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# CRAMÉR'S THEOREM IN BANACH SPACES REVISITED

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## Abstract

The text summarizes the general results of large deviations for empirical means of independent and identically distributed variables in a separable Banach space, without the hypothesis of exponential tightness. The large deviation upper bound for convex sets is proved in a nonasymptotic form; as a result, the closure of the domain of the entropy coincides with the closed convex hull of the support of the common law of the variables. Also a short original proof of the convex duality between negentropy and pressure is provided: it simply relies on the subadditive lemma and Fatou's lemma, and does not resort to the law of large numbers or any other limit theorem. Eventually a Varadhan-like version of the convex upper bound is established and embraces both results.

## 1 Introduction

Cramér's original theorem (see [11]) about the large deviations of empirical means of independent and identically distributed real-valued random variables has led to an extensive literature. Several proofs of it were given by Chernoff, Bahadur, Ranga Rao, Petrov, Hammersley, and Kingman (see [10], [3], [27], [2], [19], and [22]). The result was extended to higher dimensions by Sethuraman, Borovkov, Rogosin, Hoeffding, Sievers, Bartfai, and many others (see [31], [32], [7], [21], [33], [5]). At the same time, Sanov's theorem (see [30]) and its generalizations (see, e.g., [20]), and the study of large deviations of random processes (see, e.g., [34]) gave rise to Donsker and Varadhan's setting of large deviation principles in separable Banach spaces (see [16]). In this unifying setting, if we assume the exponential tightness of the sequence of empirical means, or equivalently the boundedness of the pressure in a neighborhood of the origin, then a full large deviation principle can be proved.

Independently, the physicist Lanford imported the subadditive argument, developed by him and Ruelle in statistical physics, into Cramér's theory (see [29] and [23]). Bahadur and Zabell (see [4]) took advantage of this new method to generalize Cramér's theory to locally convex spaces, to simplify some proofs, and to provide a good synthesis of the previous texts. By the way, they revealed that, if you replace the exponential tightness by the less restricting convex tightness, you still have the exponential decay for large deviation events associated with a convex set and the convex duality between negentropy and pressure. Among many others, the standard texts of Azencott, de Acosta, Deuschel, Stroock, Dembo, Zeitouni, and Cerf summarize the successive developments of the theory (see [1], [12], [14], [13], [8]).

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Here, we prove the general results of Cramér's theory in separable Banach spaces without assuming extra hypotheses. Our arguments rely on geometrical and topological properties of Banach spaces, in the spirit of [4] and [8], and enable to complete some known partial conclusions. The main one is the large deviation upper bound for all convex sets, which is even valid in a nonasymptotic form. We deduce that the closure of the domain of the entropy coincides with the closed convex hull of the law of the variables. Another goal of the present text is to shed a new light on the theory, providing efficient and simple proofs. For instance, to prove the convex duality between the negentropy  $-s$  and the pressure  $p$ , we prove the equality  $p = (-s)^*$  using the convex tightness of the probability measures on a Banach space and Fatou's lemma (see [15] for a similar proof when the full large deviation principle is assumed), whereas usual proofs show the dual equality  $s = -p^*$  by means of convex regularity and Cramér's theorem in  $\mathbb{R}$ , which in turn relies on an approximation by simpler variables (discrete in [10], bounded in [13]) and a limit theorem (Stirling's formula in [10], the law of large numbers in [13]). By the way, we intensively exploit the nice properties of convex sets to simplify proofs and establish the equivalence between convex regularity and convex tightness (which clarifies the appendix of [4]). It appears that our methods can be generalized to locally convex spaces, but technical points may have hidden the heart of our new proofs. We also show how Varadhan-like lemmas provide unifying results and, eventually, we prove a Varadhan-like lemma for concave functions which embraces both the nonasymptotic upper bound for convex sets and the equality  $p = (-s)^*$ .

After setting the stage and stating the results (Sect. 2), we first give a short proof of the weak large deviation principle (Sect. 3). Then we prove the large deviation upper bound for convex sets and deduce the clear identification of the closure of the domain of the entropy (Sect. 4). Section 5 is devoted to the proof of the convex duality between negentropy and pressure. Finally we prove the general convex upper bound à la Varadhan (Sect. 6). Except for the classic Fenchel-Moreau theorem (see [25]), proofs of convex analysis are provided; complementary notions can be found in general texts like [25] and [28].

## 2 Setting and results

Let  $\mathcal{X}$  be a separable Banach space,  $\mathcal{B}$  the Borel  $\sigma$ -algebra over  $\mathcal{X}$ , and  $\mu$  a probability measure on  $(\mathcal{X}, \mathcal{B})$ . Let  $(X_n)_{n \geq 1}$  be a sequence of independent and identically distributed random variables with law  $\mu$ . For all  $n \geq 1$ , let  $\bar{X}_n$  be the empirical mean  $(X_1 + X_2 + \dots + X_n)/n$ .

**Definition 1.** *The entropy of the sequence  $(\bar{X}_n)_{n \geq 1}$  is the function  $s : \mathcal{X} \rightarrow [-\infty, 0]$  defined by*

$$\forall x \in \mathcal{X} \quad s(x) := \inf_{\varepsilon > 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\bar{X}_n \in B(x, \varepsilon))$$

where  $B(x, \varepsilon)$  denotes the open ball of radius  $\varepsilon$  centered at  $x$  in  $\mathcal{X}$ .

By construction, the entropy  $s$  is the greatest function that satisfies the lower bound:

$$(LB) \text{ for all open subsets } G, \liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\bar{X}_n \in G) \geq \sup_{x \in G} s(x).$$

One says that the sequence  $(\bar{X}_n)_{n \geq 1}$  satisfies a *large deviation principle* if, in addition, it satisfies the upper bound:

$$(UB) \text{ for all closed subsets } F, \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\bar{X}_n \in F) \leq \sup_{x \in F} s(x).$$

Conditions so that (UB) be satisfied, such as exponential tightness of the sequence  $(\bar{X}_n)_{n \geq 1}$ , are given in standard texts (see [16], [4], [1], [12], [14], [13] and [8]). Here, as in [4] and [8], we are interested in weaker upper bounds that do not require additional hypotheses. For instance, the following result is well-known (see, e.g., [16] or [4]).

