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# Estimate of the validity interval for the Antimaximum Principle and application to a non-cooperative system

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## Abstract

We are concerned with the sign of the solutions of non-cooperative systems when the parameter varies near a principal eigenvalue of the system. With this aim we give precise estimates of the validity interval for the Antimaximum Principle for an equation and an example. We apply these results to a non-cooperative system. Finally a counterexample shows that our hypotheses are necessary. The Maximum Principle remains true only for a restricted positive cone.

## 1 Introduction

In this paper we use ideas concerning the Anti-Maximum Principle due to Clément and Peletier [5] and later to Arcoya Gamez [3] to obtain in Section 2 precise estimates concerning the validity interval for the Anti-maximum Principle for one equation. An example shows that this estimate is sharp. The Maximum Principle and then the Anti-Maximum Principle for the case of a single equation have been extensively studied later for cooperative elliptic systems (see the references ([1],[6],[7],[8],[10],[12])). The results in [10], are still valid for systems (with constant coefficients) involving the  $p$ -Laplacian. Some results for non-cooperative systems can be found *e.g.* in [4],[11]. Very general results concerning the Maximum Principle for equations and cooperative systems for different classes (classical, weak, very weak) of solutions were given by Amann in a long paper [2], in particular the Maximum Principle was shown to be equivalent to the positivity of the principal eigenvalue. Here in Section 3, we consider a non-cooperative  $2 \times 2$  system with constant coefficients depending on a real parameter  $\mu$  having two real principal eigenvalues  $\mu_1^- < \mu_1^+$ . We obtain some theorems of Anti-Maximum principle type concerning the behavior of different cones of couples of functions having positivity (or negativity) properties. We give several results of this type for values of  $\mu_1^- < \mu$  but close to  $\mu_1^-$  by combining the usual Maximum Principle and the results for the Anti-Maximum Principle in Section 2. Finally a counterexample is given showing that the Maximum Principle does not hold in general for non cooperative systems, but a (partial, under an additional assumption) Maximum Principle for  $\mu < \mu_1^-$  is also obtained.

## 2 Estimate of the validity interval for the anti-maximum principle

Let  $\Omega$  be a smooth bounded domain in  $\mathbb{R}^N$ . We consider the following Dirichlet boundary value problem

$$-\Delta z = \mu z + h \text{ in } \Omega, \quad z = 0 \text{ on } \partial\Omega, \quad (2.1)$$

where  $\mu$  is a real parameter. We associate to (2.1) the eigenvalue problem

$$-\Delta \varphi = \lambda \varphi \text{ in } \Omega, \quad \varphi = 0 \text{ on } \partial\Omega. \quad (2.2)$$

We denote by  $\lambda_k$ ,  $k \in \mathbb{N}^*$  the eigenvalues ( $0 < \lambda_1 < \lambda_2 \leq \dots$ ) and by  $\varphi_k$  a set of orthonormal associated eigenfunctions. We choose  $\varphi_1 > 0$ .

**Hypothesis ( $H_0$ ):** We write

$$h = \alpha\varphi_1 + h^\perp \quad (2.3)$$

where  $\int_\Omega h^\perp \varphi_1 = 0$  and we assume  $\alpha > 0$  and  $h \in L^q$ ,  $q > N$  if  $N \geq 2$  and  $q = 2$  if  $N = 1$ .

**Theorem 1 :** *We assume ( $H_0$ ) and  $\lambda_1 < \mu \leq \Lambda < \lambda_2$ . There exists a constant  $K$  depending only on  $\Omega$ ,  $\Lambda$  and  $q$  such that, for  $\lambda_1 < \mu < \lambda_1 + \delta(h)$  with*

$$\delta(h) = \frac{K\alpha}{\|h^\perp\|_{L^q}}, \quad (2.4)$$

*the solution  $z$  to (2.1) satisfies the antimaximum principle, that is*

$$z < 0 \text{ in } \Omega; \quad \partial z / \partial \nu > 0 \text{ on } \partial\Omega, \quad (2.5)$$

*where  $\partial/\partial\nu$  denotes the outward normal derivative.*

**Remark 2.1** *The antimaximum principle of Theorem 1, assuming  $\alpha > 0$ , is in the line of the version given by Arcoya Gamez [3].*

