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Cheap talk, monitoring and collusion

David Spector*

December 2018

Abstract

Many collusive agreements involve the exchange of self-reported sales data between competitors, which use them to monitor compliance with a target market share allocation. Such communication may facilitate collusion even if it is unverifiable cheap talk and the underlying information becomes publicly available with a delay. The exchange of sales information may allow firms to implement incentive-compatible market share reallocation mechanisms after unexpected swings, limiting the recourse to price wars. Such communication may allow firms to earn profits that could not be earned in any collusive, symmetric pure-strategy equilibrium without communication.

1 Introduction

The objective of this paper is to better understand the role of communication in collusive practices.

Collusion, whether tacit or explicit, requires mutual monitoring.¹ In many recent cartel cases, monitoring took place by having companies compare each other's self-reported sales with some agreed-upon quotas, with a high frequency (often, weekly or monthly). However, these sales reports were for the most part not verifiable, at least in the short run. For instance, in several cases, reliable sales information was available only with a lag of about one year.

Prima facie, this observation is puzzling. If the goal of monitoring is to deter deviations from a collusive agreement, why couldn't a firm wanting to deviate

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¹Sugaya and Wolitzky (2018a) show that this general statement is not as universally valid as is often believed, because too precise information on competitors' actions may increase a firm's incentive to deviate from a collusive equilibrium by helping it to identify profitable deviation opportunities.

simply undercut its competitors and misreport its sales at the same time? At first glance, it seems that the only constraint on a firm's incentive to deviate is the amount of time elapsing between the date a deviation occurs and the date it is bound to being revealed to competitors, as reliable sales data become public. How, then, could the exchange of sales reports facilitate mutual monitoring and collusion if it occurs long before sales data can be verified?

Answering this question would contribute to the ongoing debate on the antitrust treatment of information exchanges. In the absence of direct evidence of cartel behavior, competition authorities face a difficult tradeoff. On the one hand, an outright ban on information exchanges would deny companies and consumers the procompetitive benefits that such exchanges may entail. Conversely, a too lenient approach would allow companies to engage in practices that could facilitate collusion and harm consumers.²

For instance, in its guidelines on horizontal co-operation between undertakings,³ the European Commission states that exchanges of information on past sales are not prohibited *per se* (unlike communication on future behavior) and that they should be assessed under a case-by-case approach. According to K.-U. Kühn, a former Chief Economist of the European Commission, this case-by-case approach should focus on the 'marginal impact' of the information exchanges under scrutiny on the likelihood of collusion.⁴

Accordingly, this paper is an attempt to assess the marginal impact of the early disclosure of sales information long before it becomes public. When 'hard' information is publicly available in any case, should early disclosure be considered harmless cheap talk or a practice facilitating collusion relative to a no-communication benchmark?

We construct a model showing that, in a market where demand is uncertain and (individual or aggregate) sales data become available with a delay, early communication on sales volumes may make collusion more efficient. Communication may reduce the recourse to price wars as a disciplining device by ensuring that unexpected market share swings are swiftly identified and compensated through short phases of market share reallocation.

Our main finding is that, for some parameter values, collusion with near-

²See Kühn (2001).

³Official Journal of the European Union, C 11/91, 14.1.2011, Communication from the Commission — Guidelines on the applicability of Article 101 of the Treaty on the Functioning of the European Union to horizontal co-operation agreements.

⁴Kühn (2011) advocates "*an analysis of the marginal impact of the information exchange on monitoring or the scope for coordination in the market. If the marginal impact appears small, the case should be closed.*"

monopoly pricing in all periods along the equilibrium path can occur *only* if communication is possible, even though such communication does not increase data verifiability.

The collusive equilibrium we derive involves no need for contact between competitors beyond the exchange of sales reports: it is symmetric, which limits the need for pre-play coordination. It involves pure strategies along the equilibrium path, with no need for coordination on a public randomization device. It does not involve interfirm payments.

This suggests that competition authorities should be wary of exchanges of information on past sales, even if they appear to be mere cheap talk and there is no evidence of other interfirm contacts: by itself, such communication can make collusion more efficient and lead to higher prices.

The main features of the collusive equilibrium derived in this paper

The collusive equilibrium derived in this paper exhibits features similar to those observed in many recent cartels.⁵

- The collusive scheme is based on a target market share allocation. This is indeed the case in many cartels, especially those in markets in which prices are not easily observed, for instance because they are set bilaterally between sellers and buyers.
- Colluding firms exchange detailed information on sales volumes at a high frequency. They did so every month, in the case of the lysine,⁶ zinc phosphate⁷ and citric acid⁸ cartels, and every week in the case of the Vitamins A and E cartel⁹.
- When the exchange of self-reported sales data points to a discrepancy between actual and target market shares, which can happen as a result of demand uncertainty, companies that sold above their quotas take steps to adjust their sales so as to compensate those that sold below theirs.

⁵Harrington (2006), Levenstein and Suslow (2006).

⁶Official Journal of the European Union, L 152/24, 7.6.2001, Case COMP/36.545/F3 - Amino Acids, Decision of June 7, 2000, recital 100.

⁷Official Journal of the European Union, L 153/1, 20.6.2003, Case COMP/E- 1/37.027 - Zinc phosphate, Decision of December 11, 2001, recital 69.

⁸Official Journal of the European Union, L 239/18, 6.9.2002, Case COMP/E- 1/36.604 - Citric acid, Decision of December 5, 2001, recital 100.

⁹Official Journal of the European Union, L 6/1, 10.1.2003, Case COMP/E- 1/37.512 - Vitamins, Decision of November 21, 2001, recital 577.

According to Harrington (2006), whereas in some cases cartelists compensated market share swings by making payments to each other (often under the guise of interfirm sales), in other cases the compensation was through market share reallocations of the kind highlighted in this paper, as the continuous monitoring of sales volumes afforded colluding firms “*the opportunity to adjust their sales*”. For instance, “*the citric acid and vitamins A and E cartels engaged in ‘continuous monitoring’ to assess how sales matched up with quotas and, where a firm was at a pace to sell too much by the year’s end, the firm was expected to slow down its sales*”. Likewise, in the zinc phosphate cartel, “*customer allocation was used as a form of compensation in the event of a company not having achieved its allocated quota.*”¹⁰

- Self-reported sales volumes are not instantaneously verifiable, but they can be compared to reliable data that become public with a lag. Accordingly, firms are deterred from misreporting their sales because inaccurate reports lead to price wars once they are exposed. This information structure is in line with the facts of several cartels. Reliable information can come from companies’ annual reports, verification by independent auditors appointed by the cartelists, import statistics, the (delayed) publication of tender results that is mandatory in many settings (for instance in the context of public procurement). In several recent cases, market share information was found to become available with a delay of about one year.¹¹ For instance, in the lysine cartel, “*ADM reported its citric acid sales every month to a trade association, and every year, Swiss accountants audited those fig-*

¹⁰However, one feature of our collusive equilibrium does not coincide with the abovementioned characteristics. The collusive equilibrium we focus on prescribes firms to adjust market shares as soon as sales reports point to a discrepancy relative to target market shares. Whereas such high-frequency adjustments indeed took place in some cases, it appears that they were not prescribed by the collusive agreement, in the sense that a firm’s failure to decrease its market share right after communication revealed that it had sold too much was not considered a deviation and did not trigger a punishment phase. This might be explained by the fact that delaying the ‘rebalancing’ of market shares allows firms to average out random demand shocks and hence reduce the total amount of (often inefficient) market share reallocation.

¹¹In our model, we make the strong assumption that the data that become publicly known after a delay are accurate and detailed, whereas in reality they may be noisy and imprecise. Firms may try to conceal some sales from auditors, as noted by Harrington and Skrzypacz (2011); and firms’ annual reports, just like cross-country trade data, may not be very disaggregated. However, it must also be noted that the degree of detail that is assumed in our model is not necessary for our results to hold: the assumption that total market sales become public with a lag, without the breakdown by company, would leave our results unchanged.

ures.”¹² In the Copper Plumbing Tubes cartel,¹³ import statistics were a reliable source of information on some producers’ sales and helped cartel members to monitor the veracity of the previously exchanged reports,¹⁴ just like Japanese export statistics allowed companies to check Japanese manufacturers’ sales reports in the sorbates cartel.¹⁵ In the Citric Acid cartel, companies reported information on sales every month¹⁶ and “*the sales figures reported [were] audited by the Schweizerische Treuhandgesellschaft, Coopers & Lybrand, which deliver[ed] a regular report on its findings*”. This description does not include a mention of the frequency of these regular reports, but it suggests that it was less than every month.¹⁷ Also, it must be noted that whereas in our model, individual sales are assumed to become public after a lag, the results carry over to the case in which the only information that becomes public after a lag is the volume of aggregate, rather than individual sales.

Overview of the mechanism at stake The possibility that firms could collude more efficiently by exchanging reports on their own sales may seem surprising because such reports provide no verifiable information in addition to the one that becomes public with a delay in any case. Consider a collusive scheme characterized by a market share allocation and assume that, irrespective of whether communication takes place, reliable information on each firm’s sales becomes public with a one-year lag. At first glance, it may seem that only this one-year lag should matter when analyzing the sustainability of collusion, and that communication is irrelevant. If a company is requested to report its sales every month, it can still undercut its competitors and at the same time fail to report its increased sales. Both the undercutting and the lie will be detected

¹²*Lysine* decision, recital 100 (underlined by us).

¹³Official Journal of the European Union, L 192/21, 13.7.2006, Case COMP/E-1/38.069 - Copper Plumbing Tubes, Decision of 3 September 2004.

¹⁴“*First, SANCO manufacturers (...) reported their production and sales volume figures often on a monthly basis to the secretariat. The figures were compiled and circulated among the SANCO club members. Second, until the beginning of the 1990s, import statistics allowed the control of sales figures, since they were a reliable source for data on production volumes, at least for countries like Belgium, where only one national producer (BCZ) existed.*” (*Copper Plumbing Tubes* decision, recital 141).

¹⁵Harrington (2006), pp. 36-37.

¹⁶“*Each of the participants reported the tonnage they had sold in each region (Europe, North America, and the rest of the world) to the secretariat of the ECAMA President by the seventh (day) of each month. In the secretariat these sales figures were assembled and then reported back to the members by telephone, broken down by firm and by region. This made it possible to monitor the relative market shares continuously.*” (*Citric Acid* decision, recital 100).

¹⁷*Citric Acid* decision, recital 37.

one year later, irrespective of whether firms communicate. How could monthly reports make any difference?

The answer highlighted in this paper relies on the following difference between the deterrence of undercutting and the deterrence of misreporting: misreporting can be spotted with certainty once sales information becomes public, whereas, if demand is uncertain enough, undercutting may not be distinguishable from demand shocks.

In general, colluding firms observing market share fluctuations face a tension between two goals. On the one hand, the collusive equilibrium should deter deviations, implying that a firm gaining market share should suffer some loss thereafter. On the other hand, since market share fluctuations may be caused by demand shocks as well as by deviations, colluding firms have an interest in avoiding that such fluctuations trigger price wars.¹⁸ One possible solution is for colluding firms to set up a mechanism that compensates for market share fluctuations while prices remain collusive, through transfers or market share reallocations. This paper considers more specifically market share reallocations: in the collusive equilibrium we derive, a firm whose market share exceeded its quota is expected to reduce its sales for a while by increasing its prices, while competitors still set the collusive price.

However, it is harder to satisfy incentive-compatibility constraints during a market share reallocation phase than during a price war. Unlike a price war, it is vulnerable to deviations because the prevailing price during such a phase is collusive. A firm that is supposed to decrease its sales for a while may fail to do so and profitably undercut its competitors instead.

Therefore, in some cases, the only way to deter deviations is by having market share swings lead to price wars, even though this causes prices to fall below the monopoly level along the equilibrium path.

In contrast, the deterrence of sales misreporting involves no such tension. A lie can be detected unambiguously by comparing hard data with earlier reports. This allows for maximal punishments following lies, in the form of price wars, without any efficiency loss since lies do not occur in equilibrium.

This contrast is the main factor making communication relevant to the effectiveness of collusion.

Assume that information on sales becomes publicly available with a one-period lag, and that the possible demand patterns are so diverse that a firm

¹⁸This tension was addressed for the first time in Stigler (1964). See also Green and Porter (1984).

cannot infer its market share by observing only its own sales. Without communication, sales asymmetry in Period t is revealed at the end of Period $(t + 1)$ and the market share reallocation process cannot start before Period $(t + 2)$. In contrast, if firms can report their sales at the end of every period, the threat of a price war starting in Period $(t + 2)$ in case some firm's report on its Period t sales is revealed to be inaccurate at the end of Period $(t + 1)$ (when sales data become public) induces firms to truthfully report their sales at the end of Period t . This allows firms to identify market share asymmetries right after they arise, which allows them to launch a market share reallocation phase in Period $(t + 1)$. This possibility of quickly identifying and compensating market share swings decreases firms' incentives to undercut competitors in order to appear to benefit from an asymmetric demand shock, which facilitates collusion.

In other words, each firm's ignorance of its competitors' sales prevents a compensating market share reallocation from starting right after a firm benefitted from biased demand, because a firm in such a situation cannot know whether demand was biased in its favor (which could warrant the immediate start of a correction period) or not, unless firms find a way to pool their sales data. Efficiency mandates that firms not launch a correction period till they know, but this delay increases individual incentives to deviate.

Communication could thus facilitate the recourse to market share reallocations at collusive prices, rather than to costly price wars, as a mechanism deterring deviations.¹⁹

Relation to the literature The theoretical literature on the role of communication in collusion comprises two main sets of contributions.

Several papers address pre-play communication in the presence of private information on costs or demand.²⁰ The role of communication in these contexts is to facilitate coordination on an efficient collusive outcome.

Another branch of the literature addresses the role of communication as a

¹⁹In the *Citric Acid* decision, it is mentioned that the information exchange allowed colluding firms to 'monitor the correct implementation of [the] quotas and avoid, as far as possible, the need for compensation at the end of each year' (recital 100). In that case, the outcome that colluding firms attempted to avoid thanks to the frequent exchange of sales reports was a need for compensating interfirm payments (which are ruled out by assumption in our model) rather than a price war (as per the theory presented in this paper). Despite this difference, this quote nevertheless illustrates one of the key ideas of this paper: the role of communication on sales volumes is to allow for frequent market share adjustments so as to avoid less desirable outcomes that might occur, absent communication, if market share swings were detected with a longer delay.

