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MAXIMAL ESTIMATES FOR THE FOKKER-PLANCK OPERATOR WITH MAGNETIC FIELD

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ABSTRACT. We consider the Fokker-Planck operator with a strong external magnetic field. We show a maximal type estimate on this operator using a nilpotent approach on vector field polynomial operators and including the notion of representation of a Lie algebra. This estimate makes it possible to give an optimal characterization of the domain of the closure of the considered operator.

1. INTRODUCTION AND MAIN RESULTS

1.1. **Introduction.** The Fokker-Planck equation was introduced by Fokker [4] and Planck [14], to describe the evolution of the density of particles under Brownian motion. In recent years, global hypoelliptic estimates have experienced a rebirth through applications to the kinetic theory of gases. In this direction many authors have shown maximal estimates to deduce the compactness of the resolvent of the Fokker-Planck operator and resolvent estimates to address the issue of return to the equilibrium. F. Hérau and F. Nier in [10] have highlighted the links between the Fokker-Planck operator with a confining potential and the associated Witten Laplacian. Then, in the book of B. Helffer and F. Nier [5], this work has been extended and explained in a general way, and we refer more specifically to Chapter 9 for a demonstration of the maximal estimate.

In this article, we continue the study of the model case of the Fokker-Planck operator with an external magnetic field B_e, started in [12], and we establish a maximal-type estimate for this model, giving a characterization of the domain of its closed extension.

1.2. **Statement of the result.** For $d = 2$ or 3, we consider the Fokker-Planck operator K with an external magnetic field B_e defined on $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ with values in $\mathbb{R}^{d(d-1)/2}$ such that

$$(1) \qquad K = v \cdot \nabla_x - (v \wedge B_e) \cdot \nabla_v - \Delta_v + v^2/4 - d/2,$$

where $v \in \mathbb{R}^d$ represents the velocity, $x \in \mathbb{T}^d$ represents the space variable and $t > 0$ is the time. In the previous definition of our operator, we use $(v \wedge B_e) \cdot \nabla_v$ to

mean

$$(v \wedge B_e) \cdot \nabla_v = \begin{cases} b(x)\,(v_1 \partial_{v_2} - v_2 \partial_{v_1}) & \text{if } d = 2 \\[2mm] b_1(x)(v_2 \partial_{v_3} - v_3 \partial_{v_2}) + b_2(x)(v_3 \partial_{v_1} - v_1 \partial_{v_3}) \\ + b_3(x)(v_1 \partial_{v_2} - v_2 \partial_{v_1}) & \text{if } d = 3. \end{cases}$$

The operator K is considered as an unbounded operator on the Hilbert space $H = L^2(\mathbb{T}^d \times \mathbb{R}^d)$ whose domain is the Schwartz space $D(K) = \mathcal{S}(\mathbb{T}^d \times \mathbb{R}^d)$. We denote by

- $K_{\min}$ the minimal extension of K where $D(K_{\min})$ is the closure of $D(K)$ with respect to the graph norm on $H \times H$.
- $K_{\max}$ is the maximal extension of K whose domain $D(K_{\max})$ is given by

$$D(K_{\max}) = \{u \in L^2(\mathbb{T}^d \times \mathbb{R}^d) \,/\, Ku \in L^2(\mathbb{T}^d \times \mathbb{R}^d)\}.$$

From now on, we use the notation $\mathbf{K}$ for the operator $K_{\min}$.

The existence of a strongly continuous semi-group associated to the operator $\mathbf{K}$ is shown in [12] when the magnetic field is regular. We improve this result by considering a much lower regularity. In order to obtain the maximal accretivity, we are led to substitute the hypoellipticity argument by a regularity argument for the operators with coefficients in L^∞, which will be combined with more classical results of Rothschild-Stein in [16] for Hörmander operators of type-2 (see [11] for more details of this subject).

Theorem 1.1. *If $B_e \in L^\infty(\mathbb{T}^d, \mathbb{R}^{d(d-1)/2})$, then $\mathbf{K}$ is maximally accretive.*

This implies that the domain of the operator $\mathbf{K}$ has the following property:

$$(2) \qquad\qquad\qquad D(\mathbf{K}) = D(K_{\max}).$$

In this work, we are interested in specifying the domain of the operator $\mathbf{K}$ introduced in (2). To accomplish this, we will obtain maximal estimates for $\mathbf{K}$, using techniques developed initially for the study of hypoellipticity of invariant operators on nilpotent groups. Before we state our main result, we establish some notation.

Notation 1.2.

- $B^2(\mathbb{R}^d)$ *(or B_v^2 to indicate the name of the variables) denotes the space*

$$B^2(\mathbb{R}^d) := \{u \in L^2(\mathbb{R}^d) \,/\, \forall (\alpha, \beta) \in \mathbb{N}^{2d}, \, |\alpha| + |\beta| \le 2, \, v^\alpha \, \partial_v^\beta u \in L^2(\mathbb{R}^d)\},$$

  which is equipped with its natural Hilbertian norm

- $\tilde{B}^2(\mathbb{T}^d \times \mathbb{R}^d)$ *is the space $L^2(\mathbb{T}_x^d, B_v^2(\mathbb{R}^d))$ with the following Hilbertian norm:*

$$\tilde{B}^2(\mathbb{T}^d \times \mathbb{R}^d) \ni u \longrightarrow \|u\|_{\tilde{B}^2} = \sqrt{\sum_{|\alpha| + |\beta| \le 2} \left\| v^\alpha \, \partial_v^\beta u \right\|^2}.$$

  where $\|.\|$ is the $L^2(\mathbb{T}^d \times \mathbb{R}^d)$ norm .

- $\mathrm{Lipsch}(\mathbb{T}^d)$ *is the space of Lipschitizian functions from $\mathbb{T}^d$ with values in $\mathbb{R}^{d(d-1)/2}$, equipped with the following norm:*

$$\|u\|_{\mathrm{Lipsch}(\mathbb{T}^d)} = \|u\|_{L^\infty(\mathbb{T}^d, \mathbb{R}^{d(d-1)/2})} + \sup_{x,y \in \mathbb{T}^d, x \ne y} \frac{|u(x) - u(y)|}{d(x,y)},$$

  where d is the natural distance in $\mathbb{T}^d$ and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^{d(d-1)/2}$.

We can now state the main theorem of this article:

Theorem 1.3. *Let $d = 2$ or 3. We assume that $B_e \in \text{Lipsch}(\mathbb{T}^d)$. Then for any $C_1 > 0$, there exists some $C > 0$ such that for all B_e with $\|B_e\|_{\text{Lipsch}(\mathbb{T}^d)} \leq C_1$, and for all $u \in \mathcal{S}(\mathbb{T}^d \times \mathbb{R}^d)$, the operator K satisfies the following maximal estimate:*

$$(3) \qquad \|(v \cdot \nabla_x - (v \wedge B_e) \cdot \nabla_v)u\| + \|u\|_{\tilde{B}^2} \leq C(\|Ku\| + \|u\|),$$

where $\|.\|$ is the L^2 norm.

Given the density of $\mathcal{S}(\mathbb{T}^d \times \mathbb{R}^d)$ in the domain of $\mathbf{K}$, we obtain the following characterization of the domain of $\mathbf{K}$:

Corollary 1.4.

$$D(\mathbf{K}) = \{u \in \tilde{B}^2(\mathbb{T}^d \times \mathbb{R}^d) \,/\, (v \cdot \nabla_x - (v \wedge B_e) \cdot \nabla_v)\, u \in L^2(\mathbb{T}^d \times \mathbb{R}^d)\}.$$

Organization of the article:
In the next section, we recall the notion of maximal hypoellipticity, and more specifically, we give some results of maximal hypoellipticity for polynomial operators of vector fields with the nilpotent approach. Then, in the sections 3 and 4, we use these techniques to prove the main result Theorem 1.3, beginning with $d = 2$ and continuing with $d = 3$. Finally, in the appendix we give the proof of Theorem 1.1 .

2. Review of maximal hypoellipticity in the nilpotent approach

2.1. **Maximal hypoellipticity for polynomial operators of vector fields.**
We are interested in the polynomial operators of vector fields. We consider $p + q$ real C^∞ vector fields $(X_1, ..., X_p, Y_1, ..., Y_q)$ on an open set $\Omega \subset \mathbb{R}^d$.

Let $P(z, \zeta_1, ..., \zeta_{p+q})$ be a non-commutative polynomial of degree m in $p + q$ variables, with C^∞ coefficients on Ω, and let $\mathcal{P}$ be the differential operator

$$(4) \qquad \mathcal{P} = P(z, Z_1, ..., Z_{p+q}) = \sum_{|\alpha| \leq m} a_\alpha(z)\, Z^\alpha, \quad \forall \alpha \in \{1, ..., p + q\}^k,$$

where, for $\ell \in \{1, ..., p + q\}$, the vector field Z_ℓ, is defined by

$$Z_\ell = X_\ell, \quad \forall \ell = 1, ..., p,$$
$$Z_\ell = Y_{\ell-p}, \quad \forall \ell = p + 1, ..., p + q,$$

and where, for $\alpha = (\alpha_1, .., \alpha_k) \in \{1, ..., p + q\}^k$,

$$|\alpha| = \sum_{j=1}^k d(\alpha_j) \text{ with } d(\alpha_j) = \begin{cases} 1 & \text{if } \alpha_j \in \{1, .., p\} \\ 2 & \text{if } \alpha_j \in \{p + 1, .., p + q\}. \end{cases}$$

It is further assumed that the vector fields Z_j with $j \in \{1, ..., p + q\}$ satisfy the Hörmander condition in Ω:

Condition 2.1. *There exists an integer r such that the vector space spanned by the iterated brackets of the vector fields Z_j of length less than or equal to r, in each point z of Ω, is all of $T_z\Omega$.*

When $q = 0$ and the vector fields Z_j satisfy Condition 2.1, the operator $\mathcal{P}$ is called a differential operator of type-1. Thus the Hörmander operator $\sum_{j=1}^p X_j^2$ (case $q = 0$) is called a "**type-1 Hörmander operator**".
When $q = 1$ and the vector fields Z_j satisfy Condition 2.1, the operator $\mathcal{P}$ is

called a differential operator of type-2. The operator also studied by Hörmander $\sum_{j=1}^{p} X_j^2 + Y_1$ (case $q = 1$) is called a "**type-2 Hörmander operator**".