**Lemma 2.1** *We assume  $\lambda_1 < \mu \leq \Lambda < \lambda_2$  and  $h \in L^q$ ,  $q > N \geq 2$ . We suppose that there exists a constant  $C_1$  depending only on  $\Omega$ ,  $q$ , and  $\Lambda$  such that  $z$  satisfying (2.1) is such that*

$$\|z\|_{L^2} \leq C_1 \|h\|_{L^2}. \quad (2.6)$$

*Then there exist constants  $C_2$  and  $C_3$ , depending only on  $\Omega$ ,  $q$  and  $\Lambda$  such that*

$$\|z\|_{C^1} \leq C_2 \|h\|_{L^q} \text{ and } \|z\|_{L^q} \leq C_3 \|h\|_{L^q}. \quad (2.7)$$

**Remark 2.2** *Hypothesis 2.6 cannot hold, unless  $h$  is orthogonal to  $\varphi_1$ . Indeed, letting  $\mu$  go to  $\lambda_1$ , 2.6 implies the existence of a solution to 2.1 with  $\mu = \lambda_1$ . Note that in the proof of Theorem 1, Lemma 2.1 is used for  $h$  orthogonal to  $\varphi_1$ .*

## 2.1 Proof of Lemma 2.1

All constants in this proof depend only on  $\Omega$ ,  $\Lambda$  and  $q$ .

**Claim:**  $\|z\|_{L^q} \leq C_3 \|h\|_{L^q}$ .

If the claim is verified then, by regularity results for the Laplace operator combined with Sobolev imbeddings

$$\|z\|_{C^1} \leq C_4 \|z\|_{W^{2,q}} \leq C_5 (\Lambda \|z\|_{L^q} + \|h\|_{L^q}). \quad (2.8)$$

From the claim and regularity results we deduce (2.7).

**Proof of the claim:**

- **Step 1** We consider the sequence  $p_j = 2 + \frac{8j}{N}$  for  $j \in \mathbb{N}$ . Observe that for any  $j$ ,  $W^{2,p_j} \hookrightarrow L^{p_{j+1}}$  and that there exists a constant  $H(j)$  such that

$$\forall v \in W^{2,p_j}, \|v\|_{L^{p_{j+1}}} \leq H(j)\|v\|_{W^{2,p_j}}. \quad (2.9)$$

The relation (2.9) is obvious if  $2p_j \geq N$  and for  $2p_j < N$  we have

$$\frac{Np_j}{N-2p_j} - p_{j+1} = \frac{2p_j p_{j+1} - 8}{N-2p_j} > 0$$

and the result follows by classical Sobolev imbedding.

- **Step 2** We consider  $z$  satisfying (2.1). For  $j = 0$ , we derive from (2.6) and Hölder inequality that

$$\|z\|_{L^2} \leq C_5 \|h\|_{L^q}. \quad (2.10)$$

By induction we assume that  $z \in L^{p_j}$  with  $p_j < q$  and that

$$\|z\|_{L^{p_j}} \leq K(j) \|h\|_{L^q}. \quad (2.11)$$

By Hölder inequality,

$$\|\mu z + h\|_{L^{p_j}} \leq \Lambda \|z\|_{L^{p_j}} + |\Omega|^{\frac{q-p_j}{qp_j}} \|h\|_{L^q}.$$

By regularity results for the Laplace operator:

$$\|z\|_{W^{2,p_j}} \leq C(j)(\Lambda \|z\|_{L^{p_j}} + |\Omega|^{\frac{q-p_j}{qp_j}} \|h\|_{L^q}) \leq C(j)(\Lambda K(j) + |\Omega|^{\frac{q-p_j}{qp_j}}) \|h\|_{L^q}.$$

Using (2.9) the relation (2.11) holds for  $j+1$  and the induction is proved.

