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# A HYBRID JOSEPHY–NEWTON METHOD FOR SOLVING BOX CONSTRAINED VARIATIONAL INEQUALITY PROBLEMS VIA THE D-GAP FUNCTION

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A box constrained variational inequality problem can be reformulated as an unconstrained minimization problem through the D-gap function. Some basic properties of the affine variational inequality subproblems in the classical Josephy–Newton method are studied. A hybrid Josephy–Newton method is then proposed for minimizing the D-gap function. Under suitable conditions, the algorithm is shown to be globally convergent and locally quadratically convergent. Some numerical results are also presented.

*Keywords:* Variational inequality problem; box constraints; D-gap function; Newton’s method; unconstrained optimization; global convergence; quadratic convergence

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# 1 INTRODUCTION

Let  $F$  be a mapping from  $\mathfrak{R}^n$  into itself and  $X$  be a nonempty closed convex subset of  $\mathfrak{R}^n$ . The variational inequality problem (VIP) is to find a vector  $x^* \in X$  such that

$$\langle F(x^*), y - x^* \rangle \geq 0, \quad \forall y \in X, \quad (1)$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathfrak{R}^n$ . In this paper, we study the case that  $X$  is a box defined by

$$X = \{x \in \mathfrak{R}^n \mid l_i \leq x_i \leq u_i, \quad i = 1, \dots, n\}$$

where  $l_i \in \mathfrak{R} \cup \{-\infty\}$  and  $u_i \in \mathfrak{R} \cup \{+\infty\}$  with  $l_i < u_i$  representing the lower and upper bounds on the variables, respectively. It is not difficult to see [3, Proposition 5.7] that the box structure of the set  $X$  enables us to rewrite (1) as

$$F_i(x^*)(y_i - x_i^*) \geq 0, \quad i = 1, \dots, n, \quad \forall y \in X. \quad (2)$$

If the constraint set  $X$  is the nonnegative orthant  $\mathfrak{R}_+^n$ , then the VIP reduces to the complementarity problem (CP). This class of special VIPs has numerous important applications in various fields such as mathematical programming, economics, and engineering; see [7,13] and references therein.

A useful way to deal with VIP (1) is to reformulate it first as a system of equations or an optimization problem via a merit function, and then solve the resulting system of equations or optimization problem. Recently, much attention has been paid to the reformulation of VIPs and CPs and various merit functions have been proposed and studied [10]. Well known merit functions for VIPs include the *gap function*

$$g(x) = \sup_{y \in X} \langle F(x), x - y \rangle$$

first presented by Auslender [2] and then studied by Marcotte and Dussault [18,19], the *regularized gap function*

$$f_\alpha(x) = \max_{y \in X} \left\{ \langle F(x), x - y \rangle - \frac{\alpha}{2} \|y - x\|^2 \right\} \quad (3)$$

introduced by Fukushima [9] and Auchmuty [1], and the *D-gap function*

$$g_{\alpha\beta}(x) := f_{\alpha}(x) - f_{\beta}(x), \quad (4)$$

proposed by Peng [21] and Yamashita *et al.* [27], where  $\alpha$  and  $\beta$  are arbitrary positive parameters such that  $\alpha < \beta$ . Since a box constrained VIP is actually equivalent to a system of KKT mixed complementarity conditions, several merit functions based on the KKT system of VIP have also been proposed and explored extensively, see Qi [23] and references therein.

In this paper, we focus our attention on the D-gap function for VIP. It is not difficult to see that  $g_{\alpha\beta}(x) \geq 0$  for all  $x \in \mathbb{R}^n$ , and  $g_{\alpha\beta}(x) = 0$  if and only if  $x$  is a solution of the VIP (1). Therefore the VIP can be cast as the following unconstrained optimization problem:

$$\min_{x \in \mathbb{R}^n} g_{\alpha\beta}(x). \quad (5)$$

When the mapping  $F$  is differentiable, the D-gap function  $g_{\alpha\beta}$  is also differentiable [9,27]. However, it is not twice differentiable in general. Therefore it is not straightforward to apply conventional second-order methods to problem (5). As a remedy for this inconvenience, Sun *et al.* [25] introduced the concept of a computable generalized Hessian of the D-gap function  $g_{\alpha,\beta}$  and presented a Newton-type method for solving problem (5). Restricting themselves to the box constrained VIP, Kanzow and Fukushima [15] discussed a generalized Hessian of the D-gap function and proposed a Gauss–Newton-type method to minimize it. Further properties of the D-gap function have been investigated in [11,15,25].

This work is motivated by the recent paper [22] in which the D-gap function is used to globalize the classical Josephy–Newton method [14] for general VIPs. The main purpose of this work is to further study the algorithm proposed in [22] by restricting ourselves to box constrained VIPs, and test the effectiveness of the algorithm. In [22], the global and local convergence of the algorithm were proven under the assumptions that  $F$  is strongly monotone. By restricting to box constrained VIP, we will show in this paper that the algorithm is globally convergent and locally quadratically convergent if the mapping  $F$  is a uniform  $P$ -function. Hence the results here can also be viewed as a refinement of the results in [22].

The paper is organized as follows: In Section 2, we first introduce some basic definitions that will be used in the paper. Some sensitivity

results for box constrained affine VIP are given under certain conditions. Then, by applying these sensitivity results, we study the properties of the subproblem in the Josephy–Newton method [14]. In Section 3, we present a hybrid algorithm which uses the D-gap function to globalize Josephy–Newton method. Convergence properties of the algorithm are also discussed. In Section 4 we report some numerical results. Finally we conclude the paper with some remarks in Section 5.

## 2 SOME RESULTS FOR BOX CONSTRAINED AFFINE VIP

We first review some concepts related to the mapping  $F$ .