²⁰Athey and Bagwell (2001, 2008); see also Aoyagi (2007), Harrington (2017).

private monitoring tool in repeated games. Compte (1998) and Kandori and Matsushima (1998) derive folk theorems, showing that infinitely patient players may achieve efficiency (*i.e.*, collusion with maximum profits) if they can communicate to facilitate mutual monitoring.²¹ Our contribution differs from these on several grounds. First, the equilibria derived in these papers are highly abstract and the communication mechanisms are complex, whereas our goal is to make sense of a simple communication device, often observed in reality, namely, the exchange of own sales reports. Second, it is assumed in these papers that payments across firms are possible, whereas we rule them out. Third, these contributions do not address comparative statics, whereas we are interested in comparing the sustainability of collusion with and without communication.

Aoyagi (2002), Harrington and Skrzypacz (2011, 'HS' hereafter) and Chan and Zhang (2012, 'CZ' hereafter) are closely related to this paper since they address the role of non-verifiable sales reports. They show that there may exist collusive equilibria in which companies monitor each other by exchanging sales reports, even though these reports can never be verified.²²

This paper differs from Aoyagi (2002), HS and CZ as regards both the nature of the main result and the assumptions of the model.

Whereas these papers characterize plausible collusive equilibria involving communication under general assumptions on demand and costs, ours provides a comparative statics result (albeit at the price of highly specific assumptions), showing that for some parameter values, sustaining near-monopoly prices requires firms to engage in communication. Our result is therefore about the incremental impact on the feasibility of collusion of communication on past market outcomes.²³

Our assumptions depart from those in Aoyagi (2002), HS and CZ, regarding both the information structure and the strategy space.

Aoyagi (2002), HS and CZ assume that neither price nor sales data ever become public, whereas we assume sales data to become public with a delay.²⁴

²¹See also Obara (2009). A related paper is Rahman (2014), which describes a collusive equilibrium involving mediated or verifiable communication between firms engaged in Cournot competition.

²²In Aoyagi (2002), sales reports are imprecise, as each firm reports 'low' or 'high' sales depending on whether its sales are below or above a certain threshold.

²³Athey and Bagwell (2001, 2008) also derive comparative statics results on the role of communication. But they relate to a different type of communication in a different setting, namely, pre-play coordination to take into account private information on costs.

²⁴On the general analysis of repeated games in which information on other players' actions is revealed with lags, see Abreu, Milgrom and Pearce (1991). Fudenberg, Ishii and Kominers (2004) study the impact of cheap-talk communication in such games. Igami and Sugaya (2018)

Neither of these assumptions is inherently better than the other, since both types of information structures are found in the real world, depending on the market.

Also, HS and CZ assume that firms can make direct payments to each other, whereas we rule out such payments. Accordingly, in the collusive equilibria they derive, market share swings are compensated through interfirm payments, whereas in our model they are compensated through market share reallocations. Again, none of these features is inherently more relevant than the other, since both types of corrective mechanisms have been found to occur. The corrective market share reallocation mechanism highlighted in this paper, however, is probably better suited to the analysis of the cases for which information exchanges are the main legal question, as least from the viewpoint of policy relevance. This is because frequent interfirm payments may provide evidence of unlawful coordination that makes the analysis of information exchanges superfluous for the legal assessment.

Finally, Awaya and Krishna (2016, 'AK' hereafter, which is extended and generalized in Awaya and Krishna, 2019) prove a result related to ours, deriving conditions under which communication on past sales increases collusive profits. However, our model and AK's are relevant to different settings. AK assume that sales data remain private forever. Also, in their model, firms exchange imprecise threshold-based reports ('high' or 'low', as in Aoyagi, 2002), and collusion breaks down with a positive probability even if no firm cheats. In contrast, our model is relevant to cases characterized by the features described above: (i) private sales information becomes public after a delay (at least in aggregate form), (ii) sales reports are detailed, (iii) sales reports that reveal a discrepancy between actual and target market shares are followed by a temporary corrective reallocation phase at collusive prices, but collusion does not break down as long as firms comply with the collusive discipline.²⁵

Organization of the paper This paper is organized as follows. After presenting the model (Section 2), we derive an upper bound for expected profits in any symmetric collusive equilibrium without communication (Section 3), and we

provide a quantitative estimate of the impact of the lag of the possibility of collusion on the basis of a stylized model of the vitamins cartel.

²⁵ Another related paper is Mouraviev (2014), which shows that communication can increase collusive profits, albeit in an environment very different from ours since firms can communicate verifiable information and the only limit on communication comes from the risk of being caught colluding and having to pay a fine.

exhibit a specific equilibrium of the repeated game in which firms can exchange sales reports (Section 4). Combining these results allows us to derive a sufficient condition for such communication to expand the set of attainable profit levels (Section 5). Section 6 concludes with a few remarks on policy implications and future research.

2 The model

The main result of this paper relies on the following ingredients: (i) sales information (at the firm level, or in aggregate form) becomes public after a lag; (ii) a firm never observes other firms' prices; (iii) demand can be zero, which prevents a firm from inferring that a rival deviated from a hypothetical collusive equilibrium simply by observing its own zero sales; (iv) demand can be biased in favor of some firm, so that an asymmetric sales distribution does not necessarily allow a firm to infer that a deviation took place; and (v) aggregate demand is uncertain, so that a firm cannot infer its market share from its own sales only. The model presented below combines all these elements.

2.1 Firms

There are n ($n \geq 3$) identical firms producing a heterogeneous good at constant marginal cost $c > 0$, and facing the same rate of time preference δ . Each firm's only action, in each period, is the choice of its price (at the beginning of the period) and, in some of the games we will investigate, of its sales report (at the end of the period).

2.2 Demand

There is a continuum of consumers. Its mass is normalized to 1.

The demand function depends on the state of the world, which is drawn from some (constant) probability distribution. The draws are independent across periods. A state of the world is characterized by two parameters: total demand (which is price-inelastic), and whether demand is 'normal' or 'biased' towards one firm. Total demand can take any value Q within an interval $S = [0, S_{max})$, with $S_{max} > 0$.

There exists $V > c$ (to be interpreted as consumers' per-unit valuation of the good, except in 'biased' states of the world) such that the demand function

is as follows.

Demand in the 'normal' states of the world In a normal state of the world such that total demand is Q , consumers consider all n goods as perfect substitutes. They attribute to the homogeneous good a subjective valuation equal to V , so that if the lowest of all firms' prices does not exceed V , total sales are equal to Q and they are evenly distributed among all the firms setting the lowest price. Formally, the demand function is as follows (with p_i denoting the price set by Firm i and $D_i^N(p_1, \dots, p_n; Q)$ denoting Firm i 's sales as a function of all firms' prices, when demand is normal and total demand is Q):

- If $\text{Min}(p_1, \dots, p_n) > V$ or $p_i > \text{Min}(p_1, \dots, p_n)$ then $D_i^N(p_1, \dots, p_n; Q) = 0$;
- If $\text{Min}(p_1, \dots, p_n) \leq V$ and $p_i = \text{Min}(p_1, \dots, p_n)$ then $D_i^N(p_1, \dots, p_n; Q) = \frac{Q}{\#\{j|p_j = \text{Min}(p_1, \dots, p_n)\}}$.

Demand in the 'biased' states of the world A state of the world that is biased in favor of Firm i is one such that total demand is strictly positive and consumers have a preference for Firm i 's product: their willingness to pay for product i is some $v > V$. If demand is biased in favor of Firm i then with $D_j^{Bi}(p_1, \dots, p_n; Q; v)$ denoting Firm j 's sales as a function of all firms' prices, when total demand is Q , and consumers' valuation of good i is v , the demand function is as follows, defining $p'_i = p_i - (v - V)$, and for any $j \neq i$, $p'_j = p_j$:

- If $\text{Min}(p'_1, \dots, p'_n) > V$ or $p'_j > \text{Min}(p'_1, \dots, p'_n)$ then $D_j^{Bi}(p_1, \dots, p_n; Q; v) = 0$;
- If $\text{Min}(p'_1, \dots, p'_n) \leq V$ and $p'_j = \text{Min}(p'_1, \dots, p'_n)$ then $D_j^{Bi}(p_1, \dots, p_n; Q; v) = \frac{Q}{\#\{j|p'_j = \text{Min}(p'_1, \dots, p'_n)\}}$.

We assume that consumers' willingness to pay for product i when demand is biased in favor of Firm i is drawn from some probability distribution ν over some interval (V, V') with $V' > V$. For the sake of tractability, ν is assumed to be the uniform distribution. The ratio $(V' - c)/(V - c)$ is denoted v' . Throughout the paper we will consider cases such that $\frac{V' - V}{c}$ is close to zero.

The probability distribution over demand functions The following assumptions about the states of the world and their probability are made throughout the paper. They determine the information that firms can infer from sales

data along hypothetical collusive equilibria, and in the event of one firm deviating from such an equilibrium.

Total demand

- Notation: D_t denotes total demand in period t (when at least one firm's price is less than or equal to consumers' willingness to pay).
- Notation: D denotes total expected demand ahead of any period.
- Assumption: for simplicity, and without any loss of generality, we assume that total expected demand conditional on demand being biased is equal to total expected demand conditional on demand being normal, implying that both are equal to D .

Continuity and symmetry

- Irrespective of whether demand is normal or biased, the probability distribution over total demand is absolutely continuous.
- The demand function is symmetric in the sense of the following two assumptions: (i) the distribution of total demand conditional on demand being biased is independent of the identity of the firm in favor of which demand is biased; and (ii) the probability that demand is biased in favor of any particular firm is the same for all firms.

The possibility of a zero demand shock: assumption and implications

- Conditional on demand being normal, the distribution of total demand has a single atom at 0 and it is characterized by the probability density function $\mu^N(\cdot)$ over $(0, S_{max})$. Without loss of generality, we assume that for any $Q \in (0, S_{max})$, $\mu^N(Q) > 0$.
- The strictly positive probability that total demand is zero is denoted π^L . The possibility of a low demand shock implies that a firm selling zero cannot tell, from its own sales alone, between a low demand shock and other possible causes (such as a deviation by some competitor, or biased demand).²⁶
- Notation: for any $E \subset S$, $\mu^N(E)$ denotes the probability that total demand belongs to E conditional on demand being normal. The above

²⁶This impossibility of distinguishing between an aggregate negative demand shock and a deviation is the main element of Green and Porter's (1984) analysis.

assumptions implies that $\mu^N(E) > 0$ for any E with nonzero measure or containing 0.

Demand in biased states of the world

- Notation: conditional on demand being biased, the distribution of total demand is given by the probability density function $\mu^B(\cdot)$ over $(0, S_{max})$.
- Assumption: for all $Q \in (0, S_{max})$, $\mu^B(Q) > 0$.
- Notation: for any $E \subset S$, $\mu^B(E)$ denotes the probability that total demand belongs to E conditional on demand being biased.
- Notation: π^B denotes the probability that demand is biased. The symmetry assumption implies that the probability that demand is biased in favor of any specific set of z firms is equal to $\frac{z\pi^B}{n}$ and the above assumptions imply that $\pi^B > 0$.
- Comment about these assumptions: the assumption that the support of aggregate demand is the same for normal and for biased demand is made to simplify the proofs. The main result of this paper only requires that the intersection of these supports have a strictly positive measure. This positive-measure intersection matters because it complicates the inferences that can be made from the observation of the distribution of sales. In a normal state of the world characterized by total demand belonging to the aforementioned intersection of supports, undercutting by Firm i leads to the same distribution of sales as the absence of any deviation in the presence of biased demand (in favor of Firm i) with the same total demand. An asymmetric sales distribution can thus be explained alternatively by a deviation from a hypothetical collusive agreement or by a demand shock. This makes mutual monitoring difficult, even in the presence of public, complete information on sales.
- Assumption: $\text{Max}_{Q \in (0, S_{max})} \frac{\mu^N(Q)}{\mu^B(Q)}$ is finite. This technical assumption ensures that the informational limitation described above matters for the determination of the set of equilibria.
- Notation: A denotes $\mu^N\left(\left(0, \frac{S_{max}}{n}\right)\right)$. The above assumptions imply that $A > 0$. The possibility for demand to belong to $\left(0, \frac{S_{max}}{n}\right)$ matters for the following reason: in a hypothetical collusive equilibrium such that all

firms set the same price, a firm benefitting from biased demand with total demand smaller than $\frac{S_{max}}{n}$, and thus being the only one making nonzero sales, cannot tell after observing only its own sales whether it is lucky or whether its sales are $1/n$ -th of a larger demand pool. In other words, with a positive probability, a firm that deviated can plausibly claim that, even after observing its own sales, it could not know that sales were asymmetric. This assumption is crucial: it implies that a 'lucky' firm needs information on other firms' sales in order to know that sales were asymmetric, so that communication may facilitate collusion by accelerating the identification of sales asymmetries. Under a more general demand setup (not requiring, for instance, that the support of possible demand levels be an interval with a zero lower bound), the key assumption is simply that the set of demand levels that are equal both to some aggregate demand level under biased demand, and $1/n$ -th of a possible aggregate demand level under normal demand, have a positive probability of arising both under normal and biased demand.

- Assumption: $\text{Max}_{Q \in (0, \frac{S_{max}}{n})} \frac{\mu^N(Q)}{\mu^N(nQ)}$ is finite. This technical assumption facilitates the application of the above reasoning in the proof of our main result.
- Notations: λ_1 denotes $\frac{1}{1 - \frac{\pi^L}{1 - \pi^B}} \text{Max}_{Q \in (0, S_{max})} \frac{\mu^N(Q)}{\mu^B(Q)}$ and λ_2 denotes $\text{Max}_{Q \in (0, \frac{S_{max}}{n})} \frac{\mu^N(Q)}{\mu^N(nQ)}$. The above assumptions implies that λ_1 and λ_2 are finite. Also, if the relative probabilities of the various demand levels in S_B are independent of whether demand is normal or biased, then $\lambda_1 = 1$.

2.3 Timing of the game

The game is repeated for infinitely many periods, starting in period 1.

Each period is divided into three or four stages (depending on whether firms can make sales reports).

Stage 1. Firms simultaneously set prices.

Stage 2. The state of the world is determined at random.

Stage 3. Each firm observes the demand addressed to it and serves it.

Stage 4 (in some variants). Firms simultaneously make a statement on their own sales.

2.4 Information structure

The state of the world is never observed by any player.