**Theorem 1.** *The sequence  $(\bar{X}_n)_{n \geq 1}$  satisfies a weak large deviation principle, i.e. it satisfies the compact upper bound:*

$$(UB_k) \text{ for all compact subsets } K, \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\bar{X}_n \in K) \leq \sup_{x \in K} s(x).$$

The upper bound is known also for open convex sets (see [4]), but the proof for closed convex sets is omitted. Here we prove the better nonasymptotic versions of them.

**Theorem 2.** *The sequence  $(\bar{X}_n)_{n \geq 1}$  satisfies the nonasymptotic closed convex upper bound:*

$$(UB_{cc}) \text{ for all closed convex subsets } C \text{ and } n \geq 1, \mathbb{P}(\bar{X}_n \in C) \leq \exp\left(n \sup_{x \in C} s(x)\right);$$

*and the nonasymptotic open convex upper bound:*

$$(UB_{oc}) \text{ for all open convex subsets } C \text{ and } n \geq 1, \mathbb{P}(\bar{X}_n \in C) \leq \exp\left(n \sup_{x \in C} s(x)\right).$$

*In particular, if  $C$  is an open convex subset, we get*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\bar{X}_n \in C) = \sup_{n \geq 1} \frac{1}{n} \log \mathbb{P}(\bar{X}_n \in C) = \sup_{x \in C} s(x).$$

The proof we give here does not rely on hypothesis  $(\hat{C})$  of [14, Sect. 3.1], or assumption 6.1.2 of [13], but simply on the convex tightness of  $\mu$  introduced in [4] and it generalizes more easily<sup>1</sup>. Theorem 2 appears to be very convenient in the study of large deviations of means of independent and identically distributed random variables. For instance, consider the *domain* of the entropy  $\text{dom}(s) = \{s > -\infty\}$ . Denote by  $\text{co supp}(\mu)$  the convex hull of the support of the measure  $\mu$ .

**Theorem 3.** *The closure of the domain of the entropy  $s$  is the closed convex hull of the support of the measure  $\mu$ , i.e.*

$$\overline{\text{dom}(s)} = \overline{\text{co supp}(\mu)}.$$

The result is only partially proved in [4] and [8]. We give a complete proof. Another consequence of theorem 2 is the link between entropy and pressure. Let  $\mathcal{X}^*$  denote the topological dual of  $\mathcal{X}$  and let  $p : \mathcal{X}^* \rightarrow (-\infty, +\infty]$  be the *pressure*<sup>2</sup> of the sequence  $(\bar{X}_n)_{n \geq 1}$  defined by

$$\forall \lambda \in \mathcal{X}^* \quad p(\lambda) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}(e^{n\lambda(\bar{X}_n)}) = \log \mathbb{E}(e^{\lambda(X_1)})$$

which reduces to the log-Laplace transform of  $\mu$ .

**Theorem 4.** *The pressure  $p$  and the negentropy  $-s$  are convex-conjugate functions, i.e.*

$$\forall \lambda \in \mathcal{X}^* \quad p(\lambda) = \sup_{x \in \mathcal{X}} (\lambda(x) + s(x)) =: (-s)^*(\lambda) \tag{1}$$

*and*

$$\forall x \in \mathcal{X} \quad -s(x) = \sup_{\lambda \in \mathcal{X}^*} (\lambda(x) - p(\lambda)) =: p^*(x). \tag{2}$$

<sup>1</sup>Hypothesis  $(\hat{C})$  of [14] and assumption 6.1.2 (b) of [13] were introduced so as to complete the proofs of the appendix of [4], but they are not required to prove the first proposition of the appendix.

<sup>2</sup>Physically speaking, the function  $p$  should be interpreted as the opposite of a free energy, which is proportional to the pressure in the case of simple fluids.

Equation (2) is well-known (see, e.g., [4], [13] and [8]) and standard proofs rely on three ingredients: Chebyshev's inequality, the open half-space upper bound<sup>3</sup>, which is a particular case of (UB<sub>oc</sub>), and Cramér's theorem in  $\mathbb{R}$ . Equation (1) follows from equation (2) by proving that  $p$  is convex and lower semi-continuous (see [8], chapter 12). Here we give a simple original proof of equation (1) from which we deduce equation (2). Even in  $\mathcal{X} = \mathbb{R}$ , it provides a new proof of Cramér's theorem (see [9]). Notice that equation (1) is similar to Varadhan's lemma (remember the first definition of the pressure  $p$ ). The present proof relies on Varadhan-like versions of the lower bound and compact upper bound:

**Lemma 1.** *The sequence  $(\bar{X}_n)_{n \geq 1}$  satisfies the lower bound à la Varadhan:*

(VLB) for all lower semi-continuous functions  $f : \mathcal{X} \rightarrow [-\infty, +\infty)$ ,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}(e^{nf(\bar{X}_n)}) \geq \sup_{x \in \mathcal{X}} (f(x) + s(x)) .$$

**Lemma 2.** *The sequence  $(\bar{X}_n)_{n \geq 1}$  satisfies the compact upper bound à la Varadhan:*

(VUB<sub>k</sub>) for all upper semi-continuous functions  $f : \mathcal{X} \rightarrow [-\infty, +\infty)$  such that  $\{f > -\infty\}$  is relatively compact,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}(e^{nf(\bar{X}_n)}) \leq \sup_{x \in \mathcal{X}} (f(x) + s(x)) .$$

Interestingly enough, lemma 2 provides a Varadhan-like version of the convex upper bounds, which in turn implies theorems 2 and 4:

**Theorem 5.** *The sequence  $(\bar{X}_n)_{n \geq 1}$  satisfies the nonasymptotic convex upper bounds à la Varadhan:*

(VUB<sub>cc</sub>) for all upper semi-continuous concave functions  $f : \mathcal{X} \rightarrow [-\infty, +\infty)$ ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}(e^{nf(\bar{X}_n)}) = \sup_{n \geq 1} \frac{1}{n} \log \mathbb{E}(e^{nf(\bar{X}_n)}) \leq \sup_{x \in \mathcal{X}} (f(x) + s(x)) ;$$

and:

(VUB<sub>oc</sub>) for all concave functions  $f : \mathcal{X} \rightarrow [-\infty, +\infty)$  such that  $C = \{f > -\infty\}$  is open and  $f|_C$  is upper semi-continuous,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}(e^{nf(\bar{X}_n)}) = \sup_{n \geq 1} \frac{1}{n} \log \mathbb{E}(e^{nf(\bar{X}_n)}) \leq \sup_{x \in \mathcal{X}} (f(x) + s(x)) .$$

### 3 Proof of theorem 1

The proof of the weak large deviation principle relies on two key arguments: subadditivity and what may be called “the principle of the largest term” (see [24]). The former is the purpose of proposition 2 and the latter that of proposition 3. Beforehand, we need two very handy properties of open convex sets.

**Proposition 1.** *Let  $C$  be an open convex subset of  $\mathcal{X}$  containing 0. Then*

$$\bigcup_{t > 0} tC = \mathcal{X} , \tag{3}$$

i.e.  $C$  is an absorbing subset of  $\mathcal{X}$ , and

$$\bigcup_{\delta \in (0,1)} (1 - \delta)C = C . \tag{4}$$

<sup>3</sup>The proof is even simpler using the closed half-space upper bound, which is a particular case of (UB<sub>cc</sub>).

*Proof.* To show (3), let  $x \in \mathcal{X}$ . Since the mapping  $a \in \mathbb{R} \mapsto ax \in \mathcal{X}$  is continuous and  $C$  is a neighborhood of 0, there is  $\alpha > 0$  such that  $\alpha x \in C$ . Setting  $t = 1/\alpha$ , we get  $x \in tC$ . As for (4), let  $x \in C$ . Since the mapping  $a \in \mathbb{R} \mapsto ax \in \mathcal{X}$  is continuous and  $C$  is a neighborhood of  $x$ , there is  $\alpha > 0$  such that  $(1 + \alpha)x \in C$ . Defining  $\delta \in (0, 1)$  by  $1 - \delta = 1/(1 + \alpha)$ , we get  $x \in (1 - \delta)C$ , whence

$$C \subset \bigcup_{\delta \in (0,1)} (1 - \delta)C$$

and the converse inclusion is trivial.  $\square$

Proposition 2 below is fundamental in Cramér's theory. Here is a short proof relying on the proposition above.

**Proposition 2.** *Let  $C$  be an open convex subset of  $\mathcal{X}$ . Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\bar{X}_n \in C) = \sup_{n \geq 1} \frac{1}{n} \log \mathbb{P}(\bar{X}_n \in C).$$

*Proof.* The result is trivial if  $C = \emptyset$ . Now suppose  $0 \in C$ , otherwise consider  $(X_n - x)_{n \geq 1}$  for some  $x \in C$ . Let  $n, m \geq 1$  and write  $n = qm + r$  the Euclidean division of  $n$  by  $m$  with  $r \in \{1, 2, \dots, m\}$ . Let  $\delta \in (0, 1)$ . Using the convexity of  $C$ , the independence of  $X_1, X_2, \dots, X_n$ , and the fact that

$$\bar{X}_n = \frac{m}{n} \sum_{k=0}^{q-1} \left( \frac{1}{m} \sum_{i=km+1}^{(k+1)m} X_i \right) + \frac{1}{n} \sum_{i=mq+1}^n X_i,$$

we get

$$\mathbb{P}(\bar{X}_n \in C) \geq \mathbb{P}\left(\bar{X}_m \in \frac{n}{qm}(1 - \delta)C\right)^q \mathbb{P}\left(X_1 \in \frac{n}{r}\delta C\right)^r.$$

Since  $r \leq m$  and  $C$  is an absorbing subset of  $\mathcal{X}$  (see proposition 1 (3)),

$$\mathbb{P}\left(X_1 \in \frac{n\delta}{r}C\right)^r \geq \mathbb{P}\left(X_1 \in \frac{n\delta}{m}C\right)^m \xrightarrow{n \rightarrow \infty} 1.$$

Hence, remembering that  $qm \leq n$ ,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\bar{X}_n \in C) &\geq \liminf_{n \rightarrow \infty} \frac{q}{n} \log \mathbb{P}\left(\bar{X}_m \in \frac{n(1 - \delta)}{qm}C\right) \\ &\geq \frac{1}{m} \log \mathbb{P}(\bar{X}_m \in (1 - \delta)C) \end{aligned}$$

and the proof is completed by taking the limit when  $\delta \rightarrow 0$  (see proposition 1), and then the supremum over  $m \geq 1$ .  $\square$

Notice that proposition 2 is more generally valid for *algebraically open* convex sets that are measurable, i.e. measurable convex sets that satisfy properties (3) and (4) of proposition 1 and their translates, i.e. measurable convex sets that are equal to their algebraic interior (see [35]).

The next simple but useful result is well-known and may be called the ‘‘principle of the largest term’’ (see, e.g., [13, Lemma 1.2.15], [24], and [8]). We give its proof for the sake of completeness.

