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# Sufficient Bigness Criterion for Differences of Two Nef Classes

Dan Popovici

**Abstract.** We prove the qualitative part of Demailly's conjecture on transcendental Morse inequalities for differences of two nef classes satisfying a numerical relative positivity condition on an arbitrary compact Kähler (and even more general) manifold. The result improves on an earlier one by J. Xiao whose constant  $4n$  featuring in the hypothesis is now replaced by the optimal and natural  $n$ . Our method follows arguments by Chiose as subsequently used by Xiao up to the point where we introduce a new way of handling the estimates in a certain Monge-Ampère equation. This result is needed to extend to the Kähler case and to transcendental classes the Boucksom-Demailly-Paun-Peternell cone duality theorem if one is to follow these authors' method and was conjectured by them.

## 1 Introduction

Let  $X$  be a compact complex manifold with  $\dim_{\mathbb{C}} X = n$ . Following various authors (e.g. [Chi13]), Xiao makes in [Xia13] the following assumption:

(H) *there exists a Hermitian metric  $\omega$  on  $X$  such that*

$$\partial\bar{\partial}\omega^k = 0 \text{ for all } k = 1, 2, \dots, n-1.$$

It is clear that (H) holds if  $X$  is a Kähler manifold. It is also standard and easy to check that condition (H) is equivalent to either of the following two equivalent conditions:

$$\partial\bar{\partial}\omega = 0 \text{ and } \partial\bar{\partial}\omega^2 = 0 \iff \partial\bar{\partial}\omega = 0 \text{ and } \partial\omega \wedge \bar{\partial}\omega = 0.$$

Following Xiao's method in [Xia13], itself inspired by earlier authors, especially Chiose [Chi13], we prove the following statement in which a real Bott-Chern class of bidegree  $(1, 1)$  being *nef* means, as usual, that it contains  $C^\infty$  representatives with arbitrarily small negative parts (see inequalities (1)).

**Theorem 1.1** *Let  $X$  be a compact complex manifold with  $\dim_{\mathbb{C}} X = n$  satisfying the assumption (H). Then, for any nef Bott-Chern cohomology classes  $\{\alpha\}, \{\beta\} \in H_{BC}^{1,1}(X, \mathbb{R})$ , the following implication holds:*

$$\{\alpha\}^n - n \{\alpha\}^{n-1} \cdot \{\beta\} > 0 \implies \text{the class } \{\alpha - \beta\} \text{ contains a Kähler current.}$$

This answers affirmatively the *qualitative* part of a special version (i.e. the one for a difference of two nef classes) of Demailly's *transcendental Morse inequalities conjecture* (see [BDPP13, Conjecture 10.1, (ii)]) and will be crucial to the eventual extension of the duality theorem proved in [BDPP13, Theorem 2.2] to transcendental classes in the fairly general context of compact Kähler (not necessarily projective) manifolds. Although the method we propose here also produces a lower bound for the volume of the difference class  $\{\alpha - \beta\}$ , this bound (that we will not present here) is weaker than the lower bound  $\{\alpha\}^n - n\{\alpha\}^{n-1} \cdot \{\beta\}$  predicted in the *quantitative* part of Conjecture 10.1, (ii) in [BDPP13].

Xiao proves in [Xia13] the existence of a Kähler current in the class  $\{\alpha - \beta\}$  under the stronger assumption  $\{\alpha\}^n - 4n\{\alpha\}^{n-1} \cdot \{\beta\} > 0$  and the same assumption (H) on  $X$ . The two ingredients he uses are as follows.

**Lemma 1.2** (*Lamari's duality lemma, [Lam99, Lemme 3.3]*) *Let  $X$  be a compact complex manifold with  $\dim_{\mathbb{C}} X = n$  and let  $\alpha$  be any  $C^\infty$  real  $(1, 1)$ -form on  $X$ . Then the following two statements are equivalent.*

(i) *There exists a distribution  $\psi$  on  $X$  such that  $\alpha + i\partial\bar{\partial}\psi \geq 0$  in the sense of  $(1, 1)$ -currents on  $X$ .*

(ii)  $\int_X \alpha \wedge \gamma^{n-1} \geq 0$  for any Gauduchon metric  $\gamma$  on  $X$ .