The mapping  $F: \mathfrak{R}^n \rightarrow \mathfrak{R}^n$  is said to be a  $P_0$ -function if

$$\max_{\substack{1 \leq i \leq n \\ x_i \neq y_i}} (x_i - y_i)(F_i(x) - F_i(y)) \geq 0 \quad \forall x, y \in \mathfrak{R}^n, \quad x \neq y;$$

a  $P$ -function if

$$\max_{1 \leq i \leq n} (x_i - y_i)(F_i(x) - F_i(y)) > 0 \quad \forall x, y \in \mathfrak{R}^n, \quad x \neq y;$$

and a *uniform P-function* (with modulus  $\mu > 0$ ) if

$$\max_{1 \leq i \leq n} (x_i - y_i)(F_i(x) - F_i(y)) \geq \mu \|x - y\|^2, \quad \forall x, y \in \mathfrak{R}^n.$$

An  $n \times n$  matrix  $M$  is a  $P_0$ -matrix if

$$\max_{\substack{1 \leq i \leq n \\ z_i \neq 0}} z_i(Mz)_i \geq 0, \quad \forall z \in \mathfrak{R}^n, \quad z \neq 0,$$

and a  $P$ -matrix if

$$\max_{1 \leq i \leq n} z_i(Mz)_i > 0, \quad \forall z \in \mathfrak{R}^n, \quad z \neq 0.$$

It is easy to see that, if  $M$  is a  $P$ -matrix, then there exists a constant  $\mu > 0$  such that

$$\max_{1 \leq i \leq n} z_i(Mz)_i \geq \mu \|z\|^2, \quad \forall z \in \mathfrak{R}^n. \quad (6)$$

It is known [20, Theorem 5.8] that if  $F$  is a differentiable  $P_0$ -function, then  $\nabla F(x)^\top$  is a  $P_0$ -matrix for each  $x$ . Moreover, if  $F$  is a differentiable uniform  $P$ -function with modulus  $\mu > 0$ , then  $\nabla F(x)^\top$  is a uniform  $P$ -matrix with modulus  $\mu > 0$  in the sense that

$$\max_{1 \leq i \leq n} z_i [\nabla F(x)^\top z]_i \geq \mu \|z\|^2, \quad \forall z \in \mathfrak{R}^n, \quad \forall x \in \mathfrak{R}^n. \quad (7)$$

Since we are not aware of any explicit reference for the formula (7), we include a short proof for it.

**LEMMA 2.1** *Let  $F: \mathfrak{R}^n \rightarrow \mathfrak{R}^n$  be a differentiable uniform  $P$ -function with modulus  $\mu > 0$ . Then (7) holds.*

*Proof* Let  $x \in \mathfrak{R}^n$  and  $z \in \mathfrak{R}^n$  be arbitrary but fixed. Let  $\{t_k\} \subseteq \mathfrak{R}$  be a sequence of positive numbers converging to 0. Since  $F$  is a uniform  $P$ -function and the index set  $\{1, \dots, n\}$  is finite, there exists an index  $i_0 \in \{1, \dots, n\}$  (independent of  $k$ ) and a subsequence  $\{t_k\}_K$  such that  $z_{i_0} \neq 0$  and

$$t_k z_{i_0} [F_{i_0}(x + t_k z) - F_{i_0}(x)] \geq \mu t_k^2 \|z\|^2 \quad \forall k \in K.$$

Dividing this expression by  $t_k^2$ , taking the limit  $k \rightarrow \infty$  ( $k \in K$ ) and using the assumed differentiability of  $F$ , we obtain

$$z_{i_0} [\nabla F(x)^\top z]_{i_0} \geq \mu \|z\|^2.$$

This implies

$$\max_{1 \leq i \leq n} z_i [\nabla F(x)^\top z]_i \geq \mu \|z\|^2$$

and completes the proof.

In the classical Josephy–Newton method, we solve the following linearized VIP subproblem at each iteration: Given a current iterate  $x^k$ , find a vector  $x \in X$  such that

$$\langle F(x^k) + \nabla F(x^k)^\top (x - x^k), y - x \rangle \geq 0, \quad \forall y \in X. \quad (8)$$

Here we study the properties of the affine subproblem (8). In [26], Taji *et al.* studied the subproblem (8) for a general closed convex constraint

set  $X$  under the assumption that  $F$  is strongly monotone and continuously differentiable. Restricting ourselves to the box constrained VIP, we will investigate properties of the subproblem under weaker conditions.

To this end, we first consider a perturbation of affine VIP with box constraints. Let  $X$  be a box,  $M$  a  $P$ -matrix, and  $b \in \mathfrak{R}^n$ . For any  $v \in \mathfrak{R}^n$ , let  $x(v) \in X$  denote the unique solution of the following affine VIP:

$$\langle b + v + Mx, y - x \rangle \geq 0, \quad \forall y \in X, \quad (9)$$

or equivalently

$$(b_i + v_i + [Mx]_i)(y_i - x_i) \geq 0, \quad i = 1, \dots, n, \quad \forall y \in X. \quad (10)$$

**PROPOSITION 2.2** *Let  $x(v)$  be the unique solution of the affine VIP (9), where  $X$  is a box and  $M$  a  $P$ -matrix. Let  $\mu > 0$  be a constant satisfying (6). Then we have*

$$\|x(v) - x(v')\| \leq \frac{1}{\mu} \|v - v'\|, \quad \forall v, v' \in \mathfrak{R}^n.$$

*Proof* Let  $v, v' \in \mathfrak{R}^n$  be arbitrary. Since  $x(v) \in X$ ,  $x(v') \in X$ , and  $X$  is a box, it follows from (10) that

$$(b_i + v_i + [Mx(v)]_i)(x_i(v') - x_i(v)) \geq 0, \quad i = 1, \dots, n,$$

and

$$(b_i + v'_i + [Mx(v')]_i)(x_i(v) - x_i(v')) \geq 0, \quad i = 1, \dots, n.$$

Adding the above two inequalities, we obtain

$$\begin{aligned} (v_i - v'_i)(x_i(v') - x_i(v)) &\geq (x_i(v) - x_i(v'))[M(x(v) - x(v'))]_i, \\ i &= 1, \dots, n. \end{aligned} \quad (11)$$

Since  $M$  is a  $P$ -matrix, there exists a constant  $\mu > 0$  and some index  $i_0$  such that

$$(x_{i_0}(v) - x_{i_0}(v'))[M(x(v) - x(v'))]_{i_0} \geq \mu \|x(v) - x(v')\|^2.$$

This inequality together with (11) implies that

$$(v_{i_0} - v'_{i_0})(x_{i_0}(v') - x_{i_0}(v)) \geq \mu \|x(v) - x(v')\|^2. \quad (12)$$

Since

$$(v_{i_0} - v'_{i_0})(x_{i_0}(v') - x_{i_0}(v)) \leq \|v - v'\| \cdot \|x(v) - x(v')\|,$$

the inequality (12) yields

$$\|x(v) - x(v')\| \leq \frac{1}{\mu} \|v - v'\|.$$

This completes the proof of the proposition.