- **Step 3** Let  $J$  be such that  $p_{J+1} \geq q > p_J$ . After  $J$  iterations we get by (2.11)

$$\begin{aligned} \|z\|_{L^q} &\leq C_6 \|z\|_{L^{p_{J+1}}} \leq C_6 K(J+1) \|z\|_{W^{2,p}} \leq \\ &C_7 K(J+1) \|\mu z + h\|_{L^{p_J}} \leq C_8 (\Lambda \|h\|_{L^q} + \|h\|_{L^{p_J}}) \leq C_9 \|h\|_{L^q}, \end{aligned}$$

which is the claim. •

## 2.2 Proof of Theorem 1

- **Step 1:** We prove the following inequality:

$$\|z^\perp\|_{C^1} \leq C_2 \|h^\perp\|_{L^q}. \quad (2.12)$$

We derive from (2.3)

$$z = \frac{\alpha}{\lambda_1 - \mu} \varphi_1 + z^\perp, \quad (2.13)$$

with  $z^\perp$  solution of

$$-\Delta z^\perp = \mu z^\perp + h^\perp \text{ in } \Omega; \quad z^\perp = 0 \text{ on } \partial\Omega. \quad (2.14)$$

By the variational characterization of  $\lambda_2$ :

$$\lambda_2 \int_{\Omega} |z^\perp|^2 \leq \int_{\Omega} |\nabla z^\perp|^2 = \mu \int_{\Omega} |z^\perp|^2 + \int_{\Omega} z^\perp h^\perp.$$

Hence

$$\|z^\perp\|_{L^2} \leq \frac{1}{\lambda_2 - \mu} \|h^\perp\|_{L^2}.$$

By Lemma 2.1, we derive (2.12).

- **Step 2:** *Close to the boundary:*

We show now that on the boundary  $\frac{\partial z}{\partial \nu}(x) > 0$ . and near the boundary  $z < 0$ .

Since  $\partial\varphi_1/\partial\nu < 0$  on  $\partial\Omega$ , we set

$$A := \min_{\partial\Omega} |\partial\varphi_1/\partial\nu| > 0. \quad (2.15)$$

By a continuity argument there exists  $\varepsilon > 0$  such that

$$\text{dist}(x, \partial\Omega) < \varepsilon \Rightarrow \partial\varphi_1/\partial\nu(x) \leq -A/2. \quad (2.16)$$

Hence by (2.12) to (2.16), for any  $x \in \Omega$  such that  $\text{dist}(x, \partial\Omega) < \varepsilon$ , and if

$$0 < \mu - \lambda_1 < \frac{\alpha A}{4C_2 \|h^\perp\|_{L^q}},$$

we have

$$\frac{\partial z}{\partial \nu}(x) = \frac{\alpha}{\lambda_1 - \mu} \frac{\partial\varphi_1}{\partial \nu}(x) + \frac{\partial z^\perp}{\partial \nu}(x) \geq \frac{\alpha}{\lambda_1 - \mu} \frac{\partial\varphi_1}{\partial \nu}(x) - C_2 \|h^\perp\|_{L^q},$$

hence

$$\frac{\partial z}{\partial \nu}(x) \geq \frac{\alpha}{2(\lambda_1 - \mu)} \frac{\partial\varphi_1}{\partial \nu}(x) > 0. \quad (2.17)$$

Therefore  $\frac{\partial z}{\partial \nu}(x) > 0$  on  $\partial\Omega$ . Moreover since  $z = \varphi_1 = 0$  on  $\partial\Omega$ , we deduce from (2.17) that, for  $x \in \Omega$  with  $\text{dist}(x, \partial\Omega) < \varepsilon' \leq \varepsilon/2$  ( $\varepsilon'$  small enough),

$$z(x) \leq \frac{\alpha}{2(\lambda_1 - \mu)} \varphi_1(x) < 0,$$

where  $\varepsilon'$  does not depend on  $\mu$ .

**- Step 3:** *Inside  $\Omega$ :*

We consider now  $\Omega_{\varepsilon'} := \{x \in \Omega, \text{dist}(x, \partial\Omega) > \varepsilon'\}$ . Set

$$B := \min_{\Omega_{\varepsilon'}} \varphi_1(x) > 0.$$

We have in  $\Omega_{\varepsilon'}$  by (2.12) and (2.13)

$$z(x) = \frac{\alpha}{\lambda_1 - \mu} \varphi_1(x) + z^\perp(x) \leq \frac{\alpha}{\lambda_1 - \mu} B + C_2 \|h^\perp\|_{L^q} < 0$$

if we choose

$$\mu - \lambda_1 < \frac{\alpha \min(B, A/2)}{C_2 \|h^\perp\|_{L^q}}.$$

We derive now Theorem 1. •

### 2.3 An example

Let  $N = 1$ ,  $\Omega = ]0, 1[$  and  $h = h_1 \varphi_1 + h_2 \varphi_2$  with  $h_1 > 0$ ,  $h_2 > 0$ . We note that