As an aside, we notice that this statement, when applied to  $d$ -closed real  $(1, 1)$ -forms  $\alpha$ , translates to the pseudo-effective cone  $\mathcal{E}_X \subset H_{BC}^{1,1}(X, \mathbb{R})$  of  $X$  and the closure of the Gauduchon cone  $\overline{\mathcal{G}}_X \subset H_A^{n-1, n-1}(X, \mathbb{R})$  of  $X$  being dual under the duality between the Bott-Chern cohomology of bidegree  $(1, 1)$  and the Aeppli cohomology of bidegree  $(n-1, n-1)$ . (See [Pop13] for the definition of the Gauduchon cone.)

**Theorem 1.3** (*the Tosatti-Weinkove resolution of Hermitian Monge-Ampère equations, [TW10]*) *Let  $X$  be a compact complex manifold with  $\dim_{\mathbb{C}} X = n$  and let  $\omega$  be a Hermitian metric on  $X$ .*

*Then, for any  $C^\infty$  function  $F : X \rightarrow \mathbb{R}$ , there exist a unique constant  $C > 0$  and a unique  $C^\infty$  function  $\varphi : X \rightarrow \mathbb{R}$  such that*

$$(\omega + i\partial\bar{\partial}\varphi)^n = Ce^F \omega^n, \quad \omega + i\partial\bar{\partial}\varphi > 0 \quad \text{and} \quad \sup_X \varphi = 0.$$

As a matter of fact, Yau's classical theorem that solved the Calabi Conjecture, of which Theorem 1.3 is a generalisation to the possibly non-Kähler context, suffices for the proof of Theorem 1.1 whose assumptions imply that  $X$  must be Kähler (as already pointed out by Xiao in his situation based on [Chi13, Theorem 0.2]) although this is not used either here or in Xiao's work.

## 2 Xiao's approach

In this section, we simply reproduce Xiao's arguments (themselves inspired by earlier authors) up to the point where we will branch off in a different direction in the next section to handle certain estimates.

Let us fix a Hermitian metric  $\omega$  on  $X$  such that  $\partial\bar{\partial}\omega^k = 0$  for all  $k$ . We also fix nef Bott-Chern  $(1, 1)$ -classes  $\{\alpha\}, \{\beta\}$ . By the nef assumption, for every  $\varepsilon > 0$ , there exist  $C^\infty$  functions  $\varphi_\varepsilon, \psi_\varepsilon : X \rightarrow \mathbb{R}$  such that

$$\alpha_\varepsilon := \alpha + \varepsilon\omega + i\partial\bar{\partial}\varphi_\varepsilon > 0 \quad \text{and} \quad \beta_\varepsilon := \beta + \varepsilon\omega + i\partial\bar{\partial}\psi_\varepsilon > 0 \quad \text{on } X. \quad (1)$$

Note that  $\alpha_\varepsilon$  and  $\beta_\varepsilon$  need not be  $d$ -closed, but the property  $\partial\bar{\partial}\omega^k = 0$  yields:

$$\partial\bar{\partial}\alpha_\varepsilon^k = \partial\bar{\partial}\beta_\varepsilon^k = 0 \quad \text{and} \quad \partial\bar{\partial}(\alpha + \varepsilon\omega)^k = \partial\bar{\partial}(\beta + \varepsilon\omega)^k = 0 \quad (2)$$

for all  $k = 1, 2, \dots, n-1$ . We normalise  $\sup_X \varphi_\varepsilon = \sup_X \psi_\varepsilon = 0$  for every  $\varepsilon > 0$ .

Let us fix  $\varepsilon > 0$ . The existence of a Kähler current in the class  $\{\alpha - \beta\} = \{\alpha_\varepsilon - \beta_\varepsilon\}$  is equivalent to

$$\exists \delta > 0, \exists \text{ a distribution } \theta_\delta \text{ on } X \quad \text{such that} \quad \alpha_\varepsilon - \beta_\varepsilon + i\partial\bar{\partial}\theta_\delta \geq \delta \alpha_\varepsilon,$$

which, in view of Lamari's duality lemma 1.2, is equivalent to

$$\exists \delta > 0 \quad \text{such that} \quad \int_X (\alpha_\varepsilon - \beta_\varepsilon) \wedge \gamma^{n-1} \geq \delta \int_X \alpha_\varepsilon \wedge \gamma^{n-1}$$

for every Gauduchon metric  $\gamma$  on  $X$ . This is, of course, equivalent to

$$\exists \delta > 0 \quad \text{such that} \quad (1 - \delta) \int_X \alpha_\varepsilon \wedge \gamma^{n-1} \geq \int_X \beta_\varepsilon \wedge \gamma^{n-1}$$

for every Gauduchon metric  $\gamma$  on  $X$ .

