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► **To cite this version:**

Indranil Biswas, Sorin Dumitrescu, Henri Guenancia. A Bochner principle and its applications to Fujiki class C manifolds with vanishing first Chern class. *Communications in Contemporary Mathematics*, In press, 22 (6), 10.1142/S0219199719500512 . hal-01975962

HAL Id: hal-01975962

<https://hal.science/hal-01975962>

Submitted on 9 Jan 2019

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A BOCHNER PRINCIPLE AND ITS APPLICATION TO FUJIKI CLASS \mathcal{C} MANIFOLDS WITH VANISHING FIRST CHERN CLASS

INDRANIL BISWAS, SORIN DUMITRESCU, AND HENRI GUENANCIA

ABSTRACT. We prove a Bochner type vanishing theorem for compact complex manifolds Y in Fujiki class \mathcal{C} , with vanishing first Chern class, that admit a cohomology class $[\alpha] \in H^{1,1}(Y, \mathbb{R})$ which is numerically effective (nef) and has positive self-intersection (meaning $\int_Y \alpha^n > 0$, where $n = \dim_{\mathbb{C}} Y$). Using it, we prove that all holomorphic geometric structures of affine type on such a manifold Y are locally homogeneous on a non-empty Zariski open subset. Consequently, if the geometric structure is rigid in the sense of Gromov, then the fundamental group of Y must be infinite. In the particular case where the geometric structure is a holomorphic Riemannian metric, we show that the manifold Y admits a finite unramified cover by a complex torus with the property that the pulled back holomorphic Riemannian metric on the torus is translation invariant.

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2010 *Mathematics Subject Classification.* 53B35, 53C55, 53A55.

Key words and phrases. Holomorphic geometric structure, Fujiki class \mathcal{C} manifold, holomorphic Riemannian metric, Bochner principle, Calabi-Yau manifold.

1. INTRODUCTION

Yau's celebrated theorem [Ya], proving Calabi's conjecture, endows any compact Kähler manifold X with vanishing real first Chern class (meaning $c_1(TX) = 0$ in $H^2(X, \mathbb{R})$) with a Ricci flat Kähler metric. Such metrics constitute an extremely useful tool for studying the geometry of these manifolds, known as Calabi-Yau manifolds. For example, by the well-known Bochner principle, any holomorphic tensor on X must be parallel with respect to any Ricci flat Kähler metric [Be]. The study of the holonomy of such a Ricci flat Kähler metric furnishes, in particular, an elegant proof of the Beauville–Bogomolov decomposition theorem that asserts that, up to a finite unramified cover, a Calabi-Yau manifold X is biholomorphic to the product of a complex torus with a compact complex simply connected manifold with trivial first Chern class [Be, Bog].

In the special case where the second real Chern class of the Calabi-Yau manifold X also vanishes, any Ricci flat Kähler metric on X has vanishing sectional curvature. In that case, as a consequence of Bieberbach's theorem, X actually admits a finite unramified cover which is a complex torus [Be]. Notice that a compact Kähler manifold bearing a holomorphic affine connection on its holomorphic tangent bundle has vanishing real Chern classes [At, p. 192–193, Theorem 4], and hence it admits an unramified cover by a compact complex torus [IKO].

Using the Bochner principle, it was proved in [Du1] that holomorphic geometric structures of affine type (their definition is recalled in the paragraph following Definition 3.1) on any compact Kähler manifold X with vanishing first Chern class are in fact locally homogeneous. Consequently, if the geometric structure satisfies the condition that it is rigid in the sense of Gromov, [DG, Gr], then X admits a finite unramified cover which is a complex torus.

The aim in this paper is to generalize the above mentioned results to the broader context of Fujiki class \mathcal{C} manifolds [Fu2, Fu1]. Recall that a compact complex manifold Y is in Fujiki class \mathcal{C} if it is the image of a holomorphic map from a compact Kähler manifold. By an important result of Varouchas [Va], this is equivalent to the assertion that Y admits a surjective holomorphic map from a compact Kähler manifold such that the map is a bimeromorphism.

Let Y be a compact connected complex manifold, of complex dimension n , with trivial first (real) Chern class. We assume that Y

- lies in Fujiki class \mathcal{C} , and
- admits a numerically effective (nef) cohomology class $[\alpha] \in H^{1,1}(Y, \mathbb{R})$ that has positive self-intersection, meaning $\int_Y \alpha^n > 0$ with α being a real closed smooth $(1, 1)$ -form representing the cohomology class $[\alpha]$.

By a result of Demailly and Păun in [DP], the combination of the two conditions above is equivalent to the condition that Y admits a cohomology class $[\alpha] \in H_{\partial\bar{\partial}}^{1,1}(Y, \mathbb{R})$ which is both nef and big (see Section 2.1, e.g. Remark 2.5 and Corollary 2.6).

Under the above assumptions we prove the following Bochner type theorem for holomorphic tensors on Y (see Corollary 2.17).

Theorem A. *Let Y be a complex compact manifold with trivial first Chern class. Assume that Y admits a nef and big cohomology class $[\alpha] \in H_{\partial\bar{\partial}}^{1,1}(Y, \mathbb{R})$.*

Then, there exists a closed, positive $(1, 1)$ -current $\omega \in [\alpha]$ which induces a genuine Ricci-flat Kähler metric on a non-empty Zariski open subset $\Omega \subset Y$.

Furthermore, given any global holomorphic tensor $\tau \in H^0(Y, T_Y^{\otimes p} \otimes T_Y^{\otimes q})$ with $p, q \geq 0$, the restriction $\tau|_{\Omega}$ is parallel with respect to $\omega|_{\Omega}$.*

The existence of the singular Ricci-flat metric ω is essentially due to [BEGZ]; see Section 2.3. The statement about parallelism is new, although the techniques involved in its proof are mostly borrowed from [Gue] (see also [CP1]). Using this Bochner principle, we deduce the following result (see Theorem 3.4).

Theorem B. *Under the assumptions of Theorem A, the following holds:*

- (1) *There exists Zariski open subset $\emptyset \neq \Omega \subset Y$ such that any holomorphic geometric structure of affine type on Y is locally homogeneous on Ω .*
- (2) *If Y admits a rigid holomorphic geometric structure of affine type, then the fundamental group of Y is infinite.*

For the particular case of a holomorphic Riemannian metric, we deduce using the above mentioned Bochner principle the following result (see Theorem 3.5).

Theorem C. *Under the assumptions of Theorem A, assume additionally that Y admits a holomorphic Riemannian metric g .*

*Then, there is a finite unramified cover $\gamma : \mathbb{T} \rightarrow Y$, where \mathbb{T} is a complex torus, such that the pulled back holomorphic Riemannian metric γ^*g on the torus \mathbb{T} is translation invariant.*

The Zariski open subset Ω involved in Theorem B can be chosen to contain the smooth locus of the singular Ricci-flat metric ω from Theorem A. We would actually conjecture that we could choose $\Omega = Y$. This was proved to be true in [BD] for Moishezon manifolds (meaning manifolds that are bimeromorphic to some complex projective manifold [Mo]) using deep positivity properties proved recently in [CP2] for tensor powers of the cotangent bundle of *projective* manifolds with pseudoeffective canonical class (see also the expository article [Cl]). However it is still not known whether this holds in the set-up of Kähler manifolds. It should also be mentioned that it was proved in [BD] that a compact simply connected manifold in Fujiki class \mathcal{C} does not admit any holomorphic affine connection on its holomorphic tangent bundle. A compact complex manifold in Fujiki class \mathcal{C} bearing a holomorphic affine connection has in fact trivial real Chern classes (its proof is identical to the proof in the Kähler case [At]). We would conjecture that a compact complex manifold in Fujiki class \mathcal{C} with trivial Chern classes (in the real cohomology) admits a finite unramified cover which is a complex torus.