Next we give another result on the continuity of a solution of the box constrained affine VIP with a  $P$ -matrix. For any  $P$ -matrix  $M$  and vectors  $b, p \in \mathfrak{R}^n$ , let  $x(M) \in X$  denote the unique solution of the following affine VIP:

$$\langle b + Mp + Mx, y - x \rangle \geq 0, \quad \forall y \in X, \quad (13)$$

or equivalently

$$(b_i + [Mp]_i + [Mx]_i)(y_i - x_i) \geq 0, \quad i = 1, \dots, n, \quad \forall y \in X. \quad (14)$$

**PROPOSITION 2.3** *Suppose that  $M$  and  $N$  are  $P$ -matrices and  $X$  is a box. Let  $\mu > 0$  be a constant satisfying (6). Then we have*

$$\|x(M) - x(N)\| \leq \frac{1}{\mu} \|M - N\| \cdot \|x(N) + p\|. \quad (15)$$

*Proof* The proposition trivially holds if  $x(M) = x(N)$ . Hence we only need to consider the case where  $x(M) \neq x(N)$ . Since  $X$  is a box, it



follows from (14) and the definitions of  $x(M)$  and  $x(N)$  that

$$(b_i + [Mp]_i + [Mx(M)]_i)(x_i(N) - x_i(M)) \geq 0, \quad i = 1, \dots, n,$$

and

$$(b_i + [Np]_i + [Nx(N)]_i)(x_i(M) - x_i(N)) \geq 0, \quad i = 1, \dots, n.$$

Adding the above two inequalities, we get

$$([Mp]_i - [Np]_i + [Mx(M)]_i - [Nx(N)]_i)(x_i(N) - x_i(M)) \geq 0, \\ i = 1, \dots, n,$$

which implies that

$$[(M - N)(x(N) + p)]_i(x_i(N) - x_i(M)) \\ \geq [M(x(M) - x(N))]_i(x_i(M) - x_i(N)), \quad i = 1, \dots, n. \quad (16)$$

Since  $M$  is a  $P$ -matrix, it follows from (6) and  $x(M) - x(N) \neq 0$  that there exists an index  $i_0$  such that

$$[M(x(M) - x(N))]_{i_0}(x_{i_0}(M) - x_{i_0}(N)) \geq \mu \|x(M) - x(N)\|^2. \quad (17)$$

Then (16) and (17) imply

$$\mu \|x(M) - x(N)\|^2 \leq [(M - N)(x(N) + p)]_{i_0}(x_{i_0}(N) - x_{i_0}(M)) \\ \leq \|M - N\| \cdot \|x(N) + p\| \cdot \|x(N) - x(M)\|.$$

This completes the proof.

*Remark* In the literature on sensitivity analysis for VIP and CP, a lot of results have been obtained mainly based on the KKT system of the problem and various regularity conditions (or local assumptions), see [13, Section 5, 17]. Propositions 2.2 and 2.3 give global and simple results based only on somewhat strong assumptions on the mapping  $F$ .

Now we return to the affine subproblem of Josephy-Newton method. Let  $x$  be a given point in  $\mathfrak{R}^n$  and consider the linearized VIP of finding a

point  $z \in X$  such that

$$\langle F(x) + \nabla F(x)^\top(z - x), y - z \rangle \geq 0, \quad \forall y \in X, \quad (18)$$

or equivalently

$$(F_i(x) + [\nabla F(x)^\top(z - x)]_i)(y_i - z_i) \geq 0, \quad i = 1, \dots, n, \quad \forall y \in X. \quad (19)$$

If  $\nabla F(x)$  is a  $P$ -matrix, then problem (18) has a unique solution, which we denote  $z(x)$ . Our next result studies the properties of the solution of problem (18). The proposition is a refinement of Proposition 2.2 in [26].

**PROPOSITION 2.4** *Suppose that  $F$  is a continuously differentiable uniform  $P$ -function and  $X$  is a box. Then the solution  $z(x)$  of the affine VIP (18) is continuous as a function of  $x$ . Moreover,  $x$  is a solution of the VIP (1) if and only if  $x = z(x)$ .*

*Proof* First we note that, since  $F$  is a uniform  $P$ -function, (7) is satisfied with some constant  $\mu > 0$  independent of  $x$ . For two arbitrary points  $x, x' \in \mathfrak{R}^n$ , let  $z(x)$  and  $z(x')$  be the unique solutions of the linearized VIPs (18) at  $x$  and  $x'$ , respectively. Also let  $\bar{z}$  denote the unique solution of the affine VIP

$$\langle F(x) + \nabla F(x')^\top(z - x), y - z \rangle \geq 0, \quad \forall y \in X.$$

It then follows from Proposition 2.2 with  $v := F(x) - \nabla F(x')^\top x$ ,  $v' := F(x') - \nabla F(x')^\top x'$ ,  $b := 0$  and  $M := \nabla F(x')^\top$  that

$$\|\bar{z} - z(x')\| \leq \frac{1}{\mu} \|F(x) - F(x') + \nabla F(x')^\top(x' - x)\|.$$

On the other hand, by Proposition 2.3 with  $b := F(x)$ ,  $p := -x$ ,  $N := \nabla F(x)^\top$  and  $M := \nabla F(x')^\top$ , we have

$$\|\bar{z} - z(x)\| \leq \frac{1}{\mu} \|\nabla F(x')^\top - \nabla F(x)^\top\| \cdot \|z(x) - x\|.$$

It then follows that

$$\begin{aligned}
& \|z(x) - z(x')\| \\
& \leq \|\bar{z} - z(x')\| + \|\bar{z} - z(x)\| \\
& \leq \frac{1}{\mu} (\|F(x) - F(x') + \nabla F(x')^\top(x' - x)\| \\
& \quad + \|\nabla F(x')^\top - \nabla F(x)^\top\| \cdot \|z(x) - x\|).
\end{aligned}$$

Consequently, for any fixed  $x \in \mathfrak{X}^i$ , we obtain

$$\lim_{x' \rightarrow x} \|z(x) - z(x')\| = 0.$$

This proves the first half of the proposition.