Xiao's approach is to prove the existence of a Kähler current in the class  $\{\alpha - \beta\} = \{\alpha_\varepsilon - \beta_\varepsilon\}$  by contradiction. Suppose that no such current exists. Then, for every  $\varepsilon > 0$  and every sequence of positive reals  $\delta_m \downarrow 0$ , there exist Gauduchon metrics  $\gamma_{m,\varepsilon}$  on  $X$  such that

$$(1 - \delta_m) \int_X \alpha_\varepsilon \wedge \gamma_{m,\varepsilon}^{n-1} < \int_X \beta_\varepsilon \wedge \gamma_{m,\varepsilon}^{n-1} = 1 \quad \text{for all } m \in \mathbb{N}^*, \varepsilon > 0. \quad (3)$$

The last identity is a normalisation of the Gauduchon metrics  $\gamma_{m,\varepsilon}$  which is clearly always possible by rescaling  $\gamma_{m,\varepsilon}$  by a positive factor. This normalisation implies that for every  $\varepsilon > 0$ , the positive definite  $(n-1, n-1)$ -forms

$(\gamma_{m,\varepsilon}^{n-1})_m$  are uniformly bounded in mass, hence after possibly extracting a subsequence we can assume the convergence  $\gamma_{m,\varepsilon}^{n-1} \rightarrow \Gamma_{\infty,\varepsilon}$  in the weak topology of currents as  $m \rightarrow +\infty$ , where  $\Gamma_{\infty,\varepsilon} \geq 0$  is an  $(n-1, n-1)$ -current on  $X$ . Taking limits as  $m \rightarrow +\infty$  in (3), we get

$$\int_X \alpha_\varepsilon \wedge \Gamma_{\infty,\varepsilon} \leq 1 \quad \text{for all } \varepsilon > 0. \quad (4)$$

Note that the l.h.s. of (3) does not change if  $\alpha_\varepsilon$  is replaced with any  $\alpha_\varepsilon + i\partial\bar{\partial}u$  (thanks to  $\gamma_{m,\varepsilon}$  being Gauduchon), while  $\alpha_\varepsilon \wedge \gamma_{m,\varepsilon}^{n-1}$  is (after division by  $\gamma_{m,\varepsilon}^n$ ) the trace of  $\alpha_\varepsilon$  w.r.t.  $\gamma_{m,\varepsilon}$  divided by  $n$  (i.e. the arithmetic mean of the eigenvalues). To find a lower bound for the trace that would contradict (3), it is natural to prescribe the volume form (i.e. the product of the eigenvalues) of some  $\alpha_\varepsilon + i\partial\bar{\partial}u_{m,\varepsilon}$  by imposing that it be, up to a constant factor, the strictly positive  $(n, n)$ -form featuring in the r.h.s. of (3). More precisely, the Tosatti-Weinkove theorem 1.3 allows us to solve the Monge-Ampère equation

$$(\star)_{m,\varepsilon} \quad (\alpha_\varepsilon + i\partial\bar{\partial}u_{m,\varepsilon})^n = c_\varepsilon \beta_\varepsilon \wedge \gamma_{m,\varepsilon}^{n-1}$$

for any  $\varepsilon > 0$  and any  $m \in \mathbb{N}^*$  by ensuring the existence of a unique constant  $c_\varepsilon > 0$  and of a unique  $C^\infty$  function  $u_{m,\varepsilon} : X \rightarrow \mathbb{R}$  satisfying  $(\star)_{m,\varepsilon}$  so that

$$\tilde{\alpha}_{m,\varepsilon} := \alpha_\varepsilon + i\partial\bar{\partial}u_{m,\varepsilon} > 0, \quad \sup_X(\varphi_\varepsilon + u_{m,\varepsilon}) = 0.$$

