first inequality uses the first stopping condition, then we dominate the operator norm by the Hilbert Schmidt norm (denoted by $\|\cdot\|_{S_2}$). In the sequel we use the linearity of trace and disjointness of stopping intervals. \square

REFERENCES

- [1] C. Benea, F. Bernicot, and T. Luque. Sparse bilinear forms for Bochner Riesz multipliers and applications. *Trans. London Math. Soc.*, 4(1):110–128, 2017.
- [2] F. Bernicot, D. Frey, and S. Petermichl. Sharp weighted norm estimates beyond Calderón-Zygmund theory. *Anal. PDE*, 9(5):1079–1113, 2016.
- [3] K. Bickel, S. Petermichl, and B. Wick. Bounds for the Hilbert transform with matrix A_2 weights. *J. Funct. Anal.*, 270(5):1719–1743, 2016.
- [4] M. Christ and M. Goldberg. Vector A_2 weights and a Hardy-Littlewood maximal function. *Trans. Amer. Math. Soc.*, 353(5):1995–2002, 2001.
- [5] A. Culiuc, F. Di Plinio, and Y. Ou. Domination of multilinear singular integrals by positive sparse forms. *ArXiv e-prints*, March 2016.
- [6] A. Culiuc and S. Treil. The Carleson Embedding Theorem with matrix weights. *ArXiv e-prints*, August 2015.
- [7] S. Hukovic, S. Treil, and A. Volberg. The Bellman functions and sharp weighted inequalities for square functions. In *Complex analysis, operators, and related topics*, volume 113 of *Oper. Theory Adv. Appl.*: 97–113. Birkhäuser, Basel, 2000.
- [8] T. Hytönen, L. Roncal, and O. Tapiola. Quantitative weighted estimates for rough homogeneous singular integrals. *Israel J. Math.*, 218(1): 133–164, 2017.
- [9] T. Hytönen and C. Pérez. Sharp weighted bounds involving A_∞ . *Anal. PDE*, 6(4):777–818, 2013.
- [10] T. Hytönen, C. Pérez, S. Treil, and A. Volberg. Sharp weighted estimates for dyadic shifts and the A_2 conjecture. *J. Reine Angew. Math.*, 687:43–86, 2014.
- [11] T. Hytönen. The sharp weighted bound for general Calderón-Zygmund operators. *Ann. of Math. (2)*, 175(3):1473–1506, 2012.
- [12] T. Hytönen and M. Lacey. The A_p - A_∞ inequality for general Calderón-Zygmund operators. *Indiana Univ. Math. J.*, 61(6): 2041–2092, 2012.
- [13] J. Israelowitz, H. Kwon, and S. Pott. Matrix weighted norm inequalities for commutators and paraproducts with matrix symbols. *J. Lond. Math. Soc.*, 96(1): 243–270, 2017.
- [14] M. Lacey. An elementary proof of the A_2 Bound. *Israel J. Math.*, 2017:181–195, 2017.
- [15] M. Lacey, S. Petermichl, and M. Reguera. Sharp A_2 inequality for Haar shift operators. *Math. Ann.*, 348(1):127–141, 2010.
- [16] M. Lacey, K. Li, On $A_p - A_\infty$ type estimates for square functions. *Math. Z.*, 284(3-4):1211–1222, 2016.
- [17] A. Lerner. A simple proof of the A_2 conjecture. *Int. Math. Res. Not.*, (14):3159–3170, 2013.
- [18] A. Lerner. On pointwise estimates involving sparse operators. *New York J. Math.*, 22:341–349, 2016.

- [19] F. Nazarov, S. Petermichl, S. Treil, and A. Volberg. Convex body domination and weighted estimates with matrix weights. *Adv. Math.*, 318:279–306, 2017.
- [20] S. Petermichl. The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical A_p characteristic. *Amer. J. Math.*, 129(5):1355–1375, 2007.
- [21] S. Petermichl and S. Pott. A version of Burkholder’s theorem for operator-weighted spaces. *Proc. Amer. Math. Soc.*, 131(11):3457–3461, 2003.
- [22] S. Petermichl. The sharp weighted bound for the Riesz transforms. *Proc. Amer. Math. Soc.*, 136(4):1237–1249, 2008.
- [23] S. Petermichl and A. Volberg. Heating of the Ahlfors-Beurling operator: weakly quasiregular maps on the plane are quasiregular. *Duke Math. J.*, 112(2):281–305, 2002.
- [24] S. Treil and A. Volberg. Wavelets and the angle between past and future. *J. Funct. Anal.*, 143(2):269–308, 1997.
- [25] S. Treil. Sharp A_2 estimates of Haar shifts via Bellman function. In *Recent trends in analysis*, Theta Ser. Adv. Math.:187–208. Theta, Bucharest, 2013.
- [26] A. Volberg. Matrix A_p weights via S -functions. *J. Amer. Math. Soc.*, 10(2):445–466, 1997.
- [27] J. Wittwer. A sharp estimate on the norm of the martingale transform. *Math. Res. Lett.*, 7(1):1–12, 2000.

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