**Proposition 3.** Let  $u_1, u_2, \dots, u_r$  be  $[0, +\infty]$ -valued sequences. Then

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{i=1}^r u_i(n) = \max_{1 \leq i \leq r} \limsup_{n \rightarrow \infty} \frac{1}{n} \log u_i(n).$$

*Proof.* From the double inequality

$$\max_{1 \leq i \leq r} u_i(n) \leq \sum_{i=1}^r u_i(n) \leq r \max_{1 \leq i \leq r} u_i(n),$$

we get

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{i=1}^r u_i(n) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \max_{1 \leq i \leq r} u_i(n).$$

Moreover

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \max_{1 \leq i \leq r} u_i(n) &= \lim_{n \rightarrow \infty} \max_{1 \leq i \leq r} \left( \sup_{k \geq n} \frac{1}{k} \log u_i(k) \right) \\ &= \max_{1 \leq i \leq r} \lim_{n \rightarrow \infty} \left( \sup_{k \geq n} \frac{1}{k} \log u_i(k) \right), \end{aligned}$$

since the function  $\max : [-\infty, +\infty]^r \rightarrow [-\infty, +\infty]$  is continuous.  $\square$

*Proof of theorem 1.* Let  $K$  be a compact subset of  $\mathcal{X}$  and  $\alpha > 0$ . For all  $x \in K$ , apply proposition 2 and choose  $\varepsilon > 0$  such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\bar{X}_n \in B(x, \varepsilon)) \leq \max(s(x) + \alpha, -1/\alpha).$$

Since  $K$  is compact, there is a finite subcover  $K \subset B_1 \cup B_2 \cup \dots \cup B_r$  with  $B_i = B(x_i, \varepsilon_i)$ . Now apply propositions 3 and 2 to get

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\bar{X}_n \in K) &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{i=1}^r \mathbb{P}(\bar{X}_n \in B_i) \\ &= \max_{1 \leq i \leq r} \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\bar{X}_n \in B_i) \\ &\leq \max_{1 \leq i \leq r} \max(s(x_i) + \alpha, -1/\alpha) \\ &\leq \max \left( \sup_{x \in K} s(x) + \alpha, -1/\alpha \right) \end{aligned}$$

and finally let  $\alpha \rightarrow 0$ .  $\square$

## 4 Proofs of theorems 2 and 3

To prove the convex upper bounds, we will simply extend the compact (convex) upper bound to convex sets using the convex tightness of the measures on  $(\mathcal{X}, \mathcal{B})$ . The idea can be traced back to [4] and the proof given here is shorter and complete.

**Proposition 4.** Any probability measure  $\nu$  on  $(\mathcal{X}, \mathcal{B})$  is convex tight, i.e. for all  $\alpha > 0$ , there exists a compact convex subset  $K$  of  $\mathcal{X}$  such that  $\nu(K) > 1 - \alpha$ .

*Proof.* Let  $\nu$  be a probability measure on  $(\mathcal{X}, \mathcal{B})$  and let  $\alpha > 0$ . Since  $\mathcal{X}$  is metric, separable, and complete,  $\nu$  is *tight*, i.e. there is a compact subset  $K_1$  of  $\mathcal{X}$  such that  $\nu(K_1) > 1 - \alpha$  (see [6], theorem 1.3). Then  $K = \overline{\text{co}}(K_1)$  the closed convex hull of  $K_1$  is compact (see [17], theorem V.2.6) and satisfies  $\nu(K) > 1 - \alpha$ .  $\square$

To prove (UB<sub>cc</sub>), we also need a fact similar to proposition 2.

**Proposition 5.** *Let  $C$  be a measurable convex subset of  $\mathcal{X}$ . Then*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\overline{X}_n \in C) = \sup_{n \geq 1} \frac{1}{n} \log \mathbb{P}(\overline{X}_n \in C).$$

*Proof.* Let  $m, q \geq 1$ . Since  $C$  is convex and  $X_1, X_2, \dots, X_{qm}$  are independent,

$$\mathbb{P}(\overline{X}_{qm} \in C) \geq \mathbb{P}(\overline{X}_m \in C)^q.$$

Hence

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\overline{X}_n \in C) \geq \limsup_{q \rightarrow \infty} \frac{1}{qm} \log \mathbb{P}(\overline{X}_{qm} \in C) \geq \frac{1}{m} \log \mathbb{P}(\overline{X}_m \in C).$$

Take the supremum over  $m \geq 1$  to conclude.  $\square$

*Proof of (UB<sub>cc</sub>).* Let  $C$  be a closed convex subset of  $\mathcal{X}$  and  $N \geq 1$ . By proposition 4, the distribution of  $\overline{X}_N$  is convex tight, whence, for all  $\alpha > 0$ , there exists a compact convex subset  $K$  of  $\mathcal{X}$  such that

$$\frac{1}{N} \log \mathbb{P}(\overline{X}_N \in C) \leq \frac{1}{N} \log \mathbb{P}(\overline{X}_N \in C \cap K) + \alpha. \quad (5)$$

Applying proposition 5 to the convex  $C \cap K$  leads to

$$\frac{1}{N} \log \mathbb{P}(\overline{X}_N \in C \cap K) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\overline{X}_n \in C \cap K).$$

Finally, the application of theorem 1 to the compact  $C \cap K$  yields

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\overline{X}_n \in C \cap K) \leq \sup_{x \in C \cap K} s(x) \leq \sup_{x \in C} s(x).$$

From (5), we get

$$\frac{1}{N} \log \mathbb{P}(\overline{X}_N \in C) \leq \sup_{x \in C} s(x) + \alpha.$$

Conclude by letting  $\alpha \rightarrow 0$ .  $\square$

A detailed observation of this last proof shows that it only requires the convex tightness of  $\mu$ . Indeed, the convex tightness of  $\mu$  implies the convex tightness of the distribution of  $\overline{X}_N$ , since, if  $K$  is convex, then

$$\mathbb{P}(\overline{X}_N \in K) \geq \mathbb{P}(X_1 \in K)^N.$$

This simple remark is fruitful: it permits to establish (UB<sub>cc</sub>) in a more general context and to avoid technical hypotheses. The proof of (UB<sub>oc</sub>) is in the same vein. We only need a nice property of open convex sets.