Now we introduce the following definition.

Definition 2.2. *Let $m \in \mathbb{N}^*$. The operator $\mathcal{P}$ is maximal hypoelliptic at a point z of Ω, if there is a neighborhood ω of z, and a constant $C > 0$ such that*

$$(5) \qquad \|u\|^2_{\mathcal{H}^m(\omega)} \le C[\|\mathcal{P}u\|^2_{L^2(\omega)} + \|u\|^2_{L^2(\omega)}], \quad \forall u \in C_0^\infty(\omega),$$

where $\|\cdot\|_{\mathcal{H}^m(\omega)}$ denotes the standard norm whose square is defined by

$$(6) \qquad u \mapsto \|u\|^2_{\mathcal{H}^m(\omega)} = \sum_{|\alpha| \le m} \|Z_{\alpha_1}...Z_{\alpha_k} u\|^2_{L^2(\omega)}, \quad \forall \alpha \in \{1, ..., p\}^m.$$

It has been shown by Helffer-Nourrigat in [6] that if the Hörmander condition is verified at z, then the maximal hypoellipticity at z implies that $\mathcal{P}$ is hypoelliptic in a neighborhood of z, which justifies the terminology.

It has also been shown by Rothschild-Stein in [17] that if the homogeneous operator associated to $\mathcal{P}$ (*i.e.* $\sum_{|\alpha|=m} a_\alpha(z) Z^\alpha$) is hypoelliptic at a point z_0, in the sense introduced by L. Schwartz, then $\mathcal{P}$ is maximal hypoelliptic in a neighborhood of z_0.

2.2. **Nilpotent and graded Lie algebras.** We refer the reader to [9, Chapter 2] for more details on this subject.

Definition 2.3. *We say that a Lie algebra $\mathcal{G}$ is graded nilpotent of rank r, if it admits a decomposition of the form*

$$\mathcal{G} = \mathcal{G}_1 \oplus ... \oplus \mathcal{G}_r \,,$$
$$[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j} \quad \text{if } i+j \le r \quad \text{and} \quad [\mathcal{G}_i, \mathcal{G}_j] = 0 \quad \text{if } i+j > r \,.$$

Definition 2.4. *Let $\mathcal{G} = \mathcal{G}_1 \oplus ... \oplus \mathcal{G}_r$ be a graded nilpotent Lie algebra with rank r.*

(1) *We say $\mathcal{G}$ is stratified of type 1 (or simply stratified) if it is generated by $\mathcal{G}_1$.*
(2) *We say $\mathcal{G}$ is stratified of type 2 if it is generated by $\mathcal{G}_1 \oplus \mathcal{G}_2$.*

From now on, $\mathcal{G}$ will always refer to a graded nilpotent Lie algebra of rank r.

2.3. **Representations theory on Lie algebras.** Among the representations, the irreducible unitary representations play a crucial role. Kirillov's theory allows us to associate to every element of the dual $\mathcal{G}^*$ of $\mathcal{G}$ an irreducible representation. Moreover, this theory says that any irreducible unitary representation can be represented in this way.

To be more precise, we give a definition of an induced representation. The starting point is a subalgebra $\mathcal{H} \subset \mathcal{G}$ and a linear form ℓ on $\mathcal{G}$ such that

$$\ell([\mathcal{H}, \mathcal{H}]) = 0.$$

We will then associate a representation $\pi_{\ell, \mathcal{H}}$ of the group $G := \exp(\mathcal{G})$ in $V_\pi := L^2(\mathbb{R}^{k(\pi)})$ which is uniquely defined modulo unitary conjugation, where $k(\pi)$ is the codimension of $\mathcal{H}$ in $\mathcal{G}$. For this construction and using the nilpotent character, we can find $k = k(\pi)$ linearly independent vectors $e_1, .., e_k$ such that any $a \in \mathcal{G}$ can be written in the form:

$$(7) \qquad g := \exp(a) = h \, \exp(s_k e_k) ... \exp(s_1 e_1) \,,$$

and such that, if

$$\mathcal{A}_j = \mathcal{H} \oplus \mathbb{R}e_1 \oplus .. \oplus \mathbb{R}e_{k-j+1} \,,$$

then $\mathcal{A}_{j-1}$ is ideal of codimension one in $\mathcal{A}_j$.

With this construction, we can obtain that $g \mapsto (s, h)$ is a global diffeomorphism from $\mathcal{G}$ to $\mathbb{R}^k \times \mathcal{H}$. The induced representation is given by

$$(\pi_{\ell,\mathcal{H}}(\exp a)f)(t) = \exp i \langle \ell, h(t,a) \rangle f(\sigma(t,a)),$$

where $h(t,a)$ and $\sigma(t,a)$ are defined by the following formula:

$$\exp t_k e_k .... \exp t_1 e_1 \exp a = \exp(h(t,a)) \exp \sigma_k(t,a) e_k ... \exp \sigma_1(t,a) e_1 .$$

We also note $\pi_{\ell,\mathcal{H}}$ the representation of the associated Lie algebra defined by

(8)
$$\pi_{\ell,\mathcal{H}}(a)u = \frac{d}{ds} \left(\pi_{\ell,\mathcal{H}}(e^{sa})u \right)_{|s=0} ,$$

where the representation $\pi_{\ell,\mathcal{H}}$ can be defined on the set of $u \in V_\pi$ such that the mapping $s \mapsto \pi_{\ell,\mathcal{H}}(e^{as})u$ is of class C^1. We will actually work on the space $\mathcal{S}_\pi$ of the C^∞ representations $\pi_{\ell,\mathcal{H}}$.

More explicitly, we have

$$\pi_{\ell,\mathcal{H}}(a) = i\langle \ell, h'(t,a) \rangle + \sum_{j=1}^{k} \sigma_j'(t,a) \, \partial_{t_j},$$

where h' and σ' designate

$$h'(t,a) := \frac{d}{ds} \left(h(t,sa) \right)_{|s=0} ,$$

$$\sigma'(t,a) := \frac{d}{ds} \left(\sigma(t,sa) \right)_{|s=0} .$$

In addition, σ has the following structure:

(9)
$$\sigma_j(t_j, ..., t_1, a) = t_j + \psi_j(t_{j-1}, .., t_1, a) ,$$

where ψ_j are polynomials on $\mathbb{R}^k$, depending only on the given variables, with real coefficients.

We know from Kirillov's theory that, in the nilpotent case, the irreducible representations are associated with elements of $\mathcal{G}^*$ and that, when π is irreducible, the space V_π identifies with $L^2(\mathbb{R}^{k(\pi)})$ where $k(\pi)$ is a integer with $L^2(\mathbb{R}^0) = \mathbb{C}$ by convention. We denote by $\widehat{G}$ the set of irreducible representations of the simply connected group $G := \exp \mathcal{G}$ associated to $\mathcal{G}$. It is also important to note that in the case of an irreducible representation, $\mathcal{S}_\pi$ identifies with the Schwartz space $\mathcal{S}(\mathbb{R}^{k(\pi)})$.

Returning to the induced representations $\pi_{\ell,\mathcal{H}}$, two particular cases will interest us.

When $\ell = 0$, we obtain the standard extension of the trivial representation of the H subgroup of G. We can consider this as a representation on $L^2(G/H)$. An interesting problem (which is solved in [9]) is to characterize the maximal hypoellipticity of $\pi_{0,\mathcal{H}}(P)$ for $P \in \mathcal{U}_m(\mathcal{G})$ (elements of $\mathcal{U}(\mathcal{G})$ with degree m).

The second case is when the subalgebra $\mathcal{H} \subset \mathcal{G}$ is of maximal dimension, for a fixed form $\ell \in \mathcal{G}^*$, with the above property. In this case, we can show that the representation is irreducible. Moreover one can thus construct all the irreducible representations (up o unitary equivalence). Starting this time with an element $\ell \in \mathcal{G}^*$, we can construct a maximal subalgebra V_ℓ such that $\ell([V_\ell, V_\ell]) = 0$. We can also show that the codimension $k(\ell)$ of V_ℓ is equal to $\frac{1}{2} \operatorname{rank} B_\ell$, where B_ℓ is the 2-form defined by

$$\mathcal{G} \times \mathcal{G} \to \ell([X, Y]).$$

For $a \in \mathcal{G}$, we define by $(\operatorname{ad} a)^*$ the adjoint of $\operatorname{ad} a : b \ni \mathcal{G} \to (\operatorname{ad} a)b := [a, b]$ which is an endomorphism of $\mathcal{G}^*$ defined by

$$(\operatorname{ad} a)^* \, \ell(b) := \ell([a, b]).$$

The group G then naturally acts on $\mathcal{G}^*$ by

$$g \mapsto g\ell = \sum_{k=0}^{r} \frac{(-1)^k}{k!} \, (\operatorname{ad} a)^{*k} \, \ell,$$

with $g = \exp(a)$.