At the end of period t , a firm observes only its own sales and (except for $t = 1$) the sales made by the other firms in period $t - 1$. In other words, at the end of a period, a firm knows all its past sales and all the sales of the other firms till the penultimate period. In the variant in which firms can communicate, a firm's information at the end of a period also includes all the statements made by all firms in all previous periods and in the current one. A firm cannot observe the prices set by the other firms either in the current period or in any past period.

As is explained above, these assumptions imply that with a high probability, a firm can infer little information on the state of the world, its competitors' actions, or its market share, from observing its own sales.

This difficulty in inferring market shares from own sales information is a major driver of our results. It leaves a role for communication, since pooling sales data is the only way for firms to infer market shares, even though this is in some cases not sufficient to infer the state of the world.²⁷ This assumption is realistic, since in many cartels the exchange of sales information was meant to allow colluding firms to estimate total sales and individual market shares, in environments characterized by the lack of price transparency.

It must be stressed that all the results presented in this paper carry over to the case where the information that becomes public after a one-period lag is only the level of aggregate sales, rather than the breakdown across firms.²⁸

2.5 Strategies

Strategies when communication is not possible We describe hereafter strategies in the case where communication is not possible.

Let P denote the set of possible prices (*i.e.*, the set of nonnegative real numbers) and $\Delta(P)$ set the of probability distributions with support in P .

²⁷In contrast, Marshall and Marx (2008), who study a collusive equilibrium based on a market share allocation, assume that market share data are sufficient to allow a firm to identify deviations by competitors.

²⁸A result about the possibility (resp. impossibility) of collusion is stronger if the information available to the parties is less (resp. more) precise. Since this paper is about the incremental effect of communication, that is, about the conjunction of a possibility (with communication) and an impossibility (without communication) result, none of the two possible assumptions about the informational setup is more 'minimal' than the other, and thus conducive to a stronger result in any obvious sense.

Let $f_t = (f_t^1, \dots, f_t^n)$ denote all firms' sales at the end of period t (which become publicly observable at the end of period $t + 1$).

Let p_t^i denote the price set by Firm i in period t .

For Firm i , a strategy is an infinite set of functions $(s_1, s_2, \dots, s_t, \dots)$, where s_t maps all the information known to Firm i at the beginning of period t into the set of distributions over possible prices:

$$s_t : ((f_1, \dots, f_{t-2}), f_{t-1}^i, (p_1^i, \dots, p_{t-1}^i)) \rightarrow \text{Dist}_t(\in \Delta(P)).$$

Strategies when firms can communicate at the end of each period

When communication is possible at the end of each period, strategies are modified in the following way. There is a message set M (to be specified later). At the end of each period, a firm chooses a message from M , depending on the information known to it at the end of the period (including past observed sales, past own prices, own sales in the current period, and all firms' past messages). Likewise, at the beginning of every period, the action chosen by each firm (that is, its price) depends on past own sales and prices, the other firms' publicly observed past sales, and all firms' past messages.

2.6 Symmetric equilibria

We restrict our attention to symmetric equilibria, defined as follows.²⁹

Definition - Symmetric equilibria Consider two possible series of observed sales over t periods, $f^1 = ((f_1^{11}, \dots, f_1^{1n}), \dots, (f_t^{11}, f_t^{1n}))$ and $f^2 = ((f_1^{21}, \dots, f_1^{2n}), \dots, (f_t^{21}, \dots, f_t^{2n}))$. f^1 and f^2 are said to be symmetric with respect to $Perm$ if there exists a permutation $Perm$ of $\{1, \dots, n\}$ such that for each $t' \leq t$, and each $j \in \{1, \dots, n\}$, $f_t^{1j} = f_t^{2^{Perm(j)}}$. An equilibrium pair of strategies (S^1, S^2) is said to be symmetric if there exists a permutation $Perm$ of $\{1, \dots, n\}$ such that for any $t \geq 1$, any $j \in \{1, \dots, n\}$, any pair observed sales history (f^1, f^2) of length $Min(0, t - 2)$ that is symmetric with respect to $Perm$, any history of past own prices $P_t = (p_1, \dots, p_{t-1})$, and any period $(t - 1)$ own sales y_{t-1} :

$$s_t^{1j}(f^1, y_{t-1}, P_t) = s_t^{2^{Perm(j)}}(f^2, y_{t-1}, P_t).$$

²⁹Models of collusion often make this assumption (see the discussion in Athey, Bagwell and Sanchirico, 2004).

3 The scope for collusion without communication

In this section, we address the scope for collusion when communication is impossible. Since we are interested in equilibria that involve simple behavior, we restrict our attention to pure-strategy equilibria.³⁰ We derive in this section a sufficient condition for the expected payoffs of pure-strategy symmetric equilibria (hereafter, 'PSSE') to be bounded away from $\frac{(V-c)D}{1-\delta}$, when communication is not possible.

Proposition 1. *If the inequality*

$$1 + \delta + \delta^2 + A\delta^3 > \frac{1}{n(1-\delta)} \quad (1)$$

holds and π^L , $\pi^B \lambda_2(v'-1)$, $\frac{\pi^B}{(1-\delta)\pi^L}$ and $\frac{v'-1}{(1-\delta)\pi^B\pi^L}$ are close enough to zero, then total expected per-period profits in all PSSE are bounded away from $(V-c)D$: there exists $\alpha > 0$ such that in any PSSE, the expected sum of all firms' future discounted profits is smaller than or equal to $\frac{(V-c)D}{1-\delta} - \alpha$.

We explain hereafter the intuition behind a weaker result: if π^B , π^L , and $(v'-1)$ are close enough to zero, (1) implies that there exists no PSSE such that the prevailing price is V in all periods with probability 1 along the equilibrium path (hereafter, an 'efficient' PSSE).

The reason is that the right side of (1) is equal to expected per-firm collusive profits in an efficient PSSE (divided by $(V-c)D$) and each of the four terms in the left side is a lower bound on the profit that a firm could earn in each of the first four periods respectively by slightly undercutting its competitors, assuming these competitors set prices according to a hypothetical efficient PSSE.

An efficient (and therefore, collusive) PSSE must include a mechanism deterring deviations. This means that, after sales data revealed that only Firm i had sales above zero, outcomes must be unfavorable to Firm i for a while. The efficiency requirement mandates that, unless sales data unambiguously reveal a deviation from equilibrium behavior, such an asymmetry should be corrected not through a price war, but rather by having Firm i voluntarily withdraw from the market for some time, by setting a price strictly above V . The efficiency

³⁰Several recent papers, such as Acemoglu, Bimpikis and Ozdaglar (2009) and Gentzkow and Kamenica (2014) focus on equilibria involving pure strategies along all equilibrium paths, allowing however mixed strategies off equilibrium. However, it must be acknowledged that collusive behavior based on mixed strategies along the equilibrium path seems to be observed in some industries (Wang, 2009), in line with the theoretical results showing that mixing can improve efficiency (Kandori and Obara, 1998).

requirement constrains the timing of such correction phases in a way that allows any firm to undercut its competitors and earn monopoly profits in each of the first four periods with a high probability (assuming that the probability that demand is biased is low).

Consider a PSSE, denoted Eq^* . Symmetry and efficiency jointly imply that Eq^* prescribes all firms to set price V in Period 1.

Assume that Firm 1 deviates in Period 1 and slightly undercuts its competitors. This affords Firm 1 an expected profit arbitrarily close to $(V - c)D$ in Period 1 if demand is normal (or biased in its favor). The first term in the left side of (1) is a lower bound on the corresponding first period deviation profit.

At the end of Period 1, after such a deviation, Firm 1's competitors observe that they sold zero, but they cannot infer from this information alone that a deviation took place, because zero sales could result from zero demand, or from biased demand. According to Eq^* , a firm that sold zero at the end of Period 1 must therefore set price V at the beginning of Period 2. Having slightly undercut its competitors in Period 1, Firm 1 can thus undercut its competitors again in Period 2, which yields an expected Period 2 profit arbitrarily close to $(V - c)D$ if demand is normal. The second term in (1) is a lower bound on the corresponding second period deviation profit.

At the beginning of Period 3, after such deviations occurred in the first two periods, Firm 1's competitors have two additional pieces of information. They observe that they sold zero in Period 2, but this provides no information on Firm 1's Period 2 behavior, because zero sales could result from zero or biased demand. They also observe Firm 1's Period 1 sales at the end of Period 2. If Period 1 demand is normal, Firm 1's Period 1 sales (equal to D_1) are compatible with the possibility that Firm 1 set a price equal to V in Period 1 (if demand was zero, or biased in favor of Firm 1). Therefore, the information available to Firm 1's competitors at the beginning of Period 3 is compatible with some equilibrium path. Efficiency again implies that Firm 1's competitors set prices greater than or equal to V in Period 3, allowing Firm 1 to undercut them again and earn close to the monopoly profit if demand is normal. The third term in (1) is a lower bound on the corresponding third period deviation profit.

At the beginning of Period 4, Firm 1's competitors have two new pieces of information: their own zero sales in Period 3, and Firm 1's Period 2 sales. Their own Period 3 zero sales provide no definitive information on Firm 1's Period 3 behavior, as zero sales could have been caused by zero demand. Consider now the information conveyed by Firm 1's Period 2 sales. Let $p(Q)$ denote the price

which, according to Eq^* , Firm 1 should set in Period 2 after having sold Q in Period 1. The symmetry requirement implies that, according to Eq^* , *any* firm that sold Q in Period 1 should set $p(Q)$ in Period 2. But if $Q \in (0, \frac{S_{max}}{n})$, which is the case with probability A , demand is normal in Period 1, Firm 1's sales are compatible with those that would have arisen with normal demand, and total demand equal to nQ . The possibility of such an outcome implies that Eq^* prescribe $p(Q) = V$. In other words Eq^* prescribes Firm 1 to set price V in Period 2 with probability 1 after observing that its sales were equal to Q in Period 1, because Firm 1 could not rule out the possibility that its sales were $1/n$ -th of a large demand pool. Therefore, with probability A , Firm 1's competitors cannot infer that Firm 1 deviated simply by observing its Period 2 sales: since Firm 1 was entitled to set price V in Period 2, its 100% market share in Period 2 (observed at the end of Period 3) could be explained by luck (biased demand). With probability A , Eq^* thus prescribes that Firm 1's competitors set a price equal to V in Period 4, with probability 1. This allows Firm 1 to profitably undercut its competitors in Period 4, leading to a deviation profit that is captured by the fourth term in the left side of (1).

If (1) holds, then the abovementioned deviations in the first four periods lead to a higher expected profit than compliance with the hypothetical efficient PSSE, and such a PSSE cannot exist.

The full proof of Proposition 1 (in the appendix) follows exactly the same logic. Its technical complexity comes from the need to ensure that expected per-period profits are bounded away from, rather than merely strictly lower than $(V - c)D$. Since no step in the proof of Proposition 1 makes use of the assumption that the information that becomes public is about the breakdown of individual firms' sales in the previous period, Proposition 1 is also valid under the assumption that this public information is only about aggregate sales in the previous period. This is not surprising: with less available information, the set of collusive outcomes does not expand.³¹

³¹Sugaya and Wolitzky (2018a) construct a model with the property that more accurate information decrease the set of attainable collusive profits, because firms having more information may identify more profitable deviations. However, such a mechanism is not present here, since the deviations considered in the proof of Proposition 1 do not rely on the deviating firm's information on other firms' sales.

4 The scope for collusion with communication

4.1 An informal overview of the role played by communication

We consider the same environment as before, with one addition: at the end of every period, each firm is required to report its sales. Formally, each firm chooses a message at the end of each period. The meaning of the message is understood to be about each firm's sales.

The role of communication in a collusive equilibrium is the following. In contrast to the situation prevailing in the absence of communication (see the above discussion of Proposition 1), communication prevents a firm from enjoying a 100% market share at the collusive price during four periods. In the presence of communication, this would be possible only for three periods.

Assume that at the end of each period, each firm is expected to report its sales, and that (i) whenever a firm is found to have lied, a price war ensues; and (ii) if reported sales point to an asymmetry, this leads to a market share reallocation phase at the expense of the 'lucky' firm, for a few periods.

Consider a firm (say, Firm 1) contemplating the following deviation: slightly undercutting its competitors by pricing just below the collusive price for as long as it can. If it does so in Period 1 and then lies about its Period 1 sales, its lie will be exposed at the end of Period 2, leading to a price war at the beginning of Period 3. In the end, Firm 1 benefits from undercutting during Periods 1 and 2 only.

Alternatively, Firm 1 could decide to truthfully report its sales. In this case, all firms would know that sales were asymmetric at the end of Period 1, and the equilibrium would prescribe a market share reallocation phase starting in Period 2, during which Firm 1 would sell zero. If Firm 1 decides to deviate and slightly undercut its competitors in Period 2, then, unless demand was zero in Period 2, its deviation is revealed at the end of Period 3, leading to a price war at the beginning of Period 4. In the end, Firm 1 benefits from undercutting in Periods 1 to 3.

In other words, in the presence of communication, a firm can deviate and obtain a 100% market share slightly below the collusive price for three consecutive periods at most, whereas it could do so for four periods in the absence of communication (at least for some realizations of demand). The resulting decrease in the profitability of undercutting makes efficient collusion more likely. Also, in

equilibrium, the compensating market share reallocation following asymmetric sales outcomes (resulting from biased demand) starts one period after the asymmetric outcome, rather than two periods later in the absence of communication. This makes it easier to deter deviations, which facilitates collusion.

4.2 Description of the candidate equilibrium

We consider the following strategy profiles. There exist integers k (to be interpreted as the duration of compensating market share reallocation phases after an asymmetric sales outcome is inferred from sales reports) and k' (to be interpreted as the duration of a price war following a deviation, which cannot occur along an equilibrium path) and a price $p^w < c$ (to be interpreted as the price level during a price war); such that, at the beginning of a period, the state of the game can be any of the following ($nk + k' + 1$):

- normal collusion;
- j -th period of a price war ($1 \leq j \leq k'$), which cannot occur in equilibrium;
- correction at the expense of some firm i ($1 \leq i \leq n$), with r remaining correction periods ($1 \leq r \leq k$) (nk states of the game).³²

In a nutshell, normal collusion gives way to temporary correction phases whenever sales reports point to asymmetric sales. During a correction phase, the firm that sold more than the others in the last normal collusion period sells zero, after which firms return to normal collusion.³³ Price wars are out-of-equilibrium events that occur following the detection of a deviation (such as an inaccurate sales report or evidence that a firm failed to set the price prescribed by the equilibrium strategy profile).³⁴

The expected duration of a price war (resulting from the transition rules described hereafter) and the level of the associated below-cost price are such

³²In what follows, the firm at the expense of which a correction takes place is sometimes called the 'targeted firm'.