To prove the second half, suppose first that  $z(x) = x$ . Then it follows immediately from (18) that  $x$  solves (1). Conversely suppose that  $x$  is a solution of (1). Since  $X$  is a box and  $z(x) \in X$ , (2) yields

$$F_i(x)(z_i(x) - x_i) \geq 0, \quad i = 1, \dots, n. \quad (20)$$

Similarly, from  $x \in X$  and (19), we have

$$F_i(x)(x_i - z_i(x)) + [\nabla F(x)^\top(z(x) - x)]_i(x_i - z_i(x)) \geq 0, \quad i = 1, \dots, n. \quad (21)$$

The inequalities (20) and (21) give

$$(z_i(x) - x_i)[\nabla F(x)^\top(z(x) - x)]_i \leq 0, \quad i = 1, \dots, n. \quad (22)$$

Since  $\nabla F(x)^\top$  is a  $P$ -matrix, it follows from (7) that there exists an index  $i_0$  such that

$$(z_{i_0}(x) - x_{i_0})[\nabla F(x)^\top(z(x) - x)]_{i_0} \geq \mu \|z(x) - x\|^2. \quad (23)$$

Combining (22) with (23), we get  $z(x) = x$ .

Regularity conditions have been widely used in the study of variational inequality problems, particularly in the analysis of local convergence properties of iterative methods for VIPs. A solution  $x^*$  of the

VIP (1) is said to be *regular* in the sense of Robinson [24] (see also [13]) if there exist a neighborhood  $\Omega$  of  $x^*$  and a neighborhood  $V$  of  $0 \in \mathbb{R}^n$  such that, for every  $v \in V$ , the perturbed VIP of finding a vector  $x \in X$  such that

$$\langle \tilde{F}(x; v), y - x \rangle \geq 0, \quad \forall y \in X, \quad (24)$$

where  $\tilde{F}(x; v) := F(x^*) + v + \nabla F(x^*)^\top (x - x^*)$ , has a unique solution  $x(v) \in \Omega$  that is Lipschitz continuous as a function of  $v$ , i.e.,

$$\|x(v) - x(v')\| \leq \rho \|v - v'\|, \quad \forall v, v' \in V$$

for some  $\rho > 0$ .

In [22], a strong but simple sufficient condition for a solution  $x^*$  of the VIP to be regular was given. Here we give a different condition pertaining to the box constrained VIP. The next proposition follows directly from Proposition 2.2. The proof is omitted.

**PROPOSITION 2.5** *Assume that  $x^*$  is a solution of the VIP (1) with  $X$  being a box. If the matrix  $\nabla F(x^*)$  is a  $P$ -matrix, then  $x^*$  is a regular solution.*

*Remark* Proposition 2.5 can also be derived from a characterization of strong regularity given in [6, Theorem 3.4]. The approach in [6] is based on the equivalent KKT system of box VIP and is different from our approach here.

The classical Josephy–Newton method generates a sequence  $\{x^k\}$  by the updating rule  $x^{k+1} = z(x^k)$ . From Proposition 2.4, the method is well defined if  $F$  is a uniform  $P$ -function. Moreover, if the initial point  $x^0$  is sufficiently close to the solution point  $x^*$  and the matrix  $\nabla F(x^*)$  is a  $P$ -matrix, then by Proposition 2.5,  $x^*$  is a regular solution and hence it follows from the basic result of Josephy–Newton method [13,14] that the generated sequence  $\{x^k\}$  converges locally quadratically to  $x^*$ .

### 3 A HYBRID JOSEPHY–NEWTON METHOD

In the previous section, we have discussed the affine VIP with box constraints. In this section, we consider the method proposed in [22]

for solving the VIP (1) with general convex constraints, which is Josephy–Newton method with D-gap function globalization. Our aim is to refine the convergence results obtained in [22] by restricting ourselves to the special case where the VIP (1) is box constrained.

The algorithm is stated as follows:

ALGORITHM

*Step 0* Choose  $x^0 \in \mathfrak{R}^n$ ,  $\omega \in (0,1)$ ,  $\zeta \in (0,1)$ ,  $\delta \in (0,1)$ ,  $\sigma \in (0,1)$ , and sufficiently small  $\epsilon \geq 0$ . Let  $k := 0$ .

*Step 1* Find  $z^k \in X$  such that

$$\langle F(x^k) + \nabla F(x^k)^\top (z^k - x^k), y - z^k \rangle \geq 0, \quad \forall y \in X, \quad (25)$$

and let  $d^k := z^k - x^k$ . If

$$g_{\alpha\beta}(x^k + d^k) \leq \zeta g_{\alpha\beta}(x^k), \quad (26)$$

then let  $\lambda_k := 1$  and go to Step 3. If the linearized VIP (25) is not solvable or if  $d^k$  does not satisfy the condition

$$\langle \nabla g_{\alpha\beta}(x^k), d^k \rangle \leq -\sigma \max\{\|\nabla g_{\alpha\beta}(x^k)\|^2, \|d^k\|^2\}, \quad (27)$$

then set  $d^k := -\nabla g_{\alpha\beta}(x^k)$ .

*Step 2* If  $\|d^k\| \leq \epsilon$  then stop. Otherwise find the smallest nonnegative integer  $m_k$  satisfying

$$g_{\alpha\beta}(x^k + \omega^{m_k} d^k) - g_{\alpha\beta}(x^k) \leq \delta \omega^{m_k} \langle \nabla g_{\alpha\beta}(x^k), d^k \rangle, \quad (28)$$

and let  $\lambda_k := \omega^{m_k}$ .

*Step 3* Set  $x^{k+1} := x^k + \lambda_k d^k$  and  $k := k + 1$ . Go to Step 1.

Using similar arguments to those in the proof of the global convergence theorem in [22], one can prove the following theorem.

**THEOREM 3.1** *Suppose that the mapping  $F$  is continuously differentiable. Let  $\epsilon = 0$  and suppose that the algorithm generates an infinite sequence  $\{x^k\}$ . Then any accumulation point  $x^*$  of the sequence  $\{x^k\}$  is a stationary point of the D-gap function  $g_{\alpha\beta}$ .*

If  $F$  is a uniform  $P$ -function, then by [15, Theorem 4.1], the level sets of D-gap function  $g_{\alpha\beta}$  are bounded. Since  $\{g_{\alpha\beta}(x^k)\}$  is nonincreasing, the boundedness of level sets guarantees the boundedness of the generated sequence  $\{x^k\}$  and hence the existence of at least one accumulation point of  $\{x^k\}$ . On the other hand, by [15, Theorem 3.1], any stationary point  $\bar{x}$  of  $g_{\alpha\beta}$  such that  $\nabla F(\bar{x})$  is a  $P$ -matrix is a solution of the VIP (1). Therefore, if  $F$  is a uniform  $P$ -function, then it follows from Theorem 3.1 that any accumulation point of the generated sequence  $\{x^k\}$  solves the VIP (1). Because the box constrained VIP with a uniform  $P$ -function has a unique solution, we obtain the next corollary to Theorem 3.1.