This action is called the coadjoint action. Kirillov's theory tells us that if ℓ and $\tilde{\ell}$ are on the same orbit for the coadjoint action, then the corresponding unitary representations are equivalent. Conversely, two different orbits give two non-equivalent irreducible representations. We can thus identify $\widehat{G}$ with the set of irreducible representations of G with the set of G-orbits in $\mathcal{G}^*$:

$$\widehat{G} = \mathcal{G}^*/G.$$

In the proof of the main theorem, we find a class of representations of a Lie algebra $\mathcal{G}$ in the space $\mathcal{S}(\mathbb{R}^k)$ $(k \geq 1)$ that have the following form:

Definition 2.5. *For all $X \in \mathcal{G}$, we define the representation π as follows*

(10)
$$\pi(X) = P_1(X)\frac{\partial}{\partial y_1} + P_2(y_1; X)\frac{\partial}{\partial y_2} + ... + P_k(y_1, .., y_{k-1}; X)\frac{\partial}{\partial y_k} + i \, Q(y_1, .., y_k; X),$$

where $P_j(y; X)$ and $Q(y; X)$ are polynomials in $y \in \mathbb{R}^k$, depending only on the given variables, with real coefficients, depending linearly on $X \in \mathcal{G}$ and the linear forms $\{X \mapsto P_j(0; X)\}_{1 \leq j \leq k}$ are linearly independent in $\mathcal{G}^*$.

We note that the induced representations always have this form. Conversely, it is natural to ask if the representation defined in Definition 2.5 actually an induced representation?

The positive answer is given by Helffer-Nourrigat in [9]. By requiring, for all $X \in \mathcal{G}$

(11) $\langle \ell, X \rangle = Q(0; X),$

and denoting by $\mathcal{H}$ the subspace of $X \in \mathcal{G}$ such that $P_j(0; X) = 0$ $(j = 1, ..., k$), these authors prove the following proposition:

Proposition 2.6. *Under the assumptions above,*

i) The subspace $\mathcal{H}$ is a subalgebra of $\mathcal{G}$, and we have

(12) $\langle \ell, [X, Y] \rangle = 0 \quad \forall X, Y \in \mathcal{H} \, .$

ii) The representation π is unitarily equivalent to $\pi_{\ell, \mathcal{H}}$. There exists a unitary transform T such that

$$\pi_{\ell, \mathcal{H}}(\exp X)Tf = T\pi(\exp X)f, \, \forall X \in \mathcal{G} \, \text{ and } \forall f \in \mathcal{S}(\mathbb{R}^k),$$

where

- T is defined by

$$Tf(t) = e^{i\varphi(t)} f(\theta(t)), \, \forall f \in \mathcal{S}(\mathbb{R}^k), \forall t \in \mathbb{R}^k$$

where φ is defined by

$$\varphi(t) = \langle \ell, \gamma(t) \rangle = Q(0; \gamma(t)),$$

with $\gamma(t)$ is an element of $\mathcal{G}$ such that

$$\exp \gamma(t) = (\exp t_k X_k)...(\exp t_1 X_1).$$

- $\theta(t) = (\theta_1(t), ..., \theta_k(t))$ is global diffeomorphism of $\mathbb{R}^k$ defined by

$$\theta_1(t) = t_1$$
$$\theta_j(t) = t_j + H_j(t_1, .., t_{j-1}), \, \forall j = 2, .., k,$$

- H_j are polynomials.

In particular T sends $\mathcal{S}(\mathbb{R}^k)$ into $\mathcal{S}(\mathbb{R}^k)$.

Note that θ is a global diffeomorphism of $\mathbb{R}^k$, whose Jacobian is 1. We thus pass without any problem from one maximal inequality to the another by a change of variables which preserves Lebesgue measure.

We finish this part by proving a specific property that will be used later.

Proposition 2.7. *Let π be a representation defined by* (10). *Then for every $X \in \mathcal{G}$, the operator $\pi(X)$ is formally skew-adjoint for the usual scalar product defined on the space $L^2(\mathbb{R}^k)$, i.e.*

$$\forall u, v \in \mathcal{S}(\mathbb{R}^k), \quad \langle \pi(X) u, v \rangle = -\langle u, \pi(X) v \rangle.$$

Proof. Let $u, v \in \mathcal{S}(\mathbb{R}^k)$. For $X \in \mathcal{G}$, we have

$$\langle \pi(X)u, v \rangle = \int_{\mathbb{R}^k} \pi(X) u(y) \, \overline{v(y)} \, dy,$$

by performing an integration by parts as a function of y_j with $j = 1, ..., k$ and using the fact that $P_j(y; X)$ are polynomials with real coefficients that depend only on $y_1, .., y_{j-1}$ for all $j = 1, .., k$, we get

$$\langle \pi(X) u, v \rangle$$

$$= -\sum_{j=1}^{k} \int_{\mathbb{R}^k} u(y) \overline{P_j(y_1, ..y_{j-1}; X) \frac{\partial v}{\partial y_j}(y)} \, dy + \int_{\mathbb{R}^k} u(y) \overline{(-i \, Q(y_1, .., y_k; X) v(y))} \, dy,$$

since $Q(y; X)$ is also a polynomial with real coefficients. Then by reusing the definition (10) of the representation $\pi(X)$, we obtain

$$\langle \pi(X) u, v \rangle = -\langle u, \pi(X) v \rangle, \quad \forall u, v \in \mathcal{S}(\mathbb{R}^k),$$

which implies the result. □

Remark 2.8. *Proposition* 2.7 *is in particular true for any induced unitary representation $\pi_{\ell, \mathcal{H}}$ on $\mathcal{G}$. As noted above, the induced representations indeed satisfy* (10).

2.4. **Characterization of hypoellipticity in the case of homogeneous invariant operators on stratified groups.** The purpose of this part is to provide the necessary and sufficient conditions for a polynomial operator of vector fields to be a maximal hypoelliptic operator.

Definition 2.9. *Let $\mathcal{G}$ be a graded real Lie algebra which is stratified of type 2. We define the enveloping algebra $\mathcal{U}(\mathcal{G})$ as the noncommutative algebra of polynomial expressions of the following form:*

$$\text{(13)} \qquad P = \sum_{|\alpha| \leq m} a_\alpha Y^\alpha,$$

where $a_\alpha \in \mathbb{C}$, $Y_{i,j}$ $(i = 1, ..., p_j$ and $j = 1, 2)$ denotes a basis of $\mathcal{G}_j$, $\alpha = (\alpha_1, .., \alpha_k)$ is a k-uplet of couples (i, j) with $i \in \{1, ..., p_j\}$, and

$$Y^\alpha = Y_{\alpha_1}...Y_{\alpha_k} \;\; \text{with} \;\; |\alpha| = \sum_{l=1}^{k} j(\alpha_l).$$

When in equality (13), *we consider only terms with $|\alpha| = m$, the set of polynomial expressions of this form is denoted by $\mathcal{U}_m(\mathcal{G})$.*

It is noted in [9, Chapter 2] that the representation $\pi_{\ell, \mathcal{H}}$ of the algebra $\mathcal{G}$ naturally extends to a representation of the enveloping algebra $\mathcal{U}(\mathcal{G})$. For all $t > 0$, we define an automorphism δ_t of $\mathcal{G}$ by the condition

$$\delta_t(a) = t^j \, a \quad \text{if } a \in \mathcal{G}_j.$$

One can of course extend the definition of δ_t (called family of dilations) to the enveloping algebra $\mathcal{U}(\mathcal{G})$ by setting

$$\delta_t(P) = \sum_{|\alpha| \leq m} a_\alpha \, (\delta_t Y)^\alpha = \sum_{|\alpha| \leq m} a_\alpha \, t^{|\alpha|} \, Y^\alpha \text{ for } t > 0 \,.$$

We note that,

$$\mathcal{P} \in \mathcal{U}_m(\mathcal{G}) \text{ if and only if } \delta_t(\mathcal{P}) = \sum_{|\alpha|=m} a_\alpha \, (\delta_t Y)^\alpha = t^m \mathcal{P}, \, \forall t > 0 \,.$$

To any element Y of $\mathcal{G}$, we can associate a left-invariant vector field $\lambda(Y)$ on the group G defined by

$$(\lambda(Y) \, f)(u) = \frac{d}{dt} \, f \, (u \cdot \exp(tY))_{|t=0} \,, \, \forall f \in H^1(\mathbb{R}^k), \forall u \in G \,.$$

This correspondence makes it possible to identify the enveloping algebra $\mathcal{U}(\mathcal{G})$ with the algebra of all the polynomials of left-invariant vector fields. To $P \in \mathcal{U}(\mathcal{G})$, defined in (13), we can associate

$$\lambda(P) = \sum_{|\alpha| \leq m} a_\alpha \lambda(Y)^\alpha.$$

We recall the theorem conjectured by C. Rockland in [15], proved by him in the case of the Heisenberg group, then in the general case by R. Beals [1] for the necessary condition and by B. Helffer and J. Nourrigat in [9] for the sufficient condition. Note that the case of rank 3, which ultimately is the only one that will be useful here, was previously obtained in [6]. Combined with a result of Rothschild-Stein [17] in the particular case where the order m of the operator $\mathcal{P}$ is even and the Lie algebra is stratified of type 1 or 2, the Helffer-Nourrigat Theorem takes the following form.

Theorem 2.10. *Let $\mathcal{G}$ be a graded and stratified Lie algebra of type 1 or of type 2 and let $\mathcal{P} \in \mathcal{U}_m(\mathcal{G})$ with m even (just in the case of type 2). Then the following assertions are equivalent:*

    (1) *The operator $\mathcal{P}$ defined in (4) is hypoelliptic in G.*

    (2) *The operator $\mathcal{P}$ defined in (4) is maximal hypoelliptic in G.*

    (3) *For any π non-trivial irreducible and unitary representation in $\widehat{G}$, the operator $\pi(\mathcal{P})$ is injective in $\mathcal{S}_\pi$, where $\mathcal{S}_\pi$ denotes the space of the C^∞ vectors of the representation.*

Remark 2.11. *When the Hörmander condition 2.1 is satisfied, the condition (3) will be called the Rockland condition. To verify this condition, we observe that for any non-trivial irreducible and unitary representation π in $\widehat{G}$ it suffices to show that if u satisfies $\pi(\mathcal{P})u = 0$, then*

- *In the stratified case of Type 1,*

$$\pi(Y_j)\, u = 0\,, \quad \forall j = 1, ..., p.$$

- *In the stratified case of Type 2,*

$$\pi(X_\ell) = 0 \ and \ \pi(Y_j) = 0, \forall \ell = 1, .., p \ \ and \ \ \forall j = 1, .., q.$$

This implies indeed in the two cases that

$$\pi(Y)u = 0, \quad \forall Y \in \mathcal{G}.$$

Then, assuming that π is irreducible and not trivial in Theorem 2.10, we get $u = 0$.