$$\varphi_1(x) - s \varphi_2(x) = \sin \pi x (1 - 2s \cos \pi x) > 0 \quad (2.18)$$

in  $\Omega$  implies  $s \leq 1/2$ . For this example, taking  $\mu = \lambda_1 + \varepsilon$ ,  $\varepsilon > 0$ , we have:

$$z = \frac{h_1}{\lambda_1 - \mu} \varphi_1 + \frac{h_2}{\lambda_2 - \mu} \varphi_2 = -\frac{h_1}{\varepsilon} \left( \varphi_1 - \frac{\varepsilon h_2}{h_1(\lambda_2 - \lambda_1 - \varepsilon)} \varphi_2 \right).$$

If the Antimaximum Principle holds,  $z < 0$  in  $\Omega$ , and by (2.18), we have

$$\frac{\varepsilon h_2}{h_1(\lambda_2 - \lambda_1 - \varepsilon)} \leq \frac{1}{2},$$

hence

$$\varepsilon \leq \frac{h_1(\lambda_2 - \lambda_1)}{2h_2(1 + \frac{h_1}{2h_2})} \leq \frac{h_1(\lambda_2 - \lambda_1)}{2h_2}.$$

We obtain an estimate of  $\delta(h)$  similar to that in Theorem 1.

### 3 A non-cooperative system

Now we will consider the  $2 \times 2$  non-cooperative system depending on a real parameter  $\mu$ :

$$-\Delta u = au + bv + \mu u + f \text{ in } \Omega, \quad (S_1)$$

$$-\Delta v = cu + dv + \mu v + g \text{ in } \Omega, \quad (S_2)$$

$$u = v = 0 \text{ on } \partial\Omega. \quad (S_3)$$

or shortly

$$-\Delta U = AU + \mu U + F \text{ in } \Omega, \quad U = 0 \text{ on } \partial\Omega. \quad (S)$$

**Hypothesis** ( $H_1$ ) We assume  $b > 0, c < 0$ , and

$$D := (a - d)^2 + 4bc > 0. \quad (3.19)$$

#### 3.1 Eigenvalues of the system

As usual we say that  $\mu$  is an eigenvalue of System ( $S$ ) if ( $S_1$ ) – ( $S_3$ ) has a non trivial solution  $U = (u, v) \neq 0$  for  $F \equiv 0$  and we say that  $\mu$  is a principal eigenvalue of System ( $S$ ) if there exists  $U = (u, v)$  with  $u > 0, v > 0$  solution to ( $S$ ) with  $F \equiv 0$ .

Notice that, since ( $S$ ) is not cooperative, it is not necessarily true that there is a lowest principal eigenvalue  $\mu_1$  and that the maximum principle holds if and only if  $\mu_1 > 0$  (Amann [2]).

We seek solutions  $u = p\varphi_1, v = q\varphi_1$  to the eigenvalue problem where, as above,  $(\lambda_1, \varphi_1)$  is the principal eigenpair for  $-\Delta$  with Dirichlet boundary conditions.

Principal eigenvalues correspond to solutions with  $p, q > 0$ . The associated linear system is

$$(a + \mu - \lambda_1)p + bq = 0,$$

$$cp + (d + \mu - \lambda_1)q = 0,$$

and it follows from ( $H_1$ ) that  $(a + \mu - \lambda_1)$  and  $(d + \mu - \lambda_1)$  should have opposite signs. We should have

$$\text{Det}(A + (\mu - \lambda_1)I) = (a + \mu - \lambda_1)(d + \mu - \lambda_1) - bc = 0,$$

which implies by ( $H_1$ ) that the condition on signs is satisfied and this whatever the sign of  $\mu$  could be. (Notice that  $D > 0$  implies that both roots are real and that  $D = 0$  gives a real double root).