³³As is explained below, transition rules are a bit more complex than this summary description, to account for the possibility that during a correction phase at the expense of some firm, another firm benefitted from biased demand, in which case it becomes the target of a new correction phase.

³⁴That price wars cannot occur in equilibrium is in contrast to HS. In HS, the incentive not to misreport sales results from the existence of a function associating to each vector of sales reports the probability of a shift to a noncollusive phase. This function is such that truth-telling is an equilibrium strategy. In our model, the assumption that sales data become public with a lag allows for a simpler mechanism: lies trigger price wars once they are exposed as lies.

that a firm's expected discounted sum of future profits at the start of the first period of a price war is zero.

Prices and messages in the candidate equilibrium Equilibrium actions at the beginning of Period t depend only on the state of the game at the beginning of Period t :

- If the state of the game at the beginning of Period t is 'normal collusion', then all firms set a price equal to V ;
- If the state of the game at the beginning of Period t is 'collusion with a correction at the expense of Firm i ', then Firm i sets a price equal to $V' + 1$ whereas all other firms set a price equal to V ;
- If the state of the game at the beginning of Period t is 'price war', then all firms set a price equal to p^w .

Also, in the candidate equilibrium, each firm's message at the end of a period consists in truthfully reporting its sales after any history along which it acted according to the candidate equilibrium in the previous periods.

Transitions between states of the game in the candidate equilibrium

The state of the game at the beginning of Period 1 is 'normal collusion'. At the beginning of Period t ($t \geq 2$), the state of the game is determined as follows.

- *Evidence of lying leads to a price war.* (For $t \geq 3$). Whatever the state of the game at the beginning of Period $(t - 1)$, if the information on Period $(t - 2)$ sales, which becomes public at the end of Period $(t - 1)$, does not coincide with some firm's report (sent at the end of Period $(t - 2)$), the state of the game at the beginning of Period t is 'first period of a price war'.³⁵ Also, if the sales reported by all firms at the end of Period $(t - 1)$ do not correspond to any possible equilibrium outcome (for instance, if the sum of the reported sales does not belong to S , or if sales are neither all equal as should occur if demand is normal, nor all but one equal to zero as should occur if demand is biased), which implies that at least one firm lied, the state of the game at the beginning of Period t is 'first period of a

³⁵If the assumption about the informational setup is that only aggregate sales data become public after a one-period lag, this sentence should be replaced with 'if at the end of Period $(t - 1)$, there is a discrepancy between the sum of reported sales for Period $(t - 2)$ and observed aggregate sales for Period $(t - 2)$ '.

price war'. In other words, lying, which cannot occur along an equilibrium path, leads to a price war.

- *Price wars start over unless reported sales are equal and strictly positive.* If the state of the game at the beginning of Period $(t - 1)$ is ' r -th period of a price war' with $r \leq k'$, then (i) if reported sales at the end of Period $(t - 1)$ are equal and strictly positive and there is no evidence of lying, then the state of the game at the next period is ' $(r + 1)$ -th period of a price war' if $r < k'$ and 'normal collusion' if $r = k'$; (ii) otherwise, the state of the world at the beginning of Period t is 'first period of a price war'. This means that for firms to progress towards the end of a price war rather than starting it over, it must be the case that sales reports unambiguously point towards compliance with the prescribed actions at the previous stages of the price war. Notice that with these transition rules, firms may have to start over a price war from the first period even though no firm deviated from the prescribed price war actions: in any period of a price war, if demand is biased or low and firms set prices according to the prescribed strategies, the state of the game in the next period is 'first period of a price war'.
- *Normal collusion is followed by normal collusion if reported sales are equal.* If the state of the game at the beginning of Period $(t - 1)$ is 'normal collusion', and none of the above conditions holds (*i.e.*, there is no evidence of lying) and all firms' reported sales for Period $(t - 1)$ are equal, then the state of the game remains 'normal collusion' at the beginning of Period t .
- *Normal collusion is followed by a correction if reported sales are unequal but compatible with equilibrium behavior.* If none of the above conditions holds (*i.e.*, there is no evidence of lying), the state of the game at the beginning of Period $(t - 1)$ is 'normal collusion' and for some i , Firm i 's reported sales for Period $(t - 1)$ are some $Q \in (0, S_{max})$ whereas all other firms' reported Period $(t - 1)$ sales are zero, then the state of the game at the beginning of Period t is 'correction at the expense of Firm i , with k remaining periods'. In other words, if a firm appears to have been lucky (demand was biased in its favor), sales are reallocated at the expense of this firm for k periods.
- *Normal collusion is followed by a price war if reported sales reveal pricing that is incompatible with equilibrium behavior.* If the state of the game at

the beginning of Period $(t - 1)$ is 'normal collusion' and none of the above conditions is met, then the state of the game at the beginning of Period t is 'first period of a price war'.

- *A correction phase leads to a price war if reported sales reveal a lack of compliance.* If the state of the world at the beginning of Period $(t - 1)$ is 'collusion with a correction at the expense of Firm i ', and reported sales are neither such that Firm i sold zero while all others sold equal amounts adding up to some $Q < S_{max}$ (which should be observed if all firms complied with the candidate equilibrium and demand was normal or biased in favor of a Firm i), nor that all firms but one sold zero and one firm (different from Firm i) had sales belonging to $(0, S_{max})$ (which should be observed if all firms complied with the candidate equilibrium and demand was biased in favor of a firm other than Firm i), then the state of the world at the beginning of Period t is 'first period of a price war'.
- *As long as there is no evidence of lying nor lack of compliance, a correction phase continues, until it gives way to 'normal collusion'.* At the end of a correction period at the expense of Firm i with r remaining periods, if there is no evidence of lying nor lack of compliance, and the sales reported by all firms other than Firm i at the end of Period $(t - 1)$ are equal, then at the beginning of period t the state of the game is 'normal collusion' if $r = 1$ and 'correction at the expense of Firm i with $(r - 1)$ remaining periods' if $r > 1$. If there is no evidence of lying nor of lack of compliance, but reported sales reveal that one firm other than Firm i (labeled Firm j) had nonzero sales, whereas all others had zero sales, then at the beginning of Period t the state of the world is 'correction at the expense of Firm j , with k correction periods remaining'. In other words, if a non-targeted firm benefits from biased demand, the correction at the expense of the previously targeted firm stops (even if there remained more than one period), and a new k -period correction phase starts, targeting the firm that just benefitted from biased demand.³⁶

It can easily be verified that the above candidate equilibrium is symmetric and involves only pure strategies, and also that along the (candidate) equilibrium

³⁶This property of the candidate equilibrium is meant for the sake of tractability. There may exist other, more complex equilibria, which make efficient collusion possible for a broader set of parameters than the candidate equilibrium outlined in this paper.

path, total profits are equal to $(V - c)D$ in every period.

4.3 A sufficient condition for the candidate equilibrium to be an actual equilibrium

Proposition 2. *If π^B , π^L and $\frac{V'-V}{c}$ are close enough to zero, and there exists an integer k such that Conditions (2) and (3)*

$$\frac{\delta^k}{n(1-\delta)} > 1 + \delta \quad (2)$$

$$\frac{1}{n(1-\delta)} > 1 + \frac{\delta^{k+1}}{n(1-\delta)} \quad (3)$$

hold, then there exist a positive integer k' and a price p^w such that the above strategy profiles correspond to a PSSE (with correction phases lasting k periods and price wars lasting k' periods with price p^w).

The proof of Proposition 2 is in the appendix.³⁷ We explain hereafter the role of conditions (2)-(3).

Condition (2) implies that a firm subjected to a correction finds that it more profitable to comply with the prescribed equilibrium behavior and sell nothing for k periods, rather than to undercut its competitors and earn monopoly profits for at most two periods (since its failure to reduce its sales will be detected with a one-period lag, leading then to a price war). (2) also implies that each firm prefers to report its true sales even if this triggers a correction at its expense.

Condition (3) implies that a firm not subjected to a correction has no interest in undercutting its competitors: the left-hand term corresponds to the expected value of complying with the candidate equilibrium, whereas the right-hand term is an upper bound on the sum of the one-period deviation profit and the sum of subsequent expected profits (after the end of the correction phase).

The granularity of sales data Implementing the strategies in the above equilibrium does not require an informational assumption as strong as ours, namely, that data on individual sales become verifiable with a one-period lag. These strategies could be implemented under the weaker assumption that reliable data on total market sales become available after a one-period lag, because

³⁷The possibility of constructing a demand function satisfying the assumptions of the model, with π^B , π^L and $\frac{V'-V}{c}$ arbitrarily close to zero is established in the proof of Proposition 3 in the appendix.

such information would allow firms to identify whether any of them deviated by misreporting sales: if all firms but one, in accordance with the candidate equilibrium, truthfully report their sales, then the fact that one firm lied will become common knowledge after a one-period lag, as all firms observe that the discrepancy between actual and reported total sales become public. In such a setting, the identity of the liar would not be known to other firms, but this would have no impact since the equilibrium prescribes a symmetric price war forever after any evidence that one firm lied, rather than targeted retaliation. A similar reasoning implies that, in order to facilitate collusion through the mechanism described in this section, communication need not allow each firm to learn the detailed distribution of current-period sales. Communication through a third party, which would collect firms' own-sales reports and communicate to all firms information on total sales based on the reports it received would fulfill exactly the same function as direct communication on own sales. Such communication would allow each firm to know whether demand was biased in its favor, which is sufficient to trigger an immediate market reallocation phase.³⁸

5 Comparative statics: the marginal impact of communication

Proposition 1 provides a sufficient condition for total expected per-period profits to be bounded away from $(V - c)D$ in any PSSE without communication, and Proposition 2 provides a sufficient condition for the existence of a PSSE with communication yielding total expected per-period profits $(V - c)D$. Combining these results leads to a sufficient condition for total per-period expected profits as high as $(V - c)D$ to be attainable (or even approachable) in a PSSE *only* in the presence of communication. We find that for some parameter values, communication on self-reported sales increases the level of the collusive profits that can be attained in a PSSE - even though the underlying information structure is the same with and without communication, in the sense that the timing of sales verifiability is not altered by communication:

Proposition 3. *For each n between 3 and 10, there exist parameter values such*

³⁸This remark is at odds with the oft-made claim that “aggregating the data [on firms’ historical and current prices, costs, and output] largely removes the value of information in facilitating collusion” (Carlton, Gertner and Rosenfield, 1997). Awaya and Krishna (2018) present another mechanism through which the communication of aggregate sales data to individual firms can facilitate collusion.

that, if communication is possible, there exists a PSSE leading to a discounted sum of total expected profits $\frac{(V-c)D}{1-\delta}$, whereas if communication is not possible, this sum, considered over all PSSE, is bounded away from $\frac{(V-c)D}{1-\delta}$.

To prove this result, we limit our focus to the case in which π^B , π^L , $\frac{V'-V}{c}$, $\frac{\pi^B}{\pi^L}$ and $\frac{v'-1}{\pi^L}$ are close enough to zero.³⁹

Numerical simulations show that, if A is close enough to 1, then for each n between 3 and 10, there is a value of k and a nonempty interval of values of δ (bounded away from 1) such that equations (1), (2) and (3) are satisfied, implying that for the values of δ in this interval, there exist equilibria with communication of the kind described in Section 4 (with correction phases lasting k periods) leading to total expected per-period profits of $(V - c)D$ whereas profits in all PSSE without communication are bounded away from this value. For the sake of readability, we present these results in terms of the discount rate applied over a period, that is, the rate ρ such that $\delta = \frac{1}{1+\rho}$. As Figure 1 shows, communication substantially expands the set of discount rates compatible with the existence of a PSSE leading to total expected per-period profits as high as $(V - c)D$.⁴⁰

INSERT FIGURE 1.

6 Conclusion

The model presented in this paper casts light on the marginal impact of communication on the feasibility of collusion: the exchange of sales reports may lead to higher prices because it facilitates the recourse of colluding firms to incentive-compatible market share reallocation mechanisms, limiting the need for price wars. This effect, which is similar to the one identified by Awaya and Krishna (2016, 2019) but applies to a different information structure (sales data become public information after some delay in our model, whereas they remain private forever in theirs), may be present even though the information exchange is mere

³⁹Doing so requires one to check that it is possible to construct a stochastic demand function satisfying the assumptions of the model, and such that π^B , π^L , $\frac{V'-V}{c}$, and the ratios $\frac{\pi^B}{\pi^L}$ and $\frac{v'-1}{\pi^L}$ are close to zero while λ_1 and λ_2 remain bounded. This is done in the appendix.

⁴⁰Since Proposition 1 states a necessary condition for the existence of a symmetric collusive equilibrium approaching monopoly profits without communication, and Proposition 2 states a sufficient condition for the existence of a SECE with communication, Graph 1 is conservative. The entire set of the discount rates such that communication allows for higher collusive profits is likely to strictly include the interval shown in Graph 1. This is why the legend for the lower part of each bar contains the word “may”.

cheap talk, in the sense that it does not make sales data verifiable sooner than they would be without communication.

While the assumption that sales data become public after a delay is strong, it nevertheless is relevant to many markets. Also, our results are robust to weakening this assumption. Assume for instance that sales data become public after being audited by third parties (as was the case in several cartels mentioned in the introduction as a motivation for this paper). If a firm that undercuts its competitors and increases its sales as a result can, with a small probability, succeed in concealing some sales from its auditor, then our results should carry over, since the profitability of undercutting is a continuous function of the probabilities of the various states of the world in the subsequent periods.

Turning to the implications for antitrust enforcement, our results suggest that the exchange of sales reports should be considered suspicious if reports revealing market share swings lead to prompt compensating movements, to an extent that cannot be explained by individual firms' unilateral profit maximization behavior, given the intertemporal pattern of demand shocks. Competition authorities should not rule out the possibility that communication is meant to facilitate such compensation mechanisms, even if the data that are exchanged between firms are not verifiable when communication takes place, and communication does not affect the date at which they will become verifiable.

