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# On total $f$ -domination: polyhedral and algorithmic results

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**Abstract.** Given a graph  $G = (V, E)$  and integer values  $f_v, v \in V$ , a node subset  $D \subset V$  is a total  $f$ -dominating set if every node  $v \in V$  is adjacent to at least  $f_v$  nodes of  $D$ . Given a weight system  $c(v), v \in V$ , the minimum weight total  $f$ -dominating set problem is to find a total  $f$ -dominating set of minimum total weight. In this article, we propose a polyhedral study of the associated polytope together with a complete and compact description of the polytope for totally unimodular graphs and cycles. We also propose a linear time dynamic programming algorithm for the case of trees.

**Keywords:** total domination, polytope, tree, linear-time algorithm

## 1 Introduction

Let  $G = (V, E)$  denote a simple graph having node set  $V = \llbracket 1, n \rrbracket$  and edge set  $E$ , where  $\llbracket 1, n \rrbracket$  stands for the set of integers  $\{1, 2, \dots, n\}$ . For each  $v \in V$ , let  $d_v$  denote its degree in  $G$  and let  $f_v$  be a given nonnegative integer value. Let  $\mathcal{F}_G$  stand for the set of vectors  $\{f \in \mathbb{Z}_+^n : 0 \leq f_v \leq d_v, \forall v \in V\}$ . A node subset  $D \subseteq V$  is called an  $f$ -dominating set (resp. a total  $f$ -dominating set) if each node  $v \in V \setminus D$  (resp.  $v \in V$ ) has at least  $f_v$  neighbor(s) in  $D$ . In the special case  $f_v = 1$ , for all  $v \in V$ , node set  $D$  is called a dominating set (resp. a total dominating set), see [13,14] and [17]. We consider then the *minimum weight total  $f$ -dominating set problem*, denoted by  $[MWT_f]$ : Given a simple graph  $G = (V, E)$  with node weights  $c_v \in \mathbb{R}$ , for all  $v \in V$ , and  $f \in \mathcal{F}_G$ , find a minimum weight total  $f$ -dominating set of  $G$ , i.e. find a node subset  $D \subseteq V$  such that  $D$  is a total  $f$ -dominating set and

the weight of  $D$ :  $\sum_{v \in D} c_v$ , is minimum. This problem may be formulated as the integer program

$$(IP) \quad \left\{ \min \sum_{v \in V} c_v x_v : \sum_{u \in N(v)} x_u \geq f_v, \forall v \in V; x \in \{0, 1\}^n \right\},$$

where  $N(v) = \{u: [u, v] \in E\}$  denotes the neighboring nodes of  $v$ . Its linear relaxation (obtained replacing the constraints  $x \in \{0, 1\}^n$  by  $x \in [0, 1]^n$ ) will be denoted by  $(P)$ . Given a node subset  $S \subseteq V$ , let  $\chi^S \in \{0, 1\}^n$  denote its incidence vector:  $\chi_v^S = 1$  if  $v \in S$ , and  $\chi_v^S = 0$  otherwise. Let  $\mathcal{T}_G^f$  denote the total  $f$ -dominating set polytope, i.e. the convex hull of all the incidence vectors of the total  $f$ -dominating sets in  $G$ . Then, problem  $[MWT_f]$  can be reformulated as the linear program:  $\min\{c^t x : x \in \mathcal{T}_G^f\}$ .

Optimization problems involving dominating sets and some of their many variants arise in several important applications, in particular for the strategic placement of resources in network infrastructures (see e.g. [13,14]). Consider a graph whose node set corresponds to locations where some resource (energy, data, ...) can be made available at some cost, and whose edges represent connections allowing the distribution of this resource between pairs of locations. Then, an optimal solution to  $[MWT_f]$  may be interpreted as a set of locations where the resource is made available so that each location  $v$  can get it from at least  $f_v$  neighboring places and the total cost for locating the resource is minimized. For information on domination and many of its variants, the reader may consult both books by Haynes et al. [13,14], and for total domination we may refer to the survey by Henning [15] and to the book by Henning and Yeo [17]. Many works on total domination focus on finding the minimum cardinality of a total dominating set in a given graph  $G = (V, E)$ , i.e. the case  $f_v = c_v = 1$ , for all  $v \in V$ .

Let [DT] (resp. [DD]) stand for the decision problem associated with the minimum cardinality total dominating set problem (resp. the minimum cardinality dominating set problem). [DD] was shown to be NP-complete [12] for undirected path graphs in [5] using a reduction from the 3-dimensional matching problem. A variation of this reduction was used in [22] to prove the same result holds for [DT]. Further graph classes for which [DT] is known to be NP-complete include, e.g., split (and thus also chordal) graphs [21], line graphs of bipartite graphs [23] and circle graphs [18]. Connections between [DD] and [DT] are investigated in [20] which presents a linear time many-one reduction from [DT] to [DD]. This transformation allows the derivation of complexity results for one of the two decision problems from complexity results on the other for some particular graph families (closed for the graph transformation that is introduced there), among which the fact that the minimum cardinality total dominating set problem can be solved in polynomial time in permutation graphs, dually chordal graphs and  $k$ -polygon graphs. Laskar et al. [22] gave the first linear time algorithm to

find a minimum cardinality total dominating set in a tree. Their greedy algorithm uses a particular node labeling and iteratively processes a leaf and removes it from the current tree, which is initialized with the input graph. In this paper, we extend their result by showing  $[MWT_f]$  can be solved in linear time for trees (see Proposition 11). Other graph classes for which the minimum cardinality total dominating set problem can be solved in polynomial time include strongly chordal graphs [7] and cocomparability graphs [19]. In [3] a  $\mathcal{O}(n \log n)$  algorithm is presented for solving the minimum weight total dominating set problem in interval graphs. A notable graph family for which the complexity status of the problem [DD] differs from the one of [DT] is that of chordal bipartite graphs: when restricted to this graph family [DD] is NP-complete whereas [DT] can be solved in polynomial time [10].

Let  $\gamma_t(G)$  (resp.  $\gamma_{t,f}(G)$ ) denote the minimum cardinality of a total dominating set (resp. total  $f$ -dominating set) in a graph  $G = (V, E)$ . Given the complexity of the problem for computing  $\gamma_t(G)$ , some works focused on getting bounds. Lower and upper bounds on  $\gamma_t(G)$  appear in [8,17]. To the best of the authors' knowledge, a lower bound on  $\gamma_{t,f}(G)$  only appears in [27], while the upper bound  $\frac{6n}{7}$  is reported in [16] for the particular case  $f_v = 2$ , for all  $v \in V$ .