We have then shown directly that our system has (at least) two principal eigenvalues. Their signs will depend on the coefficients. If, for example,  $a < \lambda_1$ ,  $d < \lambda_1$ , the largest one is positive. We will denote the two principal eigenvalues by  $\mu_1^-$  and  $\mu_1^+$  where

$$\mu_1^- := \lambda_1 - \xi_1 < \mu_1^+ := \lambda_1 - \xi_2, \quad (3.20)$$

where the eigenvalues of Matrix  $A$  are:

$$\xi_1 = \frac{a + d + \sqrt{D}}{2} > \xi_2 = \frac{a + d - \sqrt{D}}{2}.$$

**Remark 3.1** *Usually the Maximum Principle holds if and only if the first eigenvalue is positive. Here by replacing  $-\Delta$  by  $-\Delta + K$  with  $K > 0$  large enough we may get  $\mu_1^- > 0$ . Nevertheless the maximum principle needs an additional condition (see Theorem 4 and its remark).*

## 3.2 Main Theorems

### 3.2.1 The case $\mu_1^- < \mu < \mu_1^+$

We assume in this subsection that the parameter  $\mu$  satisfies:

$$(H_2) \quad \mu_1^- < \mu < \mu_1^+.$$

**Theorem 2** *Assume  $(H_1)$ ,  $(H_2)$ , and*

$$(H_3) \quad d < a,$$

$$(H_4) \quad f \geq 0, g \geq 0, f, g \not\equiv 0, f, g \in L^q, q > N \text{ if } N \geq 2; q = 2 \text{ if } N = 1.$$

*Then there exists  $\delta > 0$ , independent of  $\mu$ , such that if*

$$(H_5) \quad \mu < \mu_1^- + \delta,$$

*we get*

$$u < 0, v > 0 \text{ in } \Omega; \frac{\partial u}{\partial \nu} > 0, \frac{\partial v}{\partial \nu} < 0 \text{ on } \partial\Omega.$$

**Remark 3.2** *If in the theorem above we reverse signs of  $f, g, u, v$  that is  $f \leq 0, g \leq 0, f, g \not\equiv 0$ , then for  $\mu$  satisfying  $(H_5)$ , we get*

$$u > 0, v < 0 \text{ in } \Omega; \frac{\partial u}{\partial \nu} < 0, \frac{\partial v}{\partial \nu} > 0 \text{ on } \partial\Omega.$$

*Note that the counterexample in subsection (3.3) shows that for  $f, g$  of opposite sign ( $fg < 0$ ),  $u$  or  $v$  may change sign.*

**Theorem 3** Assume  $(H_1)$ ,  $(H_2)$ , and

$$(H'_3) \quad a < d,$$

$$(H'_4) \quad f \leq 0, g \geq 0, f, g \neq 0, f, g \in L^q, q > N \text{ if } N \geq 2; q = 2 \text{ if } N = 1.$$

Then there exists  $\delta > 0$ , independent of  $\mu$ , such that if

$$(H_5) \quad \mu < \mu_1^- + \delta,$$

we obtain

$$u < 0, v < 0 \text{ in } \Omega; \frac{\partial u}{\partial \nu} > 0, \frac{\partial v}{\partial \nu} > 0 \text{ on } \partial\Omega.$$

**Remark 3.3** If in the theorem above we reverse signs of  $f, g, u, v$  that is  $f \geq 0, g \leq 0, f, g \neq 0$ , then for  $\mu$  satisfying  $(H_5)$ , we get

$$u > 0, v > 0 \text{ in } \Omega; \frac{\partial u}{\partial \nu} < 0, \frac{\partial v}{\partial \nu} < 0 \text{ on } \partial\Omega.$$

Note that, by the changes used in the proof of the theorem above, the counterexample in subsection (3.3) shows that for  $f, g$  with same sign ( $fg > 0$ ),  $u$  or  $v$  may change sign.

### 3.2.2 The case $\mu < \mu_1^-$

We assume in this Section that the parameter  $\mu$  satisfies:

$$(H'_2) \quad \mu < \mu_1^-.$$

**Theorem 4** Assume  $(H_1)$ ,  $(H'_2)$ , and

$$(H'_3) \quad a < d,$$

$$(H''_4) \quad f \geq 0, g \geq 0, f, g \neq 0, f, g \in L^2.$$

Assume also  $t^*g - f \geq 0, t^*g - f \neq 0$  with

$$t^* = \frac{d - a + \sqrt{D}}{-2c}.$$

Then

$$u > 0, v > 0 \text{ in } \Omega; \frac{\partial u}{\partial \nu} < 0, \frac{\partial v}{\partial \nu} < 0 \text{ on } \partial\Omega.$$