Also, since our results continue to hold under the assumption that the data becoming public after a lag are about total sales only, or when communication takes place through a third party that reports total sales (based on firms' own statements), they imply that the availability of reliable data on total sales, even with a lag, may facilitate efficient collusion if firms also engage in cheap-talk communication on individual sales. This confirms that the dissemination of aggregated sales data by trade associations need not be as innocuous as has been suggested sometimes.⁴¹

Admittedly, our model relies on special and restrictive assumptions about demand. This is in our view the price to pay in order to estimate an upper bound on firms' profits in *all* (pure strategy, symmetric) equilibria without communication, which then allows us to state results on the marginal effect of communication. This is because little is known yet on bounds on equilibrium

⁴¹See, *e.g.*, Carlton, Gertner and Rosenfield (1997). Notice that even in the simpler setting of Green and Porter (1984), the availability of reliable data on total sales would be enough to make efficient collusion (without price wars) possible for patient enough firms, without any need for it to be complemented by cheap-talk communication.

payoffs in infinitely repeated games under general assumptions (except at the limit when players are almost infinitely patient). However, recent advances on this topic⁴² may pave the way for results on the marginal impact of communication under less special assumptions. This should be the focus of future research.

Appendix

Proof of Proposition 1. We prove the following result, which implies Proposition 1: if the condition

$$\begin{aligned}
& (1 - h_1)(1 - \pi^B) + \delta(1 - h_2)(1 - \pi^B)^2 + \delta^2 \left(1 - \frac{\pi^L}{1 - \pi^B} - h_3\right)(1 - h_3)(1 - \pi^B)^3 \\
& + \delta^3 \left(\left(A - \frac{\pi^L}{1 - \pi^B} - \lambda_2 \frac{\pi^B(v' - 1) \left(1 + \frac{1}{\delta^4(1 - \delta)}\right)}{(1 - \pi^B)^2} - h_4 \right) \left(1 - \frac{\pi^L}{1 - \pi^B}\right) \right) (1 - h_4)(1 - \pi^B)^4 \\
& > \frac{1 + \pi^B(v' - 1)}{n(1 - \delta)} \quad (4)
\end{aligned}$$

holds, with the following notations

$$\begin{aligned}
h_1 &= \frac{\pi^B(v' - 1)}{\delta^4(1 - \delta)} \\
h_2 &= \frac{\pi^B(v' - 1) \left(1 + \frac{1}{\delta^4(1 - \delta)}\right)}{(\pi^L + \pi^B)(1 - \pi^B)} \\
h_3 &= \sqrt{\frac{n\lambda_1(v' - 1) \left(1 + \frac{1}{\delta^4(1 - \delta)}\right)}{\pi^L(1 - \pi^B)}} \\
h_4 &= n\lambda_1 \sqrt{\frac{(v' - 1) \left(1 + \frac{1}{\delta^4(1 - \delta)}\right) (\pi^L + \pi^B)}{\pi^L \pi^B \left(\pi^L(1 - \pi^B) - (\pi^B)^2 - \frac{\pi^B}{\delta^4(1 - \delta)}\right)}},
\end{aligned}$$

then total expected per-period profits in all PSSE are bounded away from $(V - c)D$: there exists $\alpha > 0$ such that in any PSSE, the expected sum of all firms' future discounted profits is smaller than or equal to $\frac{(V - c)D}{1 - \delta} - \alpha$.

We consider a hypothetical PSSE E_{q^*} such that the expectation in Period 0 of combined profits in each of the first four periods is strictly greater than $(V - c)D(1 - \sigma)$ for some small σ . The core of this proof is the calculation of a lower bound for the profits that Firm 1 could earn by following a certain strategy,

⁴²Pai, Roth and Ullman (2016); Sugaya and Wolitzky (2017, 2018b).

which can be interpreted as undercutting in each of the first four periods. In order to calculate this lower bound, we need intermediate results about the strategies followed by Firms 2 to n conditional on the information available to them. The information structure implies that the actions undertaken by Firms 2 to n in Period t depend only on J_t , which in this proof denotes the vector representing the information collectively available to these $(n - 1)$ firms, that is, the sales made by all firms (including Firm 1) till Period $(t - 2)$ and the sales made by Firms 2 to n in Period $(t - 1)$, with obvious adjustments for $t = 1$ and $t = 2$. For any such J_t , and for any set IS_t of possible such information vectors, let $Pr(J_t)$ (resp. $Pr(IS_t)$) denote the (density) probability that the information collectively available to Firms 2 to n at the beginning of Period t is J_t (resp. belongs to IS_t) conditional on all firms having behaved according to Eq^* till Period $(t - 1)$ included; and for a history h_{t-1} of prices set by Firm 1 till Period $(t - 1)$, let $Pr(J_t, h_{t-1})$ (resp. $Pr(IS_t, h_{t-1})$) denote the (density) probability that Firms 2 to n collectively have information J_t (resp. belonging to IS_t) at the beginning of Period t , conditional on Firms 2 to n having behaved according to Eq^* till Period $(t - 1)$ included and on Firm 1's having set prices according to h_{t-1} .⁴³

Step 1: Firm 1's possible deviation in Period 1. Since Eq^* is symmetric, it prescribes all firms to set the same price p_1^* in Period 1. The total expected profit induced by Eq^* in Period 1 is thus less than or equal to $D(p_1^* - c)$, implying that $p_1^* - c > (V - c)(1 - g_1)$, with $g_1 = \sigma$. If Firm 1 deviates and sets a price equal to $(V - c)(1 - g_1) + c$ in Period 1, it serves the entire demand if demand is normal or biased in its favor, *i.e.*, with a probability exceeding $1 - \pi^B$.

Step 2: Firm 1's possible deviation in Period 2. Since Eq^* is a symmetric equilibrium, it prescribes any firm that sold zero in Period 1 to set the same price $p_2^*(0)$ in Period 2. The total expected profit induced by Eq^* in Period 2 conditional on at least $(n - 1)$ firms having sold zero in Period 1 is thus less than or equal to $D((1 - \pi^B)(p_2^*(0) - c) + \pi^B(V - c)v')$. Since the probability that at least $(n - 1)$ firms sold zero in Period 1, conditional on all firms (including Firm 1) having set Period 1 prices according to Eq^* , is $\pi^L + \pi^B$, and total expected profits in any period cannot exceed $(V - c)D(1 - \pi^B + \pi^B v')$, the

⁴³These probabilities reflect the interplay between firms' actions and the realization of uncertainty regarding demand (both whether demand is normal or biased, and the level of total demand).

expected sum of Period 2 profits induced by Eq^* is less than or equal to

$$(V - c) D \left((1 - \pi^L - \pi^B) (1 - \pi^B + \pi^B v') + (\pi^L + \pi^B) \left((1 - \pi^B) \frac{p_2^*(0) - c}{V - c} + \pi^B v' \right) \right),$$

implying that $p_2^*(0) - c > (V - c)(1 - g_2)$, with $g_2 = \frac{\sigma + \pi^B(v' - 1)}{(\pi^L + \pi^B)(1 - \pi^B)}$. This implies that if Firm 1 sets a price equal to $(V - c)(1 - g_1) + c$ at the beginning of Period 1 and to $(V - c)(1 - g_2) + c$ at the beginning of Period 2, then its price is the lowest one in Period 1 with probability 1, and again in Period 2 with probability 1 if demand was not biased in Period 1 (which ensures that Firms 2 to n sell zero in Period 1 following this deviation by Firm 1). Therefore, these deviations in Periods 1 and 2 yield Firm 1 an expected profit equal to $(V - c) D(1 - g_1)$ in Period 1 with probability greater than $(1 - \pi^B)$ and equal to $(V - c) D(1 - g_2)$ in Period 2 with probability greater than $(1 - \pi^B)^2$, while all other firms sell zero in both periods.

Step 3: Firm 1's possible deviation in Period 3. We assume that Firm 1 (deviating from equilibrium) did set a price equal to $(V - c)(1 - g_1) + c$ at the beginning of Period 1 and to $(V - c)(1 - g_2) + c$ at the beginning of Period 2 and that demand was not biased in either Period 1 or Period 2 - which is the case with probability $(1 - \pi^B)^2$. Conditional on this, it is the case that with probability $1 - \frac{\pi^L}{1 - \pi^B}$, $D_1 > 0$. For each $Q > 0$, let $p_3^*(Q)$ denote the price prescribed by Eq^* in Period 3 for a firm having observed that (i) it sold zero in Period 2; and (ii) in Period 1, one of the firms (not itself) sold Q while all others sold 0. We also define $g_3(Q)$ by the identity $p_3^*(Q) - c = (V - c)(1 - g_3(Q))$. Let $J_3(Q)$ denote the following information collectively available to Firms 2 to n at the beginning of Period 3: Firm 1 sold Q in Period 1, and Firms 2 to n sold zero in Periods 1 and 2. The assumption that Eq^* is symmetric implies that $Pr(J_3(Q)) = \frac{\pi^B}{n} \left(\frac{\pi^B}{n} + \pi^L \right) \mu^B(Q)$. The same reasoning as in Step 2 implies that the expected sum of Period 3 profits induced by Eq^* is less than or equal to

$$(V - c) D \left(\left(1 - \frac{\pi^B}{n} \left(\frac{\pi^B}{n} + \pi^L \right) \right) (1 - \pi^B) + \pi^B v' \right) + \frac{\pi^B}{n} \left(\frac{\pi^B}{n} + \pi^L \right) (1 - \pi^B) \int_{(0, S_{max})} \mu^B(Q) (p_3^*(Q) - c) dQ.$$

The assumption that this must be strictly greater than $(V - c) D(1 - \sigma)$ implies that $\int_{(0, S_{max})} \mu^B(Q) g_3(Q) dQ < \frac{\sigma + \pi^B(v' - 1)}{\frac{\pi^B}{n} \left(\frac{\pi^B}{n} + \pi^L \right) (1 - \pi^B)} < n \frac{\sigma + \pi^B(v' - 1)}{\pi^B \pi^L (1 - \pi^B)}$ so that $\int_{(0, S_{max})} \mu^N(Q) g_3(Q) dQ < \lambda_1 \left(1 - \frac{\pi^L}{1 - \pi^B} \right) n \frac{\sigma + \pi^B(v' - 1)}{\pi^B \pi^L (1 - \pi^B)}$. We define $g_3 =$

$\sqrt{\lambda_1 n \frac{\sigma + \pi^B (v' - 1)}{\pi^B \pi^L (1 - \pi^B)}}$. The last inequality implies the existence of $S_{B3} \subset (0, S_{max})$ such that (i) $\pi^N(S_{B3}) > \pi^N((0, S_{max})) - g_3 = 1 - \frac{\pi^L}{1 - \pi^B} - g_3$ and (ii) $\forall Q \in S_{B3}, g_3(Q) < g_3$.

Therefore, if Firm 1 did set a price equal to $(V - c)(1 - g_1) + c$ at the beginning of Period 1, to $(V - c)(1 - g_2) + c$ at the beginning of Period 2, and to $(V - c)(1 - g_3) + c$ at the beginning of Period 3, then with a probability strictly greater than $(1 - \pi^B)^3 \left(1 - \frac{\pi^L}{1 - \pi^B} - g_3\right)$, demand is normal in each of the first three periods and Firm 1's price is the lowest of all firms' in Period 1, Period 2 and Period 3, yielding Firm 1 an expected Period 3 profit of at least $(V - c)D(1 - g_3)(1 - \pi^B)^3 \left(1 - \frac{\pi^L}{1 - \pi^B} - g_3\right)$.

Step 4: Firm 1's Period 2 prices according to Eq^* after it observed that its own Period 1 nonzero sales.

Eq^* prescribes all firms to set the same price in Period 1. Since Eq^* prescribes pure strategies, this implies that for all such $Q \in (0, S_{max})$ there exists a price $p_2^*(Q)$ such that a firm having sold Q in Period 1 sets a price equal to $p_2^*(Q)$ in Period 2. We define the set S' as follows: $S' = \{Q \text{ s.t. } Q \in (0, \frac{S_{max}}{n}) \text{ and } p_2^*(Q) > V\}$. With probability density $(1 - \pi^B) \mu^N(nS')$ (using the obvious notation nS' for the set of all elements of S' multiplied by n), demand is normal and each firm sells some $Q \in S'$ in Period 1, implying that all firms set a price strictly above V in Period 2, leading to expected Period 2 profits smaller than or equal to $\pi^B(V - c)Dv'$. The assumption that expected Period 2 profits are greater than $(V - c)D(1 - \sigma)$ implies that

$$((1 - (1 - \pi^B) \mu^N(nS')) (1 - \pi^B + \pi^B v') + (1 - \pi^B) \pi^B v' \mu^N(nS')) > (1 - \sigma),$$

or $\mu^N(nS') < \frac{\sigma + \pi^B (v' - 1)}{(1 - \pi^B)^2}$, implying $\mu^N(S') < \lambda_2 \frac{\sigma + \pi^B (v' - 1)}{(1 - \pi^B)^2}$.

Step 5: the probability, according to Eq^* , that Firm 1's sales belong to $(0, \frac{S_{max}}{n}) \setminus S'$ in Period 1 and to $(0, S_{max})$ in Period 2 while all other firms sell zero in Periods 1 and 2. According to Eq^* , the probability that Firm 1's Period 1 sales belong to $(0, \frac{S_{max}}{n}) \setminus S'$ whereas all other firms' Period 1 sales are zero is $\frac{\pi^B}{n} (1 - \mu^B(S') - \mu^B(\frac{S_{max}}{n}, S_{max}))$. If Firm 1's Period 1 sales belong to $(0, \frac{S_{max}}{n}) \setminus S'$ whereas all other firms' Period 1 sales are zero, then according to Eq^* , in Period 2 Firm 1 sets a price $p_2^*(Q) \leq V$ and all other firms set a price $p_2^*(0) > (V - c)(1 - g_2) + c$ (see Step 2 above), so that $p_2^*(Q) - p_2^*(0) < (V - c)g_2$, implying that the entirety of Period 2 demand goes to Firm 1 if demand is biased in Firm 1's favor and consumers' valuation v of Firm 1's product is such that $\frac{v - c}{V - c} - 1 > g_2$, which is the case with

probability $\frac{\pi^B}{n} \nu(((V-c)(1+g_2)+c, V')) = \frac{\pi^B}{n} \left(1 - \frac{g_2}{v'-1}\right)$ if $g_2 < v' - 1$, or $\frac{\sigma + \pi^B(v'-1)}{(\pi^L + \pi^B)(1 - \pi^B)} < v' - 1$. From here onwards, we assume that $\frac{\pi^B}{\pi^L}$ is close enough to zero and that σ is small enough relative to $v' - 1$, so that $g_2 < v' - 1$. The probability according to E_{q^*} that firms other than Firm 1 sell zero in the first and the second period whereas Firm 1's sales in these periods belong respectively to $(0, \frac{S_{max}}{n}) \setminus S'$ and $(0, S_{max})$ is given by the probability density function $\left(\frac{\pi^B}{n}\right)^2 \mu^B(Q_1) \mu^B(Q_2) \nu((V + p_2^*(Q) - p_2^*(0), V'))$ over the set of pairs $(Q_1, Q_2) \in ((0, \frac{S_{max}}{n}) \setminus S') \times (0, S_{max})$. Notice that the latter expression is greater than $\left(\frac{\pi^B}{n}\right)^2 \mu^B(Q_1) \mu^B(Q_2) \left(1 - \frac{g_2}{v'-1}\right)$.