**Proposition 6.** *Let  $C$  be an open convex subset of  $\mathcal{X}$  containing 0. Then,*

$$\bigcup_{\delta \in (0,1)} (1-\delta)\overline{C} = C.$$

*Proof.* Given proposition 1, it remains to show that, for all  $\delta \in (0,1)$ ,  $(1-\delta)\overline{C} \subset C$ . Let  $\delta \in (0,1)$  and let  $x \in (1-\delta)\overline{C}$ . Defining  $\alpha > 0$  by  $1+\alpha = 1/(1-\delta)$ , we have  $(1+\alpha)x \in \overline{C}$ . Since  $-C$  is a neighborhood of 0,  $((1+\alpha)x - \alpha C) \cap C \neq \emptyset$ , whence  $x \in C$ .  $\square$

Proposition 6 implies:

**Proposition 7.** *Any probability measure  $\nu$  on  $(\mathcal{X}, \mathcal{B})$  is convex inner regular, i.e. for all open convex subsets  $C$  of  $\mathcal{X}$  and for all  $\alpha > 0$ , there exists a compact convex subset  $K$  of  $C$  such that  $\nu(K) > \nu(C) - \alpha$ .*

*Proof.* Let  $\nu$  be a probability measure on  $\mathcal{X}$ , let  $C$  be an open convex subset of  $\mathcal{X}$ , and let  $\alpha > 0$ . Using proposition 4, there is a compact subset  $K_1$  of  $\mathcal{X}$  such that  $\nu(K_1) > 1 - \alpha/2$ . Using proposition 6, we can choose  $\delta \in (0,1)$  such that  $\nu((1-\delta)\overline{C}) > \nu(C) - \alpha/2$ . Finally,  $K = K_1 \cap (1-\delta)\overline{C}$  is a compact convex subset of  $C$  such that  $\nu(K) > \nu(C) - \alpha$ .  $\square$

To sum up the previous proof, the convex inner regularity of a measure is equivalent to its convex tightness (in a general topological vector space). In a more general context, this argument completes the proof of [4, appendix, proposition 1] and gives a simpler condition than hypothesis  $(\hat{C})$  of [14, Sect. 3.1] or assumption 6.1.2 of [13].

*Proof of  $(\text{UB}_{\text{oc}})$ .* In inequality (5) of the proof of  $(\text{UB}_{\text{cc}})$ , replace  $C \cap K$  by a compact convex subset  $K$  of  $C$  given by proposition 7 to obtain

$$\frac{1}{N} \log \mathbb{P}(\overline{X}_N \in C) \leq \sup_{x \in C} s(x).$$

The last remark of theorem 2 then follows from (LB).  $\square$

To prove theorem 3, we show two intermediate and useful results. Remember that the *support* of the measure  $\mu$  is the subset of  $\mathcal{X}$  defined by

$$\text{supp}(\mu) = \{x \in \mathcal{X} ; \forall \varepsilon > 0, \mu(B(x, \varepsilon)) > 0\}.$$

**Proposition 8.** *For any open ball  $B$  in  $\mathcal{X}$ ,*

$$B \cap \text{supp}(\mu) \neq \emptyset \iff \mu(B) > 0.$$

*Proof.* The direct implication is a mere consequence of the definition of  $\text{supp}(\mu)$ . And the converse one stems from the fact that  $\mathcal{X}$  is second countable, so that we have  $\mu(\text{supp}(\mu)) = 1$  (see [26], theorem 2.1). We provide another proof that relies on the convex inner regularity of  $\mu$ . Consider an open ball  $B$  such that  $B \cap \text{supp}(\mu) = \emptyset$ . Let  $\alpha > 0$ . Use the convex inner regularity of  $\mu$  to find a compact subset  $K$  of  $B$  such that  $\mu(K) > \mu(B) - \alpha$ . For all  $x \in K$ , there exists  $\varepsilon > 0$  such that  $\mu(B(x, \varepsilon)) = 0$ . Extract a finite subcover  $K \subset B_1 \cup B_2 \cup \dots \cup B_r$  with  $B_i = B(x_i, \varepsilon_i)$ . Finally,

$$\mu(B) \leq \mu(K) + \alpha \leq \sum_{i=1}^r \mu(B_i) + \alpha = \alpha$$

and let  $\alpha \rightarrow 0$ .  $\square$

**Proposition 9.** *The entropy  $s$  is upper semi-continuous and concave.*

*Proof.* To show that  $s$  is upper semi-continuous, take  $t \in \mathbb{R}$  and  $x \in \mathcal{X}$  such that  $s(x) < t$ . By the very definition of  $s$ , there is  $\varepsilon > 0$  such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\bar{X}_n \in B(x, \varepsilon)) < t.$$

For all  $y \in B(x, \varepsilon)$ , take  $\delta$  such that  $B(y, \delta) \subset B(x, \varepsilon)$  and write

$$s(y) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\bar{X}_n \in B(y, \delta)) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\bar{X}_n \in B(x, \varepsilon)) < t.$$

So  $s$  is upper semi-continuous. Now we prove that  $s$  is concave. Let  $x, y \in \mathcal{X}$  and set  $z = (x + y)/2$ . Let  $\varepsilon > 0$  and set  $B_z = B(z, \varepsilon)$ ,  $B_x = B(x, \varepsilon/2)$ , and  $B_y = B(y, \varepsilon/2)$ . For all  $n \geq 1$ ,

$$\mathbb{P}(\bar{X}_{2n} \in B_z) \geq \mathbb{P}(\bar{X}_n \in B_x) \mathbb{P}(\bar{X}_n \in B_y)$$

whence

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{2n} \log \mathbb{P}(\bar{X}_{2n} \in B_z) &\geq \lim_{n \rightarrow \infty} \frac{1}{2n} \log (\mathbb{P}(\bar{X}_n \in B_x) \mathbb{P}(\bar{X}_n \in B_y)) \\ &\geq \frac{s(x) + s(y)}{2}. \end{aligned}$$