Note that  $c_\varepsilon$  is independent of  $m$  since we must have

$$c_\varepsilon = \int_X \tilde{\alpha}_{m,\varepsilon}^n = \int_X (\alpha + \varepsilon\omega)^n \downarrow \int_X \alpha^n := c_0 > 0, \quad (5)$$

where the non-increasing convergence is relative to  $\varepsilon \downarrow 0$ . Indeed, the second identity in (5) follows from  $\partial\bar{\partial}(\alpha + \varepsilon\omega)^k = 0$  for all  $k = 1, 2, \dots, n-1$  (cf. (2)). Thus, it is significant that  $c_\varepsilon$  does not change if we add any  $i\partial\bar{\partial}u$  to  $\alpha$ , i.e.  $c_\varepsilon$  depends only on the Bott-Chern class  $\{\alpha\}$ , on  $\omega$  and on  $\varepsilon$ . Analogously, one defines

$$M_\varepsilon := \int_X \tilde{\alpha}_{m,\varepsilon}^{n-1} \wedge \beta_\varepsilon = \int_X (\alpha + \varepsilon\omega)^{n-1} \wedge (\beta + \varepsilon\omega) \downarrow \int_X \alpha^{n-1} \wedge \beta := M_0 \geq 0, \quad (6)$$

where the non-increasing convergence is relative to  $\varepsilon \downarrow 0$ . Clearly,  $M_\varepsilon$  is independent of  $m$  and depends only on the Bott-Chern classes  $\{\alpha\}$ ,  $\{\beta\}$ , on  $\omega$  and on  $\varepsilon$ . Note that the second integral in (6) equals  $\int_X (\alpha + \varepsilon\omega + i\partial\bar{\partial}\varphi_\varepsilon)^{n-1} \wedge (\beta + \varepsilon\omega + i\partial\bar{\partial}\psi_\varepsilon)$  which is positive since  $\alpha_\varepsilon, \beta_\varepsilon > 0$  by (1). Since  $M_0 \geq 0$ , the hypothesis  $c_0 - nM_0 > 0$  made in Theorem 1.1 implies  $c_0 > 0$ . This justifies the final claim in (5).

### 3 Estimates in the Monge-Ampère equation

We now propose an approach to the details of these estimates that differs from that of Xiao. We start with a very simple, elementary (and probably known) observation.

**Lemma 3.1** *For any Hermitian metrics  $\alpha, \beta, \gamma$  on a complex manifold, the following inequality holds at every point:*

$$(\Lambda_\alpha \beta) \cdot (\Lambda_\beta \gamma) \geq \Lambda_\alpha \gamma. \quad (7)$$

*Proof.* Since (7) is a pointwise inequality, we fix an arbitrary point  $x$  and choose local coordinates about  $x$  such that

$$\beta(x) = \sum_j idz_j \wedge d\bar{z}_j, \quad \alpha(x) = \sum_j \alpha_j idz_j \wedge d\bar{z}_j \quad \text{and} \quad \gamma(x) = \sum_{j,k} \gamma_{j\bar{k}} idz_j \wedge d\bar{z}_k.$$

Then  $\alpha_j > 0$  and  $\gamma_{j\bar{j}} > 0$  for every  $j$ . If we denote by the same symbol any  $(1, 1)$ -form and its coefficient matrix in the chosen coordinates, we have

$$\alpha^{-1} \gamma = \left( \frac{1}{\alpha_j} \gamma_{j\bar{k}} \right)_{j,k}, \quad \text{hence} \quad \text{Tr}(\alpha^{-1} \gamma) = \sum_j \frac{1}{\alpha_j} \gamma_{j\bar{j}}.$$

Thus (7) translates to  $\left( \sum_j \frac{1}{\alpha_j} \right) \sum_k \gamma_{k\bar{k}} \geq \sum_j \frac{1}{\alpha_j} \gamma_{j\bar{j}}$  which clearly holds since  $\sum_{j \neq k} \frac{1}{\alpha_j} \gamma_{k\bar{k}} > 0$  because all the  $\alpha_j$  and all the  $\gamma_{k\bar{k}}$  are positive.  $\square$

Our main observation is the following statement.