The polyhedral structures of polytopes related to domination problems seem to have received little attention. With respect to the classical domination concept relevant works on such aspects are namely [6,11]. Let  $\mathcal{D}_G$  denote the dominating set polytope, i.e. the convex hull of the incidence vectors of the dominating sets in  $G$ . Farber's work [11] gives a complete description of  $\mathcal{D}_G$  for strongly chordal graphs, while Bouchakour and Mahjoub's paper [6] provides properties and characterizations of facet-defining inequalities, and also presents a peculiar decomposition result which may be formulated as follows. If  $G = (V, E)$  is the 1-sum of the graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  (i.e.  $V = V_1 \cup V_2$ ,  $E = E_1 \cup E_2$  and  $|V_1 \cap V_2| = 1$ ), then a complete formulation of  $\mathcal{D}_G$  can be deduced from the ones of  $\mathcal{D}_{G_1}$  and  $\mathcal{D}_{G_2}$ . We proceed to similar investigations w.r.t.  $\mathcal{T}_f^G$ , which, to our knowledge, do not appear elsewhere in the literature.

The paper is organized as follows. In Section 2, we present basic polyhedral results on  $\mathcal{T}_G^f$ . In Section 3, we prove that if the graph  $G$  has an articulation point  $u$  whose degree equals the number of connected components of the graph induced by  $V \setminus \{u\}$ , then an extended formulation of  $\mathcal{T}_G^f$  can be obtained from complete formulations related to these components. In Section 4, complete formulations of  $\mathcal{T}_G^f$  for some special graph families are given, namely: totally unimodular graphs and cycles. Then, in Section 5, a linear time dynamic programming algorithm to solve  $[MWT_f]$  for trees is presented, before we conclude in Section 6.

## 2 Basic polyhedral results on $\mathcal{T}_G^f$

Let  $G = (V, E)$  denote a simple undirected graph, and let  $f \in \mathcal{F}_G$  such that  $f_v < d_v$ , for all  $v \in V$ . In this section we give basic polyhedral properties like dimension and facet-defining inequalities of  $\mathcal{T}_G^f$ .

**Proposition 1.** *The following statements hold.*

- (i) *The polytope  $\mathcal{T}_G^f$  has dimension  $n$ , i.e. it is full dimensional.*
- (ii) *The trivial inequality  $x_v \geq 0$  is facet-defining for  $\mathcal{T}_G^f$  if and only if  $f_w < d_w - 1$ , for all  $w \in N(v)$  such that  $d_w \geq 2$ .*
- (iii) *The inequality  $x_v \leq 1$  is facet-defining for  $\mathcal{T}_G^f$ , for all  $v \in V$ .*

*Proof.* Result (i) follows from the affine independence of the incidence vectors of the following total  $f$ -dominating sets:  $V$  and  $V \setminus \{v\}$ , for all  $v \in V$ . Statement (iii) can be deduced from the affine independence of the incidence vectors of the sets:  $V$  and  $V \setminus \{w\}$ , for all  $w \in V \setminus \{v\}$ . We now prove (ii). Let  $F_u^\alpha = \mathcal{T}_G^f \cap \{x \in \mathbb{R}^n : x_u = \alpha\}$  for  $u \in V$ ,  $\alpha \in \{0, 1\}$ .

[ $\Rightarrow$ ] In case  $f_w = d_w - 1$  for some  $w \in N(v)$  with  $d_w \geq 2$ , then necessarily  $F_v^0 \subset \cap_{u \in N(w) \setminus \{v\}} F_u^1$ , thus the inequality  $x_v \geq 0$  cannot define a facet of  $\mathcal{T}_G^f$ .

[ $\Leftarrow$ ] The incidence vectors of the  $n$  total  $f$ -dominating sets:  $V \setminus \{v\}$  and  $V \setminus \{v, w\}$ , for all  $w \in V \setminus \{v\}$  are affinely independent and they all belong to  $F_v^0$ .  $\square$

In what follows, given a node subset  $S \subseteq V$ , its *open neighborhood* is the set  $N(S) = \{v \in V \setminus S : \exists u \in S \text{ such that } [u, v] \in E\}$ , and its *closed neighborhood* is the set  $N[S] = N(S) \cup S$ .

We now provide a simple sufficient condition for an inequality

$$\sum_{z \in N(v)} x_z \geq f_v \tag{1}$$

of (P) to be facet-defining (where  $v \in V$ ).

**Proposition 2.** *Let  $u \in V$  such that  $f_u \geq 1$ , and assume that  $|N(w) \setminus N[u]| \geq f_w$ , for all  $w \in N(u)$ , and  $|N(w) \setminus N[u]| > f_w$ , for all  $w \in V \setminus N[u]$ . Then the inequality (1) (with  $u$  taking the role of  $v$  in this expression) is facet-defining for  $\mathcal{T}_G^f$ .*

*Proof.* Assume that all the mentioned conditions are satisfied. Let  $F$  denote the face of  $\mathcal{T}_G^f$  induced by (1) and assume that  $F$  is contained in a facet  $\bar{F}$  of  $\mathcal{T}_G^f$  that is defined by the inequality  $a^t x \geq b$ , with  $(a, b) \in (\mathbb{R}^n \setminus \{0\} \times \mathbb{R})$ . We prove that  $a^t x \geq b$  corresponds to inequality (1), up to multiplication by a positive scalar.

Let  $S \subset N(u)$  such that  $|S| = f_u$ . Since the node subsets  $(V \setminus N[u]) \cup S$  and  $(V \setminus N(u)) \cup S$  are total  $f$ -dominating sets whose incidence vectors satisfy (1) with equality, we deduce  $a_u = 0$ .

Let  $w \in V \setminus N[u]$  and let  $S \subset N(u)$  such that  $|S| = f_u$ . Since the node sets  $(V \setminus N[u]) \cup S$  and  $((V \setminus N(u)) \cup S) \setminus \{w\}$  are total  $f$ -dominating sets, both satisfying (1) with equality we deduce  $a_w = 0$ .

We now show  $a_v = a_z$ , for all  $(v, z) \in N(u)^2$ . Let  $S \subset N(u)$  such that  $|S| = f_u$ . Let  $(v, z) \in S \times (N(u) \setminus S)$ . Then the incidence vectors of the node subsets  $(V \setminus N[u]) \cup S$  and  $(V \setminus N[u]) \cup (S \cup \{z\}) \setminus \{v\}$  both satisfy (1) with equality. We deduce:  $a_v = a_z$ .