2. FUJIKI CLASS \mathcal{C} MANIFOLDS AND BOCHNER PRINCIPLE

2.1. Positivity property of $(1, 1)$ -classes. Let X be a compact complex manifold of dimension n . We set $d^c := \frac{\sqrt{-1}}{2\pi}(\bar{\partial} - \partial)$ the real operator so that $dd^c = \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}$. We shall denote by $H_{\partial\bar{\partial}}^{1,1}(X)$ the $\partial\bar{\partial}$ -cohomology, in other words,

$$H_{\partial\bar{\partial}}^{1,1}(X) = \{\alpha \in \mathcal{C}^\infty(X, \Omega_{X,\mathbb{C}}^{1,1}) \mid d\alpha = 0\} / \{\partial\bar{\partial}u \mid u \in \mathcal{C}^\infty(X, \mathbb{C})\}.$$

Moreover, set

$$H_{\partial\bar{\partial}}^{1,1}(X, \mathbb{R}) := H_{\partial\bar{\partial}}^{1,1}(X) \cap H^2(X, \mathbb{R}).$$

Definition 2.1. Let X be a compact complex manifold of dimension n , and let $[\alpha] \in H_{\partial\bar{\partial}}^{1,1}(X, \mathbb{R})$ be a cohomology class represented by a smooth, closed $(1, 1)$ -form α . We recall the standard terminology:

- (1) $[\alpha]$ is *nef* if for any $\varepsilon > 0$, there exists a smooth representative $\omega_\varepsilon \in [\alpha]$ such that $\omega_\varepsilon \geq -\varepsilon\omega_X$, where ω_X is some fixed (independent of ε) hermitian metric on X .
- (2) $[\alpha]$ has *positive self-intersection* if $\int_X \alpha^n > 0$.
- (3) $[\alpha]$ is *big* if there exists a Kähler current $T \in [\alpha]$, i.e., if there exists a closed $(1, 1)$ -current $T = \alpha + dd^c u \in [\alpha]$ such that $T \geq \omega_X$ in the sense of currents, where $u \in L^1(X)$ and ω_X is some hermitian metric on X .

If X is a compact Kähler manifold or more generally a Fujiki manifold (see Definition 2.4), then the $\partial\bar{\partial}$ -cohomology coincides with the usual $\bar{\partial}$ -cohomology (also called Dolbeault cohomology). Moreover, Demailly and Păun proved the following fundamental theorem which is the key result towards their numerical characterization of the Kähler cone:

Theorem 2.2 ([DP, p. 1259, Theorem 2.12]). *Let X be a compact Kähler manifold, and let $[\alpha] \in H_{\partial\bar{\partial}}^{1,1}(X, \mathbb{R})$ be a nef class with positive self-intersection. Then, $[\alpha]$ is a big class.*

Its converse is true as well, meaning all nef and big classes on a compact Kähler manifold have positive self-intersection [Bou1, p. 1054, Lemma 4.2]. Let us now recall a few basic facts about the behavior of these notions under bimeromorphisms.

Lemma 2.3. *Let $f : X \rightarrow Y$ be a surjective, bimeromorphic morphism between two compact complex manifolds. Let $[\alpha] \in H_{\partial\bar{\partial}}^{1,1}(Y, \mathbb{R})$ and $[\beta] \in H_{\partial\bar{\partial}}^{1,1}(X, \mathbb{R})$ be cohomology classes.*

- (1) *If $[\beta]$ is big, then so is $f_*[\beta]$.*
- (2) *If $[\alpha]$ is nef and has positive self-intersection, then $f^*[\alpha]$ is also nef and has positive self-intersection.*
- (3) *If $[\alpha]$ is big and X is Kähler, then $f^*[\alpha]$ is big.*

Proof. Let ω_Y be a hermitian metric on Y , and let ω_X be a hermitian metric on X such that $\omega_X \geq f^*\omega_Y$.

Proof of 1: If $[\beta]$ is big, then there exists a closed positive current $T \in [\beta]$ and also a real number $\varepsilon_0 > 0$, such that $T \geq \varepsilon_0 \omega_X \geq \varepsilon_0 f^* \omega_Y$. Then, $f_* T \in f_* [\beta]$ satisfies the condition $f_* T \geq \varepsilon_0 \omega_Y$, and consequently $f_* [\beta]$ is big.

Proof of 2: If $[\alpha]$ is nef, then for every $\varepsilon > 0$, there exists a smooth form $\omega_\varepsilon \in [\alpha]$ such that $\omega_\varepsilon \geq -\varepsilon \omega_Y$. Therefore, the pullback $f^* \omega_\varepsilon \in f^* [\alpha]$ satisfies the condition

$$f^* \omega_\varepsilon \geq -\varepsilon f^* \omega_Y \geq -\varepsilon \omega_X.$$

From this it follows that $f^* [\alpha]$ is nef. Moreover, we have

$$\int_X (f^* \alpha)^n = \int_Y \alpha^n > 0.$$

Proof of 3: Take $[\alpha]$ to be big. Let $T \in [\alpha]$ be a Kähler current. In view of Demailly's regularization theorem, we may assume that T has analytic singularities (see Section 2.2). Since T can locally be written as $\alpha + dd^c \varphi$ with φ quasi-plurisubharmonic, we may set the pullback of T to be $f^* T = f^* \alpha + dd^c(\varphi \circ f)$. Then, $f^* T$ is a positive current lying in the class $f^* [\alpha]$, and $f^* T$ is a Kähler metric on a Zariski open set of X . Now from [Bou1, p. 1057, Theorem 4.7] it follows that $f^* [\alpha]$ is big. \square

Definition 2.4. A compact complex manifold Y of dimension n is said to be in Fujiki class \mathcal{C} if there exist a compact Kähler manifold X and a surjective meromorphic map

$$f : X \longrightarrow Y$$

([Fu2, Fu1], [Va, p. 50, Definition 3.1]). By [Va, p. 51, Theorem 5], for any Y in Fujiki class \mathcal{C} , the above pair (X, f) can be so chosen that the map f from the compact Kähler manifold X is a bimeromorphism. In other words, a Fujiki class \mathcal{C} manifold admits a Kähler modification.

Remark 2.5. A manifold Y in Fujiki class \mathcal{C} admits a Kähler current (big class). Indeed, if $f : X \longrightarrow Y$ is a Kähler modification, and ω is a Kähler form on X , then $f_* \omega$ is a Kähler current on Y (see proof of Lemma 2.3 (1)). In the opposite direction, it was proved in [DP, p. 1250, Theorem 0.7] that any complex compact manifold Y admitting a Kähler current is necessarily in Fujiki class \mathcal{C} .

Corollary 2.6. *Let Y be a compact complex manifold in Fujiki class \mathcal{C} . Let $[\alpha] \in H_{\partial\bar{\partial}}^{1,1}(Y, \mathbb{R})$ be a nef class. Then, $[\alpha]$ is big if and only if $[\alpha]$ has positive self-intersection.*

Proof. Let $f : X \longrightarrow Y$ be a Kähler modification.

Assume that $[\alpha]$ has positive self-intersection. Then the pullback $f^* [\alpha]$ is nef and has positive self-intersection by Lemma 2.3 (2). Now Theorem 2.2 says that $f^* [\alpha]$ is big. Therefore, Lemma 2.3 (1) implies that $[\alpha]$ is big.

Conversely, assume that $[\alpha]$ is big. By Lemma 2.3 (3), the pullback $f^* [\alpha]$ is big as well and it follows from [Bou1, p. 1054, Lemma 4.2] that $f^* [\alpha]$ has positive self-intersection. As $\int_Y \alpha^n = \int_X (f^* \alpha)^n > 0$, the result is proved. \square

Remark 2.7. Demailly and Păun conjectured the following ([DP, p. 1250, Conjecture 0.8]): If a complex compact manifold Z possesses a nef cohomology class $[\alpha]$ which has positive self-intersection, then Z lies in the Fujiki class \mathcal{C} . This conjecture would imply that a nef class $[\alpha] \in H_{\partial\bar{\partial}}^{1,1}(Z, \mathbb{R})$ on a compact complex manifold Z is big if and only if $[\alpha]$ has positive self-intersection (see Corollary 2.6).