**Lemma 3.2** *For every  $m \in \mathbb{N}^*$  and every  $\varepsilon > 0$ , we have:*

$$\left( \int_X \tilde{\alpha}_{m,\varepsilon} \wedge \gamma_{m,\varepsilon}^{n-1} \right) \cdot \left( \int_X \tilde{\alpha}_{m,\varepsilon}^{n-1} \wedge \beta_\varepsilon \right) \geq \frac{1}{n} \int_X \tilde{\alpha}_{m,\varepsilon}^n = \frac{c_\varepsilon}{n}. \quad (8)$$

*Proof.* Let  $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ , resp.  $0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_n$ , be the eigenvalues of  $\tilde{\alpha}_{m,\varepsilon}$ , resp.  $\beta_\varepsilon$ , w.r.t.  $\gamma_{m,\varepsilon}$ . We have:

$$\tilde{\alpha}_{m,\varepsilon}^n = \lambda_1 \dots \lambda_n \gamma_{m,\varepsilon}^n \quad \text{and} \quad \tilde{\alpha}_{m,\varepsilon} \wedge \gamma_{m,\varepsilon}^{n-1} = \frac{1}{n} (\Lambda_{\gamma_{m,\varepsilon}} \tilde{\alpha}_{m,\varepsilon}) \gamma_{m,\varepsilon}^n = \frac{\lambda_1 + \dots + \lambda_n}{n} \gamma_{m,\varepsilon}^n.$$

$$\text{Similarly, } \beta_\varepsilon \wedge \gamma_{m,\varepsilon}^{n-1} = \frac{1}{n} (\Lambda_{\gamma_{m,\varepsilon}} \beta_\varepsilon) \gamma_{m,\varepsilon}^n = \frac{\mu_1 + \dots + \mu_n}{n} \gamma_{m,\varepsilon}^n.$$

Thus, the Monge-Ampère equation  $(\star)_{m,\varepsilon}$  translates to

$$\lambda_1 \dots \lambda_n = c_\varepsilon \frac{\mu_1 + \dots + \mu_n}{n}. \quad (9)$$

In particular, the normalisation  $\int_X \beta_\varepsilon \wedge \gamma_{m,\varepsilon}^{n-1} = 1$  reads

$$\frac{1}{c_\varepsilon} \int_X \lambda_1 \dots \lambda_n \gamma_{m,\varepsilon}^n = \int_X \frac{\mu_1 + \dots + \mu_n}{n} \gamma_{m,\varepsilon}^n = 1. \quad (10)$$

Note that we also have

$$\tilde{\alpha}_{m,\varepsilon}^{n-1} \wedge \beta_\varepsilon = \frac{1}{n} (\Lambda_{\tilde{\alpha}_{m,\varepsilon}} \beta_\varepsilon) \tilde{\alpha}_{m,\varepsilon}^n = \frac{1}{n} (\Lambda_{\tilde{\alpha}_{m,\varepsilon}} \beta_\varepsilon) \lambda_1 \dots \lambda_n \gamma_{m,\varepsilon}^n. \quad (11)$$

Putting all of the above together, we get:

$$\begin{aligned} & \left( \int_X \tilde{\alpha}_{m,\varepsilon} \wedge \gamma_{m,\varepsilon}^{n-1} \right) \cdot \left( \int_X \tilde{\alpha}_{m,\varepsilon}^{n-1} \wedge \beta_\varepsilon \right) \\ &= \left( \int_X \frac{1}{n} (\Lambda_{\gamma_{m,\varepsilon}} \tilde{\alpha}_{m,\varepsilon}) \gamma_{m,\varepsilon}^n \right) \cdot \left( \int_X \frac{1}{n} (\Lambda_{\tilde{\alpha}_{m,\varepsilon}} \beta_\varepsilon) \lambda_1 \dots \lambda_n \gamma_{m,\varepsilon}^n \right) \\ &\stackrel{(a)}{\geq} \frac{1}{n^2} \left( \int_X \left[ (\Lambda_{\gamma_{m,\varepsilon}} \tilde{\alpha}_{m,\varepsilon}) (\Lambda_{\tilde{\alpha}_{m,\varepsilon}} \beta_\varepsilon) \right]^{\frac{1}{2}} (\lambda_1 \dots \lambda_n)^{\frac{1}{2}} \gamma_{m,\varepsilon}^n \right)^2 \\ &\stackrel{(b)}{\geq} \frac{1}{n^2} \left( \int_X (\Lambda_{\gamma_{m,\varepsilon}} \beta_\varepsilon)^{\frac{1}{2}} (\lambda_1 \dots \lambda_n)^{\frac{1}{2}} \gamma_{m,\varepsilon}^n \right)^2 \stackrel{(c)}{=} \frac{1}{n^2} \left( \int_X \frac{\sqrt{n}}{\sqrt{c_\varepsilon}} \lambda_1 \dots \lambda_n \gamma_{m,\varepsilon}^n \right)^2 \\ &= \frac{1}{n c_\varepsilon} \left( \int_X \tilde{\alpha}_{m,\varepsilon}^n \right)^2 \stackrel{(d)}{=} \frac{1}{n c_\varepsilon} \left( \int_X c_\varepsilon \beta_\varepsilon \wedge \gamma_{m,\varepsilon}^{n-1} \right)^2 \stackrel{(e)}{=} \frac{c_\varepsilon}{n}. \end{aligned}$$