It follows that the inequality  $a^t x \geq b$  must correspond, up to multiplication by a positive scalar, to inequality (1).  $\square$

*Remark 1.* The conditions of Proposition 2 are not necessary for (1) to be facet-defining, and it remains open whether one can formulate a simple characterization. Nontrivial relations may be induced by inequalities of the form (1) related to other nodes than  $u$  and whose neighborhood intersects  $N(u)$ . A simple necessary condition (still assuming  $f_u \geq 1$ ), but that is not sufficient, is given by:  $f_w < f_u + |N(w) \setminus N(u)|$  if  $N(w) \neq N(u)$ , and  $f_w \leq f_u$  otherwise, for all  $w \in V \setminus \{u\}$ .

### 3 Decomposition results

In this section we provide results that allow us, in some cases, to decompose the search for a formulation of  $\mathcal{T}_G^f$  into several such searches but on graphs having smaller order.

We start by introducing some simple auxiliary properties. Firstly, we show that when there exists a node for which the domination requirement equals its degree, in order to get a complete formulation of  $\mathcal{T}_G^f$ , we can easily reduce the situation to the case when  $f_v < d_v$ , for all  $v \in V$ .

**Proposition 3.** *Let  $f \in \mathcal{F}_G$  and assume there exists some node  $v \in V$  such that  $f_v = d_v$ . Define  $f' \in \mathcal{F}_G$  as follows:  $f'_w = f_w$ , for all  $w \in V \setminus \{v\}$  and  $f'_v = 0$ . Then, a complete formulation of  $\mathcal{T}_G^f$  can be obtained by adding to a complete formulation of  $\mathcal{T}_G^{f'}$  the set of equations  $\{x_u = 1 : u \in N(v)\}$ .*

*Proof.* Let  $Q$  denote the polytope defined by a complete formulation of  $\mathcal{T}_G^{f'}$  with the set of equations  $\{x_u = 1 : u \in N(v)\}$  added. Note that  $Q$  is a face of  $\mathcal{T}_G^{f'}$  contained in  $\mathcal{T}_G^f$ , and since any point of  $\mathcal{T}_G^f$  satisfies the system defining  $Q$ , we deduce  $Q = \mathcal{T}_G^f$ .  $\square$

The next auxiliary result shows that edges whose endpoints have zero domination requirements can be ignored when looking for a complete formulation of  $\mathcal{T}_G^f$ .

**Proposition 4.** Let  $G = (V, E)$  denote an undirected graph, and let  $f \in \mathcal{F}_G$  be such that  $f_u = f_v = 0$  for some edge  $[u, v] \in E$ . Let  $G' = (V, E \setminus [u, v])$  denote the graph obtained from  $G$  by deleting edge  $[u, v]$ . Then  $\mathcal{T}_G^f = \mathcal{T}_{G'}^f$ .

Trivially, if the graph  $G$  is not connected, then a complete formulation of  $\mathcal{T}_G^f$  is obtained by aggregating the complete formulations corresponding to its connected components. So, in what follows, we can assume w.l.o.g. that  $G$  is connected. Given a graph  $G = (V, E)$  and a node subset  $S \subseteq V$ , let  $G[S]$  denote the subgraph of  $G$  that is induced by  $S$ , i.e.  $G[S] = (S, E')$ , where  $E'$  stands for the subset of edges in  $E$  having both endpoints in  $S$ .

**Proposition 5.** Let  $G = (V, E)$  be an undirected graph,  $f \in \mathcal{F}_G$ , and assume that, for some node  $u \in V$ , the following holds:  $f_u = d_u$  and  $f_v \leq |N(v) \cap N(u)|$ , for all  $v \in N(u)$ . Let  $\tilde{G} = (\tilde{V}, \tilde{E}) = G[V \setminus N[u]]$  and let  $\tilde{f}_v = f_v$ , for all  $v \in \tilde{V} \setminus S$  and  $\tilde{f}_v = f_v - |N(v) \cap N(u)|$ , for all  $v \in S$  with  $S = N(N[u])$ . Then, a complete description of  $\mathcal{T}_G^f$  is obtained by adding to a complete description of  $\mathcal{T}_{\tilde{G}}^{\tilde{f}}$  the equations  $x_v = 1$ , for all  $v \in N(u)$  and the trivial inequalities  $0 \leq x_u \leq 1$ .

*Proof.* Let  $Q \subset \mathbb{R}^V$  denote the polytope whose complete formulation is obtained from that of  $\mathcal{T}_{\tilde{G}}^{\tilde{f}}$ , adding the equations  $x_v = 1$ , for all  $v \in N(u)$  and the trivial inequalities  $0 \leq x_u \leq 1$ .

Note that the restriction to the nodes in  $\tilde{V}$  of any total  $f$ -dominating set in  $G$  is a total  $\tilde{f}$ -dominating set in  $\tilde{G}$ . Thus, all the incidence vectors of total  $f$ -dominating sets in  $G$  satisfy all the constraints defining  $Q$  and we have:  $\mathcal{T}_G^f \subseteq Q$ .

We now show  $Q \subseteq \mathcal{T}_G^f$ . For, let  $y \in Q$  denote an extreme point of  $Q$ . Remark that since the variable  $x_u$  occurs only in the trivial inequalities in the description of  $Q$ , necessarily:  $y_u \in \{0, 1\}$ . Let  $\tilde{y} \in \mathbb{R}^{\tilde{V}}$  denote the restriction of  $y$  to its entries in  $\tilde{V}$ . Then,  $\tilde{y} \in \mathcal{T}_{\tilde{G}}^{\tilde{f}}$  and it can be expressed as a convex combination of incidence vectors of total  $\tilde{f}$ -dominating sets in  $\tilde{G}$ :  $\tilde{y} = \sum_{j=1}^q \lambda_j z^j$ , with  $q$  a positive integer,  $\lambda_j \geq 0$ , for all  $j$ ,  $\sum_j \lambda_j = 1$ , and where  $z^j$  denotes an extreme point of  $\mathcal{T}_{\tilde{G}}^{\tilde{f}}$ .

Now, for each  $j \in \llbracket 1, q \rrbracket$ , define  $\tilde{z}^j \in \mathbb{R}^V$  as follows:  $(\tilde{z}^j)_v = (z^j)_v$ , for all  $v \in \tilde{V}$ ,  $(\tilde{z}^j)_v = 1$ , for all  $v \in N(u)$  and  $(\tilde{z}^j)_u = y_u$ . Note that for each  $j \in \llbracket 1, q \rrbracket$ ,  $\tilde{z}^j$  is the incidence vector of a total  $f$ -dominating set in  $G$ , and we have  $y = \sum_{j=1}^q \lambda_j \tilde{z}^j$ . Thus  $y \in \mathcal{T}_G^f$ .  $\square$

The next result holds for the case when the node  $u$  used for decomposition is such that the number of connected components in  $G[V \setminus \{u\}]$  equals  $d_u$ .