Taking the infimum in  $\varepsilon$ , we get  $s((x + y)/2) \geq (s(x) + s(y))/2$  and the concavity of  $s$  follows, since  $s$  is upper semi-continuous.  $\square$

*Proof of theorem 3.* Since  $s$  is concave (see proposition 9),  $\text{dom}(s)$  is a convex subset of  $\mathcal{X}$ , so we only need to prove

$$\text{supp}(\mu) \subset \overline{\text{dom}(s)} \tag{6}$$

and

$$\text{dom}(s) \subset \overline{\text{co supp}(\mu)}. \tag{7}$$

Let  $x \notin \overline{\text{dom}(s)}$  and  $\varepsilon > 0$  such that  $B(x, \varepsilon) \cap \text{dom}(s) = \emptyset$ . The bound (UB<sub>oc</sub>) implies  $\log \mu(B(x, \varepsilon)) = -\infty$ . With proposition 8, we get  $B(x, \varepsilon) \cap \text{supp}(\mu) = \emptyset$ , so inclusion (6) is proved. Now, let  $x \in \text{dom}(s)$  and  $\varepsilon > 0$ . Showing that  $B(x, \varepsilon) \cap \text{co supp}(\mu) \neq \emptyset$  is enough to prove inclusion (7). There is  $n \geq 1$  such that  $\mathbb{P}(\bar{X}_n \in B(x, \varepsilon/2)) > 0$ , i.e.  $\mu^{\otimes n}(C) > 0$  where

$$C = \left\{ (u_1, u_2, \dots, u_n) \in \mathcal{X}^n; \frac{u_1 + u_2 + \dots + u_n}{n} \in B(x, \varepsilon/2) \right\}.$$

Let  $Q$  be a countable dense subset of  $\mathcal{X}$ . Since  $C$  is an open subset of  $\mathcal{X}^n$ ,  $Q^n \cap C$  is a dense subset of  $C$ , whence

$$C \subset \bigcup_{(u_1, \dots, u_n) \in Q^n \cap C} \prod_{i=1}^n B(u_i, \varepsilon/2).$$

Since the union is countable and  $\mu^{\otimes n}(C) > 0$ , there is  $(u_1, u_2, \dots, u_n) \in C$  such that, for all integers  $i \in \{1, 2, \dots, n\}$ ,  $\mu(B(u_i, \varepsilon/2)) > 0$ . So, by proposition 8, for all integers  $i \in \{1, 2, \dots, n\}$ , there is  $y_i \in B(u_i, \varepsilon/2) \cap \text{supp}(\mu)$ . Hence,

$$y := \frac{y_1 + y_2 + \dots + y_n}{n} \in B\left(\frac{u_1 + u_2 + \dots + u_n}{n}, \varepsilon/2\right) \subset B(x, \varepsilon)$$

and  $y \in \text{co supp}(\mu)$ .  $\square$

Note that theorem 3 implies theorem 2.4 (a) and (b) of [4] and results 9.7 and 9.8 of [8].

## 5 Proof of theorem 4

The *Fenchel-Legendre transform* of a function  $g : \mathcal{X} \rightarrow [-\infty, +\infty]$  is the function on the dual space  $g^* : \mathcal{X}^* \rightarrow [-\infty, +\infty]$  defined by

$$\forall \lambda \in \mathcal{X}^* \quad g^*(\lambda) = \sup_{x \in \mathcal{X}} (\lambda(x) - g(x)) .$$

Similarly, the Fenchel-Legendre transform of a function  $h : \mathcal{X}^* \rightarrow [-\infty, +\infty]$  is the function  $h^* : \mathcal{X} \rightarrow [-\infty, +\infty]$

$$\forall x \in \mathcal{X} \quad h^*(x) = \sup_{\lambda \in \mathcal{X}^*} (\lambda(x) - h(\lambda)) .$$

We say that the functions  $g : \mathcal{X} \rightarrow [-\infty, +\infty]$  and  $h : \mathcal{X}^* \rightarrow [-\infty, +\infty]$  are *convex conjugate functions* if  $g^* = h$  and  $h^* = g$ .

**Proposition 10** (Fenchel-Moreau theorem). *A function  $g : \mathcal{X} \rightarrow (-\infty, +\infty]$  satisfies  $g^{**} = g$  if and only if  $g$  is lower semi-continuous and convex.*

*Proof.* See, e.g., [25] 5.d. □

*Proof of theorem 4.* Knowing that  $s$  is upper semi-continuous and concave (see proposition 9), and applying proposition 10, we only need to prove  $p = (-s)^*$ . The classic proof of the inequality  $p \geq (-s)^*$ , or its equivalent  $s \leq -p^*$ , relies on Chebyshev's inequality (see, e.g., [4] theorem 3.1). Another proof consists in applying lemma 1 (the proof of which is given below) to the continuous functions  $f = \lambda \in \mathcal{X}^*$ . The other inequality  $p \leq (-s)^*$ , or its equivalent  $s \geq -p^*$ , is usually proved via the open half-space upper bound and Cramér's theorem in  $\mathbb{R}$  (see, e.g., [4] part 3). Let us see how we can get it via lemma 2 (the proof of which is given below). Let  $\lambda \in \mathcal{X}^*$  and let  $\alpha > 0$ . Since  $\mu$  is convex tight (see proposition 4) and using Fatou's lemma, there exists a compact convex subset  $K$  of  $\mathcal{X}$  such that

$$\min (\log \mathbb{E}(e^{\lambda(X_1)}) - \alpha, 1/\alpha) \leq \log \mathbb{E}(e^{\lambda(X_1)} \mathbf{1}_K(X_1)) .$$