We will finish this part by quoting another result, from Helffer-Nourrigat in [8], which appears in the proof of their theorem and which will be very useful to us.

Theorem 2.12. *If $\mathcal{P} \in \mathcal{U}_m(\mathcal{G})$ is a maximal hypoelliptic operator, then there exists a strictly positive constant C such that, for any induced representation $\pi = \pi_{\ell, \mathcal{H}}$, for all u in $\mathcal{S}(\mathbb{R}^{k(\pi)})$, we have the following maximal estimate:*

$$\sum_{|\alpha| \leq m} \|\pi(Y^\alpha)u\|^2_{L^2(\mathbb{R}^k)} \leq C \left(\|\pi(\mathcal{P})u\|^2_{L^2(\mathbb{R}^k)} + \|u\|^2_{L^2(\mathbb{R}^k)} \right).$$

2.5. **Application to the maximal hypoellipticity of vector fields.** We assume that the fields X_i and Y_j for $i \in \{1, .., p\}$ and $j \in \{1, .., q\}$ satisfy the Hörmander condition (2.1) in z_0. To the operator defined in (4), we first associate in z_0, an element of the enveloping algebra $\mathcal{U}(\mathcal{G})$ where $\mathcal{G}$ denotes the free nilpotent Lie algebra with $p+q$ generators $(Z_1, ..., Z_{p+q})$ with rank r. Here, we follow Rothschild-Stein's approach in [17]. We associate to the operator P an element $\mathcal{P}_{z_0}$ of $\mathcal{U}_m(\mathcal{G})$ defined by

$$\mathcal{P}_{z_0} = \sum_{|\alpha|=m} a_\alpha(z_0)\, Z^\alpha\,,$$

and we recall a result based on the articles [16], [7] and [2]:

Theorem 2.13 (Theorem 0.7 in [16])**.** *Let $\mathcal{P}$ be the operator defined in (4) satisfying the Hörmander condition 2.1 at $z_0 \in \Omega$. If $\mathcal{P}_{z_0}$ satisfies the Rockland's Criterion then the operator $\mathcal{P}$ is maximal hypoelliptic in a neighborhood of z_0.*

The inverse is not true in general. It is the whole purpose of the book [9] to give necessary and sufficient conditions for this maximal hypoellipticity.

3. Proof of Theorem 1.3 when $d = 2$

The proof consists of constructing $\mathcal{G}$, a graded and stratified algebra of type 2, and, at any point $x \in \mathbb{T}^2$, an element $\mathcal{K}_x$ of $\mathcal{U}_2(\mathcal{G})$ which is hypoelliptic. The maximal estimate obtained for each $\mathcal{K}_x$ will then be combined to give a maximal estimate for the operator K.

To define $\mathcal{K}_x$, we replace $B_e(x)$ by a fixed constant $\mathbf{b} \in \mathbb{R}$, and we will show a global estimate for the following model:

$$K_{\mathbf{b}} = v \cdot \nabla_x + \mathbf{b}(v_1 \partial_{v_2} - v_2 \partial_{v_1}) - \Delta_v + v^2/4 - 1, \text{ in } \mathbb{R}^2 \times \mathbb{R}^2,$$

which appears as the image by an induced representation of $\mathcal{K}_x$.

Then we will use a partition of unity and check the errors coming from the localization of the estimates.

3.1. **Application of Proposition 2.6.** We will show that we can put ourselves within the framework of this proposition. We therefore look for a graded Lie algebra $\mathcal{G}$ of type 2, a subalgebra $\mathcal{H}$, an element $\tilde{K}_{\mathbf{b}}$ in $\mathcal{U}_2(\mathcal{G})$ and a linear form ℓ such that

$$\pi_{\ell, \mathcal{H}}(\tilde{K}_{\mathbf{b}}) = K_{\mathbf{b}}.$$

For this, one determines the necessary conditions on the brackets between the generating elements of this algebra. In the writing of $K_{\mathbf{b}}$ we can see the differential operators of degree 1 with the following polynomial coefficients:

$$(14) \qquad\qquad X'_{1,1} = \partial_{v_1} \qquad\qquad\qquad\qquad X''_{1,1} = iv_1$$

$$(15) \qquad\qquad X'_{2,1} = \partial_{v_2} \qquad\qquad\qquad\qquad X''_{2,1} = iv_2$$

$$(16) \qquad\qquad X_{1,2} = v.\nabla_x.$$

$K_{\mathbf{b}}$ is indeed written as a polynomial of these five differential operators

$$K_{\mathbf{b}} = X_{1,2} - \sum_{k=1}^{2} \left((X'_{k,1})^2 + \frac{1}{4}(X''_{k,1})^2 - \frac{i}{4}(X'_{k,1}X''_{k,1} - X''_{k,1}X'_{k,1}) \right)$$

$$(17) \qquad\qquad - i\,\mathbf{b}\left(X'_{1,1}X''_{2,1} - X'_{2,1}X''_{1,1} \right).$$

We now look at the Lie algebra generated by these five operators and their brackets. This leads us to introduce three new elements that verify the following relations :

$$X_{2,2} := [X'_{1,1}, X''_{1,1}] = [X'_{2,1}, X''_{2,1}] = i,$$
$$X_{1,3} := [X_{1,2}, X'_{1,1}] = \partial_{x_1}, \ X_{2,3} := [X_{1,2}, X'_{2,1}] = \partial_{x_2}.$$

We also observe that we have the following properties:

$$[X'_{1,1}, X'_{2,1}] = [X''_{1,1}, X''_{2,1}] = 0,$$
$$[X'_{j,1}, X_{k,3}] = [X''_{j,1}, X_{k,3}] = [X_{k,3}, X_{2,2}] = ... = 0 \qquad , \forall j, k = 1, 2.$$

We then construct a graded Lie algebra $\mathcal{G}$ verifying the same commutator relations. More precisely, $\mathcal{G}$ is stratified of type 2, nilpotent of rank 3, its underlying vector space is $\mathbb{R}^8$, and $\mathcal{G}_1$ is generated by $Y'_{1,1}, Y'_{2,1}, Y''_{1,1}$ and $Y''_{2,1}$, $\mathcal{G}_2$ is generated by $Y_{1,2}$ and $Y_{2,2}$ and $\mathcal{G}_3$ is generated by $Y_{1,3}$ and $Y_{2,3}$. The laws of algebra are given

by

$$(18) \qquad Y_{2,2} = [Y_{1,1}', Y_{1,1}''] = [Y_{2,1}', Y_{2,1}''], \ Y_{1,3} = [Y_{1,2}, Y_{1,1}'], \ Y_{2,3} = [Y_{1,2}, Y_{2,1}'],$$

$$(19) \qquad [Y_{1,1}', Y_{2,1}'] = [Y_{1,1}'', Y_{2,1}''] = 0,$$

$$(20) \qquad [Y_{j,1}', Y_{k,3}] = [Y_{j,1}'', Y_{k,3}] = [Y_{k,3}, Y_{2,2}] = ... = 0 \ \forall j, k = 1, 2.$$

We check that the mapping π (with the convention that if $\diamond = \emptyset$ there is no exponent) defined on its basis by

$$(21) \qquad \pi(Y_{i,j}^{\diamond}) = X_{i,j}^{\diamond} \text{ with } i = 1, 2, j = 1, 2, 3 \text{ and } \diamond \in \{\emptyset, \prime, \prime\prime\}.$$

defines a representation of the Lie algebra $\mathcal{G}$.

We now see that our representation π can be rewritten in the following form

$$\pi(Y) = P_1(Y)\partial_{v_1} + P_2(v_1; Y)\partial_{v_2} + P_3(v_1, v_2; Y)\partial_{x_1} + P_4(v_1, v_2, x_1; Y)\partial_{x_2}$$
$$+ iQ(v_1, v_2, x_1, x_2; Y),$$

where for all $Y \in \mathcal{G}$, there are $a, b, c, d, \alpha, \beta, \gamma,$ and $\delta \in \mathbb{R}$ such that

$$Y = aY_{1,1}' + bY_{2,1}' + cY_{1,1}'' + dY_{2,1}''$$
$$+ \alpha Y_{1,2} + \beta Y_{2,2}$$
$$+ \gamma Y_{1,3} + \delta Y_{2,3},$$

and the polynomials P_j with $j = 1, .., 4$ and Q are defined by

$$P_1(Y) = a, \ P_2(v_1; Y) = b, \ P_3(v_1, v_2; Y) = \alpha v_1 + \gamma$$
$$P_4(v_1, v_2, x_1; Y) = \alpha v_2 + \delta \text{ and } Q(v, x; Y) = cv_1 + dv_2 + \beta.$$

We can now apply Proposition 2.6. We then obtain $\pi = \pi_{\ell, \mathcal{H}}$ with

$$\ell \in \mathcal{G}^*, \ \langle \ell, Y \rangle = Q(0; Y) = \beta,$$
$$\mathcal{H} := \{Y \in \mathcal{G} / P_1(0; Y) = ... = P_4(0; Y) = 0\} = \text{Vect}(Y_{1,1}'', Y_{2,1}'', Y_{1,2}, Y_{2,2}),$$