**COROLLARY 3.2** *Suppose that  $F$  is a continuously differentiable uniform  $P$ -function and  $X$  is a box. Then for any starting point  $x^0 \in \mathbb{R}^n$ , the sequence  $\{x^k\}$  generated by the algorithm converges to the unique solution of the VIP (1).*

To study the convergence rate of the algorithm, we need the following results concerning an error bound property of the D-gap function. We denote by  $y_\alpha(x)$  the unique maximizer on the right-hand side in the defining equation (3) of the regularized gap function  $f'_\alpha$ . Note that  $y_\alpha(x) = \prod_X(x - \alpha^{-1}F(x))$ , where  $\prod_X$  denotes the projection operator on  $X$ . We define  $R_\alpha(x) := x - y_\alpha(x)$ . Moreover  $y_\beta(x)$  and  $R_\beta(x)$  are defined similarly. Let  $B(\Delta)$  denote the closed sphere centered at  $x^*$  with radius  $\Delta > 0$ , i.e.,

$$B(\Delta) := \{x \in \mathbb{R}^n \mid \|x - x^*\| \leq \Delta\}.$$

Following the proof of Lemma 5.1 in [15], we have the next lemma.

**LEMMA 3.3** *Let  $x^*$  be a solution of the VIP (1) with  $X$  being a box. Suppose that  $F$  is a uniform  $P$ -function with modulus  $\mu$ . Suppose also that  $F$  is Lipschitz continuous with constant  $\kappa > 0$  on  $B(\Delta)$  for some  $\Delta > 0$ . Then there exists a constant  $\eta > 0$  such that*

$$\|x - x^*\| \leq \eta \|R_\beta(x)\|, \quad \forall x \in B(\Delta), \quad (29)$$

where  $\eta = (\kappa + \beta)/\mu$ .

The next lemma shows that the D-gap function provides a local error bound for the VIP (1) under suitable assumptions. This result will be useful in establishing the quadratic convergence of the proposed algorithm.

LEMMA 3.4 *Let  $x^*$  be a solution of the VIP (1) with  $X$  being a box. Suppose that  $F$  is a uniform  $P$ -function with modulus  $\mu$ . Suppose also that  $F$  is Lipschitz continuous with constant  $\kappa > 0$  on  $B(\Delta)$  for some  $\Delta > 0$ . Then there exist constants  $c_1, c_2 > 0$  such that*

$$c_1 \|x - x^*\|^2 \leq g_{\alpha\beta}(x) \leq c_2 \|x - x^*\|^2, \quad \forall x \in B(\Delta). \quad (30)$$

*Proof* By [27, Proposition 3.1], we have

$$\frac{\beta - \alpha}{2} \|R_\beta(x)\|^2 \leq g_{\alpha\beta}(x) \leq \frac{\beta - \alpha}{2} \|R_\alpha(x)\|^2, \quad \forall x \in \mathfrak{R}^n. \quad (31)$$

It follows from Lemma 3.3 and the left part of the above inequality that

$$g_{\alpha\beta}(x) \geq \frac{\beta - \alpha}{2\eta^2} \|x - x^*\|^2, \quad \forall x \in \mathfrak{R}^n,$$

where  $\eta = (\kappa + \beta)/\mu$ , which shows that the left inequality in (30) is true.

Next observe that

$$\begin{aligned} \|R_\alpha(x)\| &= \|x - y_\alpha(x) - x^* + y_\alpha(x^*)\| \\ &\leq \|x - x^*\| + \left\| \prod_X(x - \alpha^{-1}F(x)) - \prod_X(x^* - \alpha^{-1}F(x^*)) \right\| \\ &\leq \|x - x^*\| + \|x - \alpha^{-1}F(x) - x^* + \alpha^{-1}F(x^*)\| \\ &\leq \left(2 + \frac{\kappa}{\alpha}\right) \|x - x^*\| \end{aligned}$$

for all  $x \in B(\Delta)$ , where the equality follows from the definition of  $R_\alpha$  and the fact that  $R_\alpha(x^*) = 0$ , the first inequality follows from the triangle inequality and the definition of  $y_\alpha$ , the second inequality follows from the nonexpansiveness of the projection operator  $\prod_X$ , and the last inequality follows from the Lipschitz continuity of  $F$ . The right inequality in (30) then follows from the right inequality of (31). The proof is complete.

We are ready to prove quadratic convergence of the proposed algorithm.

**THEOREM 3.5** *Suppose that  $F$  is continuously differentiable and  $X$  is a box. Let  $x^*$  be an accumulation point of the sequence  $\{x^k\}$  generated by the algorithm. If  $F$  is a uniform  $P$ -function and  $\nabla F$  is locally Lipschitzian, then  $x^*$  is a solution of the VIP (1) and the sequence  $\{x^k\}$  converges quadratically to  $x^*$ .*

*Proof* By Theorem 3.1,  $x^*$  is a stationary point of the D-gap function  $g_{\alpha\beta}$ . Since  $\nabla F(x^*)$  is a  $P$ -matrix by Lemma 2.1, it follows from [15, Theorem 3.1] that  $x^*$  is already a solution of the VIP (1). Moreover Proposition 2.5 shows that  $x^*$  is a regular solution. Since  $F$  is differentiable, it is locally Lipschitzian; hence there exists a  $\Delta_1 > 0$  such that  $F$  is Lipschitz continuous on  $B(\Delta_1)$ . Hence by Lemma 3.4, we have

$$c_1 \|x - x^*\|^2 \leq g_{\alpha\beta}(x) \leq c_2 \|x - x^*\|^2, \quad \forall x \in B(\Delta_1) \quad (32)$$

for some  $c_1, c_2 > 0$ . Moreover, by choosing a smaller  $\Delta_1$  if necessary, we may assume that  $\nabla F$  is Lipschitz continuous on  $B(\Delta_1)$ . Since  $x^*$  is a regular solution, by the basic result on Josephy–Newton’s method for the VIP [13,14], there exists a  $\Delta_2 > 0$  such that for any initial point chosen from  $B(\Delta_2)$ , the Newton iteration is well-defined and

$$\|z(x) - x^*\| \leq c_3 \|x - x^*\|^2, \quad \forall x \in B(\Delta_2) \quad (33)$$

holds for some constant  $c_3 > 0$ . Let  $\Delta_3 := \min(\Delta_1, \Delta_2)$ . Then it follows from (32) and (33) that

$$g_{\alpha\beta}(z(x)) \leq c_2 c_3 \|x - x^*\|^4, \quad \forall x \in B(\Delta_3). \quad (34)$$