Since  $K$  is convex, for all  $n \geq 1$ , the conjunction of  $X_1 \in K$ ,  $X_2 \in K$ ,  $\dots$ , and  $X_n \in K$  implies  $\bar{X}_n \in K$ . Hence, using the independence of the  $X_i$ 's, we get

$$\begin{aligned} \log \mathbb{E}(e^{\lambda(X_1)} \mathbf{1}_K(X_1)) &\leq \inf_{n \geq 1} \frac{1}{n} \log \mathbb{E}(e^{n\lambda(\bar{X}_n)} \mathbf{1}_K(\bar{X}_n)) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}(e^{n(\lambda + \chi_K)(\bar{X}_n)}) \end{aligned}$$

where

$$\chi_K = \log \mathbf{1}_K$$

is the characteristic function of the convex set  $K$ . Finally, we apply lemma 2 to the upper semi-continuous function  $f = \lambda + \chi_K$  for which  $\{f > -\infty\} = K$  is compact and we get

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}(e^{n(\lambda + \chi_K)(\bar{X}_n)}) \leq \sup_{x \in \mathcal{X}} (\lambda(x) + \chi_K(x) + s(x)) \leq (-s)^*(\lambda) .$$

Conclude the proof by letting  $\alpha \rightarrow 0$ . □

*Proof of lemma 1.* Let  $f : \mathcal{X} \rightarrow [-\infty, +\infty]$  be a lower semi-continuous function. Let  $x \in \mathcal{X}$  and let  $\alpha > 0$ . There is  $\varepsilon > 0$  such that, for all  $y \in B(x, \varepsilon)$ ,

$$f(y) \geq \min(f(x) - \alpha, 1/\alpha) .$$

Hence,

$$\begin{aligned}
\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}(e^{nf(\bar{X}_n)}) &\geq \liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}(e^{nf(\bar{X}_n)} \mathbf{1}_{B(x, \varepsilon)}(\bar{X}_n)) \\
&\geq \min(f(x) - \alpha, 1/\alpha) + \liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\bar{X}_n \in B(x, \varepsilon)) \\
&\geq \min(f(x) - \alpha, 1/\alpha) + s(x).
\end{aligned}$$

Taking the limit when  $\alpha \rightarrow 0$  and the supremum over  $x \in \mathcal{X}$ , we get

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}(e^{nf(\bar{X}_n)}) \geq \sup_{x \in \mathcal{X}} (f(x) \dagger s(x))$$

where  $\dagger$  is the natural extension of the addition verifying  $(-\infty) \dagger (+\infty) = -\infty$ . The result reduces to (VLB) when  $\{f = +\infty\} = \emptyset$ .  $\square$

*Proof of lemma 2.* Let  $f : \mathcal{X} \rightarrow [-\infty, +\infty]$  be an upper semi-continuous function such that  $K := \{f > -\infty\}$  is relatively compact. Let  $\alpha > 0$ . For all  $x \in \mathcal{X}$ , there is  $\varepsilon > 0$  such that, for all  $y \in B(x, \varepsilon)$ ,

$$f(y) \leq \max(f(x) + \alpha, -1/\alpha).$$

By the definition of  $s(x)$  and proposition 2, should we reduce  $\varepsilon$ , we may suppose that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\bar{X}_n \in B(x, \varepsilon)) \leq \max(s(x) + \alpha, -1/\alpha).$$

Extract a finite subcover  $\bar{K} \subset B_1 \cup B_2 \cup \dots \cup B_r$  with  $B_i = B(x_i, \varepsilon_i)$ . For all  $n \geq 1$ ,

$$\begin{aligned}
\frac{1}{n} \log \mathbb{E}(e^{nf(\bar{X}_n)}) &\leq \frac{1}{n} \log \sum_{i=1}^r \mathbb{E}(e^{nf(\bar{X}_n)} \mathbf{1}_{B_i}(\bar{X}_n)) \\
&\leq \frac{1}{n} \log \sum_{i=1}^r e^{n \max(f(x_i) + \alpha, -1/\alpha)} \mathbb{P}(\bar{X}_n \in B_i).
\end{aligned}$$

Taking the limit superior when  $n \rightarrow \infty$  and applying the principle of the largest term (proposition 3), we get

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}(e^{nf(\bar{X}_n)}) &\leq \max_{1 \leq i \leq r} (\max(f(x_i) + \alpha, -1/\alpha) + \max(s(x_i) + \alpha, -1/\alpha)) \\
&\leq \sup_{x \in \mathcal{X}} (\max(f(x) + \alpha, -1/\alpha) + \max(s(x) + \alpha, -1/\alpha)).
\end{aligned}$$

Letting  $\alpha \rightarrow 0$ , we get

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}(e^{nf(\bar{X}_n)}) \leq \sup_{x \in \mathcal{X}} (f(x) \dagger s(x))$$

where  $\dagger$  is the natural extension of the addition such that  $(-\infty) \dagger (+\infty) = +\infty$ . The result reduces to (VUB<sub>k</sub>) when  $\{f = +\infty\} = \emptyset$ .  $\square$

## 6 Proof of theorem 5

The proof of theorem 5 is a slight variant of that of theorem 2. We need here a complete version of the subadditive lemma due to Fekete (see [18]). It is very well known when  $u$  is finite valued with a proof similar to that of propositions 2 and 5.

**Proposition 11.** *Let  $u$  be a  $[-\infty, +\infty]$ -valued sequence. Suppose that  $u$  is subadditive, i.e. for all  $m, n \geq 1$ ,  $u(m+n) \leq u(m) \dot{+} u(n)$ , where  $\dot{+}$  is the natural extension of the addition such that  $(-\infty) \dot{+} (+\infty) = +\infty$ . Then*

$$\liminf_{n \rightarrow \infty} \frac{u(n)}{n} = \inf_{n \geq 1} \frac{u(n)}{n}. \quad (8)$$

If  $u$  is also controlled, i.e. there is  $N \geq 1$  such that, for all  $n \geq N$ ,  $u(n) < +\infty$ , then

$$\lim_{n \rightarrow \infty} \frac{u(n)}{n} = \inf_{n \geq 1} \frac{u(n)}{n}. \quad (9)$$