Step 6: The Period 4 prices set by all other firms after having sold zero in Periods 1 to 3 and observing that Firm 1's sales belong to $(0, \frac{S_{max}}{n}) \setminus S'$ in Period 1 and $(0, S_{max})$ in Period 2. For any $(Q_1, Q_2) \in ((0, \frac{S_{max}}{n}) \setminus S') \times (0, S_{max})$, let $p_4^*(Q_1, Q_2)$ denote the price prescribed by E_{q^*} in Period 4 for a firm having observed that (i) it sold zero in Period 3; and (ii) some other firm (say Firm 1) sold Q_1 in Period 1 and Q_2 in Period 2, while all other firms sold zero in the first two periods. Step 5 implies that the probability that a firm makes this observation is given by a density function over $((0, \frac{S_{max}}{n}) \setminus S') \times (0, S_{max})$ that takes a value greater than $\pi^L \left(\frac{\pi^B}{n}\right)^2 \mu^B(Q_1) \mu^B(Q_2) \left(1 - \frac{g_2}{v'-1}\right)$ at (Q_1, Q_2) . The same reasoning as in Steps 2 and 3 implies that the expected sum of Period 4 profits induced by E_{q^*} is less than or equal to

$$(V-c) D \left(\pi^B v' + (1 - \pi^B) \left(1 - \pi^L \left(\frac{\pi^B}{n} \right)^2 \left(1 - \frac{g_2}{v'-1} \right) (1 - \mu^B(S') - \mu^B(\frac{S_{max}}{n}, S_{max})) \right) \right) \\ + \pi^L \left(\frac{\pi^B}{n} \right)^2 \left(1 - \frac{g_2}{v'-1} \right) (1 - \pi^B) \int_{((0, \frac{S_{max}}{n}) \setminus S') \times (0, S_{max})} \mu^B(Q_1) \mu^B(Q_2) (p_4^*(Q_1, Q_2) - c) dQ_1 dQ_2$$

With the notation $p_4^*(Q_1, Q_2) = (V-c)(1 - g_4(Q_1, Q_2)) + c$, this implies

$$\int_{((0, \frac{S_{max}}{n}) \setminus S') \times (0, S_{max})} \mu^B(Q_1) \mu^B(Q_2) g_4(Q_1, Q_2) dQ_1 dQ_2 \\ < n^2 \frac{\sigma + \pi^B(v'-1)}{\pi^L (\pi^B)^2 \left(1 - \frac{g_2}{v'-1}\right) (1 - \pi^B)}$$

implying

$$\begin{aligned} & \int_{((0, \frac{S_{max}}{n}) \setminus S') \times (0, S_{max})} \mu^N(Q_1) \mu^N(Q_2) g_4(Q_1, Q_2) dQ_1 dQ_2 \\ & < n^2 \lambda_1^2 \left(1 - \frac{\pi^L}{1 - \pi^B}\right)^2 \frac{\sigma + \pi^B(v' - 1)}{\pi^L (\pi^B)^2 \left(1 - \frac{g_2}{v' - 1}\right) (1 - \pi^B)} \end{aligned}$$

This inequality and the identity $\int_{((0, \frac{S_{max}}{n}) \setminus S') \times (0, S_{max})} \mu^N(Q_1) \mu^N(Q_2) dQ_1 dQ_2 = \mu^N((0, \frac{S_{max}}{n}) \setminus S') \mu^N(0, S_{max}) = \left(A - \frac{\pi^L}{1 - \pi^B} - \mu^N(S')\right) \left(1 - \frac{\pi^L}{1 - \pi^B}\right)$ implies that, if we define $g_4 = n \lambda_1 \sqrt{\frac{\sigma + \pi^B(v' - 1)}{\pi^L (\pi^B)^2 \left(1 - \frac{g_2}{v' - 1}\right) (1 - \pi^B)}}$, then there exists $S_4 \subset ((0, \frac{S_{max}}{n}) \setminus S') \times (0, S_{max})$ such that

$$\int_{S_4} \mu^N(Q_1) \mu^N(Q_2) dQ_1 dQ_2 > \left(A - \frac{\pi^L}{1 - \pi^B} - \mu^N(S')\right) \left(1 - \frac{\pi^L}{1 - \pi^B}\right) - g_4$$

and $\forall (Q_1, Q_2) \in S_4, g_4(Q_1, Q_2) < g_4$.

Step 7: Firm 1's possible deviation in Period 4. Assume that Firm 1 sets a price equal to $(V - c)(1 - g_1) + c$ at the beginning of Period 1, to $(V - c)(1 - g_2) + c$ at the beginning of Period 2, to $(V - c)(1 - g_3) + c$ at the beginning of Period 3, and to $(V - c)(1 - g_4) + c$ at the beginning of Period 4, and that in addition demand is normal in Periods 1 to 3. Step 5 implies that if $(D_1, D_2) \in S_4$, which occurs with a probability greater than $\left(A - \frac{\pi^L}{1 - \pi^B} - \mu^N(S')\right) \left(1 - \frac{\pi^L}{1 - \pi^B}\right) - g_4$, then Firm 1's price $(V - c)(1 - g_4) + c$ is strictly lower than all other firms', implying that it serves the entire Period 4 demand with a probability of at least $1 - \pi^B$.

Step 8: Firm 1's possible deviations in Periods 1 to 4 and the corresponding expected profit. Steps 1 to 7 imply that the deviations described above (Step 6) afford Firm 1 expected profits exceeding $(V - c)D(1 - g_1)(1 - \pi^B)$ in Period 1, $(V - c)D(1 - g_2)(1 - \pi^B)^2$ in Period 2, $(V - c)D(1 - g_3)(1 - \pi^B)^3 \left(1 - \frac{\pi^L}{1 - \pi^B} - g_3\right)$ in Period 3 and $(V - c)D \left(\left(A - \frac{\pi^L}{1 - \pi^B} - \lambda_2 \frac{\sigma + \pi^B(v' - 1)}{(1 - \pi^B)^2}\right) \left(1 - \frac{\pi^L}{1 - \pi^B}\right) - g_4\right) (1 - \pi^B)^4 (1 - g_4)$ in Period 4.

Step 9: a necessary condition for combined profits in any PSSE to be strictly above $VD(1 - \sigma)$ in each of the first four periods. For each firm, the discounted sum of future expected profits in a PSSE cannot exceed $\frac{(V - c)D(1 + \pi^B(v' - 1))}{n(1 - \delta)}$. A

necessary condition for profits in a PSSE to be strictly above $(V - c)D(1 - \sigma)$ in each of the first four periods is that the deviations described above do not yield the deviating firm a discounted sum of expected profits strictly greater than $\frac{(V-c)D(1+\pi^B(v'-1))}{n(1-\delta)}$. This condition can be written as

$$\begin{aligned} & (1 - g_1) \left(1 - \pi^B\right) + \delta (1 - g_2) \left(1 - \pi^B\right)^2 + \delta^2 \left(1 - \frac{\pi^L}{1 - \pi^B} - g_3\right) (1 - g_3) \left(1 - \pi^B\right)^3 \\ & + \delta^3 \left(\left(A - \frac{\pi^L}{1 - \pi^B} - \lambda_2 \frac{\sigma + \pi^B(v' - 1)}{(1 - \pi^B)^2}\right) \left(1 - \frac{\pi^L}{1 - \pi^B}\right) - g_4 \right) \left(1 - \pi^B\right)^4 (1 - g_4) \\ & \leq \frac{1 + \pi^B(v' - 1)}{n(1 - \delta)} \end{aligned}$$

Step 10: a sufficient condition for the expected sum of future combined profits in any PSSE to be bounded away from $(V - c)D/1 - \delta$. In any period, expected combined profits cannot exceed total demand times the expected willingness to pay for the most attractive good, which is less than $(V - c)D(1 + \pi^B(v' - 1))$. Consider a PSSE leading to an expected sum of future combined profits above $\frac{(V - c)D}{1 - \delta} - \eta$ for some $\eta > 0$ and let Y_t denote the corresponding expectation of combined Period t profits. The two following inequalities hold:

$$\begin{aligned} \sum_{t \geq 1} \delta^{t-1} \left(\frac{Y_t}{(V - c)D} - 1 \right) & > -\eta \\ \forall t \geq 1, \frac{Y_t}{(V - c)D} - 1 & < \pi^B(v' - 1), \end{aligned}$$

jointly implying that for any t between 1 and 4, $\frac{Y_t}{(V - c)D} - 1 > -\frac{\eta}{\delta^4} - \frac{\pi^B(v' - 1)}{\delta^4(1 - \delta)}$.

Therefore, a sufficient condition for the expected sum of future discounted profits in any PSSE to be less than $\frac{(V - c)D}{1 - \delta} - \eta$ is that the inequality at the end of Step 9 not be satisfied when g_1, g_2, g_3 and g_4 defined as the above functions of σ with $\sigma = \frac{\eta}{\delta^4} + \frac{\pi^B(v' - 1)}{\delta^4(1 - \delta)}$, for values of η close enough to zero. By continuity, a sufficient condition is that this inequality not be satisfied for $\eta = 0$, *i.e.*, $\sigma = \frac{\pi^B(v' - 1)}{\delta^4(1 - \delta)}$:

$$\begin{aligned}
& (1 - h_1)(1 - \pi^B) + \delta(1 - h_2)(1 - \pi^B)^2 + \delta^2 \left(1 - \frac{\pi^L}{1 - \pi^B} - h_3\right)(1 - h_3)(1 - \pi^B)^3 \\
& + \delta^3 \left(A - \frac{\pi^L}{1 - \pi^B} - \lambda_2 \frac{\pi^B(v' - 1) \left(1 + \frac{1}{\delta^4(1 - \delta)}\right)}{(1 - \pi^B)^2} \right) \left(1 - \frac{\pi^L}{1 - \pi^B} - h_4\right)(1 - h_4)(1 - \pi^B)^4 \\
& > \frac{1 + \pi^B(v' - 1)}{n(1 - \delta)},
\end{aligned}$$

with the following notations:

$$\begin{aligned}
h_1 &= \frac{\pi^B(v' - 1)}{\delta^4(1 - \delta)} \\
h_2 &= \frac{\pi^B(v' - 1) \left(1 + \frac{1}{\delta^4(1 - \delta)}\right)}{(\pi^L + \pi^B)(1 - \pi^B)} \\
h_3 &= \sqrt{\frac{n\lambda_1(v' - 1) \left(1 + \frac{1}{\delta^4(1 - \delta)}\right)}{\pi^L(1 - \pi^B)}} \\
h_4 &= n\lambda_1 \sqrt{\frac{(v' - 1) \left(1 + \frac{1}{\delta^4(1 - \delta)}\right)}{\pi^L\pi^B \left(1 - \frac{h_2}{v' - 1}\right) (1 - \pi^B)}} = n\lambda_1 \sqrt{\frac{(v' - 1) \left(1 + \frac{1}{\delta^4(1 - \delta)}\right) (\pi^L + \pi^B)}{\pi^L\pi^B \left(\pi^L(1 - \pi^B) - (\pi^B)^2 - \frac{\pi^B}{\delta^4(1 - \delta)}\right)}}.
\end{aligned}$$

Proof of Proposition 2. Preamble: the parameters of the price war. We assume that condition (2) holds (which implies $\delta > \frac{1}{2}$), and that $c > 5(V' - V)$ (which is not a restriction since the result is stated for $\frac{V' - V}{c}$ close enough to zero). We want to find a price war level and a price war duration such that (i) any firm's expectation of the sum of its future discounted profits at the start of a price war's first period is 0 if all firms are expected to follow the strategies described above (including the transition rules) and (ii) these strategies cannot be individually improved upon. Let $PW_r(p, k')$ denote a firm's expectation of the discounted sum of its future payoffs at the start of the r -th period of a price war lasting k' periods ($1 \leq r \leq k'$) with a price level of p ($p < V$), assuming all firms act according to the candidate equilibrium. The price war strategies and transition rules described above imply that at the beginning of the r -th period of a price war lasting a total of k' periods, each firm's current period expected profit is $\frac{(p-c)D}{n}$, and with probability $(\pi^B + \pi^L)$, the state of the world in the next period is the first period of a price war, whereas with probability $1 - (\pi^B + \pi^L)$, it is ' $(r + 1)$ -th period of a price war' if $r < k'$ and 'normal collusion' if $r = k'$. This implies the following equalities:

$$\begin{aligned}
PW_r(p, k') &= \frac{(p-c)D}{n} + \delta(1 - \pi^B - \pi^L)PW_{r+1}(p, k') + \delta(\pi^B + \pi^L)PW_1(p, k') \text{ if } r < k' \\
PW_{k'}(p, k') &= \frac{(p-c)D}{n} + \delta(1 - \pi^B - \pi^L)\frac{(V-c)D}{n(1-\delta)} + \delta(\pi^B + \pi^L)PW_1(p, k'),
\end{aligned}$$

implying

$$\begin{aligned}
PW_1(p, k') &= \frac{(p-c)D}{n} \sum_{1 \leq t \leq k'} (\delta(1 - \pi^B - \pi^L))^t, \\
&\quad + \delta(\pi^B + \pi^L)PW_1(p, k') \sum_{1 \leq t \leq k'} (\delta(1 - \pi^B - \pi^L))^{t-1} \\
&\quad + (\delta(1 - \pi^B - \pi^L))^{k'} \frac{(V-c)D}{n(1-\delta)}
\end{aligned}$$

or equivalently

$$\begin{aligned}
&PW_1(p, k') \left(1 - \delta(\pi^B + \pi^L) \frac{(1 - (\delta(1 - \pi^B - \pi^L))^{k'})}{1 - \delta(1 - \pi^B - \pi^L)} \right) \\
&= \frac{(p-c)D}{n} \frac{(1 - (\delta(1 - \pi^B - \pi^L))^{k'})}{1 - \delta(1 - \pi^B - \pi^L)} + \frac{(V-c)D}{n(1-\delta)} (\delta(1 - \pi^B - \pi^L))^{k'}.
\end{aligned}$$