2.2. Non-Kähler locus of a cohomology class.

Definition 2.8. Let X be a complex manifold and $U \subset X$ an open subset.

- (1) A plurisubharmonic function (psh for short) φ on U is said to have analytic singularities if there exist holomorphic functions $f_1, \dots, f_r \in \mathcal{O}_X(U)$, a smooth function $\psi \in \mathcal{C}^\infty(U)$, and $a \in \mathbb{R}_+$, such that

$$\varphi = a \log(|f_1|^2 + \dots + |f_r|^2) + \psi.$$

- (2) Let T be a closed, positive $(1,1)$ -current on X . Then T is said to have analytic singularities if it can be expressed, locally, as $T = dd^c\varphi$, where φ is a psh function with analytic singularities. The singular set for T is denoted by $E_+(T)$; it is a proper Zariski closed subset of X .

It follows from the fundamental regularization theorems of Demailly, [De], that any big class $[\alpha]$ in a compact complex manifold X contains a Kähler current T with analytic singularities. Note that such a current T is smooth on a non-empty Zariski open subset of X , and T induce a Kähler metric on this Zariski open subset. Following [Bou2], we define:

Definition 2.9. Let X be a compact complex manifold, and let $[\alpha] \in H_{\partial\bar{\partial}}^{1,1}(X, \mathbb{R})$ be a big cohomology class. The *non-Kähler locus* of $[\alpha]$ is

$$E_{nK}([\alpha]) := \bigcap_{T \in [\alpha]} E_+(T),$$

where the intersection is taken over all positive currents with analytic singularities. The *ample locus* $\text{Amp}([\alpha])$ of $[\alpha]$ is the complement of the non-Kähler locus, meaning

$$\text{Amp}([\alpha]) := X \setminus E_{nK}([\alpha]).$$

Boucksom proved the following: Given any big class $[\alpha] \in H_{\partial\bar{\partial}}^{1,1}(X, \mathbb{R})$, there exists a positive current $T \in [\alpha]$ with analytic singularities such that $E_+(T) = E_{nK}([\alpha])$; in particular, $E_{nK}([\alpha])$ is a proper Zariski closed subset of X (see [Bou2, p. 59, Theorem 3.17]).

The following Proposition builds upon a result of Collins and Tosatti [CT, p. 1168, Theorem 1.1] showing that the non-Kähler locus of a nef and big class $[\alpha]$ on a Fujiki manifold X coincides with its null locus $\text{Null}([\alpha])$ defined as the reunion of all irreducible subvarieties $V \subseteq X$ such that $\int_V \alpha^{\dim V} = 0$.

Proposition 2.10. *Let $f : X \rightarrow Y$ be a bimeromorphic morphism between two compact complex manifolds belonging to the Fujiki class \mathcal{C} . Let $[\alpha] \in H_{\partial\bar{\partial}}^{1,1}(X, \mathbb{R})$ be a nef and big cohomology class. Then*

$$E_{nK}(f^*[\alpha]) = f^{-1}(E_{nK}([\alpha])) \cup \text{Exc}(f),$$

where $\text{Exc}(f)$ is the exceptional locus of f , i.e., the singular locus of the Jacobian of f .

Proof. By [CT, p. 1168, Theorem 1.1], it is enough to prove the analogous result for null loci. Let $n := \dim_{\mathbb{C}} X$.

Let $E \subset \text{Exc}(f)$ be an irreducible component; it has dimension $n - 1$, while $\dim f(E) \leq n - 2$. Therefore, $\int_E (f^*\alpha)^{n-1} = \int_{f(E)} \alpha^{n-1} = 0$ and $E \subset E_{nK}(f^*[\alpha])$. Next, if $V \subset E_{nK}([\alpha])$ is a k -dimensional subvariety not included in $f(\text{Exc}(f))$, let \tilde{V} be its strict transform. We have $f^{-1}(V) = \tilde{V} \cup F$ with $F \subset \text{Exc}(f)$ and f inducing a bimeromorphic morphism $f|_{\tilde{V}} : \tilde{V} \rightarrow V$. From the identity $\int_{\tilde{V}} (f^*\alpha)^k = \int_V \alpha^k$ it follows that

$$\tilde{V} \subset E_{nK}(f^*[\alpha]).$$

In summary, $f^{-1}(E_{nK}([\alpha])) \cup \text{Exc}(f) \subset E_{nK}(f^*[\alpha])$.

Now, let $W \subset E_{nK}(f^*[\alpha])$ be an irreducible k -dimensional subvariety not included in $\text{Exc}(f)$. The morphism f induces a bimeromorphic morphism

$$f|_W : W \rightarrow f(W),$$

and $\int_{f(W)} \alpha^k = \int_W (f^*\alpha)^k = 0$. Therefore, $f(W) \subset E_{nK}([\alpha])$, which completes the proof of the Proposition. \square

2.3. Singular Ricci-flat metrics. Let Y be a compact complex manifold of dimension n such that the first Chern class $c_1(T_Y)$ vanishes in $H^2(Y, \mathbb{R})$. We assume that Y admits a big cohomology class $[\alpha] \in H_{\partial\bar{\partial}}^{1,1}(Y, \mathbb{R})$. In particular, Y lies in Fujiki class \mathcal{C} (see Remark 2.5). We fix once and for all a Kähler modification

$$f : X \rightarrow Y.$$

It follows from Lemma 2.3 (3) that the class $f^*[\alpha]$ is big. The Jacobian of f induces the following identity

$$K_X = f^*K_Y + E, \tag{2.1}$$

where

$$E = \sum_{i=1}^r a_i E_i \tag{2.2}$$

is an effective \mathbb{Z} -divisor on X contracted by f , meaning $\text{codim}_Y f(E) \geq 2$, and each E_i is irreducible. More precisely, $\text{Supp}(E)$ coincides with the exceptional locus of f , that is the complement of the locus of points on X in a neighborhood of which f induces a local biholomorphism.

Since $c_1(K_Y) = 0 \in H^2(Y, \mathbb{R})$, it follows from (2.1) that K_X is numerically equivalent to the effective divisor E . Moreover, the class $f^*[\alpha] \in H^{1,1}(X, \mathbb{R})$ is big by Lemma 2.3 (3). From [BEGZ, p. 200, Theorem A] we know that there exists a unique closed positive current

$$T \in f^*[\alpha] \quad (2.3)$$

with finite energy such that

$$-\text{Ric}(T) = [E].$$

In terms of Monge–Ampère equations, this means that the non-pluripolar Monge–Ampère measure of T , denoted by $\langle T^n \rangle$, can be expressed as

$$\langle T^n \rangle = |s_E|_h^2 dV,$$

where

- h is a smooth hermitian metric on $\mathcal{O}_X(E)$ with Chern curvature tensor denoted by $\Theta_h(E)$,
- $s_E \in H^0(X, \mathcal{O}_X(E))$ satisfies the condition $\text{div}(s_E) = E$, and
- dV is the smooth volume form on X satisfying $\text{Ric}(dV) = -\Theta_h(E)$ and normalized such that $\int_X |s_E|_h^2 dV = \text{vol}(f^*\alpha) = \text{vol}(\alpha)$.

Proposition-Definition 2.11. *Let Y be a compact complex manifold with trivial first Chern class endowed with a big cohomology class $[\alpha] \in H_{\partial\bar{\partial}}^{1,1}(Y, \mathbb{R})$. Let $f : X \rightarrow Y$ be a Kähler modification, and let $T \in f^*[\alpha]$ be the associated singular Ricci-flat metric. Then*

- (1) *There exists a closed, positive (1, 1)-current $\omega \in [\alpha]$ on Y such that $T = f^*\omega$.*
- (2) *The current ω is independent of the choice of the Kähler modification of Y .*

We call ω the singular Ricci-flat metric in $[\alpha]$.