*Proof.* Let  $u$  be a subadditive  $[-\infty, +\infty]$ -valued sequence. For  $m \geq 1$ , we have

$$\liminf_{n \rightarrow \infty} \frac{u(n)}{n} \leq \liminf_{q \rightarrow \infty} \frac{u(qm)}{qm} \leq \frac{u(m)}{m}$$

and equation (8) follows by taking the infimum over  $m \geq 1$ . Now suppose that  $u$  is also controlled. Let  $m \geq N$ . For all  $n \geq m$ , write  $n = qm + r$  the Euclidean division of  $n$  by  $m$  with  $r \in \{m, m+1, \dots, 2m-1\}$  and

$$u(n) \leq qu(m) \dot{+} u(r) \leq \frac{n}{m}u(m) \dot{+} \max_{m \leq i < 2m} u(i).$$

Since, for all  $i \geq m$ ,  $u(i) < +\infty$ , we get

$$\limsup_{n \rightarrow \infty} \frac{u(n)}{n} \leq \frac{u(m)}{m}$$

and equation (9) follows by taking the infimum over  $m \geq 1$ . □

We immediately deduce the useful property:

**Proposition 12.** *Let  $f : \mathcal{X} \rightarrow [-\infty, +\infty]$  be a  $\dot{+}$ -concave function, i.e. for all  $x, y \in \mathcal{X}$  and  $t \in (0, 1)$ ,*

$$f((1-t)x + ty) \geq (1-t)f(x) \dot{+} tf(y),$$

where  $\dot{+}$  is the natural extension of the addition verifying  $(-\infty) \dot{+} (+\infty) = -\infty$ . Then

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}(e^{nf(\bar{X}_n)}) = \sup_{n \geq 1} \frac{1}{n} \log \mathbb{E}(e^{nf(\bar{X}_n)}). \quad (10)$$

If, moreover,  $C = \{f > -\infty\}$  is open, then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}(e^{nf(\bar{X}_n)}) = \sup_{n \geq 1} \frac{1}{n} \log \mathbb{E}(e^{nf(\bar{X}_n)}). \quad (11)$$

*Proof.* Let  $f : \mathcal{X} \rightarrow [-\infty, +\infty]$  be a  $\dagger$ -concave function. For all integers  $m, n \geq 1$ , since  $(m+n)f(\bar{X}_{m+n}) \geq mf(\bar{X}_m) \dagger nf((X_{m+1} + \dots + X_{m+n})/n)$ , we get

$$\mathbb{E}(e^{(m+n)f(\bar{X}_{m+n})}) \geq \mathbb{E}(e^{mf(\bar{X}_m)})\mathbb{E}(e^{nf(\bar{X}_n)}),$$

so  $u(n) := -\log \mathbb{E}(e^{nf(\bar{X}_n)})$  is a subadditive sequence and equation (10) stems from proposition 11. Suppose that  $C = \{f > -\infty\}$  is open. Then, either, for all  $n \geq 1$ ,  $u(n) = +\infty$  and equation (11) is trivial; or there exists  $m \geq 1$  such that  $u(m) < +\infty$ . Then  $\mathbb{P}(\bar{X}_m \in C) > 0$ . Using proposition 2, we find that there exists  $N \geq 1$  such that, for all  $n \geq N$ ,  $\mathbb{P}(\bar{X}_n \in C) > 0$ , whence  $u(n) < +\infty$ . So  $u$  is controlled and equation (11) stems from proposition 11.  $\square$

*Proof of (VUB<sub>cc</sub>).* Let  $f : \mathcal{X} \rightarrow [-\infty, +\infty]$  be an upper semi-continuous  $\dagger$ -concave function (if  $\{f = +\infty\} = \emptyset$ ,  $f$  is simply upper semi-continuous and concave). The first equality stems from proposition 12. Let us prove the inequality. Let  $\alpha > 0$ . Choose  $N \geq 1$  such that

$$\min \left( \sup_{n \geq 1} \frac{1}{n} \log \mathbb{E}(e^{nf(\bar{X}_n)}) - \alpha, 1/\alpha \right) \leq \frac{1}{N} \log \mathbb{E}(e^{Nf(\bar{X}_N)}).$$

Let  $\beta > 0$ . By proposition 4, the distribution of  $\bar{X}_N$  is convex tight. Using Fatou's lemma, there exists a compact convex subset  $K$  of  $\mathcal{X}$  such that

$$\min \left( \frac{1}{N} \log \mathbb{E}(e^{Nf(\bar{X}_N)}) - \beta, 1/\beta \right) \leq \frac{1}{N} \log \mathbb{E}(e^{Nf(\bar{X}_N)} \mathbf{1}_K(\bar{X}_N)). \quad (12)$$

Applying proposition 12 to the  $\dagger$ -concave function  $f \dagger \chi_K$ , we get

$$\frac{1}{N} \log \mathbb{E}(e^{Nf(\bar{X}_N)} \mathbf{1}_K(\bar{X}_N)) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}(e^{n(f \dagger \chi_K)(\bar{X}_n)}).$$

Finally, we apply lemma 2 (more precisely the slight generalization appearing in its proof) to the upper semi-continuous function  $f \dagger \chi_K$  and get

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}(e^{n(f \dagger \chi_K)(\bar{X}_n)}) \leq \sup_{x \in \mathcal{X}} (f(x) \dagger s(x)).$$

Conclude by letting  $\alpha, \beta \rightarrow 0$ .  $\square$

*Proof of (VUB<sub>oc</sub>).* Let  $f : \mathcal{X} \rightarrow [-\infty, +\infty]$  be a  $\dagger$ -concave function such that  $C = \{f > -\infty\}$  is open and  $f|_C$  is upper semi-continuous. The first equality stems from proposition 12. To prove the inequality, suppose that, in inequality (12),  $K$  is a compact convex subset of  $C$  (see proposition 7) and notice that  $f \dagger \chi_K$  is upper semi-continuous.  $\square$

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