**Proposition 6.** Let  $G = (V, E)$  be an undirected graph,  $f \in \mathcal{F}_G$ , such that  $f_v < d_v$ , for all  $v \in V$ . Consider some node  $u \in V$ , let  $C_1, C_2, \dots, C_p$  denote all the different connected components of  $G[V \setminus \{u\}]$  and assume that  $|C_i \cap N(u)| = 1$ , for all  $i \in \llbracket 1, p \rrbracket$  (i.e. all the neighbors of  $u$  belong

to different connected components of  $G[V \setminus \{u\}]$ ). For each  $i \in \llbracket 1, p \rrbracket$ , let  $C_i = (V_i, E_i)$ , define  $\widehat{C}_i = G[V_i \cup \{u\}] = (\widehat{V}_i, \widehat{E}_i)$  and  $\widehat{f}^i \in \mathcal{F}_{\widehat{C}_i}$  such that  $(\widehat{f}^i)_w = f_w$ , for all  $w \in V_i$  and  $(\widehat{f}^i)_u = 0$ . Then, for any  $\delta \in \{0, 1\}$ , the polytope  $Q^\delta$  defined by the aggregation of complete formulations of the polytopes  $\mathcal{T}_{\widehat{C}_i}^{\widehat{f}^i}$  for  $i = 1, 2, \dots, p$ , together with the constraints  $\sum_{v \in N(u)} x_v \geq f_u$  and  $x_u = \delta$  is integral.

*Proof.* We do the proof for  $\delta = 0$  (the case  $\delta = 1$  can be treated similarly). Let  $F^0$  denote the face of  $\mathcal{T}_G^f$  that is defined by  $x_u \geq 0$ :  $F^0 = \mathcal{T}_G^f \cap \{x \in \mathbb{R}^n : x_u = 0\}$ . Since all the incidence vectors of the total  $f$ -dominating sets in  $G$  not containing  $u$  satisfy the constraints defining  $Q^0$ , we have  $F^0 \subseteq Q^0$ . We now show  $Q^0 \subseteq F^0$ .

Given any extreme point  $y$  of  $Q^0$ , for each  $i \in \llbracket 1, p \rrbracket$ , let  $y^i \in \mathbb{R}^{\widehat{V}_i}$  denote the restriction of  $y$  to its components corresponding to  $\widehat{V}_i$ . From the definition of  $Q^0$  we have  $y^i \in \mathcal{T}_{\widehat{C}_i}^{\widehat{f}^i}$ . This namely implies the existence of positive coefficients  $(\lambda_j^i)_{j=1}^{r_i}$ , for some positive integer  $r_i$  such that  $\sum_{j=1}^{r_i} \lambda_j^i = 1$  and  $y^i = \sum_{j=1}^{r_i} \lambda_j^i z^{i,j}$ , where for each  $j \in \llbracket 1, r_i \rrbracket$ ,  $z^{i,j}$  denotes the incidence vector in  $\mathbb{R}^{|\widehat{V}_i|}$  of a total  $\widehat{f}^i$ -dominating set in  $\widehat{C}_i$ . We now prove that  $y$  can be expressed as a convex combination of incidence vectors of total  $f$ -dominating sets in  $G$ .  $Q^0$  is a rational polyhedron, all the entries of  $y$  are rational and we can assume all the coefficients  $\lambda_j^i$  above are rational. For our purposes, we shall now express all the coefficients  $\lambda_j^i$  using a common denominator, i.e. under the form  $\lambda_j^i = \frac{\alpha_j^i}{D}$ , where  $\alpha_j^i$  is a positive integer for all  $i, j$  and  $D$  is a positive integer. Consider a partition of the interval  $[0, 1]$  into subintervals of equal length  $\frac{1}{D}$ :  $I_k = [\frac{k-1}{D}, \frac{k}{D}[$  for  $k \in \llbracket 1, D-1 \rrbracket$  and  $I_D = [\frac{D-1}{D}, 1]$ . For each  $k \in \llbracket 1, D \rrbracket$  and each  $i \in \llbracket 1, p \rrbracket$ , our objective is now to determine a total  $\widehat{f}^i$ -dominating set of  $\widehat{C}_i$ , denoted  $S_k^i$ , that we will associate with the interval  $I_k$  and such that the following three properties, denoted by (*PROPER1*), hold.

1. The incidence vector of  $S_k^i$  corresponds to one of the vectors  $z^{i,j}$  that is associated with a positive  $\alpha_j^i$  in the expression of  $y^i$ . Let  $\bar{j} \in \llbracket 1, r_i \rrbracket$  denote this index value.
2. The total number of intervals among  $(I_q)_{q=1}^D$  that are associated with  $S_k^i$  equals  $\alpha_{\bar{j}}^i$ .
3. For each  $k \in \llbracket 1, D \rrbracket$ ,  $|\cup_{i=1}^p S_k^i \cap N(u)| \geq f_u$ , i.e.  $\cup_{i=1}^p S_k^i$  is a total  $f$ -dominating set in  $G$ .

The sum divided by  $D$  of the incidence vectors of the sets  $(\cup_{i=1}^p S_k^i)_{1 \leq k \leq D}$  coincides with the point  $y \in Q^0$ . It remains to show that such sets  $(S_k^i)_{\substack{1 \leq i \leq p \\ 1 \leq k \leq D}}$  do exist.

Consider the auxiliary directed and arc-capacitated graph  $H = (\{s\} \cup V_H \cup L_H \cup \{t\}, E_H)$  which we define as follows. The node set is composed of

- $s$ : a source node,
- $t$ : a target (or destination) node,

- a set  $V_H$  containing a node  $v^i$  for each  $i \in \llbracket 1, p \rrbracket$ , representing the total  $\widehat{f}^i$ -dominating sets that contain a neighbor of  $u$  and that are associated with a positive coefficient in the expression of  $y^i$  above,
- a set  $L_H$  containing a node  $l_k$  for each interval  $I_k$ ,  $k \in \llbracket 1, D \rrbracket$ .