Proof. For the first item, let us write $T = f^*\alpha + dd^c u$ for some $(f^*\alpha)$ -psh function u on X . The restriction of u to any fiber of f is psh, but the fibers of f are connected. By the maximum principle, u is constant on the fibers of f , hence can be written as $u = \pi^*v$ for some α -psh function v on Y . Then, $T = f^*\omega$ with $\omega = \alpha + dd^c v$.

Now, let $f' : X' \rightarrow Y$ be another Kähler modification, and let $T' := (f')^*\omega'$ be the associated singular Ricci-flat current. Let Z be a desingularization of $X' \times_Y X$, so that we have the following Cartesian square where all maps are bimeromorphic

$$\begin{array}{ccc} Z & \xrightarrow{g'} & X \\ g \downarrow & & \downarrow f \\ X' & \xrightarrow{f'} & Y, \end{array}$$

Let $h := f \circ g' = f' \circ g$. By [BEGZ, p. 201, Theorem B] and [BEGZ, Proposition 1.12], both currents g^*T' and $(g')^*T$ have minimal singularities in $h^*[\alpha]$ and are solutions of the non-pluripolar Monge–Ampère equation $-\text{Ric}(\cdot) = [K_{Z/Y}]$. By uniqueness of the solution of that equation [BEGZ, Theorem A], it follows that

$h^*\omega = h^*\omega'$. Taking the direct image of the previous equality, it is deduced that $\omega = \omega'$. \square

The regularity properties of the current T (or ω) are quite mysterious in general. For instance, it is not known whether T is smooth on a Zariski open set. However, things become simpler once we assume additionally that the cohomology class $[\alpha]$ is nef.

Proposition-Definition 2.12. *In the set-up of Proposition-Definition 2.11, assume additionally that the cohomology class $[\alpha]$ is nef. Then, the singular Ricci-flat current $\omega \in [\alpha]$ is smooth on a non-empty Zariski open subset of Y .*

We define Ω to be the largest Zariski open subset in restriction of which ω is a genuine Kähler form. Then

$$\emptyset \neq f(\text{Amp}(f^*[\alpha])) \subseteq \Omega \subseteq \text{Amp}([\alpha])$$

for any Kähler modification $f : X \rightarrow Y$ (see Definition 2.9).

Proof. Let $f : X \rightarrow Y$ be a Kähler modification. As $[\alpha]$ is nef, $f^*[\alpha]$ is nef as well (see Lemma 2.3 (2)) and the proof of [BEGZ, p. 201, Theorem C] applies verbatim to give that T is smooth outside $E_{nK}(f^*[\alpha]) \setminus \text{Supp}(E)$. Note that Proposition 2.10 shows that this locus is just $E_{nK}(f^*[\alpha])$. In particular, this shows that the current $\omega \in [\alpha]$ on Y is a genuine Ricci-flat Kähler metric on the non-empty Zariski open set

$$f(\text{Amp}(f^*[\alpha])) \subseteq \text{Amp}([\alpha]),$$

which concludes the proof. \square

We will need a refinement of the result above which will be explained next.

Let $t, \varepsilon > 0$; by Yau's theorem (Calabi's conjecture) [Ya], there exists a unique Kähler metric $\omega_{t,\varepsilon} \in f^*[\alpha] + t[\omega_X]$ solving the equation

$$\text{Ric}(\omega_{t,\varepsilon}) = t\omega_X - \theta_\varepsilon, \tag{2.4}$$

where $\theta_\varepsilon \in c_1(E)$ is a regularization of the current defined by integration along E . For instance, such a smooth closed $(1, 1)$ -form θ_ε can be constructed as follows: pick smooth hermitian metrics h_i on $\mathcal{O}_X(E_i)$ (see (2.2)), and choose a holomorphic section $s_i \in H^0(X, \mathcal{O}_X(E_i))$ for each i cutting out E_i in the sense that $\text{div}(s_i) = E_i$; then set

$$\theta_\varepsilon = \sum_{i=1}^r a_i \left(\Theta_{h_i}(E_i) + dd^c \log(|s_i|_{h_i}^2 + \varepsilon^2) \right),$$

where $\Theta_{h_i}(E_i)$ is the Chern curvature of the hermitian line bundle $(\mathcal{O}_X(E_i), h_i)$.

Now, the proof of [BEGZ, p. 201, Theorem C] can be adapted without any significant changes to obtain the following result.

Theorem 2.13 ([BEGZ, Theorem C]). *In the set-up of Proposition-Definition 2.11, assume furthermore that $[\alpha]$ is nef, and let $\omega_{t,\varepsilon} = f^*[\alpha] + t[\omega_X]$ be the Kähler form*

solving Eq. (2.4). When $t \rightarrow 0$ and $\varepsilon \rightarrow 0$, the form $\omega_{t,\varepsilon}$ converges to the current T in (2.3) in the weak topology of currents, and also in the $\mathcal{C}_{\text{loc}}^\infty(\text{Amp}(f^*[\alpha]))$ topology.

2.4. Flatness of tensors. It will be convenient in this section to refer to the following set-up.

Setting 2.14. Let Y be a compact complex manifold with trivial first Chern class endowed with a nef and big cohomology class $[\alpha] \in H_{\partial\bar{\partial}}^{1,1}(Y, \mathbb{R})$. We denote by ω the singular Ricci-flat metric from Proposition-Definition 2.11, and we let Ω be its smooth locus; see Proposition-Definition 2.12.

Theorem 2.15. *In Setting 2.14, let $\tau \in H^0(Y, T_Y^{*\otimes p})$ with $p \geq 0$.*

Then, the restriction $\tau|_\Omega$ to Ω in (2.12) is parallel with respect to the Ricci-flat Kähler metric $\omega|_\Omega$.

Remark 2.16. Since $c_1(T_Y) = 0$ in $H^2(Y, \mathbb{R})$, and Y is in Fujiki class \mathcal{C} , from [To, Theorem 1.5] we know that the holomorphic line bundle K_Y is of finite order. Therefore, there exists a finite unramified cover $\pi : \widehat{Y} \rightarrow Y$ such that $K_{\widehat{Y}}$ is holomorphically trivial.

The previous remark leads to the following application of Theorem 2.15, valid for any holomorphic tensors.

Corollary 2.17. *In Setting 2.14, let $\tau \in H^0(Y, T_Y^{\otimes p} \otimes T_Y^{*\otimes q})$ with $p, q \geq 0$. Then, the restriction $\tau|_\Omega$ to Ω in (2.12) is parallel with respect to the Ricci-flat Kähler metric $\omega|_\Omega$.*

In particular, the evaluation map

$$\begin{array}{ccc} \text{ev}_y : H^0(Y, T_Y^{\otimes p} \otimes T_Y^{*\otimes q}) & \longrightarrow & (T_Y^{\otimes p} \otimes T_Y^{*\otimes q})_y \\ & \tau & \longmapsto \tau(y) \end{array}$$

is injective for all $y \in \Omega$ and all integers $p, q \geq 0$.

Proof of Corollary 2.17. Let τ be any holomorphic tensor on Y , not necessarily contravariant. Let $\pi : \widehat{Y} \rightarrow Y$ be the finite unramified cover from Remark 2.16. As π is a local biholomorphism, the pull-back $\pi^*\tau$ on \widehat{Y} is well-defined. We can interpret $\pi^*\tau$ as a holomorphic contravariant tensor on \widehat{Y} . Indeed, a holomorphic trivialization of $K_{\widehat{Y}}$ produces a holomorphic isomorphism $T_{\widehat{Y}} \simeq \bigwedge^{n-1} T_{\widehat{Y}}^*$, where $n = \dim_{\mathbb{C}} Y$.