**Remark 3.4** As above we can reverse signs of  $f, g, u, v$ .

### 3.3 Counterexample: $a > d$

We consider the system in 1 dimension

$$-u'' = 4u + v + \mu u + f \text{ in } I := ]0; \pi[,$$

$$-v'' = -u + v + \mu v + g \text{ in } I,$$

$$u(0) = u(\pi) = v(0) = v(\pi) = 0.$$

$\lambda_1 = 1$  and  $\lambda_2 = 4$ ;  $\varphi_1 = \sin x$ ,  $\varphi_2 = \sin 2x$ . We compute  $\mu_1^- = 1 - \frac{5+\sqrt{5}}{2}$ . Choose  $f = \varphi_1 - \frac{1}{2}\varphi_2 \geq 0$  and  $g = kf$  with  $k \neq 0$  to be determined later. We obtain

$$u = u_1\varphi_1 + u_2\varphi_2 \text{ and } v = v_1\varphi_1 + v_2\varphi_2,$$

where

$$u_1 = \frac{k - \mu}{\mu^2 + 3\mu + 1}, \quad u_2 = \frac{\mu - k - 3}{2(\mu^2 - 3\mu + 1)},$$

1/ Choosing  $\mu = -3 < \mu_1^-$ , we get  $v_1 = -1$  and  $v_2 = \frac{1-3k}{38}$ . Therefore

$$-v = \varphi_1 + \frac{3k-1}{38}\varphi_2,$$

and for  $\frac{3k-1}{38} > \frac{1}{2}$ ,  $v$  changes sign. Hence Maximum Principle does not hold.

2/ Choosing  $\mu_1^- < \mu = \mu_1^- + \epsilon$ ,  $k = \mu_1^- + \epsilon^2$ , we have

$$\frac{u_2}{u_1} = \left( \frac{\mu - k - 3}{k - \mu} \right) \left( \frac{\mu^2 + 3\mu + 1}{2(\mu^2 - 3\mu + 1)} \right) = \left( \frac{3 + \epsilon}{\epsilon} \right) \left( \frac{\sqrt{5} - \epsilon}{(9 + 3\sqrt{5}) - (6 + \sqrt{5})\epsilon + \epsilon^2} \right).$$

So that  $\frac{u_2}{u_1} \rightarrow \infty$  as  $\epsilon \rightarrow 0$ . Hence for these  $f > 0$ ,  $g < 0$ ,  $u$  changes sign. •

## 3.4 Proofs of the main results

### 3.4.1 Some computations and associate equation

In the following we introduce

$$\gamma_1 = \frac{1}{2}(a + d + 2\mu - \sqrt{D}) = \lambda_1 + \mu - \mu_1^+; \quad (3.21)$$

$$\gamma_2 = \frac{1}{2}(a + d + 2\mu + \sqrt{D}) = \lambda_1 + \mu - \mu_1^-, \quad (3.22)$$

and some auxiliary results used in the proofs of our results.

**Lemma 3.1** *We have*

$$(L1) \quad \mu < \mu_1^+ \Leftrightarrow \gamma_1 < \lambda_1.$$

$$(L2) \quad \mu_1^- < \mu \Leftrightarrow \lambda_1 < \gamma_2.$$

$$(L3) \quad \sqrt{D} < a - d \Leftrightarrow d + \mu < \gamma_1 < \gamma_2 < a + \mu.$$

$$(L4) \quad \sqrt{D} < d - a \Leftrightarrow a + \mu < \gamma_1 < \gamma_2 < d + \mu.$$

$$(L5) \quad \mu < \mu_1^+ + \delta \Leftrightarrow \gamma_1 < \lambda_1 + \delta.$$

$$(L6) \quad \mu < \mu_1^- + \delta \Leftrightarrow \gamma_2 < \lambda_1 + \delta.$$

### 3.4.2 Proofs of Theorems 2 and 3

**Proof of Theorem 2,  $a > d$ :**

We introduce now

$$w = u + tv, \tag{3.23}$$

with

$$t = \frac{a - d + \sqrt{D}}{-2c} = \frac{2b}{a - d - \sqrt{D}} \tag{3.24}$$

so that

$$-\Delta w = \gamma_1 w + f + tg \text{ in } \Omega; w|_{\partial\Omega} = 0. \tag{3.25}$$