Let $F(p, k')$ denote the right-hand side of this equation, multiplied by n/D :

$$F(p, \kappa) = \frac{(p-c)(1 - (\delta(1 - \pi^B - \pi^L))^\kappa)}{1 - \delta(1 - \pi^B - \pi^L)} + \frac{(V-c)(\delta(1 - \pi^B - \pi^L))^\kappa}{(1-\delta)}.$$

If $\delta > \frac{1}{2}$, and π^B, π^L and $\frac{V'-V}{c}$ are close enough to zero, then $F(c - (V' - V), 1) > 0$. Also, $\lim_{\kappa \rightarrow \infty} F(c - (V' - V), \kappa) < 0$. Therefore, if $\delta > \frac{1}{2}$ and π^B, π^L and $\frac{V'-V}{c}$ are close enough to zero, there exists an integer $k' \geq 1$ such that $F(c - (V' - V), k' + 1) < 0 < F(c - (V' - V), k')$. We show now that $F(c - 5(V' - V), k') < 0$.

$$\begin{aligned}
&F(c - 5(V' - V), k') < 0 \\
\iff &\frac{5(V' - V) \left(1 - (\delta(1 - \pi^B - \pi^L))^{k'} \right)}{1 - \delta(1 - \pi^B - \pi^L)} > \frac{(V-c)(\delta(1 - \pi^B - \pi^L))^{k'}}{(1-\delta)}.
\end{aligned}$$

But

$$\begin{aligned}
& F(c - (V' - V), k' + 1) < 0 \\
& \iff \frac{(V' - V) \left(1 - (\delta(1 - \pi^B - \pi^L))^{k'+1}\right)}{1 - \delta(1 - \pi^B - \pi^L)} > \frac{(V - c) (\delta(1 - \pi^B - \pi^L))^{k'+1}}{(1 - \delta)} \\
& \Rightarrow \frac{5(V' - V) \left(1 - (\delta(1 - \pi^B - \pi^L))^{k'}\right) (1 - \delta)}{(V - c) (\delta(1 - \pi^B - \pi^L))^{k'} (1 - \delta(1 - \pi^B - \pi^L))} > 5\delta(1 - \pi^B - \pi^L) \frac{1 - (\delta(1 - \pi^B - \pi^L))^{k'}}{1 - (\delta(1 - \pi^B - \pi^L))^{k'+1}} \\
& > \frac{5\delta(1 - \pi^B - \pi^L)}{1 + (\delta(1 - \pi^B - \pi^L))^{k'}},
\end{aligned}$$

which is greater than 1 if $\delta > \frac{1}{2}$ and π^B, π^L are both close to zero, implying that $F(c - 5(V' - V), k') < 0$. By continuity, there exists some price p^w between $c - 5(V' - V)$ and $c - (V' - V)$ such that $PW_1(p^w, k') = F(p^w, k') = 0$.

We now prove that at the beginning of any stage of a price war, it is a best response for a firm to set price p^w and truthfully report its sales, given that all firms are expected to follow the strategies described above. First, the Bellman equation above implies that for all r such that $1 < r \leq k'$, $PW_r(p^w, k') > PW_1(p^w, k') = 0$. Consider a firm at the start of the r -th period of a price war. Complying with the candidate equilibrium strategy yields an expectation of the sum of future discounted profits equal to $PW_r(p^w, k') \geq 0$. Let $BR_r(p^w, k') \geq 0$ denote the expectation of the sum of future discounted profits of a firm (say, Firm 1) at the beginning of the r -th period of a price war, assuming that the firm, from period 1 onwards, maximizes the expectation of the discounted sum of its future profits, and that all other firms act according to the candidate equilibrium. First, notice that at the beginning of any price war period, a firm cannot earn a strictly positive profit: this would require a positive margin, hence a price above c and therefore greater $p^w + (V' - V)$, implying zero sales even if demand were biased in favor of the deviating firm. Second, if Firm 1 sets a price different from p^w , then with probability 1, the state of the world in the next period is 'first period of a price war'. This is because when not all firms set the same price, it is impossible that all firms have identical nonzero sales. Two cases must be distinguished. If one or several of the other firms sell(s) zero, at least one sales report is zero, which causes the next period to be the first period of a price war. If all firms other than Firm 1 have nonzero sales, then Firm 1 has zero sales with probability 1 and whatever its sales report, it is different from the other firms' with probability 1 since Firm 1

cannot 'guess' the value of other firms' sales (the distribution of demand levels is atomless if demand is nonzero). Therefore, whatever Firm 1's announcement after deviating, with probability 1 it is not the case that all firms report identical nonzero sales and the next period is the first period of a price war. This implies the following inequality: $BR_1(p^w, k') \leq \delta BR_1(p^w, k')$ implying $BR_1(p^w, k') \leq 0$ and therefore $BR_1(p^w, k') = 0$. It is therefore a best response in Period 1 to set a price equal to p^w . If the best response in the r -th period of a price war ($r > 1$) involved a price different from p^w , then the above analysis implies that the inequality $BR_r(p^w, k') \leq \delta BR_1(p^w, k') = 0 < BR_r(p^w, k')$, which is a contradiction. Therefore, the price war as described above is indeed an equilibrium. The price war mentioned in the remainder of this proof is the one described above, with $PW_1(p^w, k') = 0$.

Step 1: the expected sum of future discounted profits for a firm according to the state of the game. Let $W_{c,r}$, $W_{c-,r}$ and W denote the expected sum of future discounted profits of a firm at the beginning of, respectively, a correction period at its own expense with r remaining periods ($1 \leq r \leq k$), a correction period at another firm's expense with r remaining periods, and a normal collusion period (assuming that all firms behave according to the candidate equilibrium). Since, at the beginning of a normal collusion period, the distribution of future sales is symmetric across firms, and along any equilibrium path total expected profits add up to $(V - c)D$, it follows that $W = \frac{(V-c)D}{n(1-\delta)}$. Also, since ahead of any period along any equilibrium path, n or $(n - 1)$ firms are in a symmetric situation, $W_{c-,r} \leq \frac{(V-c)D}{(n-1)(1-\delta)}$.

In any correction period, there is a probability $\frac{(n-1)\pi^B}{n}$ that one of the non-targeted firms will benefit from biased demand, implying that in the following period, the expected sum of the previously targeted firm' future flow of discounted profits will be $W_{c-,k}$. This implies that for any r between 1 and k , $W_{c,r} = \delta \left[\left(1 - \frac{(n-1)\pi^B}{n}\right) W_{c,r-1} + \frac{(n-1)\pi^B}{n} W_{c-,k} \right]$ (with the notation $W_{c,0} = W$). These equalities imply that for any r between 1 and k ,

$$W_{c,r} = \delta^r \left(1 - \frac{(n-1)\pi^B}{n}\right)^r W + \frac{\frac{\delta(n-1)\pi^B}{n}}{1 - \delta \left(1 - \frac{(n-1)\pi^B}{n}\right)} \left[1 - \delta^r \left(1 - \frac{(n-1)\pi^B}{n}\right)^r\right] W_{c-,k}.$$

The above results imply that for any number of firms $n \geq 2$, the following

inequalities hold:

$$\frac{\delta^r(V-c)D}{(1-\delta)n} \left(1 - \frac{(n-1)\pi^B}{n}\right)^r \leq W_{c,r} \leq \frac{\delta^r(V-c)D}{(1-\delta)n} \left(1 - \frac{(n-1)\pi^B}{n}\right)^r + \frac{\delta\pi^B(V-c)D}{n(1-\delta)^2} \quad (5)$$

We prove now that for any r ($1 \leq r \leq k$), $W_{c,r} \leq W \leq W_{c-,r}$. First, the above Bellman equations characterizing each $W_{c,r}$ imply, by induction, that either for all r , $W_{c,r} \leq W$, or for all r , $W_{c,r} \geq W$. Assume (by contradiction) that $W_{c,k} > W$. Since expected profits add up to $(V-c)D$ ahead of any period in the candidate equilibrium, the equality $W_{c,k} + (n-1)W_{c-,k} = nW$ holds, implying that $W_{c-,k} < W$. But this inequality, together with the Bellman equation characterizing $W_{c,1}$ implies that $W_{c,1} < W$, which is a contradiction.

Finally, notice that the inequality $W_{c,r} \leq W$ implies that for any r ($2 \leq r \leq k$), $W_{c,r+1} \leq W_{c,r}$ and $W_{c-,r+1} \geq W_{c-,r}$.

Step 2. Whatever the state of the world at the beginning of Period t , if Firm i complied with the strategies prescribed by the candidate equilibrium in all previous periods and it assumes that all other firms behave according to the candidate equilibrium, and Firm i did set a price at the beginning of Period t in accordance with the candidate equilibrium, or the observation of its own sales combined with the knowledge of the price it set at the beginning of the period allows it to know that the distribution of current period sales, when observed at the end of the following period, will not reveal any deviation, then reporting sales truthfully is a best response for Firm i , assuming that all other firms behave according to the candidate equilibrium.

Proof. Since misreporting at the end of Period t is detected at the latest at the end of Period $(t+1)$, leading to a price war from Period $(t+2)$ onwards, and expected per-period profits cannot exceed total profits $(V'-c)D$, misreporting would allow Firm i to earn at most $V'D$ in Period $(t+1)$ and an expected discounted sum of subsequent profits equal to 0. In contrast, complying with the strategy prescribed by the candidate equilibrium would lead, at the beginning of Period $(t+1)$, to an expected sum of future discounted profits greater than or equal to $W_{c,k}$. Since (5) implies the inequality $W_{c,r} \geq (V'-c)D$ for all r if π^B is close enough to zero and v' is close enough to 1 (in which case it boils down to $\delta^k \geq n(1-\delta)$), truthful sales reporting is a best response for a firm that behaved as prescribed by the candidate equilibrium.

Step 3. At the beginning of a correction period, it is a best response for all firms to set the price prescribed by the candidate equilibrium.

Proof. Assume the state of the game at the beginning of Period t is 'cor-

rejection at the expense of Firm 1 (without loss of generality), with r remaining periods'. We prove first that it is optimal for Firm 1 to behave as prescribed by the candidate equilibrium, that is, by setting $p_t^1 = V' + 1$ and truthfully reporting its zero sales. We showed in Step 2 of this proof that conditional of setting a price equal to $V' + 1$ it is optimal for Firm 1 to truthfully report its zero sales. Assume that Firm 1 sets a price $p_t^1 \neq V' + 1$. Any price greater than V' yields the same zero profits as the equilibrium strategy in Period t and the same information to other firms. Therefore, such prices lead to exactly the same payoff distribution for Firm 1 in Period t and subsequent periods as $p_t^1 = V' + 1$. Consider now a price $p \in (V, V')$. Such a price leads to exactly the same outcome as $p_t^1 = V' + 1$ unless demand is biased in favor of Firm 1, with a valuation $v \geq p$. In this latter case, if Firm 1 sets $p_t^1 = V' + 1$, it earns zero in Period t and, since its best response is then to truthfully report its zero sales (as proved in Step 2), its expected sum of future profits is at least $\delta W_{c,r-1}$ (this is the expected sum if Firm 1 plans to comply with the candidate equilibrium strategies in the future, which Firm 1 can do); whereas if it sets $p_t^1 = p$, its deviation will be detected at the end of the following period, leading to profits below $(V - c)D[(1 + \delta)v']$. (2) implies the inequality $\delta W_{c,r-1} > (V - c)D[(1 + \delta)v']$ if π^B is close enough to zero and v' is close enough to 1 (in which case it boils down to $\delta^k \geq n(1 - \delta^2)$), making such a deviation unprofitable. Consider now a price $p \leq V$. Unless demand is zero or biased, setting such a price leads Firm 1 to being 'exposed' after two periods at most. The corresponding expected sum of future discounted profits is thus less than $(V - c)D[(1 + \delta)v' + (\pi^L + \pi^B)v']$. If π^B and π^L are close enough to zero and v' is close enough to 1, (2) implies that this expression is less than $W_{c,k}$, that is, less than $W_{c,r}$ for any $r \leq k$. Therefore, it is optimal for Firm 1 to follow the strategy prescribed by the candidate equilibrium, which yields it an expected sum of future discounted profits equal to $W_{c,r}$.

Consider now a firm other than Firm 1, say, Firm 2 (without loss of generality). Complying with the actions prescribed by the candidate equilibrium leads for Firm 2 to an expected sum of future discounted profits equal to $W_{c-,r}$. Assume now that Firm 2 deviates and sets a price $p_t^2 \neq V$. If $p_t^2 > V$, then Firm 2 earns zero in Period t and whatever it reports at the end of Period t , its deviation is detected at the end of Period $(t + 1)$ unless demand in Period t is zero, because a distribution of sales such that total sales are nonzero while two firms (Firms 1 and 2) have zero sales is incompatible with equilibrium. This, together with the fact that a firm's per-period expected profit cannot exceed $(V - c)Dv'$, implies that the corresponding expected sum of future discounted

profits is less than or equal to $(V - c)Dv' \left[\delta + \frac{\pi^L \delta^2}{1 - \delta} \right]$. If π^B and π^L are close enough to zero and v' is close enough to 1, (3) implies that this expression is less than $W_{c,k}$, and therefore less than $W_{c-,r}$. A price $p_t^2 > V$ therefore cannot improve upon the behavior prescribed by the candidate equilibrium for Firm 2.