Now, let $f : X \rightarrow Y$ be a Kähler modification, and let us set $T := f^*\omega$ and $\Omega' := f(\text{Amp}(f^*[\alpha])) \subset \Omega$. As Ω' is dense in Ω for the usual topology, and both τ and ω are smooth on Ω , it suffices to prove that $\tau|_{\Omega'}$ is parallel with respect to $\omega|_{\Omega'}$. Since f is an isomorphism over Ω' , we may pull back $\tau|_{\Omega'}$ by f over this locus. This way we reduce the question to proving that $f^*\tau|_{\Omega'}$ is parallel with respect to $T|_{f^{-1}(\Omega')}$.

Let $\widehat{X} := X \times_Y \widehat{Y}$, so that we have a Cartesian square

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\widehat{f}} & \widehat{Y} \\ \widehat{\pi} \downarrow & & \downarrow \pi \\ X & \xrightarrow{f} & Y, \end{array}$$

The morphism $\widehat{\pi}$ is finite unramified (in particular, \widehat{X} is compact Kähler) and \widehat{f} is birational. By the observation at the beginning of the proof, we may apply Theorem 2.15 to show that $\widehat{f}^* \pi^* \tau$ is parallel with respect to the singular Ricci-flat Kähler metric $T_{\widehat{X}} \in \widehat{f}^* \pi^* [\alpha]$ on the locus $\text{Amp}(\widehat{f}^* \pi^* [\alpha])$. As $T_{\widehat{X}} \in \widehat{\pi}^* f^* [\alpha]$, the functoriality property of Kähler-Einstein metrics with respect to finite morphisms (see e.g. [GGK, Proposition 3.5]) shows that $T_{\widehat{X}} = \widehat{\pi}^* T$. Over Ω' , the following identity holds

$$\widehat{f}^* \pi^* \tau|_{\Omega'} = \widehat{\pi}^* f^* \tau|_{\Omega'}.$$

Therefore, $\widehat{\pi}^* f^* \tau|_{\Omega'}$ is parallel with respect to $\widehat{\pi}^* T|_{f^{-1}(\Omega')}$, hence $f^* \tau|_{\Omega'}$ is parallel with respect to $T|_{f^{-1}(\Omega')}$. The Corollary now follows easily. \square

Let us now prove Theorem 2.15. The arguments and computations in the proof are extensively borrowed from [Gue] (see also [CP1]).

Proof of Theorem 2.15. Let us fix a Kähler modification $f : X \rightarrow Y$, and let us set $\Omega' := f(\text{Amp}(f^*[\alpha])) \subset \Omega$. By the same arguments as in the proof of Corollary 2.17, it is sufficient to prove that $\tau|_{\Omega'}$ is parallel with respect to $\omega|_{\Omega'}$ or, equivalently, that the restriction of $\sigma := f^* \tau \in H^0(X, T_X^{\otimes p})$ to $f^{-1}(\Omega')$ is parallel with respect to $(f^* \omega)|_{f^{-1}(\Omega')}$.

Let $\mathcal{E} := T_X^{\otimes p}$, and let $h = |\cdot|$ be the hermitian metric on \mathcal{E} induced by the Kähler metric $\omega_{t,\varepsilon}$ introduced in Eq. (2.4). Let $D = D' + \bar{\partial}$ be the corresponding Chern connection for (\mathcal{E}, h) . The curvature of this Chern connection for (\mathcal{E}, h) will be denoted by $\Theta_h(\mathcal{E})$. The following holds:

$$dd^c \log(|\sigma|^2 + 1) = \frac{1}{|\sigma|^2 + 1} \left(|D'\sigma|^2 - \frac{|\langle D'\sigma, \sigma \rangle|^2}{|\sigma|^2 + 1} - \langle \Theta_h(\mathcal{E})\sigma, \sigma \rangle \right). \quad (2.5)$$

Wedging (2.5) with $\omega_{t,\varepsilon}^{n-1}$ and then integrating it on X yields:

$$\int_X \frac{\langle \Theta_h(\mathcal{E})\sigma, \sigma \rangle}{|\sigma|^2 + 1} \wedge \omega_{t,\varepsilon}^{n-1} = \int_X \frac{1}{|\sigma|^2 + 1} \left(|D'\sigma|^2 - \frac{|\langle D'\sigma, \sigma \rangle|^2}{|\sigma|^2 + 1} \right) \wedge \omega_{t,\varepsilon}^{n-1}.$$

Since $|\langle D'\sigma, \sigma \rangle| \leq |D'\sigma| \cdot |\sigma|$, we obtain the inequality

$$\int_X \frac{\langle \Theta_h(\mathcal{E})\sigma, \sigma \rangle}{|\sigma|^2 + 1} \wedge \omega_{t,\varepsilon}^{n-1} \geq \int_X \frac{|D'\sigma|^2}{(|\sigma|^2 + 1)^2} \wedge \omega_{t,\varepsilon}^{n-1}. \quad (2.6)$$

First let us introduce a notation: if V is a complex vector space of dimension n , $1 \leq p \leq n$ is an integer, and $t \in \text{End}(V)$, then we denote by $t^{\otimes p}$ the endomorphism

of $V^{\otimes p}$ defined by

$$t^{\otimes p}(v_1 \otimes \cdots \otimes v_p) := \sum_{i=1}^p v_1 \otimes \cdots \otimes v_{i-1} \otimes t(v_i) \otimes v_{i+1} \otimes \cdots \otimes v_p.$$

it may be noted that if V has an hermitian structure, and if t is hermitian semi-positive, then $t^{\otimes p}$ is also hermitian semipositive for the hermitian structure on $V^{\otimes p}$ induced by the hermitian structure on V ; also, the inequality $\text{tr}(t^{\otimes p}) \leq n^p \text{tr}(t)$ holds.

Now we can easily check the following identity:

$$n\Theta_h(\mathcal{E}) \wedge \omega_{t,\varepsilon}^{n-1} = -(\sharp \text{Ric } \omega)^{\otimes p} \omega^n,$$

where $\sharp \text{Ric } \omega$ is the endomorphism of T_X^* induced by $\text{Ric } \omega_{t,\varepsilon}$ via $\omega_{t,\varepsilon}$. As $\text{Ric } \omega_{t,\varepsilon} = -\theta_\varepsilon$, we deduce that

$$\int_X \frac{\langle \Theta_h(\mathcal{E})\sigma, \sigma \rangle}{|\sigma|^2 + 1} \wedge \omega_{t,\varepsilon}^{n-1} = - \int_X \frac{\langle (\sharp \theta_\varepsilon)^{\otimes p} \sigma, \sigma \rangle}{|\sigma|^2 + 1} \omega^n$$

(the operator \sharp is defined above).

We can write $\theta = \sum a_i \theta_{i,\varepsilon}$, where $\theta_{i,\varepsilon} := a_i \left(\frac{\varepsilon^2 |D's_i|^2}{(|s|^2 + \varepsilon^2)^2} + \frac{\varepsilon^2 \Theta_{h_i}(E_i)}{|s|^2 + \varepsilon^2} \right)$. In order to simplify the notation, we drop the index i and set

$$\beta = \frac{\varepsilon^2 |D's|^2}{(|s|^2 + \varepsilon^2)^2} \quad \text{and} \quad \gamma = \frac{\varepsilon^2 \Theta_{h_i}(E_i)}{|s|^2 + \varepsilon^2},$$

so that $\theta_{i,\varepsilon} = a(\beta + \gamma)$; remember that these forms are smooth as long as $\varepsilon > 0$.