The set  $E_H$  is composed of the arcs:

- $(s, v^i)$  with capacity  $Dy_{w(i)}^i$ , for all  $i \in \llbracket 1, p \rrbracket$ , where  $w(i)$  stands for the unique neighbor of  $u$  in  $\widehat{C}_i$ ,
- $(v^i, l_k)$  with capacity 1, for all  $i \in \llbracket 1, p \rrbracket$ , for all  $k \in \llbracket 1, D \rrbracket$ ,
- $(l_k, t)$  with capacity  $f_u$ , for all  $k \in \llbracket 1, D \rrbracket$ .

**Claim.** *There exist sets  $(S_k^i)_{\substack{1 \leq i \leq p \\ 1 \leq k \leq D}}$  satisfying (PROPER1) if and only if there exists an integral feasible flow in  $H$  with value  $Df_u$ .*

*Proof of the claim.*  $[\Rightarrow]$  Let  $(S_k^i)_{\substack{1 \leq i \leq p \\ 1 \leq k \leq D}}$  denote sets satisfying (PROPER1). Starting from a zero flow, we iteratively build a flow in  $H$  that will have value  $Df_u$ . For each  $k \in \llbracket 1, D \rrbracket$ , we arbitrarily select exactly  $f_u$  sets among  $\{S_k^i : 1 \leq i \leq p \text{ and } S_k^i \cap N(u) \neq \emptyset\}$ . Let  $v^{k,l} \in V_H$ ,  $l = 1, 2, \dots, f_u$  denote the corresponding nodes. We sequentially increase by one unit the flow on the paths  $(s, v^{k,l}, l_k, t)$ , for  $l = 1, 2, \dots, f_u$ .

$[\Leftarrow]$  Given an integral feasible flow in  $H$  with value  $Df_u$ , we consider, for each  $k \in \llbracket 1, D \rrbracket$  the set of the edges  $\mathcal{U}_k = \{(v, l_k) \in E_H : \text{flow value on } (v, l_k) \text{ is } 1\}$  which has cardinality  $f_u$  (by the construction of  $H$  and since the flow has value  $Df_u$ ). For each edge of the form  $(v^i, l_k) \in \mathcal{U}_k$ , we associate a total  $\widehat{f}^i$ -dominating set of  $\widehat{C}_i$  that corresponds to a vector of the form  $z^{i,j}$  and containing a neighbor of  $u$  (that will be assigned to  $S_k^i$ ). This is done such that each such total  $\widehat{f}^i$ -dominating set is associated with at most  $\alpha_j^i$  arcs in  $\cup_{k=1}^D \mathcal{U}_k$ .

For each  $k \in \llbracket 1, D \rrbracket$ , there are exactly  $f_u$  sets of the form  $(S_k^i)_{1 \leq i \leq p}$  that have been assigned and each one of them corresponds to some total  $\widehat{f}^i$ -dominating set of  $\widehat{C}_i$  containing a neighbor of  $u$ ; so that the third condition in (PROPER1) is satisfied by this partial assignment. For each  $i \in \llbracket 1, p \rrbracket$  we can then assign (order is arbitrary, we just have to take care of the number of times some set occurs) total  $\widehat{f}^i$ -dominating set of  $\widehat{C}_i$  to unassigned sets of the form  $(S_k^i)_{1 \leq k \leq D}$  such that each total  $\widehat{f}^i$ -dominating set of  $\widehat{C}_i$  that is associated with  $z^{i,j}$  in the expression of  $y^i$  is represented exactly  $\alpha_j^i$  times among the sets  $(S_k^i)_{1 \leq k \leq D}$ .  $\square$

To conclude the proof of Proposition 6, we now prove that a flow as mentioned in the Claim above does exist, by showing the minimum cut value of  $H$  is  $Df_u$ .

Given a node subset  $S \subseteq \{s\} \cup V_H \cup L_H$ , let  $c(\delta(S))$  denote the capacity of the  $s$ - $t$  cut defined by  $S$ , i.e. the sum of the capacities of the arcs of the form  $(a, b) \in E_H$  such that  $a \in S, b \notin S$ . Let  $r$  denote the number of nodes of  $V_H$  which are contained in  $S$ . We may consider two cases:

- Case  $r \geq f_u$ : If some node of the form  $l_k$  does not belong to  $S$ , then adding such a node to  $S$  we obtain a cut with capacity  $c(\delta(S \cup \{l_k\})) = c(\delta(S)) + f_u - r \leq c(\delta(S))$ .
- Case  $r < f_u$ : If some node of the form  $l_k$  belongs to  $S$ , then removing such a node from  $S$  we obtain a cut with capacity  $c(\delta(S \setminus \{l_k\})) = c(\delta(S)) + r - f_u < c(\delta(S))$ .

From the former, we deduce there exists a minimum capacity  $s$ - $t$  cut  $\delta(S)$  satisfying  $S \cap L_H \in \{\emptyset, L_H\}$ . Let  $S \subseteq \{s\} \cup V_H$  define a  $s$ - $t$  cut. If  $S$  contains a node of the form  $v^i$ , then removing this node from  $S$  we obtain a cut with capacity  $c(\delta(S \setminus \{v^i\})) = c(\delta(S)) - D + Dy_{w(i)}^i \leq c(\delta(S))$ . It follows that the minimum  $s$ - $t$  cut not containing  $L_H$  is given by  $S = \{s\}$  and the corresponding capacity is  $\sum_{i=1}^p Dy_{w(i)}^i = \sum_{v \in N(u)} Dy_v \geq Df_u$ . For any  $s$ - $t$  cut  $\delta(S)$  with  $S$  containing  $L_H$  we have  $c(\delta(S)) \geq Df_u$ , since the cut  $\delta(S)$  contains all the arcs of the form  $(l_k, t)$ . This leads to the result that the minimum cut value in  $H$  is  $Df_u$ .  $\square$

If the conditions for its application are satisfied, Proposition 6 together with Balas' result [2] allows the derivation of an extended formulation of  $\mathcal{T}_G^f$  from complete formulations related to the components  $(\widehat{C}_i)_{i=1}^p$ .