Let

$$\Delta_4 := \min \left\{ \Delta_3, \sqrt{\frac{\zeta c_1}{c_2 c_3}} \right\}.$$

Then it follows from (34) that for any  $x \in B(\Delta_4)$

$$g_{\alpha\beta}(z(x)) \leq \zeta c_1 \|x - x^*\|^2 \leq \zeta g_{\alpha\beta}(x),$$



where the first inequality follows from the choice of  $\Delta_4$  and the second inequality follows from the left inequality in (32). This implies that, when  $x^k \in B(\Delta_4)$ , we have  $d^k = z(x^k) - x^k$  and the step size  $\lambda_k = 1$  is accepted, i.e.,  $x^{k+1} = z(x^k)$ . Consequently it follows from (33) that the sequence  $\{x^k\}$  converges to  $x^*$  quadratically.

Similarly we may prove superlinear convergence of the algorithm under slightly weaker assumptions. The proof is omitted.

**THEOREM 3.6** *Suppose that  $F$  is continuously differentiable and  $X$  is a box. Let  $x^*$  be an accumulation point of the sequence  $\{x^k\}$  generated by the algorithm. If  $F$  is a uniform  $P$ -function, then  $x^*$  is a solution of the VIP (1) and the sequence  $\{x^k\}$  converges superlinearly to  $x^*$ .*

## 4 NUMERICAL RESULTS

We implemented the hybrid Josephy–Newton method suggested in this paper in MATLAB and run it on a SUN SPARC 10 station. We first give a brief description of the implementation: Let

$$r(x) := x - \text{Proj}_{[l,u]}(x - F(x))$$

denote the natural residual of the box constrained variational inequality problem. We terminate our method if

$$\|r(x^k)\| \leq \epsilon_1 \quad \text{or} \quad g_{\alpha\beta}(x^k) \leq \epsilon_2 \quad (35)$$

for some iterate  $x^k$ , where

$$\epsilon_1 := 10^{-6} \quad \text{and} \quad \epsilon_2 := 10^{-11}.$$

In addition, the iteration was stopped if  $k > k_{\max}$  with  $k_{\max} = 100$ .

For the D-gap function  $g_{\alpha\beta}$ , we used the parameters

$$\alpha = 0.9 \quad \text{and} \quad \beta = 1.1.$$

In the line search rule (28), we used

$$\omega = 0.5 \quad \text{and} \quad \delta = 10^{-4}.$$

However, we replaced the standard (monotone) Armijo-rule by a nonmonotone variant, see Grippo *et al.* [12] for details.

As a solver for the linearized variational inequality problems, we used the semismooth Newton-type method from [6]. In contrast to what is said in the description of our algorithm, however, we always accept the corresponding search direction  $d^k$  whenever it satisfies the descent test

$$\nabla g_{\alpha\beta}(x^k)^\top d^k < 0;$$

note that this guarantees that the Armijo line search is well-defined. In particular, we accept this search direction  $d^k$  even if we were not able to solve the corresponding linearized variational inequality problem. In this way, we try to overcome the problem that we have to take too many gradient steps in a row which is obviously not very desirable.

In order to improve the efficiency of our algorithm, however, we also used a preprocessor; more precisely, we first try to solve our test examples by using the recently proposed method from Kanzow and Fukushima [16]. This is a nonsmooth Newton-type method applied to the residual equation

$$r(x) = 0$$

and globalized by the D-gap function  $g_{\alpha\beta}$ , see [16] for details. The motivation for doing this is quite simple: The method from [16] works extremely well whenever it solves a problem successfully. Unfortunately, it does not seem to be very robust unless relatively strong assumptions are satisfied.

So we first apply the nonsmooth Newton-type method from [16] in order to solve a test example, but we stop this preprocessing iteration if either the termination criterion (35) is satisfied or if a certain test indicates that the preprocessor runs into difficulties. In the latter case, we switch to the hybrid Newton method introduced in this paper which is not as efficient as the method from [16], but which seems to be considerably more reliable.

Basically, our criterion for switching from the preprocessor to the hybrid Newton method is as follows: If

$$t_k \leq t_{\min} \quad \text{or} \quad \|\nabla g_{\alpha\beta}(x^k)\| \leq c g_{\alpha\beta}(x^k), \quad (36)$$

then terminate the preprocessing iteration and go to the hybrid Newton method using the previous iterate  $x^{k-1}$  as a starting point. The actual parameters used in (36) are

$$t_{\min} = 10^{-4} \quad \text{and} \quad c = 10^{-2}.$$

If the preprocessor is successful and converges to a solution of the box constrained variational inequality problem which satisfies the standard regularity conditions used in [16] for the local convergence theory, then  $t_k = 1$  for all  $k$  sufficiently large and  $g_{\alpha\beta}(x^k) = O(\|\nabla g_{\alpha\beta}(x^k)\|^2)$ , so none of the tests in (36) will be satisfied.

We applied the method just described to all test problems from the MCPLIB and GAMS LIB libraries, see [4,8], using all the different starting points which are available within the MATLAB environment.

We report the numerical results in Table I for the MCPLIB test problems and in Table II for the GAMS LIB test problems. The columns in these tables have the following meanings:

Problem:	name of the test problem in MCPLIB
$n$ :	number of variables
$m$ :	number of (finite) bounds on the variables $x_i$
SP:	starting point
P-steps:	number of iterations used in the preprocessing phase
N-steps:	number of Newton steps used in the hybrid Newton phase
G-steps:	number of gradient steps used in the hybrid Newton phase
$F$ -eval.:	number of function evaluations
$g_{\alpha\beta}(x^f)$ :	value of $g_{\alpha\beta}(x)$ at the final iterate $x = x^f$
$\ r(x^f)\ $ :	value of $\ r(x)\ $ at the final iterate $x = x^f$ .

Looking at Tables I and II, we see that we have just a few failures on some difficult test problems, whereas the overall behavior of our method is quite good. Although many of the simple problems were solved by the preprocessor (i.e., there are no N- and no G-steps), the hybrid Josephy–Newton method introduced in this paper was necessary in order to solve a number of other test examples.