We remark that

$$t = \frac{b}{\gamma_1 - d - \mu} = \frac{b}{a + \mu - \gamma_2} = \frac{\gamma_1 - a - \mu}{c} = \frac{d + \mu - \gamma_2}{c}. \tag{3.26}$$

Note first that Hypothesis  $(H_3)$  implies  $t > 0$  and  $a - d > \sqrt{D}$ . By  $(H_2)$ ,  $(H_4)$ , and  $(L1)$  in Lemma 3.1,  $\gamma_1 < \lambda_1$ , and we apply the Maximum Principle which gives  $w > 0$  on  $\Omega$  and  $\frac{\partial w}{\partial \nu} < 0$  on  $\partial\Omega$ . We compute

$$a + \mu - \frac{b}{t} = a + d + 2\mu - \gamma_1 = \gamma_2, \tag{3.27}$$

and since  $v = (w - u)/t$ , we derive

$$-\Delta u = \left(a + \mu - \frac{b}{t}\right)u + \frac{b}{t}w + f = \gamma_2 u + \frac{b}{t}w + f,$$

where  $\frac{b}{t}w + f > 0$ . From  $(H_5)$  and  $(L6)$ ,  $\gamma_2 \leq \lambda_1 + \delta_1$ , where

$$\delta_1 := \delta\left(\frac{b}{t}w + f\right), \quad (3.28)$$

we deduce from the Antimaximum Principle that  $u < 0$  on  $\Omega$  and  $\frac{\partial u}{\partial \nu} > 0$  on  $\partial\Omega$ . Hence  $cu + g > 0$ .

Now  $(H_2)$ ,  $(L_1)$  and  $(L_3)$  imply  $d + \mu < \gamma_1 < \lambda_1$  and the Maximum Principle applied to  $(S_2)$  gives  $v > 0$  on  $\Omega$  and  $\frac{\partial v}{\partial \nu} < 0$  on  $\partial\Omega$ .

We apply now Section 1 to estimate  $\delta_1$ .

$$h := \frac{b}{t}w + f = (\gamma_1 - d - \mu)w + f = \sigma\varphi_1 + h^\perp. \quad (3.29)$$

First we compute  $\sigma$ : Here we show that this is not the case for non-cooperative systems (with maybe  $\mu_1^- < 0$ ).

In this paper we use ideas concerning the Anti-Maximum Principle due to Clément and Peletier [5] (see also [9]) in order to study non-cooperative  $2 \times 2$  systems. In Section 2 we obtain precise estimates concerning the validity interval for the Anti-maximum Principle for one equation. We include an example.

In Section 3, we consider a non-cooperative  $2 \times 2$  system with constant coefficients depending on a real parameter  $\mu$  having two real principal eigenvalues  $\mu_1^- < \mu_1^+$ . We obtain some theorems concerning the behavior of different cones of couples of functions having positivity (or negativity) properties. We give several results of this type for values of  $\mu_1^- < \mu$  but close to  $\mu_1^-$  by combining the usual Maximum Principle and the results for the Anti-Maximum Principle in Section 2. We actually prove only one of such theorems, all the others are proved just by making suitable changes of variables. A (partial, under an additional assumption) Maximum Principle for  $\mu < \mu_1^-$  is also obtained.

Set  $f = \alpha\varphi_1 + f^\perp$ ,  $g = \beta\varphi_1 + g^\perp$ ,  $w = \kappa\varphi_1 + w^\perp$ . Since

$$-\Delta w = \gamma_1 w + f + \frac{b}{\gamma_1 - d - \mu} g,$$

we calculate:

$$\sigma = \alpha + (\gamma_1 - d - \mu)\kappa = \alpha \frac{\lambda_1 - d - \mu}{\lambda_1 - \gamma_1} + \beta \frac{b}{\lambda_1 - \gamma_1}.$$

Now we estimate  $\|h^\perp\|_{L^2}$ .

$$-\Delta w^\perp = \gamma_1 w^\perp + f^\perp + \frac{b}{\gamma_1 - d - \mu} g^\perp.$$

The variational characterization of  $\lambda_2$  gives

$$(\lambda_2 - \gamma_1)\|w^\perp\|_{L^2} \leq \|f^\perp\|_{L^2} + \frac{b}{\gamma_1 - d - \mu}\|g^\perp\|_{L^2}.$$