Consider now the possibility of a price $p_t^2 < V$. Such a price would yield Firm 2 at most $(V - c)D$ in expectation in Period t . If $D_t = 0$, which happens with probability π^L , Period t sales are identical to what they would be absent a deviation by Firm 2. Therefore, as shown in Step 2, it would be optimal for Firm 2 in this case to report zero sales, leading in Period $(t + 1)$ to either 'normal collusion' (if $r = 1$) or to 'correction at the expense of Firm 1, with $(r - 1)$ remaining periods' (if $r \geq 2$). This would lead at the beginning of Period $(t + 1)$ to an expected sum of future discounted profits equal to $W_{c-,r-1}$ (with the slight abuse of notation $W_{c-,0} = W$). If demand is normal, a deviation leads to a sales profile that is compatible with equilibrium behavior (with demand biased in favor of Firm 2). As shown in Step 2, it is then optimal for Firm 2 to truthfully reveal its sales, leading at the beginning of Period $(t + 1)$ to an expected sum of future discounted profits equal to $W_{c,k}$. Finally, if demand is biased, then with probability 1 one of the firms (Firm 2 or another one) serves the entire demand, leading to an expected sum of future discounted profits below $W_{c-,k}$. Therefore, a deviation with $p_t^2 < V$ would lead at the beginning of Period t to an expected sum of Firm 2's future discounted profits less than or equal to $(V - c)D + \delta (\pi^L W_{c-,r-1} + ((1 - \pi^B) - \pi^L) W_{c,k} + \pi^B W_{c-,k})$, whereas in the absence of deviation this expected sum is equal to $W_{c-,r}$. The difference between this expected sum in the absence and in the presence of such a deviation is thus greater than or equal to

$$\begin{aligned}
& W_{c-,r} - (V - c)D - \delta (\pi^L W_{c-,r-1} - ((1 - \pi^B) - \pi^L) W_{c,k} + \pi^B W_{c-,k}) \\
= & \quad W + \delta \pi^L (W_{c-,r} - W_{c-,r-1}) + (1 - \delta \pi^L) (W_{c-,r} - W) - \\
& \quad - (V - c)D - \delta (\pi^L W' - ((1 - \pi^B) - \pi^L) W_{c,k} + \pi^B W_{c-,k}) \\
\geq & \quad W - (V - c)D - \delta (\pi^L W + ((1 - \pi^B) - \pi^L) W_{c,k} + \pi^B W_{c-,k}) \\
\geq & \quad W - (V - c)D \left[1 + \delta \left((1 - \pi^B) \frac{W_{c,k}}{(V - c)D} + \frac{\pi^B}{(n - 1)(1 - \delta)} \right) \right] \\
\geq & \quad \frac{(V - c)D}{n(1 - \delta)} - (V - c)D \left[1 + \delta (1 - \pi^B) \frac{\delta^k}{n(1 - \delta)} \right] \\
& \quad - (V - c)D \left[\delta \left((1 - \pi^B) \frac{\delta \pi^B}{n(1 - \delta)^2} + \frac{\pi^B}{(n - 1)(1 - \delta)} \right) \right]
\end{aligned}$$

If π^B and π^L are close enough to zero and v' is close enough to 1, condition (2) implies that this difference is positive, so that it is a best response for Firm 2 to follow the strategy prescribed by the candidate equilibrium.

Step 4. If the state of the world at the beginning of Period t is 'normal collusion' and all firms followed the strategy prescribed by the candidate equi-

librium in all previous periods, then it is a best response for Firm 1 (without loss of generality) to set $p_t^1 = V$.

Proof. Consider a subgame starting in period t , such that the state of the game at the beginning of Period t is 'normal collusion' and all firms followed the strategy prescribed by the candidate equilibrium in all previous periods. Firm 1's expected sum of future discounted profits is W if it sets price as prescribed by the candidate equilibrium (that is, $p_t^1 = V$).

We prove hereafter that, if conditions (2)-(3) hold, then in a 'normal collusion' period, neither setting a price $p_t^1 < V$ nor setting one such that $p_t^1 > V$ leads to an expected sum of future discounted profits strictly greater than W . We prove this by contradiction.

We assume that there exists a best response such that, with a positive probability, Firm 1 sets a price $p_t^1 < V$ in some 'normal collusion' state. Let W' denote Firm 1's expected sum of future discounted profits at the beginning of any such 'normal collusion' state, given this best response. Since all possible best responses must yield the same expected sum of future discounted profits at the beginning of any 'normal collusion' period, W' is also equal to Firm 1's expected sum of future discounted profits at the beginning of any subsequent 'normal collusion' period.

In Period t , setting a price $p_t^1 < V$ allows Firm 1 to serve the entire demand unless demand is biased in favor of some other firm, with a bias strong enough to offset the price difference between that firm and Firm 1. If $D_t = 0$, then the state of the world is 'normal collusion' again in Period $(t + 1)$, leading to an expected sum of future discounted profits equal to W' at the beginning of Period $(t + 1)$. If demand is biased, then with probability 1 one of the firms (Firm 1 or another one) serves the entire demand, leading to an expected sum of future discounted profits below $W_{c-,k}$. If demand is normal and nonzero, the state of the world at the beginning of Period $(t + 1)$ is 'correction at the expense of Firm 1, with k remaining periods', leading to an expected sum of future discounted profits of $W_{c,k}$ (by Step 2). Therefore, a deviation with $p_t^1 < V$ would lead at the beginning of Period t to an expected sum of future discounted profits less than or equal to $(V - c)D + \delta (\pi^L W' + \pi^B W_{c-,k} + (1 - \pi^B - \pi^L) W_{c,k})$:

$$W' \leq (V - c)D + \delta (\pi^L W' + \pi^B W_{c-,k} + (1 - \pi^B - \pi^L) W_{c,k}),$$

implying

$$\begin{aligned} \frac{(1 - \pi^L \delta) W'}{(V - c)D} &\leq 1 + \delta (1 - \pi^B - \pi^L) \frac{W_{c,k}}{(V - c)D} + \delta \pi^B \frac{W_{c-,k}}{(V - c)D} \\ &\leq 1 + \delta (1 - \pi^B - \pi^L) \left(\frac{\delta^k}{(1 - \delta)n} + \frac{\delta \pi^B}{n(1 - \delta)^2} \right) + \frac{\delta \pi^B}{(1 - \delta)(n - 1)} \end{aligned}$$

If π^B and π^L are close enough to zero and v' is close enough to 1, condition (3) implies that the right-hand term of this inequality is less than W , and therefore that $W' < W$. It is thus not a best response for a firm to undercut its competitors in a normal collusion period.

We show now that there exists no best response such that, with a positive probability, $p_t^1 > V$. If $p_t^1 > V$ and demand is neither biased nor zero, then Firm 1 earns zero in Period t . In this case, whatever it reports at the end of Period t , its deviation is detected at the end of Period $(t + 1)$, because a distribution of sales such that total sales are nonzero while only one firm has zero sales is incompatible with equilibrium. This, together with the fact that a firm's per-period expected profit cannot exceed $(V - c)Dv'$, implies that the corresponding expected sum of future discounted profits is less than or equal to $(V - c)Dv' \left[\left(\frac{\pi^B}{n} + \delta \right) + \pi^L \frac{\delta^2}{1 - \delta} \right]$. If π^B and π^L are close enough to zero and v' is close enough to 1, condition (2) implies that this expression is less than W . A price $p_t^1 > V$ therefore cannot improve upon the behavior prescribed by the candidate equilibrium for Firm 1 in a 'normal collusion' period.

Step 5. The above steps imply that the strategy profile under consideration is an equilibrium strategy profile if π^B and π^L are close enough to zero and v' is close enough to 1, and conditions (2)-(3) are satisfied. By construction, this equilibrium is symmetric and involves only pure strategies, and along the equilibrium path, the prevailing price is V in all periods with probability one.

Proof of Proposition 3. Step 1. For each n between 3 and 10, one can check numerically that there exists a value of k (the duration of the correction phase in the equilibria described in Section 4) and a nonempty interval of values of δ that is bounded away from 1, such that conditions (1)-(3) are satisfied when A is replaced with 1 in (1). By continuity, this implies the existence of A^* such that conditions (1)-(3) are satisfied if $A = A^*$. The corresponding values of k and δ are displayed in Figure 1 (rather than δ , Figure 1 displays the equivalent discount rate ρ , defined by $\delta = \frac{1}{1 + \rho}$). Let $\delta^* (< 1)$ denote the upper bound of the eight intervals mentioned above.

Step 2. We show hereafter that for each n between 3 and 10, it is possible to

construct a stochastic demand function satisfying the assumptions of the model, and such that π^L , $\pi^B \lambda_2(v' - 1)$, $\frac{V' - V}{c}$, $\frac{\pi^B}{(1 - \delta^*)\pi^L}$, and $\frac{v' - 1}{(1 - \delta^*)\pi^B \pi^L}$ are arbitrarily close to zero, and λ_1 , λ_2 are bounded, so that the conditions expressed in Propositions 1 and 2 are simultaneously valid, which proves Proposition 3.

We choose some $D > 0$, $V > 0$ and $c \in (0, V)$. Let ε denote a small positive number. We define $\pi^{L^*}(\varepsilon) = \text{Min}\left(\varepsilon, \frac{1 - A^*}{2}\right)$, $\pi^{B^*}(\varepsilon) = (1 - \delta^*)\varepsilon \pi^{L^*}(\varepsilon) \leq (1 - \delta^*)\varepsilon^2$ and we consider a stochastic demand function defined as follows (with reference to the notations of Section 2):

- With probability $(1 - \pi^{B^*}(\varepsilon))$, demand is normal. In this case, aggregate demand is zero with probability $\pi^{L^*}(\varepsilon)$. Define $R^*(\varepsilon) = \frac{2}{(1 - \pi^{L^*}(\varepsilon))(A^* + (n+1)(1 - A^* - \pi^{L^*}(\varepsilon)))}$. With probability A^* demand is drawn from the uniform distribution over $(0, R^*(\varepsilon)D)$, and with probability $(1 - A^* - \pi^{L^*}(\varepsilon))$ it is drawn from the uniform distribution over $(R^*(\varepsilon)D, nR^*(\varepsilon)D)$. It can be checked that expected demand conditional on demand being normal is equal to D and that $\mu^N\left(\left(0, \frac{S_{max}}{n}\right)\right) = \mu^N\left((0, R^*(\varepsilon)D)\right) = A^*$.
- With probability $\pi^{B^*}(\varepsilon)$, demand is biased. Define $B^*(\varepsilon)$ by the identity $\frac{B^*(\varepsilon)R^*(\varepsilon)}{2} + \frac{(1 - B^*(\varepsilon))(n+1)R^*(\varepsilon)}{2} = 1$. Conditional on demand being biased, aggregate demand is drawn with probability $B^*(\varepsilon)$ from the uniform distribution over $(0, R^*(\varepsilon)D)$ and with probability $(1 - B^*(\varepsilon))$ from the uniform distribution over $(R^*(\varepsilon)D, nR^*(\varepsilon)D)$. It can be checked that expected demand conditional on demand being biased is equal to D .

This demand function is such that $\lambda_1 = \lambda_1^*(\varepsilon) = \text{Max}\left(\frac{B^*(\varepsilon)}{A^*}, \frac{1 - B^*(\varepsilon)}{1 - A^* - \pi^{L^*}(\varepsilon)}\right) \leq \text{Max}\left(\frac{B^*(\varepsilon)}{A^*}, \frac{2(1 - B^*(\varepsilon))}{1 - A^*}\right)$ and $\lambda_2 = \lambda_2^*(\varepsilon) = \frac{A^*}{1 - A^* - \pi^{L^*}(\varepsilon)} \leq \frac{2A^*}{1 - A^*}$.

We then define $v'^*(\varepsilon) = 1 + \varepsilon \text{Min}\left(1, \frac{\lambda_2^*(\varepsilon)}{\pi^{B^*}(\varepsilon)}, (1 - \delta^*)\pi^{B^*}(\varepsilon)\pi^{L^*}(\varepsilon), \frac{c}{V - c}\right)$.

This demand function satisfies all the assumptions of the model, and it is such that π^L , $\pi^B \lambda_2(v' - 1)$, $\frac{V' - V}{c}$, $\frac{\pi^B}{(1 - \delta^*)\pi^L}$, and $\frac{v' - 1}{(1 - \delta^*)\pi^B \pi^L}$ are all smaller than or equal to ε .

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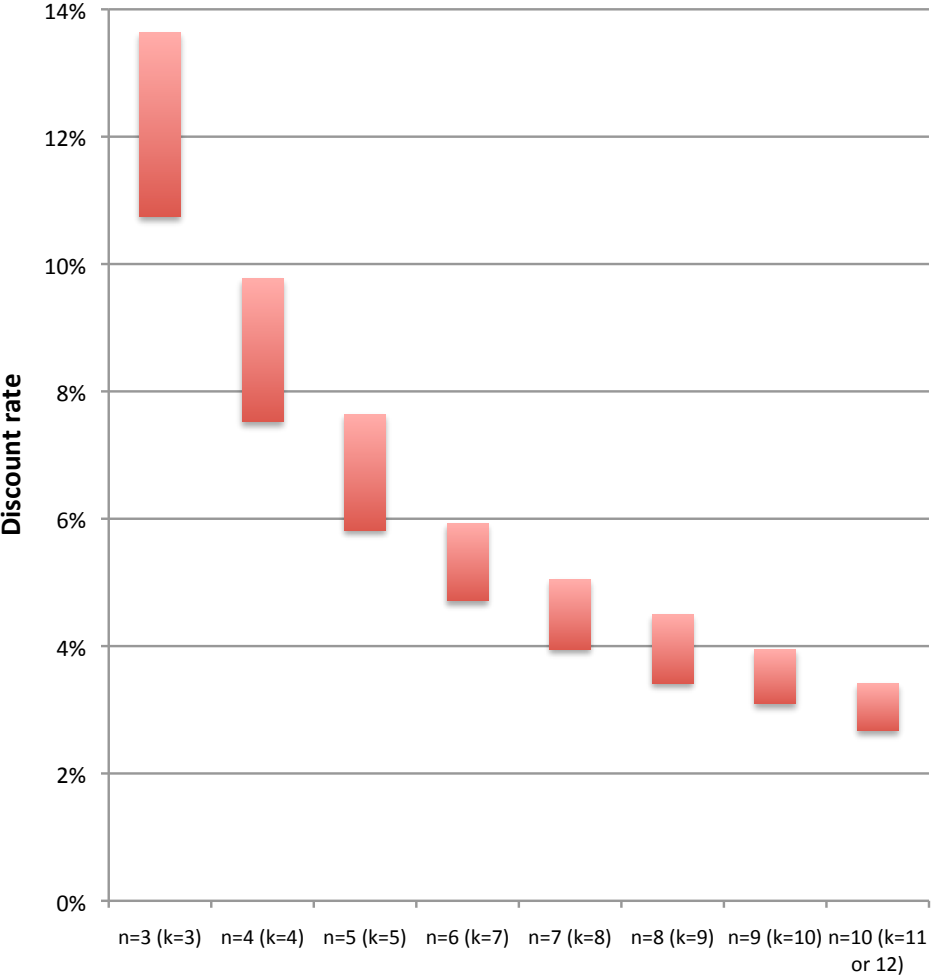
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Figure 1. The existence of a simple collusive equilibrium leading to near-monopoly profits (for the demand functions described in the proof of Proposition 3)



■ Collusion with near-monopoly profits requires communication (with correction phases lasting k periods)