Let's go back to our operator $K_{\mathbf{b}}$ which was written as a polynomial of the vector fields $\{X_{j,2}, X_{j,3}, X_{j,1}', X_{j,1}''\}_{j=1,2}$ in (17). We then define $\tilde{K}_{\mathbf{b}}$ as the same polynomial but in this time the operator is function of the vector fields $\{Y_{j,2}, Y_{j,3}, Y_{j,1}', Y_{j,1}''\}_{j=1,2}$ defined as follows

$$(22) \qquad \tilde{K}_{\mathbf{b}} = Y_{1,2} - \sum_{k=1}^{2} \left((Y_{k,1}')^2 + \frac{1}{4}(Y_{k,1}'')^2 - \frac{i}{4}(Y_{k,1}'Y_{k,1}'' - Y_{k,1}''Y_{k,1}') \right)$$
$$- i\,\mathbf{b}\left(Y_{1,1}'Y_{2,1}'' - Y_{2,1}'Y_{1,1}'' \right)$$

such that

$$\pi_{\ell, \mathcal{H}}(\tilde{K}_{\mathbf{b}}) = K_{\mathbf{b}}.$$

Note that with the notation used in the introduction to the section, we have:

$$\mathcal{K}_x = \tilde{K}_{B_e(x)}.$$

3.2. **Verification of Rockland's Criterion.**

To prove the maximal hypoellipticity, we must check the Rockland criterion (see Theorem 2.10) for $\tilde{K}_{\mathbf{b}}$. Let π be a unitary irreducible non-trivial representation of G in V_π. We will show that the operator $\pi(\tilde{K}_{\mathbf{b}})$ is an injective operator in the space $\mathcal{S}_\pi$, which identifies when π is irreducible to $\mathcal{S}(\mathbb{R}^{k(\pi)})$. Let $u \in \mathcal{S}(\mathbb{R}^{k(\pi)})$ such that

$$\pi(\tilde{K}_{\mathbf{b}})u = 0.$$

On the one hand, we have

$$\mathrm{Re}\langle \pi(\tilde{K}_{\mathbf{b}})u, u \rangle = 0.$$

On the other hand, by integration by parts and by using that the operator $\pi(Y)$ is a formally skew-adjoint operator (see Proposition 2.7), we obtain

$$\mathrm{Re}\langle \pi(\tilde{K}_{\mathbf{b}})u, u \rangle = \underbrace{\mathrm{Re}\langle \pi(Y_{1,2})u, u \rangle}_{\mathrm{I}} + \sum_{k=1}^{2}\left(\|\pi(Y_{k,1}^{'})u\|^2 + \|\pi(Y_{k,1}^{''})u\|^2 \right)$$

$$- \mathbf{b}\, \underbrace{\mathrm{Re}\left\langle i\left(\pi(Y_{1,1}^{'})\pi(Y_{2,1}^{''}) - \pi(Y_{2,1}^{'})\pi(Y_{1,1}^{''}) \right)u, u \right\rangle}_{\mathrm{II}}.$$

First, we will calculate the term I. Using the fact that the operator $\pi(Y_{1,2})$ is skew-adjoint according to Proposition 2.7, we obtain

$$\langle \pi(Y_{1,2})u, u \rangle = -\langle u, \pi(Y_{1,2})u \rangle = -\overline{\langle \pi(Y_{1,2})u, u \rangle}.$$

Therefore, we have

$$\langle \pi(Y_{1,2})u, u \rangle + \overline{\langle \pi(Y_{1,2})u, u \rangle} = 0,$$

so

$$\mathrm{I} = \mathrm{Re}\langle \pi(Y_{1,2})u, u \rangle = \left(\langle \pi(Y_{1,2})u, u \rangle + \overline{\langle \pi(Y_{1,2})u, u \rangle} \right)/2 = 0.$$

Then we go to calculating the term II. Using that π is a representation and the relations of the given commutators in (19), we get

$$[\pi(\tilde{Y}_{1,1}^{'}), \pi(\tilde{Y}_{2,1}^{''})] = \pi\left([\tilde{Y}_{1,1}^{'}, \tilde{Y}_{2,1}^{''}] \right) = 0$$

$$[\pi(\tilde{Y}_{2,1}^{'}), \pi(\tilde{Y}_{1,1}^{''})] = \pi\left([\tilde{Y}_{2,1}^{'}, \tilde{Y}_{1,1}^{''}] \right) = 0.$$

Then the operators $i\pi(Y_{1,1}^{'})\pi(Y_{2,1}^{''})$ and $i\pi(Y_{2,1}^{'})\pi(Y_{1,1}^{''})$ are skew-adjoint on the space $\mathcal{S}_\pi$ with the scalar product of V_π for any representation π. By integration by parts, we have

$$\left\langle i\pi(Y_{1,1}^{'})\pi(Y_{2,1}^{''})u, u \right\rangle = -\overline{\langle i\pi(Y_{1,1}^{'})\pi(Y_{2,1}^{''})u, u \rangle},$$

$$\left\langle i\pi(Y_{2,1}^{'})\pi(Y_{1,1}^{''})u, u \right\rangle = -\overline{\langle i\pi(Y_{2,1}^{'})\pi(Y_{1,1}^{''})u, u \rangle},$$

and then,

$$\mathrm{Re}\left\langle i\pi(Y_{1,1}^{'})\pi(Y_{2,1}^{''})u, u \right\rangle = \mathrm{Re}\left\langle i\pi(Y_{2,1}^{'})\pi(Y_{1,1}^{''})u, u \right\rangle = 0.$$

Therefore, we have

$$\mathrm{II} = \mathrm{Re}\left\langle i\left(\pi(Y_{1,1}^{'})\pi(Y_{2,1}^{''}) - \pi(Y_{2,1}^{'})\pi(Y_{1,1}^{''}) \right)u, u \right\rangle = 0.$$

The identity

$$\mathrm{Re}\langle \pi(\tilde{K}_{\mathbf{b}})u, u \rangle = \sum_{k=1}^{2}\left(\|\pi(Y_{k,1}^{'})u\|^2 + \|\pi(Y_{k,1}^{''})u\|^2 \right),$$

implies

$$\pi(Y'_{j,1})u = \pi(Y''_{j,1})u = 0 \, , \quad \forall j = 1, 2. \tag{23}$$

It remains to consider $Y_{1,2}$, for which we can notice that

$$Y_{1,2} = \tilde{K}_\mathbf{b} + \sum_{k=1}^{2} \left((Y'_{k,1})^2 + \frac{1}{4}(Y''_{k,1})^2 - \frac{i}{4}[Y'_{k,1}, Z''_{k,1}] \right)$$
$$+ i\mathbf{b} \left(Y'_{1,1} Y''_{2,1} - Y'_{2,1} Y''_{1,1} \right).$$

Applying π and by action on u, we have

$$\pi(Y_{1,2})u = \pi(\tilde{K}_\mathbf{b})u + \sum_{k=1}^{2} \pi \left((Y'_{k,1})^2 + \frac{1}{4}(Y''_{k,1})^2 - \frac{i}{4}[Y'_{k,1}, Y''_{k,1}] \right) u$$
$$+ i\mathbf{b} \, \pi \left(Y'_{1,1} Y''_{2,1} - Y'_{2,1} Y''_{1,1} \right) u$$
$$= 0 \, .$$

From Remark 2.11 in the stratified case of type 2, we deduce that

$$\pi(Y)u = 0 \, , \, \forall Y \in \mathcal{G} \, ,$$

which implies, π being assumed to be non trivial, $u = 0$.

The operator $\pi(\tilde{K}_\mathbf{b})$ is therefore injective in the $\mathcal{S}_\pi$ for any irreducible and non-trivial representation π. Therefore, according to Theorem 2.10 the operator $\tilde{K}_\mathbf{b}$ is maximal hypoelliptic in the group G. By applying Theorem 2.12 with $K_\mathbf{b} = \pi_{\ell, \mathcal{H}}(\tilde{K}_\mathbf{b})$, we obtain the existence of $C > 0$ such that

$$\|X_{1,2}u\| + \sum_{k=1}^{2} \left(\|(X'_{k,1})^2 u\| + \|(X''_{k,1})^2 u\| \right) + \sum_{k,\ell=1}^{2} \|X'_{k,1} X''_{\ell,1} u\|$$
$$\leq C \left(\|K_\mathbf{b}u\| + \|u\| \right) , \tag{24}$$

the previous inequality is verified.

3.3. Proof of Theorem 1.3.

3.3.1. *First step.* We fix $x_0 \in \mathbb{R}^2$ and we write $B_e(x_0) = \mathbf{b}_0 \in \mathbb{R}$. We recall that

$$K = v \cdot \nabla_x - (v \wedge \mathcal{B}_e) \cdot \nabla_v - \Delta_v + v^2/4 - 1.$$

We begin with $u \in \mathcal{S}(\mathbb{R}^2 \times \mathbb{R}^2)$ with $\mathrm{P}_x(\mathrm{Supp}\,u) \subset B(x_0, \varepsilon)$, where P_x is the standard projection with respect to x variables and $\varepsilon > 0$ a constant which will be chosen later, we have

$$Ku = (K - K_{\mathbf{b}_0})u + K_{\mathbf{b}_0}u.$$

On the one hand, we have by definition of K and $K_{\mathbf{b}_0}$

$$\|(K - K_{\mathbf{b}_0})u\| = \|(\mathcal{B}_e - \mathcal{B}_e(x_0))(v_1 \partial_{v_2} - v_2 \partial_{v_1})u\|$$
$$\leq \|\nabla_x B_e\|_{L^\infty} |x - x_0| \|(v_1 \partial_{v_2} - v_2 \partial_{v_1})u\| \tag{25}$$
$$\leq \varepsilon \|\nabla_x B_e\|_{L^\infty} \|(v_1 \partial_{v_2} - v_2 \partial_{v_1})u\| \, .$$

We then obtain the following estimate:

$$\|(K - K_{\mathbf{b}_0})u\| \leq \varepsilon \|B_e\|_{\mathrm{Lipsch}(\mathbb{T}^2)} \|u\|_{\tilde{B}^2} \, .$$

On the other hand, by the inequality (24) and Theorem 2.13 at the point x_0, we obtain that the operator $K_{\mathbf{b}_0}$ is maximal hypoelliptic in $B(x_0, \varepsilon)$. So the operator

$K_{\mathbf{b}_0}$ verifies, for a constant $C > 0$ large enough, which depends on the coefficients of the polynomial operator of vector fields $K_{\mathbf{b}_0}$, the following estimate:

(26)

$$\|K_{\mathbf{b}_0} u\| + \|u\| \geq \frac{1}{C}\|X_{1,2}u\| + \frac{1}{C}\sum_{k=1}^{2}\left(\|(X_{k,1}')^2 u\| + \|(X_{k,1}'')^2 u\|\right) + \frac{1}{C}\sum_{k,\ell=1}^{2}\|X_{k,1}' X_{\ell,1}'' u\|,$$

Finally, we observe that

$$\|Ku\| \geq \|K_{\mathbf{b}_0} u\| - \|(K - K_{\mathbf{b}_0})u\|,$$

Using the inequalities (25) and (26) and choosing $0 < \varepsilon < \dfrac{1}{C\,\|B_e\|_{\mathrm{Lipsch}(\mathbb{T}^2)}}$, we get $\tilde{C} > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^2 \times \mathbb{R}^2)$ with $\mathrm{P}_x(\mathrm{Supp}\,u) \subset B(x_0, \varepsilon)$

$$\|X_{1,2}u\| + \sum_{k=1}^{2}\left(\|(X_{k,1}')^2 u\| + \|(X_{k,1}'')^2 u\|\right) + \sum_{k,\ell=1}^{2}\|X_{k,1}' X_{\ell,1}'' u\|$$

(27)
$$\leq \tilde{C}\left(\|Ku\| + \|u\|\right).$$

3.3.2. *Second step.* We have shown that for each $x_0 \in \mathbb{T}^2$ there exists a $B(x_0, \varepsilon(x_0))$ such that (27) applies. $\mathbb{T}^2$ being compact, then there exists a smooth partition of unity $(\varphi_j)_{j=1}^{n_0}$ in the variable $x \in \mathbb{T}^2$ such that

$$\varphi_j \in C_0^\infty(\mathbb{T}^2) \text{ and } \sum_{j=0}^{n_0} \varphi_j^2 = 1 \text{ on } \mathbb{T}^2,$$

and the inequality (27) holds on each $\mathrm{Supp}\,\varphi_j$ (with a uniform constant since the collection of φ_j is finite). More precisely, we take $\varepsilon = \inf\{\varepsilon_j,\ j = 1, .., n_0\} > 0$, where ε_j is the radius of the ball centered at a point x_j whose (27) holds, such that

$$\mathrm{Supp}\,\varphi_j \subset B(x_j, \varepsilon) \subset B(x_j, \varepsilon_j) \quad \forall j \in \{1, .., n_0\},$$

where the ball $B(x_j, \varepsilon)$ is defined by for all $j = 1, .., n_0$

$$B(x_j, \varepsilon) := \{x \in \mathbb{T}^2 / d(x, x_j) < \varepsilon\}.$$

Intermediate step. The purpose of this step is to show the following two estimates:

(28) $\forall \eta > 0, \exists C_\eta > 0$ such that $\forall u \in \tilde{B}^2, \|u\|_{\tilde{B}^1}^2 \leq \eta\|u\|_{\tilde{B}^2}^2 + C_\eta\|u\|^2,$

(29) $\forall \varphi \in C^\infty(\mathbb{T}_x^2), \exists C_\varphi > 0$ such that $\|[K, \varphi]u\| \leq C_\varphi\|u\|_{\tilde{B}^1},\ \forall u \in \tilde{B}^1,$

where $\tilde{B}^1$ is the space $L_x^2 \widehat{\otimes} B_v^1$ with the following Hilbertian norm:

$$\|u\|_{\tilde{B}^1}^2 := \sum_{|\alpha|+|\beta|\leq 1}\|v^\alpha \partial_v^\beta u\|^2,\ \forall u \in \tilde{B}^1.$$

According to the definition of $\|\cdot\|_{B_v^1}$, we have for $u \in \mathcal{S}(\mathbb{T}^2 \times \mathbb{R}^2)$ and $x \in \mathbb{T}^2$

$$\|u(x,.)\|_{B_v^1}^2 = \sum_{|\alpha|+|\beta|\leq 1}\|v^\alpha \partial_v^\beta u(x,.)\|_{L^2(\mathbb{R}^2)}^2$$

$$= \sum_{|\alpha|+|\beta|\leq 1}\langle v^\alpha \partial_v^\beta u(x,.), v^\alpha \partial_v^\beta u(x,.)\rangle.$$

By an integration by parts with respect to v and by applying the Cauchy-Schwarz inequality, then there exists a constant $C > 0$ such that

$$\|u(x,.)\|_{B_v^1}^2 \leq C\|u(x,.)\| \, \|u(x,.)\|_{B_v^2} \,.$$

Finally, by the Young inequality we obtain that for all $\eta > 0$, there exists $C_\eta > 0$ such that

$$\forall \eta > 0, \, \|u(x,.)\|_{B_v^1}^2 \leq \eta\|u(x,.)\|_{B_v^2}^2 + C_\eta\|u(x,.)\|^2 \,.$$

Then, by integrating in x on the torus $\mathbb{T}^2$, we deduce the estimate (28).

Now, we note that the commutator is

$$[K,\varphi] = [v \cdot \nabla_x, \varphi] = v \cdot \nabla_x \varphi \,,$$

according to the previous equality and applying Hölder's inequality ($p = 1$ and $q = \infty$), we obtain

$$\begin{aligned}
\|[K,\varphi]u\| = \|v \cdot \nabla_x \varphi u\| &\leq \|\nabla_x \varphi\|_{L^\infty(\mathbb{T}^2)} \, \|v \, u\|_{L^2(\mathbb{T}^2 \times \mathbb{R}^2)} \\
&\leq \|\nabla_x \varphi\|_{L^\infty(\mathbb{T}^2)} \, \|u\|_{\tilde{B}^1} \\
&\leq C_\varphi \|u\|_{\tilde{B}^1} \,.
\end{aligned}$$

End of the proof. Let $B_e \in \mathrm{Lipsch}(\mathbb{T}^2)$ and $u \in \mathcal{S}(\mathbb{T}^2 \times \mathbb{R}^2)$. By the partition of unity and by application of the inequality (27) with $w = u\varphi_j$, we obtain that there exists $C > 0$ such that

$$(30) \qquad \|u\|_{\tilde{B}^2}^2 = \sum_{j=1}^{n_0} \|\varphi_j u\|_{\tilde{B}^2}^2 \leq C \sum_{j=1}^{n_0} \left(\|K\varphi_j u\|^2 + \|\varphi_j u\|^2 \right),$$

Using the definition of the commutators, we get

$$\sum_{j=1}^{n_0} \|K\varphi_j u\|^2 \leq 2 \sum_{j=1}^{n_0} \left(\|[\varphi_j, K]u\|^2 + \|\varphi_j K u\|^2 \right) \,.$$

According to the estimates (28) and (29), we obtain, for every $\eta > 0$, the existence of $C_\eta > 0$ such that

$$(31) \qquad \sum_{j=1}^{n_0} \|K\varphi_j u\|^2 \leq 2\|Ku\|^2 + 2\,\eta\|u\|_{\tilde{B}^2}^2 + C_\eta\|u\|^2 \,.$$

By inserting inequality (31) into inequality (30), we deduce that, for every $\eta > 0$, there exists $C_\eta > 0$ such that

$$\|u\|_{\tilde{B}^2}^2 \leq 2\,C\|Ku\|^2 + 2\,\eta\,C\|u\|_{\tilde{B}^2}^2 + C_\eta\|u\|^2,$$

Then, using the same techniques, we have

$$\begin{aligned}
\|v \cdot \nabla_x u\|^2 &= \sum_{j=1}^{n_0} \|\varphi_j(v \cdot \nabla_x u)\|^2 \\
&\leq 2 \sum_{j=1}^{n_0} \left(\|[\varphi_j, v \cdot \nabla_x]u\|^2 + \|(v \cdot \nabla_x)(\varphi_j u)\|^2 \right)
\end{aligned}$$

Using the inequalities (31) and (27), we get $C' > 0$ and for all $\eta' > 0$, $C_{\eta'} > 0$ such that

$$
\begin{aligned}
\|v \cdot \nabla_x u\|^2 &\leq \sum_{j=1}^{n_0} \left(C' \|\varphi_j \, Ku\|^2 + \eta' \|\varphi_j u\|_{\tilde{B}^2}^2 + C_{\eta'} \|\varphi_j u\|^2 \right) \\
&\leq C' \|Ku\|^2 + \eta' \|u\|_{\tilde{B}^2}^2 + C_{\eta'} \|u\|^2.
\end{aligned}
$$

It remains to estimate the following term:

$$
\|B_e(v_1 \partial_{v_2} - v_2 \partial_{v_1}) u\|^2 = \sum_{j=1}^{n_0} \|B_e(v_1 \partial_{v_2} - v_2 \partial_{v_1})(\varphi_j u)\|^2.
$$

By direct application of the inequalities (27) and (31), we obtain that there exists $C, C'' > 0$ and for all $\eta'' > 0$, $C_{\eta''} > 0$ such that

$$
\begin{aligned}
\|B_e(v_1 \partial_{v_2} - v_2 \partial_{v_1}) u\|^2 &\leq C \sum_{j=1}^{n_0} \left(\|K(\varphi_j u)\|^2 + \|u\|^2 \right) \\
&\leq C'' \|Ku\|^2 + \eta'' \|u\|_{\tilde{B}^2}^2 + C_{\eta''} \|u\|^2.
\end{aligned}
$$

Choosing η, η' and $\eta'' > 0$ such that $2C\,\eta + \eta' + \eta'' < 1$, we thus obtain the existence of a constant $\tilde{C} > 0$ such that

$$
\|(v \cdot \nabla_v - B_e(v_1 \partial_{v_2} - v_2 \partial_{v_1})) u\| + \|u\|_{\tilde{B}^2} \leq \tilde{C}(\|Ku\| + \|u\|), \quad \forall u \in \mathcal{S}(\mathbb{T}^2 \times \mathbb{R}^2).
$$