**Proposition 7.** *In the setting of Proposition 6,  $\mathcal{T}_G^f$  is the projection onto the  $x$ -space of variables of the polytope defined by the following system.*

$$\begin{cases} A^i y^{k,i} \geq \lambda_k b^i, \forall i \in \llbracket 1, p \rrbracket, \forall k \in \llbracket 1, 2 \rrbracket, \\ \sum_{v \in N(u)} y_v^k \geq \lambda_k f_u, \forall k \in \llbracket 1, 2 \rrbracket, \\ y_u^1 = 0, y_u^2 = \lambda_2, \lambda_1 + \lambda_2 = 1, \\ x = y^1 + y^2, \\ (x, y^1, y^2) \in (\mathbb{R}^V)^3, \lambda \in \mathbb{R}_+^2, \end{cases}$$

where  $y^{k,i} \in \mathbb{R}^{\widehat{V}_i}$  denotes the restriction of  $y^k$  to the entries indexed on  $\widehat{V}_i$ , and such that  $\mathcal{T}_{\widehat{C}_i}^{\widehat{f}_i} = \{z \in \mathbb{R}^{\widehat{V}_i} : A^i z \geq b^i\}$ , for all  $i \in \llbracket 1, p \rrbracket$ .

In the more particular case when, in addition to the setting of Proposition 6, we also have  $f_v = 0$ , for all  $v \in N(u)$ , then a complete formulation of  $\mathcal{T}_G^f$  in the original space of variables can be easily determined. The proof is similar to the one of Proposition 6, and thus omitted.

**Proposition 8.** *In the setting of Proposition 6, assume in addition that  $f_v = 0$ , for all  $v \in N(u)$ . For each  $i \in \llbracket 1, p \rrbracket$ , let  $f^i$  stand for the restriction of  $f$  to its entries indexed by  $V_i$ . Then, a complete formulation of  $\mathcal{T}_G^f$  is given by the aggregation of complete formulations of the polytopes  $\mathcal{T}_{C_i}^{f^i}$  for  $i = 1, 2, \dots, p$ , together with the constraints  $\sum_{v \in N(u)} x_v \geq f_u, 0 \leq x_u \leq 1$ .*

## 4 Complete formulations of $\mathcal{T}_G^f$ for some special graph families

In this section we characterize the graphs for which the trivial and neighborhood inequalities give a complete formulation of  $\mathcal{T}_G^f$  for all  $f \in \mathcal{F}_G$ , and then we provide complete formulations for cycles. Given a graph  $G = (V, E)$ , its adjacency matrix is the matrix  $A \in \{0, 1\}^{n \times n}$  satisfying  $A_{uv} = 1$  if and only if  $[u, v] \in E$ . A graph is said to be *totally unimodular* [1] if its adjacency matrix is totally unimodular. Given  $f \in \mathcal{F}_G$ , let  $P(f)$  denote the polytope corresponding to the feasible region of  $(P)$ , i.e.  $P(f) = \{x \in \mathbb{R}^n : Ax \geq f \text{ and } x \in [0, 1]^n\}$ , where  $A \in \{0, 1\}^{n \times n}$  stands for the adjacency matrix of  $G$ . The next proposition characterizes the graphs for which  $P(f)$  is integral for all  $f \in \mathcal{F}_G$ . Its proof is similar to the one given in [26] on a characterization of totally unimodular  $0, \pm 1$  matrices, and thus omitted.

**Proposition 9.** *The polytope  $P(f)$  is integral for all  $f \in \mathcal{F}_G$  if and only if  $G$  is totally unimodular.*

We now provide a complete description of  $\mathcal{T}_G^f$ , when  $G$  is a cycle, for any  $f \in \{0, 1\}^V$ .

**Proposition 10.** *Let  $G = (V, E)$  denote a cycle with node set  $V = \llbracket 0, n-1 \rrbracket$ , edge set  $E = \{[i, i+1 \bmod n] : i \in V\}$ , and let  $f \in \{0, 1\}^V$ . Then, a complete description of  $\mathcal{T}_G^f$  is given by the system  $(S_{\text{odd}})$  (resp.  $(S_{\text{even}})$ ) if  $n$  is odd (resp. even) with:*

$$(S_{\text{odd}}) \begin{cases} \sum_{v \in V} x_v \geq \lceil \frac{n}{2} \rceil \min_{v \in V} f_v, \\ \sum_{u \in N(v)} x_u \geq f_v, \forall v \in V, \\ x \in [0, 1]^n, \end{cases} \quad \text{and} \quad (S_{\text{even}}) \begin{cases} \sum_{k=0}^{\frac{n}{2}-1} x_{2k} \geq \lceil \frac{n}{4} \rceil \min_{k \in \llbracket 0, \frac{n}{2}-1 \rrbracket} f_{2k+1}, \\ \sum_{k=0}^{\frac{n}{2}-1} x_{2k+1} \geq \lceil \frac{n}{4} \rceil \min_{k \in \llbracket 0, \frac{n}{2}-1 \rrbracket} f_{2k}, \\ \sum_{u \in N(v)} x_u \geq f_v, \forall v \in V, \\ x \in [0, 1]^n. \end{cases}$$

*Proof.* Note that by Proposition 1, the polytope  $\mathcal{T}_G^f$  is full dimensional. Let  $Q$  denote the polytope defined by  $S_{\text{odd}}$  if  $n$  is odd and  $S_{\text{even}}$  otherwise. Trivially,  $\mathcal{T}_G^f \subseteq Q$  and any integer vector in  $Q$  is the incidence vector of a total  $f$ -dominating set. Consider firstly the case  $n$  is odd. We distinguish between two cases.

- Case 1: there exists  $u \in V$  such that  $f_u = 0$ . W.l.o.g. assume that  $u = n-1$ . Let  $A \in \{0, 1\}^{(n-1) \times n}$  denote the restriction of the adjacency matrix to the rows indexed by  $V' = V \setminus \{n-1\}$ . Note that  $Q = \{x \in [0, 1]^n : Ax \geq f'\}$ , where  $f'$  denotes the restriction of  $f$  to its first  $n-1$  components. We define a partition  $(B, V' \setminus B)$  of  $V'$  with  $B = \cup_{i=0}^{\frac{n-1}{4}-1} (\{4i\} \cup \{4i+1\})$ , if  $|V'| \bmod 4 = 0$  and  $B = \cup_{i=0}^{\lfloor \frac{n-1}{4} \rfloor} \{4i+1\} \cup (\cup_{i=0}^{\lfloor \frac{n-1}{4} \rfloor - 1} \{4i+2\})$ , otherwise. Each column with index in  $V' \setminus \{n-2, 0\}$  has exactly two nonzero entries, one of which belongs to a row with index in  $B$  and the other to a row with index in  $V' \setminus B$ . The column corresponding to node

0 (resp.  $n - 2$ ) has exactly one nonzero entry which belongs to a row with index in  $B$  (resp.  $V' \setminus B$ ). This implies (see Corollary 2.8 in [24]) that the matrix  $A$  is totally unimodular, and thus that  $Q$  is integral.