Let us start with γ : there exists a constant $C > 0$ such that

$$\pm \gamma \leq C\varepsilon^2 / (|s|^2 + \varepsilon^2) \omega_X.$$

As both of the two operations \sharp and p -th tensor power preserve positivity, we get that

$$\pm (\sharp \gamma)^{\otimes p} \omega_{t,\varepsilon}^n \leq C\varepsilon^2 / (|s|^2 + \varepsilon^2) (\sharp \omega_X)^{\otimes p} \omega_{t,\varepsilon}^n.$$

But $\sharp \omega_X$ is a positive endomorphism whose trace is $\text{tr}_{\omega_{t,\varepsilon}} \omega_X$, and therefore we have $(\sharp \omega_X)^{\otimes p} \leq n^p \text{tr}_{\omega_{t,\varepsilon}}(\omega_X) \cdot \text{Id}$. Consequently,

$$\begin{aligned} \frac{\pm \langle (\sharp \gamma)^{\otimes p} \sigma, \sigma \rangle}{|\sigma|^2 + 1} \omega_{t,\varepsilon}^n &\leq \frac{C\varepsilon^2}{|s|^2 + \varepsilon^2} \cdot \frac{|\sigma|^2}{|\sigma|^2 + 1} \omega_X \wedge \omega_{t,\varepsilon}^{n-1} \\ &\leq \frac{C\varepsilon^2}{|s|^2 + \varepsilon^2} \omega_X \wedge \omega_{t,\varepsilon}^{n-1} \end{aligned}$$

for some $C > 0$ which is independent of t and ε . From Lemma 2.18 below and using the dominated convergence theorem, we deduce that the integral

$$\int_X \frac{\langle (\sharp \gamma)^{\otimes p} \sigma, \sigma \rangle}{|\sigma|^2 + 1} \omega_{t,\varepsilon}^n$$

converges to 0 when ε goes to zero.

We now have to estimate the term involving β . We know that β is non-negative, so $(\sharp\beta)^{\otimes p}\omega_{t,\varepsilon}^n \leq n^{p+1} \beta \wedge \omega_{t,\varepsilon}^{n-1} \text{Id}$, and hence

$$\begin{aligned} 0 &\leq \int_X \frac{\langle (\sharp\beta)^{\otimes p}\sigma, \sigma \rangle}{|\sigma|^2 + 1} \omega_{t,\varepsilon}^n \leq C \int_X \frac{|\sigma|^2}{|\sigma|^2 + 1} \cdot \beta \wedge \omega_{t,\varepsilon}^{n-1} \\ &\leq C \int_X \beta \wedge \omega_{t,\varepsilon}^{n-1} \\ &= C \left(\int_X (\beta + \gamma) \wedge \omega_{t,\varepsilon}^{n-1} - \int_X \gamma \wedge \omega_{t,\varepsilon}^{n-1} \right) \\ &= C \left(\{\theta\} \cdot \{\omega\}^{n-1} - \int_X \gamma \wedge \omega_{t,\varepsilon}^{n-1} \right). \end{aligned}$$

We already observed that the second integral converges to 0 when $\varepsilon \rightarrow 0$. As for the first term, it is cohomological (independent of ε), equal to $t^{n-1}(E_i \cdot [\omega_X]^{n-1})$ since $f^*[\alpha]$ is orthogonal to E_i , and thus it converges to 0 as t goes to 0.

Now recall (2.6):

$$\int_X \frac{\langle \Theta_h(\mathcal{E})\sigma, \sigma \rangle}{|\sigma|^2 + 1} \wedge \omega_{t,\varepsilon}^{n-1} \geq \int_X \frac{|D'\sigma|^2}{(|\sigma|^2 + 1)^2} \wedge \omega_{t,\varepsilon}^{n-1} \geq 0.$$

When ε and t both go to 0, $\omega_{t,\varepsilon}$ converges weakly to $f^*\omega$. Moreover, $f^*\omega$ is smooth on $f^{-1}(\Omega')$ (defined at the beginning of the proof), and the convergence is smooth on the compact subsets of Ω' by Theorem 2.13. Therefore, by Fatou lemma, we deduce that $D'\sigma = 0$ on this locus, which was required to be shown (here D' denotes the Chern connection on \mathcal{E} associated to $f^*\omega$ on this open subset Ω'). \square

The proof of Theorem 2.15 involved the following result, which is proved in much greater generality in [Gue, Lemma 3.7]. We provide a simpler proof more suited to the present set-up.

Lemma 2.18. *For every fixed $t > 0$, and any section $s \in \mathcal{O}_X(E_i)$ for some component E_i of E , the integral*

$$\int_X \frac{\varepsilon^2}{|s|^2 + \varepsilon^2} \omega_X \wedge \omega_{t,\varepsilon}^{n-1}$$

converges to 0 when ε goes to 0.

Proof. By [Ya, p. 360, Proposition 2.1], there is a constant $C_t > 0$ independent of $\varepsilon > 0$ such that $\omega_{t,\varepsilon} \leq C_t \omega_X$. The lemma thus follows from Lebesgue's dominated convergence theorem. \square

3. GEOMETRIC STRUCTURES ON MANIFOLDS IN CLASS \mathcal{C}

In this section we give two applications of Theorem 2.15 for manifolds in Fujiki class \mathcal{C} bearing a holomorphic geometric structure.

3.1. Holomorphic geometric structures. Let us first recall the definition of a holomorphic (rigid) geometric structure as given in [DG, Gr].

Let Y be a complex manifold of complex dimension n and $k \geq 1$ an integer. Denote by $R^k(Y) \rightarrow Y$ the holomorphic principal bundle of k -frames of Y : it is the bundle of k -jets of local holomorphic coordinates on Y . Recall that the structure group of $R^k(Y)$ is the group D^k of k -jets of local biholomorphisms of \mathbb{C}^n fixing the origin. This D^k is a complex algebraic group.

Definition 3.1. A *holomorphic geometric structure* ϕ (of order k) on Y is a holomorphic D^k -equivariant map from $R^k(Y)$ to a complex algebraic manifold Z endowed with an algebraic action of the algebraic group D^k .

A holomorphic geometric structure ϕ as in Definition 3.1 is said to be of *affine type* if Z in Definition 3.1 is a complex affine manifold.

Notice that holomorphic tensors are holomorphic geometric structures of affine type of order one. Holomorphic affine connections are holomorphic geometric structures of affine type of order two [DG]. In contrast, while holomorphic foliations and holomorphic projective connections are holomorphic geometric structure in the sense of Definition 3.1, they are not of affine type.

A holomorphic tensor which is the complex analog of a Riemannian metric is defined in the following way:

Definition 3.2. A *holomorphic Riemannian metric* on a complex manifold Y of complex dimension n is a holomorphic section

$$g \in H^0(Y, S^2((T_Y)^*)),$$

where S^i stands for the i -th symmetric product, such that for every point $y \in Y$ the complex quadratic form $g(y)$ on $T_y Y$ is of (maximal) rank n .

Take a holomorphic Riemannian manifold (Y, g) as above. The real part of g is a pseudo-Riemannian metric h of signature (n, n) on the real manifold of dimension $2n$ underlying the complex manifold Y .

As in the set-up of (pseudo-)Riemannian manifolds, there exists a unique torsion-free holomorphic connection ∇ on the holomorphic tangent T_Y such that g is parallel with respect to ∇ . It is called the Levi-Civita connection for g .

Given (Y, g) as above, consider the curvature of the holomorphic Levi-Civita connection ∇ for g . This curvature tensor vanishes identically if and only if g is locally isomorphic to the standard flat (complex Euclidean) model $(\mathbb{C}^n, dz_1^2 + \dots + dz_n^2)$. In this flat case the real part h of g is also flat and it is locally isomorphic to $(\mathbb{R}^{2n}, dx_1^2 + \dots + dx_n^2 - dy_1^2 - \dots - dy_n^2)$. For more details about the geometry of holomorphic Riemannian metrics the reader is referred to [Du2, Gh2].

A natural notion of (local) infinitesimal symmetry is the following (the terminology comes from the standard Riemannian setting).

Definition 3.3. A (local) holomorphic vector field θ on Y is a (local) *Killing field* for a holomorphic geometric structure of order k

$$\phi : R^k(Y) \longrightarrow Z$$

if the flow for the canonical lift of θ to $R^k(Y)$ preserves each of the fibers of the map ϕ .