4. Proof of Theorem 1.3 when $d = 3$

4.1. **Preliminaries.** We now present the proof of Theorem 1.3 when $d = 3$, and explain the differences with the proof in the case $d = 2$. We first replace $B_e(x)$ with a constant vector $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$ fixed, and we show an overall estimate for the following model:

$$
(32) \qquad\qquad K_{\mathbf{b}} = v \cdot \nabla_x + (v \wedge \mathbf{b}) \cdot \nabla_v - \Delta_v + v^2/4 - 3/2.
$$

We define the transformation

$$
V_M : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3, \quad (x, v) \mapsto V_M(x, v) := (Mx, (M^{-1})^t v),
$$

where M is the rotation matrix that is obtained by multiplying the two matrices of rotations $R(\theta_1)$ and $R(\theta_2)$ around the axes (Ox) and (Oz) with rotation angles θ_1 and θ_2 respectively

$$
R(\theta_1) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix} \qquad R(\theta_2) := \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

where both angles are defined by

$$
\theta_1 = \arctan(b_1 \,/\, b_2) \quad, \quad \theta_2 = \arctan(\sqrt{b_1^2 + b_2^2} \,/\, b_3).
$$

We note that the following differential operators are invariant by conjugation by the transformation V_M (by orthogonality of matrix M that is to say $M^{-1} = M^t$):

$$
V_M (v \cdot \nabla_x) V_{M^{-1}} = (M^{-1})^t \, v \cdot (M^{-1})^t \nabla_x = (M^{-1})^t \, M^{-1} \, v \cdot \nabla_x = v \cdot \nabla_x
$$

$$V_M \left(-\Delta_v + v^2/4 - 3/2\right) V_{M^{-1}} = V_M \left(-\nabla_v + v/2\right) \cdot \left(\nabla_v + v/2\right) V_{M^{-1}}$$
$$= (M^{-1})^t \left(-\nabla_v + v/2\right) \cdot M^t \left(-\nabla_v + v/2\right)$$
$$= M\, M^t \left(-\Delta_v + v^2/4 - 3/2\right)$$
$$= -\Delta_v + v^2/4 - 3/2.$$

By construction of the matrix M, where $|\mathbf{b}|$ is the Euclidiean norm of $\mathbf{b} \in \mathbb{R}^3$, conjugation of the magnetic operator gives

$$V_M(v \wedge \mathbf{b}) \cdot \nabla_v\, V_{M^{-1}} = |\mathbf{b}|(v_1 \partial_{v_2} - v_2 \partial_{v_1}).$$

Therefore, the conjugation of the Fokker-Planck-magnetic operator defined in (32), by the canonical transformation operator V_M, gives us the following operator:

$$(33) \qquad Q_{|\mathbf{b}|} := V_M\, K_{\mathbf{b}}\, V_{M^{-1}} = v \cdot \nabla_x + |\mathbf{b}| \left(v_1 \partial_{v_2} - v_2 \partial_{v_1}\right) - \Delta_v + v^2/4 - 3/2,$$

Note that the space $\tilde{B}^2$ is invariant by rotation. The maximal estimates for $K_{\mathbf{b}}$ are therefore equivalent to the maximal estimates for $Q_{|\mathbf{b}|}$ (with uniform control of constants).

As in the case $d = 2$, the proof consists in constructing a graded Lie algebra $\mathcal{G}$ of rank 3, and, in every point $x \in \mathbb{T}^3$, an element $\mathcal{Q}_{|\mathbf{b}|}$ of $\mathcal{U}_2(\mathcal{G})$ hypoelliptic with $\mathbf{b} = B_e(x)$. We deduce from the maximal estimate obtained for each $\mathcal{Q}_{|\mathbf{b}|}$, a maximal estimate for the operator $Q_{|\mathbf{b}|}$.

By conjugation by the canonical transformation $V_{M^{-1}}$ of the operator $Q_{|\mathbf{b}|}$, we then have

$$\|X_{1,2} u\| + \sum_{k=1}^{3} \left(\|(X'_{k,1})^2 u\| + \|(X''_{k,1})^2 u\|\right) + \sum_{k,\ell=1}^{3} \|X'_{k,1} X''_{\ell,1} u\|$$
$$(34) \qquad\qquad \leq C \left(\|K_{\mathbf{b}} u\| + \|u\|\right), \quad \forall u \in S(\mathbb{R}^3 \times \mathbb{R}^3).$$

4.2. **End of the proof.** For the rest of the proof, there is no difference with the case $d = 2$, we use a partition of unity and we control the error, in order to obtain the maximal estimate (3) for the initial operator K in the space $\mathcal{S}(\mathbb{T}^3 \times \mathbb{R}^3)$. For more details on a similar result without a magnetic field but with an electric potential, we refer the reader to chapter 9 of the book of B. Helffer and F. Nier [5].

Appendix A. Maximal accretivity of magnetic Fokker-Planck operator with low regularity

A.1. **Preliminary remark.** By following the steps of the proof of the maximal accretivity for Kramers-Fokker-Planck operator without a magnetic field, given in [5, Proposition 5.5], we need some local hypoelliptic regularity of the following operator:

$$v \cdot \nabla_x - (v \wedge B_e) \cdot \nabla_v - \Delta_v = Y_0 + \sum_{j=1}^{d} Y_j^2,$$

where

$$Y_0 = v \cdot \nabla_x - (v \wedge B_e) \cdot \nabla_v \ \text{ and } \ Y_j = \partial_{v_j}, \forall j = 1, .., d.$$

In the case where B_e is C^∞, this regularity results immediately from the hypoellipticity argument of Hörmander operators. The difficulty in our weakly regular case is that the vector field Y_0 has coefficients in L^∞. We can only hope for a weaker Sobolev type of regularity.

A.2. **Review of Sobolev regularity for Hörmander operators of type-2.**
In this part, we will recall a result of Sobolev regularity for a relevant class of
differential operators. We consider the differential operator of type-2 of Hörmander
given by

$$\mathcal{L} = \sum_{j=1}^{n} X_j^2 + X_0 \,,$$

where the vector fields $X_0, .., X_n$ are real and C^∞ on an open set $\Omega \subset \mathbb{R}^n$. We
suppose further that the X_j with $j = 0, 1, .., n$ satisfy Hörmander condition 2.1.
The hypoellipticity of these operators has been studied by L. Hörmander in [11].

We denote by $\mathcal{H}_{loc}^m(\mathbb{R}^{2d})$ the space of functions locally in $\mathcal{H}^m(\mathbb{R}^{2d})$ defined in (6).
(See [17] for more details of this subject). We recall from Rothschild-Stein in [17]
the following result.

Theorem A.1 (Theorem 18 in [17]). *If $f, \mathcal{L} f \in L_{loc}^2(\Omega)$ then $f \in \mathcal{H}_{loc}^2(\Omega)$.*

This theorem is proven under the assumption that the polynomial operators of
vector fields are with C^∞ coefficients, a hypothesis that will not be satisfied in our
case. The case with weakly regular coefficients appears less often in the literature.
We note, for example, that the case of the operator of the following form:

$$\sum_{i,j=1}^{n} a_{i,j} \, X_i \, X_j + X_0 \,,$$

with non-regular coefficients $a_{i,j}$ was studied by C.J. Xu in [19] and M. Bramanti
and L. Brandolini in [3], who prove results of regularity as in Hölder and Sobolev
spaces. Readers are referred to the article [18] for the study of Hölder regularity for
the particular case of Kolmogorov operators with Hölder coefficients. None of these
theorems apply directly. Moreover, they require a hypothesis of Hölder regularity
which will not made here.

A.3. **Proof of the Sobolev regularity.** To prove Theorem 1.1, we will show the
Sobolev regularity associated to the following problem

$$K^* f = g \text{ with } f, g \in L_{loc}^2(\mathbb{R}^{2d}) \,,$$

where K^* is the formal adjoint of K:

$$(35) \qquad K^* = -v \cdot \nabla_x - \Delta_v + (v \wedge B_e) \cdot \nabla_v + v^2/4 - d/2 \,.$$

The result of Sobolev regularity is the following:

Theorem A.2. *Let $d = 2$ or 3. We suppose that $B_e \in L^\infty(\mathbb{R}^d, \mathbb{R}^{d(d-1)/2})$. Then
for all $f \in L_{loc}^2(\mathbb{R}^{2d})$, such that $K^* f = g$ with $g \in L_{loc}^2(\mathbb{R}^{2d})$, then $f \in \mathcal{H}_{loc}^2(\mathbb{R}^{2d})$.*

Before proving Theorem A.2, it is important to reduce our problem to a problem
with regular coefficients, in order to prove partial regularity in v for the following
family of operators:

Proposition A.3. *Let $c_j \in L^\infty(\mathbb{R}^d, L_{loc}^2(\mathbb{R}^d))$, $\forall j = 1, ..., d$ such that*

$$(36) \qquad \partial_{v_j} \left(c_j(x, v) \right) = 0 \ \ in \ \mathcal{D}'(\mathbb{R}^{2d}) \,, \forall j = 1, .., d \,.$$

Let P_0 be the Kolmogorov operator

$$(37) \qquad P_0 := -v \cdot \nabla_x - \Delta_v \,.$$

If $h \in L^2_{loc}(\mathbb{R}^{2d})$ satisfies

(38)
$$\begin{cases} P_0 h = \sum\limits_{j=1}^{d} c_j(x,v) \, \partial_{v_j} h_j + \tilde{g} \\ h_j, \tilde{g} \in L^2_{loc}(\mathbb{R}^{2d}), \, \forall j = 1, ..., d, \end{cases}$$

then $\nabla_v h \in L^2_{loc}(\mathbb{R}^{2d}, \mathbb{R}^d)$.