In fact, we made the following observation during the testing phase for our algorithm: Both the preprocessor from [16] and the hybrid Newton-type method discussed in this paper try to minimize the D-gap

TABLE I Numerical results for MCPLIB test problems

<i>Problem</i>	<i>n</i>	<i>m</i>	SP	P-steps	N-steps	G-steps	F-eval.	$g_{\alpha\beta}(x^f)$	$\ r(x^f)\ $
bertsekas	15	15	1	0	4	0	11	7.5e-15	2.7e-7
bertsekas	15	15	2	1	4	0	12	7.5e-15	2.7e-7
bertsekas	15	15	3	1	4	0	12	7.5e-15	2.7e-7
bert_oc	5000	2000	1	4	0	0	6	3.8e-28	6.1e-14
billups	1	1	1	—	—	—	—	—	—
bratu	5625	11250	1	13	0	0	29	1.0e-20	1.9e-10
choi	13	26	1	4	0	0	5	4.4e-15	2.1e-7
colvdual	20	20	1	—	—	—	—	—	—
colvdual	20	20	2	—	—	—	—	—	—
colvnlp	15	15	1	0	3	0	10	2.0e-13	1.4e-6
colvnlp	15	15	2	1	3	0	20	3.5e-12	5.9e-6
cycle	1	1	1	3	0	0	5	2.3e-21	1.5e-10
ehl_k40	41	40	1	12	8	0	115	9.6e-23	3.1e-11
ehl_k60	61	60	1	22	8	0	221	1.7e-17	1.3e-8
ehl_k80	81	80	1	24	8	0	233	3.1e-17	1.7e-8
ehl_kost	101	100	1	28	8	0	273	6.6e-16	8.1e-8
ehl_kost	101	100	2	28	8	0	273	6.6e-16	8.1e-8
ehl_kost	101	100	3	28	8	0	273	6.6e-16	8.1e-8
explcp	16	16	1	15	0	0	31	0	0
freebert	15	10	1	0	3	0	10	2.4e-16	4.8e-8
freebert	15	10	2	1	3	0	16	6.6e-23	2.6e-11
freebert	15	10	3	0	3	0	10	2.4e-16	4.8e-8
freebert	15	10	4	0	4	0	11	4.4e-21	2.1e-10
freebert	15	10	5	1	3	0	9	3.2e-19	1.8e-9
gafni	5	10	1	13	0	0	46	7.5e-19	2.7e-9
gafni	5	10	2	12	0	0	44	7.5e-19	2.7e-9
gafni	5	10	3	13	0	0	46	7.5e-19	2.7e-9
hanskoop	14	14	1	0	4	0	13	2.3e-14	4.8e-7
hanskoop	14	14	2	0	6	0	17	2.1e-14	4.5e-7
hanskoop	14	14	3	0	5	0	14	5.3e-14	7.3e-7
hanskoop	14	14	4	4	4	0	22	3.2e-14	5.7e-7
hanskoop	14	14	5	0	7	0	24	6.0e-15	2.4e-7
hydroc06	29	11	1	5	0	0	7	1.1e-25	1.1e-12
hydroc20	99	39	1	8	0	0	10	3.7e-14	6.0e-7
jel	6	6	1	8	0	0	16	1.4e-15	1.2e-7
josephy	4	4	1	10	0	0	23	2.9e-17	1.7e-8
josephy	4	4	2	7	0	0	15	4.5e-22	6.7e-11
josephy	4	4	3	11	0	0	24	2.9e-17	1.7e-8
josephy	4	4	4	4	0	0	5	4.2e-15	2.1e-7
josephy	4	4	5	3	0	0	4	4.3e-15	2.1e-7
josephy	4	4	6	6	0	0	12	2.2e-15	1.5e-7
kojshin	4	4	1	9	0	0	22	1.2e-20	3.5e-10
kojshin	4	4	2	7	0	0	14	1.3e-23	1.2e-11
kojshin	4	4	3	10	0	0	23	1.2e-20	3.5e-10
kojshin	4	4	4	1	0	0	2	0	0
kojshin	4	4	5	3	0	0	4	4.3e-15	2.1e-7
kojshin	4	4	6	5	0	0	7	6.3e-16	7.59e-8
mathinum	3	3	1	22	0	0	47	4.1e-14	6.4e-7
mathinum	3	3	2	4	0	0	5	3.4e-16	5.8e-8
mathinum	3	3	3	32	0	0	76	4.3e-14	6.5e-7

TABLE I (Continued)

<i>Problem</i>	<i>n</i>	<i>m</i>	SP	P-steps	N-steps	G-steps	<i>F-eval.</i>	$g_{\alpha\beta}(x^f)$	$\ r(x)^f\ $
mathinum	3	3	4	5	0	0	6	$2.7e-14$	$5.1e-7$
mathisum	4	4	1	4	0	0	6	$2.0e-22$	$4.4e-11$
mathisum	4	4	2	5	0	0	6	$5.4e-15$	$2.3e-7$
mathisum	4	4	3	26	0	0	53	$3.7e-14$	$6.1e-7$
mathisum	4	4	4	5	0	0	6	$8.2e-21$	$2.8e-10$
methan08	31	15	1	4	0	0	5	$6.3e-24$	$7.9e-12$
nash	10	10	1	6	0	0	7	$3.8e-17$	$1.9e-8$
nash	10	10	2	12	0	0	39	$6.1e-15$	$2.5e-7$
obstacle	2500	5000	1	10	0	0	11	$3.3e-31$	$2.3e-15$
opt_cont31	1024	1024	1	5	0	0	12	$3.6e-30$	$4.3e-15$
opt_cont127	4096	4096	1	8	0	0	32	$1.4e-29$	$9.0e-15$
opt_cont255	8192	8192	1	11	0	0	51	$2.9e-29$	$1.2e-14$
opt_cont511	16384	16384	1	12	0	0	52	$6.0e-29$	$1.6e-14$
pgvon105	105	105	1	20	0	0	91	$2.7e-12$	$5.2e-6$
pgvon106	106	106	1	—	—	—	—	—	—
pies	42	52	1	20	6	0	153	$2.1e-16$	$4.6e-8$
powell	16	16	1	4	3	0	17	$7.5e-20$	$8.6e-10$
powell	16	16	2	6	4	0	22	$1.8e-16$	$4.3e-8$
powell	16	16	3	0	12	0	35	$2.3e-21$	$1.5e-10$
powell	16	16	4	0	7	0	21	$1.7e-17$	$1.3e-8$
powell_mcp	8	0	1	6	0	0	7	$4.2e-24$	$6.5e-12$
powell_mcp	8	0	2	7	0	0	8	$4.7e-25$	$2.2e-12$
powell_mcp	8	0	3	8	0	0	9	$2.2e-16$	$4.7e-8$
powell_mcp	8	0	4	7	0	0	8	$1.7e-15$	$1.3e-7$
scarfanum	13	13	1	0	4	0	20	$8.2e-13$	$2.9e-6$
scarfanum	13	13	2	0	5	0	20	$1.3e-16$	$3.7e-8$
scarfanum	13	13	3	9	0	0	12	$5.2e-20$	$7.2e-10$
scarfasum	14	14	1	4	0	0	6	$3.2e-18$	$5.6e-9$
scarfasum	14	14	2	0	3	0	16	$8.8e-13$	$3.0e-6$
scarfasum	14	14	3	9	0	0	12	$5.2e-20$	$7.2e-10$
scarfbnum	39	39	1	—	—	—	—	—	—
scarfbnum	39	39	2	—	—	—	—	—	—
scarfbsum	40	40	1	13	0	0	62	$5.3e-14$	$7.3e-7$
scarfbsum	40	40	2	26	0	0	66	$9.6e-14$	$9.8e-7$
sppe	27	27	1	23	3	0	60	$1.4e-22$	$3.7e-11$
sppe	27	27	2	21	4	0	59	$3.1e-21$	$1.7e-10$
tobin	42	42	1	15	0	0	46	$9.6e-23$	$3.1e-11$
tobin	42	42	2	22	0	0	83	$4.0e-21$	$2.0e-10$