Consequently, the (local) flow of a Killing vector field for ϕ preserves ϕ . It is evident that the Killing vector fields for ϕ form a Lie algebra with respect to the operation of Lie bracket.

The holomorphic geometric structure ϕ is called *locally homogeneous* on an open subset Ω of Y if the holomorphic tangent bundle T_Y is spanned by local Killing vector fields of ϕ in the neighborhood of every point in Ω . This implies that for any pair of points $o, o' \in \Omega$, there exists a (local) biholomorphism, from a neighborhood of o to a neighborhood of o' , that preserves ϕ and also sends o to o' .

A holomorphic geometric structure ϕ is called *rigid* of order l in the sense of Gromov, [Gr], if any local biholomorphism f preserving ϕ is determined uniquely by the l -jet of f at any given point.

Holomorphic affine connections are rigid of order one in the sense of Gromov [DG, Gr]. Their rigidity comes from the fact that local biholomorphisms fixing a point and preserving a connection actually linearize in exponential coordinates around the fixed point, so they are completely determined by their differential at the fixed point.

A holomorphic Riemannian metric g is also a rigid holomorphic geometric structure, because local biholomorphisms preserving g also preserve the associated Levi-Civita connection. In contrast, holomorphic symplectic structures and holomorphic foliations are non-rigid geometric structures [DG, Gr].

3.2. A criterion for local homogeneity.

Theorem 3.4. *Let Y be a compact complex manifold in Fujiki class \mathcal{C} with trivial first Chern class ($c_1(T_Y) = 0$ in $H^2(Y, \mathbb{R})$). Assume that there exists a nef cohomology class $[\alpha] \in H^{1,1}(Y, \mathbb{R})$ of positive self-intersection. Then the following two hold:*

- (1) *There exists a non-empty Zariski open subset $\Omega \subset Y$ such that any holomorphic geometric structure of affine type on Y is locally homogeneous on Ω .*
- (2) *If Y admits a rigid holomorphic geometric structure of affine type, then the fundamental group of Y is infinite.*

Proof. (1): In view of Corollary 2.6, the assumptions of Theorem 2.15 are satisfied by Y . Corollary 2.17 implies that there exists a Zariski open subset $\Omega \subset Y$ such that any holomorphic tensor on Y is parallel with respect to some Kähler metric on Ω . In particular, any holomorphic tensor on Y vanishing at some point of Ω must

be identically zero. Then Lemme 3.2 in [Du1, p. 565] gives that any holomorphic geometric structure of affine type on Y is locally homogeneous on Ω .

(2): To prove by contradiction, assume that the fundamental group of Y is finite. Substituting the universal cover of Y in place of Y , and considering the pull-back, to the universal cover, of the geometric structure on Y , we may assume Y in the theorem to be simply connected.

Now K_Y is trivial because Y is simply connected; see Remark 2.16. A result, first proved by Nomizu in the Riemannian setting [No], and subsequently generalized by Amores [Am] and Gromov [Gr], gives the following: the condition that Y is simply connected implies that any local (holomorphic) Killing field of a rigid holomorphic geometric structure on Y extends to a global (holomorphic) Killing field (see also a nice exposition of it in [DG]).

In particular, using statement (1) in the theorem, we obtain that at any point of $z \in \Omega$, the fiber $T_z\Omega$ of the holomorphic tangent bundle T_Y is spanned by globally defined holomorphic vector fields on Y . It was noted in the proof of (1) that any holomorphic tensor on Y that vanishes at some point of Ω must be identically zero. Combining these we conclude that there are n global holomorphic vector fields on Y , where $n = \dim_{\mathbb{C}} Y$, that span T_{Ω} .

Fix n global holomorphic vector fields X_1, \dots, X_n on Y that span T_{Ω} . Also, fix a nontrivial holomorphic section vol of the trivial canonical bundle K_Y . Then $vol(X_1, \dots, X_n)$ is a global holomorphic function on Y . This function must be constant (by the maximum principle) and nonzero at points in Ω . Since $vol(X_1, \dots, X_n)$ is nowhere vanishing on Y , it follows that X_1, \dots, X_n span the holomorphic tangent bundle T_Y at all points of Y .

In other words, Y is a parallelizable manifold. Hence by a theorem of Wang, [Wa, p. 774, Theorem 1], the complex manifold Y must be biholomorphic to a quotient of a connected complex Lie group by a co-compact lattice in it. In particular, Y is not simply connected. This gives the contradiction that we are seeking. \square

3.3. A non-affine type example. It should be mentioned that statement (1) in Theorem 3.4 is not valid in general for holomorphic geometric structures of non-affine type. To see such an example, first recall that Ghys constructed in [Gh1] codimension one holomorphic foliations on complex tori which are not translation invariant. Such a foliation can be obtained in the following way. Consider a complex torus $T = \mathbb{C}^n/\Lambda$, with Λ a lattice in \mathbb{C}^n and assume that there exists a linear form $\tilde{\pi} : \mathbb{C}^n \rightarrow \mathbb{C}$ sending Λ to a lattice Λ' in \mathbb{C} . Then $\tilde{\pi}$ descends to a holomorphic fibration $\pi : T \rightarrow \mathbb{C}/\Lambda'$ over the elliptic curve \mathbb{C}/Λ' . Choose a non-constant meromorphic function u on the elliptic curve \mathbb{C}/Λ' and consider the meromorphic closed one-form $\Omega = \pi^*(udz) + \omega$ on T , where ω is any (translation invariant) holomorphic one-form on T and dz is a nontrivial holomorphic section of the canonical bundle of \mathbb{C}/Λ' . It is easy to see that the foliation given by the kernel of Ω extend on all of T as a nonsingular codimension one holomorphic foliation \mathcal{F} . This foliation is not invariant by all translations in T , more precisely, it is invariant only by those

translations that are spanned by vectors lying in the kernel of $\tilde{\pi}$. The subgroup of translations preserving \mathcal{F} is a subtorus T' of complex codimension one in T [Gh1].

On the other hand, since the holomorphic tangent bundle of T is trivial, we have a family of global (commuting) holomorphic vector fields X_1, X_2, \dots, X_n on T which span the holomorphic tangent bundle T_T at any point in T .

The holomorphic geometric structure $\phi = (\mathcal{F}, X_1, \dots, X_n)$, obtained by juxtaposing Ghys' foliation \mathcal{F} and the vector fields X_i is a holomorphic rigid geometric structure of non-affine type [DG, Gr]. Local Killing fields of ϕ commute with all X_i , so they are linear combinations of X_i ; so they extend as globally defined holomorphic vector fields on T . Since the local Killing fields of ϕ are translations which must also preserve \mathcal{F} , they span the subtorus T' of T . In particular, the Killing algebra of ϕ has orbits of complex codimension one in T and therefore ϕ is not locally homogeneous on any nontrivial open subset of T .

On the contrary, for holomorphic geometric structure of affine type, we think that the non-empty Zariski open set Ω in Theorem 3.4 is all of the manifold. This was proved to be true in [BD] for Moishezon manifolds (these are manifolds bimeromorphic to some complex projective manifold [Mo]).

3.4. Holomorphic Riemannian metric.

Theorem 3.5. *Let Y be a compact complex manifold in Fujiki class \mathcal{C} admitting a holomorphic Riemannian metric g . Assume that there exists a cohomology class $[\alpha] \in H_{\partial\bar{\partial}}^{1,1}(Y, \mathbb{R})$ that is nef and has positive self-intersection. Then there is a finite unramified cover $\gamma : \mathcal{T} \rightarrow Y$, where \mathcal{T} is a complex torus, such that the pulled back holomorphic Riemannian metric γ^*g on the torus \mathcal{T} is translation invariant.*

Proof. Assume that Y admits a holomorphic Riemannian metric g . We have $T_Y = T_Y^*$, because g gives a holomorphic isomorphism of T_Y^* with T_Y . This implies that the first Chern class of T_Y vanishes. So Theorem 2.15 holds for Y because of Corollary 2.6. By Theorem 2.15, there exists a non-empty Zariski open subset $\Omega \subset Y$ endowed with a Kähler metric ω , such that the restriction of the holomorphic tensor g to Ω is parallel with respect to the Levi-Civita connection on T_Ω for the Kähler metric ω .