This proves (8). Inequality (a) is an application of the Cauchy-Schwarz inequality, inequality (b) has followed from (7), identity (c) has followed from (9), identity (d) has followed from  $\tilde{\alpha}_{m,\varepsilon}^n = c_\varepsilon \beta_\varepsilon \wedge \gamma_{m,\varepsilon}^{n-1}$  (which is nothing but the Monge-Ampère equation  $(\star)_{m,\varepsilon}$ ), while identity (e) has followed from the normalisation  $\int_X \beta_\varepsilon \wedge \gamma_{m,\varepsilon}^{n-1} = 1$  (cf. (3)). The proof of Lemma 3.2 is complete.  $\square$

*End of proof of Theorem 1.1.* Now,  $\tilde{\alpha}_{m,\varepsilon} = \alpha_\varepsilon + i\partial\bar{\partial}u_{m,\varepsilon}$  and  $\partial\bar{\partial}\gamma_{m,\varepsilon}^{n-1} = 0$ , so

$$\int_X \tilde{\alpha}_{m,\varepsilon} \wedge \gamma_{m,\varepsilon}^{n-1} = \int_X \alpha_\varepsilon \wedge \gamma_{m,\varepsilon}^{n-1} \longrightarrow \int_X \alpha_\varepsilon \wedge \Gamma_{\infty,\varepsilon} \leq 1 \quad \text{for all } \varepsilon > 0, \quad (12)$$

where the above arrow stands for convergence as  $m \rightarrow +\infty$  and the last inequality is nothing but (4) (which, recall, is a consequence of the assumption that no Kähler current exists in  $\{\alpha - \beta\}$  — an assumption that we are going to contradict). On the other hand, the second factor on the l.h.s. of (8) is precisely  $M_\varepsilon$  defined in (6), so in particular it is independent of  $m$ . Fixing any  $\varepsilon > 0$ , taking limits as  $m \rightarrow +\infty$  in (8) and using (12), we get

$$M_\varepsilon \geq \frac{c_\varepsilon}{n} \quad \text{for every } \varepsilon > 0. \quad (13)$$

Taking now limits as  $\varepsilon \downarrow 0$  and using (6) and (5), we get

$$M_0 \geq \frac{c_0}{n}, \quad \text{i.e.} \quad \{\alpha\}^{n-1} \cdot \{\beta\} \geq \frac{\{\alpha\}^n}{n}.$$

The last identity means that  $\{\alpha\}^n - n \{\alpha\}^{n-1} \cdot \{\beta\} \leq 0$  which is impossible if we suppose that  $\{\alpha\}^n - n \{\alpha\}^{n-1} \cdot \{\beta\} > 0$ . This is the desired contradiction proving the existence of a Kähler current in the class  $\{\alpha - \beta\}$  under the assumption  $\{\alpha\}^n - n \{\alpha\}^{n-1} \cdot \{\beta\} > 0$ .  $\square$

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Institut de Mathématiques de Toulouse, Université Paul Sabatier,  
118 route de Narbonne, 31062 Toulouse, France  
Email : popovici@math.univ-toulouse.fr