We derive from (3.29)

$$\|h^\perp\|_{L^2} \leq \|f^\perp\|_{L^2} + (\gamma_1 - d - \mu)\|w^\perp\|_{L^2} \leq \frac{\lambda_2 - d - \mu}{\lambda_2 - \gamma_1}\|f^\perp\|_{L^2} + \frac{b}{\lambda_2 - \gamma_1}\|g^\perp\|_{L^2}.$$

Reasoning as in Lemma 2.1, we show that there exists a constant  $C_3$  such that

$$\|h^\perp\|_{L^q} \leq C_3 \left( \frac{\lambda_2 - d - \mu}{\lambda_2 - \gamma_1}\|f^\perp\|_{L^q} + \frac{b}{\lambda_2 - \gamma_1}\|g^\perp\|_{L^q} \right). \quad (3.30)$$

In fact for proving (3.30) we use the same sequence than that in Lemma 2.1 and we show by induction that

$$\|z^\perp\|_{L^{p_j}} \leq K(j) \left( \|f^\perp\|_{L^q} + \|g^\perp\|_{L^q} \right).$$

Now we apply the antimaximum principle to the equation

$$-\Delta u = \gamma_2 u + h.$$

This is possible since by (L6) in Lemma 3.1,  $\lambda_1 < \gamma_2 < \lambda_1 + \delta_2 = \lambda_1 + \delta(h)$  where, as in Theorem 1,  $\delta(h) = \frac{K\sigma}{\|h^\perp\|_{L^q}}$ .

Moreover we notice that  $\lambda_1 - \gamma_1 = \mu_1^+ - \mu \leq \mu_1^+ - \mu_1^-$  and therefore, since  $\alpha > 0$  and  $\beta > 0$  by (H4),

$$\sigma = \alpha \frac{\lambda_1 - d - \mu}{\lambda_1 - \gamma_1} + \beta \frac{b}{\lambda_1 - \gamma_1} \geq \mathcal{A} := \alpha \frac{\lambda_1 - d - \mu_1^+}{\mu_1^+ - \mu_1^-} + \beta \frac{b}{\mu_1^+ - \mu_1^-},$$

and from (3.30), we obtain

$$\|h^\perp\|_{L^q} \leq \mathcal{B} := C_3 \left( \frac{\lambda_2 - d - \mu_1^-}{\lambda_2 - \lambda_1}\|f^\perp\|_{L^q} + \frac{b}{\lambda_2 - \lambda_1}\|g^\perp\|_{L^q} \right).$$

From the computation above we can choose  $\delta_2 = \frac{K\mathcal{A}}{\mathcal{B}}$  which does not depend on  $\mu$ , and the result follows. •

**Proof of Theorem 3:**  $a < d$ . We deduce this theorem from Theorem 2 by change of variables. Set  $\hat{a} = d$ ,  $\hat{d} = a$ ,  $\hat{u} = v$ ,  $\hat{v} = -u$  and  $\hat{f} = g$ ,  $\hat{g} = -f$ .  $\hat{f} \geq 0$ ,  $\hat{g} \geq 0$ , imply  $\hat{u} < 0$ ,  $\hat{v} > 0$ . We get Theorem 3. •

### 3.4.3 Proof of Theorem 4

Since  $a < d$ , we have  $t^* = \frac{d-a+\sqrt{D}}{-2c} > 0$ . With now the change of variable  $w = -u + t^*v$ , as in [4] (see also [11]), we can write the system as

$$-\Delta u = \gamma_1 u + (b/t^*)w + f \text{ in } \Omega, \quad (3.31)$$

$$-\Delta v = \gamma_1 v - cw + g \text{ in } \Omega \quad (3.32)$$

$$-\Delta w = \gamma_2 w + (t^*g - f) \text{ in } \Omega, \quad (3.33)$$

$$u = v = w = 0 \text{ on } \partial\Omega.$$

Now  $\mu < \mu_1^-$ , and it follows from (L2) in Lemma 3.1 that  $\gamma_1 < \gamma_2 < \lambda_1$ . From (3.33) it follows from the Maximum Principle that  $w > 0$ . Then in (3.32)  $-cw + g > 0$ , and again by the Maximum Principle  $v > 0$ . Finally, since  $(b/t^*)w + f > 0$  in (3.31), again by the Maximum Principle  $u > 0$ . •

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