The following lemma proves that g and ω are flat.

Lemma 3.6. *Let U be an open subset of \mathbb{C}^n in Euclidean topology, and let ω' be a Kähler metric on U . Suppose that there exists a holomorphic Riemannian metric g' on U such that the tensor g' is parallel with respect to the Levi-Civita connection for the Kähler metric ω' . Then the following three hold:*

- (1) *The Kähler metric ω' is flat.*
- (2) *The holomorphic Levi-Civita connection for g' is flat.*
- (3) *The tensor ω' is flat with respect to the holomorphic Levi-Civita connection for g' .*

Proof. Take any $u \in U$. Using de Rham's local splitting theorem, there exists a local decomposition of an open neighborhood $U^u \subset U$ of u in \mathbb{C}^n such that (U^u, ω') is a Riemannian product

$$(U^u, \omega') = (U_0, \omega_0) \times \dots \times (U_p, \omega_p), \quad (3.1)$$

where (U_0, ω_0) is a flat Kähler manifold and (U_i, ω_i) is an irreducible Kähler manifold for every $1 \leq i \leq p$ (the reader is referred to [GGK, Proposition 2.9] for more details on this local Kähler decomposition).

For any $v \in U^u$, let Q_v be the complex bilinear form associated to the quadratic form $g'(v)$ on $T_v U^u$. Write $v = (v_0, v_1, \dots, v_p)$ using (3.1). Since g' is parallel with respect to ω' , it follows that for all $0 \leq i, j \leq p$,

$$Q_v(w_i, w_j) = Q_v(h_i \cdot w_i, h_j \cdot w_j) \quad (3.2)$$

for all $w_i \in T_{v_i} U_i$, $w_j \in T_{v_j} U_j$, for any $h_i \in \text{GL}(T_{v_i} U_i)$ in the holonomy group for (U_i, ω_i) , and for any $h_j \in \text{GL}(T_{v_j} U_j)$ in the holonomy group for (U_j, ω_j) . Assume that $i \neq j$. So at least one of i and j is different from zero. Assume that $j \neq 0$. From (3.2) it follows that

$$Q_v(w_i, h_j \cdot w_j - h'_j \cdot w_j) = Q_v(w_i, w_j - w_j) = 0 \quad (3.3)$$

for all h_j, h'_j in the holonomy group for (U_j, g_j) (set $h_i = \text{Id}$ in (3.2)). Since $j > 0$, the holonomy group for (U_j, g_j) is irreducible, which implies that the vector subspace of $T_{v_j} U_j$ generated by all elements of the form $h_j \cdot w_j - h'_j \cdot w_j$, where $w_j \in T_{v_j} U_j$ and h_j, h'_j are elements of the holonomy group for (U_j, ω_j) , is entire $T_{v_j} U_j$. Using this, from (3.3) it follows that TU_i and TU_j are g' -orthogonal for any $i \neq j$. Consequently, g' is non-degenerate when restricted to all (U_i, ω_i) , $0 \leq i \leq p$.

To prove the first statement of the lemma it suffices to show that $(U^u, \omega') = (U_0, \omega_0)$, meaning $p = 0$ in (3.1).

To prove $p = 0$ by contradiction, assume that ω' admits an (irreducible) factor (U_1, ω_1) . The parallel transport for the Levi-Civita connection for ω_1 must preserve the restriction g_1 of g' to TU_1 and also preserve the real part h_1 of g_1 . The real part h_1 of g_1 is a pseudo-Riemannian metric of signature (n_1, n_1) , where n_1 is the complex dimension of U_1 . Consider the positive and the negative eigenspaces of h_1 with respect to ω_1 . Since the parallel transport for the Levi-Civita connection for ω_1 preserves g_1 , the holonomy of ω_1 preserves the positive and the negative eigenspaces of h_1 with respect to ω_1 . This is a contradiction, because the factor (U_1, ω_1) is irreducible. This proves the first statement of the lemma.

To prove the second statement, since the Kähler metric ω' is flat, there exists local holomorphic coordinates with respect to which ω' is the standard hermitian metric on \mathbb{C}^n . Take such a holomorphic coordinate function on an open subset $U' \subset U$. Therefore, on U' parallel transports for the Levi-Civita connection for ω' are just translations in \mathbb{C}^n in terms of this holomorphic coordinate function on U' . Consequently, on U' the holomorphic Riemannian metric g' must be translation invariant (for the holomorphic coordinate function), because g' is invariant under

the parallel transports for the Levi-Civita connection for ω' . Hence the holomorphic Levi-Civita connection for g' coincides with the standard affine connection on \mathbb{C}^n in terms of the holomorphic coordinate function on U' . In particular, the holomorphic Levi-Civita connection for g' is flat; this proves (2).

Since the holomorphic Levi-Civita connection for g' coincides with the standard affine connection on \mathbb{C}^n in terms of a holomorphic coordinate function on U' with respect to which ω' is the standard Kähler form on \mathbb{C}^n , it follows immediately that ω' is flat with respect to the holomorphic Levi-Civita connection for g' . This completes the proof of the lemma. \square

Continuing with the proof of Theorem 3.5, let ∇^g be the holomorphic Levi-Civita connection on Y for the holomorphic Riemannian metric g . The C^∞ connection on $\Omega_Y^{1,1} = \Omega_Y^{1,0} \otimes \Omega_Y^{0,1}$ induced by ∇^g is flat because ∇^g is flat by Lemma 3.6(2). Note that from Lemma 3.6(3) it follows immediately that the section ω of this flat bundle $\Omega_\Omega^{1,1}$ is flat (covariant constant).

Since Ω is a non-empty Zariski open subset of Y , the natural homomorphism

$$\pi_1(\Omega, y_0) \longrightarrow \pi_1(Y, y_0) \quad (3.4)$$

of fundamental groups is surjective, where $y_0 \in \Omega$. Using this it can be deduced that the above flat section ω of $\Omega_\Omega^{1,1}$ for the connection ∇^g extends to a flat section of $\Omega_Y^{1,1}$. Indeed, this follows immediately from the fact that the flat sections of a flat vector bundle are precisely the invariants of the monodromy representation. Note that for a $\pi_1(Y, y_0)$ -module V , we have $V^{\pi_1(Y, y_0)} = V^{\pi_1(\Omega, y_0)}$, because the homomorphism in (3.4) is surjective. The flat section of $\Omega_Y^{1,1}$ (for the connection on it induced by ∇^g) obtained by extending ω will be denoted by $\widehat{\omega}$.

Now consider the \mathbb{C} -linear homomorphism

$$\widehat{\omega}' : T_Y^{1,0} \longrightarrow \Omega_Y^{0,1}$$

given by $\widehat{\omega}$. It is connection preserving (for the connections induced by ∇^g), because the section ω is flat. Since $\widehat{\omega}'$ is an isomorphism over Ω (as ω is a Kähler form), and $\widehat{\omega}'$ is connection preserving, it follows that $\widehat{\omega}'$ is an isomorphism over the entire Y .

Hence $\widehat{\omega}'$ defines a (nonsingular) hermitian structure on Y . This hermitian structure is Kähler because its restriction to Ω is Kähler. This Kähler structure on Y is flat because its restriction to Ω is flat by Lemma 3.6(1). Therefore, Y admits a finite unramified cover by a complex torus \mathcal{T} such that the pull-back of g to \mathcal{T} is translation invariant [Be], [Bog]. \square

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