- Case 2:  $f_v = 1$ , for all  $v \in V$ . Let  $a^t x \geq b$  denote a facet-defining inequality for  $\mathcal{T}_G^f$  that is not trivial (i.e., different from  $x_v \geq 0$ ,  $x_v \leq 1$ ,  $v \in V$ ) and different from an inequality (1). Necessarily, (this can be easily shown for any such facet)  $(a, b) \in (\mathbb{R}_+^n \setminus \{\mathbf{0}\}) \times (\mathbb{R}_+ \setminus \{0\})$ . Let  $v(1) \in V$  such that  $a_{v(1)} = \max(a_v : v \in V \text{ and } v \text{ odd})$ . Let  $\{v(0), v(1), v(2), v(3)\}$  denote a sequence of consecutive nodes on the cycle containing  $v(1)$  in second position. Let  $S \subseteq V$  denote a total  $\mathbf{1}$ -dominating set in  $G$  such that  $a^t \chi^S = b$  and  $N(v(0)) \subseteq S$ , where  $\mathbf{1}$  stands for the  $n$ -dimensional all-ones vector. (The existence of such a set follows from our assumptions on the constraint  $a^t x \geq b$ : if such a set  $S$  would not exist, then the face defined by the inequality  $a^t x \geq b$  would be contained in the one defined by  $\sum_{v \in N(v(0))} x_v \geq 1$ ). We may then consider two subcases.

- *Case 2.1:*  $v(0) \notin S$ . Then,  $v(2) \in S$ . If  $v(3) \in S$  then, necessarily,  $a_{v(1)} = 0$  (because  $S \setminus \{v(1)\}$  is also a total  $\mathbf{1}$ -dominating set), thus implying  $a_v = 0$  for all odd  $v \in V$ , from the definition of  $v(1)$ . Now, for the case  $v(3) \notin S$ , since  $(S \cup \{v(3)\}) \setminus \{v(1)\}$  is a total  $\mathbf{1}$ -dominating set in  $G$ , we deduce  $a_{v(3)} \geq a_{v(1)}$ .
- *Case 2.2:*  $v(0) \in S$ . If we had  $v(3) \in S$ , then  $S \setminus \{v(1)\}$  would be a total  $\mathbf{1}$ -dominating set, thus leading to  $a_v = 0$  for all odd  $v \in V$ . For the case when  $v(3) \notin S$  and since  $(S \cup \{v(3)\}) \setminus \{v(1)\}$  is a total  $\mathbf{1}$ -dominating set we deduce  $a_{v(3)} \geq a_{v(1)}$ .

In both subcases, we can deduce  $a_{v(1)} = a_{v(3)}$ . Since  $n$  is odd, applying iteratively the former reasoning leads to  $a_v = \alpha$ , for all  $v \in V$  for some scalar  $\alpha > 0$ . And thus the constraint  $a^t x \geq b$  corresponds, up to multiplication by a positive scalar to the constraint  $\sum_{v \in V} x_v \geq \lceil \frac{n}{2} \rceil$ .

If  $n$  is even, let  $Q_1$  (resp.  $Q_2$ ) denote the projection of  $Q$  onto the space of the variables having an odd (resp. even) index. Note that  $Q$  corresponds to the Cartesian product of polytopes  $Q_1 \times Q_2$ , up to permutation of the variable indices. Let  $A' \in \{0, 1\}^{\frac{n}{2} \times \frac{n}{2}}$  be the matrix whose rows and columns are indexed by  $\llbracket 0, \frac{n}{2} - 1 \rrbracket$  and such that  $A'_{ij} = 1 \iff (2j + 1) \in N(2i)$ . Observe that  $A'$  is the incidence matrix of a cycle having  $\frac{n}{2}$  nodes and  $Q_1 = \left\{ z \in [0, 1]^{\frac{n}{2}} : \sum_{k=0}^{\frac{n}{2}-1} z_k \geq \lceil \frac{n}{4} \rceil \min_{k \in \llbracket 0, \frac{n}{2} - 1 \rrbracket} f'_k \text{ and } A'z \geq f' \right\}$ , with  $f' \in \{0, 1\}^{\frac{n}{2}}$  such that  $f'_k = f_{2k}$ , for all  $k \in \llbracket 0, \frac{n}{2} - 1 \rrbracket$  (variable  $z_k$  in  $Q_1$  corresponds to  $x_{2k+1}$  in  $Q$ ). If  $\frac{n}{2}$  is even, the first inequality in the formulation of  $Q_1$  above is redundant,  $A'$  is balanced and by Theorem 6.13 in [9],  $Q_1$  is integral. If  $\frac{n}{2}$  is odd and there exists some node  $u$  with an even index such that  $f_u = 0$ , then the matrix obtained from  $A'$  by removing the  $\frac{n}{2}$ th row is balanced and it follows that  $Q_1$  is integral. For the remaining case when  $\frac{n}{2}$  is odd and  $f_v = 1$ , for all  $v \in V$ , the integrality of  $Q_1$  can be deduced from the fact that  $A'$  is near-perfect

(see [25]). The same approach can be used to prove the integrality of  $Q_2$ , thus implying that  $Q$  is integral and  $\mathcal{T}_G^f = Q$ .  $\square$

## 5 The minimum weight total $f$ -dominating set problem in trees

We now present a dynamic programming algorithm for the problem  $[MWT_f]$  for the particular case when  $G$  is a tree. Select an arbitrary node of the graph that will be considered as the root of the tree. The nodes are numbered in a breadth first search order with integer values in the set  $\llbracket 1, n \rrbracket$ , the root node being numbered with value 1. Given some node  $v \in V$ , let  $T_v$  denote the subtree with root  $v$ . For each node  $v \in V \setminus \{1\}$  we define four values which give the contribution of  $T_v$  to the solution of  $[MWT_f]$  with additional restrictions, and will allow us to calculate the optimal solution. For the case when the added restrictions imply that no solution exists, by convention, we consider its value to be  $+\infty$ . The set of all the children of node  $v$  (i.e. neighbors of  $v$  associated with a larger integer value) is denoted by  $F_v$ . We now describe the four considered values.