function  $g_{\alpha\beta}$ . Now, the D-gap function might have a local minimum which does not correspond to a solution of the box constrained variational inequality problem. In that case, we would expect both algorithms to run into difficulties by converging to one of these local minima, in particular, since the search directions computed by both methods are based on some local information of the variational inequality problem. In fact, this difficulty arises, e.g., for the billups example. In general, however, our observation is that the method from [16] tends to converge

TABLE II Numerical results for GAMSLIB test problems

<i>Problem</i>	<i>n</i>	<i>m</i>	SP	P-steps	N-steps	G-steps	F-eval.	$g_{\alpha,\beta}(x^f)$	$\ r(x^f)\ $
cafemge	101	101	1	10	0	0	29	$3.3e-16$	$5.7e-8$
cammcp	242	242	1	6	0	0	8	$1.1e-16$	$3.3e-8$
cirimge	9	6	1	6	0	0	53	$2.0e-32$	$4.4e-16$
co2mge	208	208	1	2	0	0	15	$1.6e-15$	$1.3e-7$
dmcmge	170	170	1	147	33	8	1658	$1.5e-14$	$3.9e-7$
ers82mcp	232	0	1	5	0	0	6	$1.8e-24$	$4.2e-12$
etamge	114	114	1	15	0	0	42	$1.5e-14$	$3.8e-7$
hansmcp	43	43	1	33	11	16	787	$2.4e-13$	$1.6e-6$
hansmge	43	43	1	5	5	0	62	$2.8e-13$	$1.7e-6$
harkmcp	32	32	1	26	0	0	60	$1.0e-14$	$3.2e-7$
harmge	11	9	1	1	6	0	18	$1.4e-13$	$1.2e-6$
kehomge	9	9	1	12	0	0	18	$4.3e-23$	$2.1e-11$
kormcp	78	0	1	3	0	0	5	$1.8e-25$	$1.3e-12$
mr5mcp	350	350	1	1	9	0	51	$7.3e-22$	$8.5e-11$
nsmge	212	212	1	5	14	0	59	$2.3e-17$	$1.5e-8$
oligomcp	6	6	1	6	0	0	9	$1.0e-20$	$3.2e-10$
scarfmcp	18	18	1	0	4	0	14	$5.4e-12$	$7.3e-6$
scarfmge	18	18	1	0	6	0	18	$1.4e-15$	$1.2e-7$
transmcp	11	11	1	0	1	0	3	$2.0e-17$	$1.4e-8$
two3mcp	6	6	1	8	0	0	16	$1.4e-15$	$1.2e-7$
unstmge	5	5	1	8	0	0	11	$2.3e-18$	$4.7e-9$
vonthmcp	125	125	1	—	—	—	—	—	—
vonthmge	80	80	1	200	8	1	1263	$1.4e-13$	$1.2e-6$
wallmcp	6	0	1	2	0	0	3	$3.5e-21$	$1.9e-10$

to a local minimum of  $g_{\alpha,\beta}$  much more often than the method discussed here. This seems to indicate that, from a global point of view, the search direction computed by our hybrid Newton-type method is a much better search direction than the one computed by the nonsmooth Newton method in [16].

It is therefore our feeling that the robustness of many existing solvers can be improved by using the search direction from our hybrid Josephy–Newton method whenever the underlying solver does not seem to converge.

Finally, let us shortly mention what happens if we do not use our preprocessor. Obviously, this changes the numerical results. In general, however, the changes are not very dramatic, for example, the three *gafni* examples can be solved in just 3–4 Josephy–Newton steps (instead of 12–13 linear system solves when using the preprocessor), the four *powell\_mcp* problems can be solved in 5–8 Josephy–Newton iterations (compared to 6–8 linear system solves), and the *cafemge*

example is solved in 6 Josephy–Newton iterations (compared to 10 iterations in the preprocessor). The differences are more dramatic for the josephy and kojshin test problems, mainly due to the fact that the linearized subproblems quite often do not have a solution for these examples. For instance, it takes 97 Josephy–Newton iterations (14 N-steps and 83 G-steps) in order to solve the josephy problem when using the fourth starting point.

## 5 CONCLUDING REMARKS

The variational inequality problem is reformulated as an unconstrained minimization problem by using the D-gap function  $g_{\alpha\beta}$ . Some properties of the affine VI subproblem of the classical Josephy–Newton method are studied. A hybrid Josephy–Newton method is then proposed to minimize the function  $g_{\alpha\beta}$ . Under mild conditions, the proposed method is shown to be globally convergent. If some additional assumptions are satisfied, then the sequence converges quadratically or superlinearly to a solution of the original variational inequality problem. A sufficient condition is given for a solution  $x^*$  of the VIP to be regular. This condition is only concerned with the mapping  $F$ , unlike the conditions in [6,17,24].

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