- $CID(v)$ : the contribution of the nodes in  $T_v$  to the cost of an optimal solution of  $[MWT_f]$  with the following restriction: node  $v$  belongs to the total  $f$ -dominating set and it is dominated by (at least)  $f_v$  of its children.
- $CIU(v)$ : the contribution of the nodes in  $T_v$  to the cost of an optimal solution of  $[MWT_f]$  with the following restriction: node  $v$  belongs to the total  $f$ -dominating set and it is dominated by (at least)  $f_v - 1$  of its children. (Note that we always have  $CIU(v) \leq CID(v)$  since the domination requirement on the node  $v$  for the subproblem corresponding to  $CID(v)$  is higher than that for the subproblem corresponding to  $CIU(v)$ .)
- $COD(v)$ : the contribution of the nodes in  $T_v$  to the cost of an optimal solution of  $[MWT_f]$  with the following restriction: node  $v$  does not belong to the total  $f$ -dominating set and it is dominated by (at least)  $f_v$  of its children.
- $COU(v)$ : the contribution of the nodes in  $T_v$  to the cost of an optimal solution of  $[MWT_f]$  with the following restriction: node  $v$  does not belong to the total  $f$ -dominating set and it is dominated by (at least)  $f_v - 1$  of its children.

Formulas relating these values associated with some node  $v \in V \setminus \{1\}$  and the ones associated with its children are as follows.

$$CID(v) = c_v + \min_{\substack{J \subseteq F_v \\ |J|=f_v}} \sum_{j \in J} CID(j) + \sum_{j \in F_v \setminus J} \min(CIU(j), COU(j))$$

$$CIU(v) = c_v + \min_{\substack{J \subseteq F_v \\ |J|=f_v-1}} \sum_{j \in J} CIU(j) + \sum_{j \in F_v \setminus J} \min(CIU(j), COU(j))$$

$$COD(v) = \min_{\substack{J \subseteq F_v: \\ |J|=f_v}} \sum_{j \in J} CID(j) + \sum_{j \in F_v \setminus J} \min(CID(j), COD(j))$$

$$COU(v) = \min_{\substack{J \subseteq F_v: \\ |J|=f_v-1}} \sum_{j \in J} CID(j) + \sum_{j \in F_v \setminus J} \min(CID(j), COD(j))$$

For the root node (i.e. the node with number 1) we just consider the values  $CID(1)$  and  $COD(1)$ , the minimum of which gives the optimal objective value of  $[MWT_f]$ . The values  $CID(v)$ ,  $CIU(v)$ ,  $COD(v)$  and  $COU(v)$  are determined using a “bottom-up” approach (i.e. for decreasing  $v$  from  $n$  to 2). Then,  $CID(1)$  and  $COD(1)$  are computed, leading to the optimal objective value of  $[MWT_f]$ . Finally, by tracking back the calculations already performed, an optimal solution to  $[MWT_f]$  is determined.

The next lemma shows that, for each node  $v \in V$ , computing the four values  $CID(v)$ ,  $CIU(v)$ ,  $COD(v)$ ,  $COU(v)$ , assuming those of all its children are available, can be done efficiently. This leads to the fact that solving  $[MWT_f]$  on trees can be done in linear time, thus extending the results by Laskar et al. [22] who gave the first linear time (greedy) algorithm to solve the minimum cardinality total dominating set in trees.

**Lemma 1.** *For each node  $v \in V$ , if the values  $CID(u)$ ,  $CIU(u)$ ,  $COD(u)$  and  $COU(u)$  are available for each children  $u$  of  $v$ , then any of the four values  $CID(v)$ ,  $CIU(v)$ ,  $COD(v)$  or  $COU(v)$  can be computed in time  $\mathcal{O}(H_v)$ , where  $H_v$  denotes the number of children of  $v$ .*

*Proof.* Consider the following problem denoted by  $\mathcal{P}_h$ , where  $h$  stands for a positive integer, and that gives a common framework for computing the quantities  $CID(v)$ ,  $CIU(v)$ ,  $COD(v)$  and  $COU(v)$ . Assume that we are given  $h$  items, each of which is associated with two real numbers: item  $i$  is associated with the values  $a_i$  and  $b_i$ . Let  $k < h$  denote some given integer. The objective is then to determine a subset  $J \subseteq \llbracket 1, h \rrbracket$  such that  $|J| = k$  and the following quantity is minimized (over all such sets):

$$Z_J = \sum_{i \in J} a_i + \sum_{i \in \llbracket 1, h \rrbracket \setminus J} \min(a_i, b_i).$$

Let  $Q$  denote the subset of the  $k$  indices in  $\llbracket 1, h \rrbracket$  corresponding to the  $k$  smallest quantities among  $\{a_i - b_i : i \in \llbracket 1, h \rrbracket\}$ . Formally,  $Q = \operatorname{argmin}_J \{\sum_{j \in J} (a_j - b_j) : J \subseteq \llbracket 1, h \rrbracket \text{ and } |J| = k\}$ .

*Claim.* *The set  $Q$  defines an optimal solution of  $\mathcal{P}_h$ .*

*Proof of the claim.* Firstly, note that if  $|\{i : b_i < a_i\}| \leq h - k$ , then the result is trivial. So, assume that there are at least  $h - k + 1$  items satisfying  $b_i < a_i$ . Then, in any optimal solution  $J^*$  of  $\mathcal{P}_n$  the following holds:  $b_i < a_i$ , for all  $i \in \llbracket 1, h \rrbracket \setminus J^*$ . Now, let  $A_Q = \sum_{i \in \llbracket 1, h \rrbracket \setminus Q} (a_i - b_i)$ ,

$A_{J^*} = \sum_{i \in [1, h] \setminus J^*} (a_i - b_i)$ . Then,  $\sum_{i=1}^h a_i = Z_{J^*} + A_{J^*} = Z_Q + A_Q$ . And since by construction  $A_Q \geq A_{J^*}$ , we deduce  $Z_Q \leq Z_{J^*}$  and the statement of the claim follows.  $\square$

Since determining the  $k$  smallest of  $h$  elements can be done in time  $\mathcal{O}(h)$ , problem  $\mathcal{P}_h$  can be solved in time  $\mathcal{O}(h)$  (see, e.g., [4]). Then, identifying  $h$  with  $|H_v|$  and  $k$  with  $f_v$  (or  $f_v - 1$  depending on the quantity to be computed), the result follows.  $\square$

**Proposition 11.** *Problem  $[MWT_f]$  can be solved in time  $\mathcal{O}(n)$  for trees.*

## 6 Conclusion

In this paper, we presented compact and complete descriptions of the total  $f$ -dominating set polytope  $\mathcal{T}_G^f$  for totally unimodular graphs and cycles. A linear-time dynamic programming algorithm solving the minimum weight total  $f$ -dominating set problem in trees was also described. Further research work may be directed towards the polyhedral structure of  $\mathcal{T}_G^f$  for other graph families such as cacti and (strongly) chordal graphs.

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