Proof. Let h satisfy (38). We can work near a point $(x_0, v_0) \in \mathbb{R}^{2d}$ *i.e.* in the ball $B((x_0, v_0), r_0)$ for some $r_0 > 0$. We show that $\nabla_v h$ belongs to $L^2(B((x_0, v_0), r_0/2), \mathbb{R}^d)$.

Step 1. Regular solution with compact support.
We begin by assuming that $h \in H^2(\mathbb{R}^{2d})$ is supported in $B((x_0, v_0), r_0)$ with $r_0 > 0$ and we will establish a priori estimates. We multiply Equation (38) by h and integrate it with respect to x and v, we obtain

(39)
$$\langle P_0 h, h \rangle = \sum_{j=1}^{d} \langle c_j \, \partial_{v_j} h_j, h \rangle + \langle \tilde{g}, h \rangle,$$

where $\langle . \, , . \rangle$ denotes the Hilbertian scalar product on the real Hilbert space $L^2(\mathbb{R}^{2d})$. Let's start by calculating the left side of the previous equality

$$\begin{aligned} \langle P_0 h, h \rangle &= \langle (v \cdot \nabla_x - \Delta_v) \, h, h \rangle \\ &= \int_{\mathbb{R}^{2d}} (v \cdot \nabla_x \, h) \, h \, dx \, dv - \int_{\mathbb{R}^{2d}} (\Delta_v \, h) \, h \, dx \, dv. \end{aligned}$$

Then performing an integrations by parts with respect to x and v in the previous equality, we get

$$\langle P_0 h, h \rangle = \| \nabla_v \, h \|^2_{L^2(\mathbb{R}^{2d}, \mathbb{R}^d)},$$

we used that the operator $v \cdot \nabla_x$ is formally skew-adjoint in $L^2(\mathbb{R}^{2d})$.

We now estimate each term in the right hand side of the equality (39). Using Assumption (36) and performing integration by parts with respect to v_j, we have

$$\sum_{j=1}^{d} \langle c_j \, \partial_{v_j} h_j, h \rangle = - \sum_{j=1}^{d} \langle c_j \, h_j, \partial_{v_j} \, h \rangle.$$

Then applying the Cauchy-Schwarz inequality to the scalar product, we get

$$\left| \sum_{j=1}^{d} \langle c_j(x,v) \, \partial_{v_j} h_j, h \rangle \right| \le \sum_{j=1}^{d} \| c_j \, h_j \|_{L^2(\mathbb{R}^{2d})} \, \| \partial_{v_j} \, h \|_{L^2(\mathbb{R}^{2d})}.$$

Then we obtain that for all $\eta > 0$, there exists $C_\eta > 0$ such that

$$\left| \sum_{j=1}^{d} \langle c_j(x,v) \, \partial_{v_j} h_j, h \rangle \right| \le \eta \, \| \nabla_v h \|^2_{L^2(\mathbb{R}^{2d}, \mathbb{R}^d)} + C_\eta \sum_{j=1}^{d} \| c_j \, h_j \|^2_{L^2(\mathbb{R}^{2d})}.$$

Similarly, we get

$$|\langle \tilde{g}, h \rangle| \le \frac{1}{2} \| \tilde{g} \|^2_{L^2(\mathbb{R}^{2d})} + \frac{1}{2} \| h \|^2_{L^2(\mathbb{R}^{2d})}.$$

By choosing $\eta < 1$, we obtain the existence of a constant $C > 0$ such that for any h and h_j satisfy the problem (38)

$$(40) \qquad \|\nabla_v\, h\|^2_{L^2(\mathbb{R}^{2d},\mathbb{R}^d)} \leq C \left(\sum_{j=1}^{d} \|c_j\, h_j\|^2_{L^2(\mathbb{R}^{2d})} + \|\tilde{g}\|^2_{L^2(\mathbb{R}^{2d})} + \|h\|^2_{L^2(\mathbb{R}^{2d})} \right).$$

Step 2. General solution.

I. **General framework and useful remark.**

To have the regularity stated in Theorem A.3 for a solution $h \in L^2_{loc}(\mathbb{R}^{2d})$ of (38), we introduce two cut-off functions χ_1 and χ_2 such that

$$\begin{cases} \chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^{2d}), \quad \mathrm{Supp}\,\chi_1 \subset B((x_0,v_0),r_0) \\ \chi_1 = 1 \text{ in } B((x_0,v_0),\frac{r_0}{2}) \text{ and } \chi_1 = 0 \text{ in } \mathbb{R}^{2d} \setminus B((x_0,v_0),r_0) \\ \chi_1\,\chi_2 = \chi_1\,. \end{cases}$$

Where $r_0 > 0$ is fixed at the beginning of the proof.

Let h_δ be the function defined on $\mathbb{R}^{2d}$ by

$$(41) \qquad h_\delta(x,v) = \chi_2\,(1 - \delta^2\,\Delta_{x,v})^{-1}\,\chi_1\,h\,, \quad \text{for } \delta > 0\,,$$

where the operator $(1 - \delta^2\,\Delta_{x,v})^{-1}$ is defined via the Fourier transform.

Note that $h_\delta \in H^2(\mathbb{R}^{2d})$ is supported in $B((x_0,v_0),r_0)$. Using the dominated convergence theorem one can show

$$(42) \qquad h_\delta \to \chi_1\,h \text{ in } L^2(\mathbb{R}^{2d}) \text{ when } \delta \to 0\,.$$

We will now show that there is a constant C and δ_0 such that for all $\delta \in (0,\delta_0]$

$$(43) \qquad \|\nabla_v\, h_\delta\|_{L^2(\mathbb{R}^{2d},\mathbb{R}^d)} \leq C\,.$$

Before giving the proof, we make the following simple remark:

Remark A.4. *If Q_1 and Q_2 are two differential operators of order k_1 and k_2 respectively with coefficients in C_0^∞ such that*

$$k_1 + k_2 \leq k \leq 2\,,$$

then there exists a constant $C > 0$ such that for all $\delta > 0$

$$(44) \qquad \|Q_1\,(1 - \delta^2\,\Delta_{x,v})^{-1}\,Q_2\,u\|_{L^2(\mathbb{R}^{2d})} \leq C\,\delta^{-k}\|u\|_{L^2(\mathbb{R}^{2d})}, \; \forall u \in L^2(\mathbb{R}^{2d})\,.$$

II. **Reformulation and relations of commutators.**

Let's go back to the proof of inequality (43). We recall that h verifies

$$(45) \qquad P_0\,h = \sum_{j=1}^{d} c_j(x,v)\,\partial_{v_j}\,h_j + \tilde{g}\,.$$

The goal is to follow the approach of the first step for h_δ with however some differences. From (45), let's look for the equation verified by h_δ. We have

$$(46) \qquad P_0\,h_\delta = \chi_2\,(1 - \delta^2\Delta_{x,v})^{-1}\,\chi_1\,P_0\,h + [\chi_2\,(1 - \delta^2\Delta_{x,v})^{-1}\,\chi_1, P_0]\,h\,.$$

To control the commutator in the right-hand side, we will use the following relations of commutators:

$$(47) \quad [\varphi, P_0] = -\Delta_v \, \varphi + v \cdot \nabla_x \, \varphi - 2\nabla_v \varphi \cdot \nabla_v \, , \forall \varphi \in C_0^\infty(\mathbb{R}^{2d}) \, ,$$

$$(48) \quad [\varphi, \partial_{v_j}] = -\partial_{v_j} \, \varphi \, , \ \forall \varphi \in C_0^\infty(\mathbb{R}^{2d}) \, , \forall j = 1, .., d \, ,$$

$$(49) \quad [(1 - \delta^2 \, \Delta_{x,v}), P_0] = \delta^2 \, [\Delta_{x,v}, v \cdot \nabla_x] = 2 \, \delta^2 \sum_{j=1}^d \, \partial_{v_j} \partial_{x_j} \, ,$$

$$(50) \quad [(1 - \delta^2 \, \Delta_{x,v})^{-1}, P_0] = -(1 - \delta^2 \, \Delta_{x,v})^{-1} \, [(1 - \delta^2 \, \Delta_{x,v}), P_0] \, (1 - \delta^2 \, \Delta_{x,v})^{-1}.$$

Using the relations of the previous commutators and equalities (45) and (46), we rewrite $P_0 \, h_\delta$ in the following form:

$$
\begin{aligned}
(51) \quad P_0 \, h_\delta = & \sum_{j=1}^d \chi_2 \, (1 - \delta^2 \, \Delta_{x,v})^{-1} \, \chi_1 \, c_j \, \partial_{v_j} \, h_j + B_0 \, \tilde{g} \\
& + B_1 \, h + 2 \, \chi_2 \, (1 - \delta^2 \, \Delta_{x,v})^{-1} \, \nabla_v \, \chi_1 \cdot \nabla_v \, h \\
& + B_2 \, h + 2 \, \nabla_v \chi_2 \cdot \nabla_v \, \left((1 - \delta^2 \, \Delta_{x,v})^{-1} \, \chi_1 \, h \right) \, .
\end{aligned}
$$

Here the B_j for $j = 0, 1, 2$ are the uniformly bounded operators with respect to δ on $L^2(\mathbb{R}^{2d})$ (according to (44)).

$$(52) \qquad B_0 = \chi_2 \, (1 - \delta^2 \, \Delta_{x,v})^{-1} \, \chi_1 \, ,$$

$$(53) \qquad B_1 = \chi_2 \, (1 - \delta^2 \, \Delta_{x,v})^{-1} \, (\Delta_{x,v} \chi_1) + \chi_2 \, (1 - \delta^2 \, \Delta_{x,v})^{-1} \, (v \cdot \nabla_x \chi_1) \, ,$$

$$(54) \qquad B_2 = \chi_2 \, (1 - \delta^2 \, \Delta_{x,v})^{-1} \, \delta^2 \, R \, (1 - \delta^2 \, \Delta_{x,v})^{-1} \, \chi_1 \, ,$$

where R is the second-order operator

$$(55) \qquad R = [\Delta_{x,v}, v \cdot \nabla_x] = 2 \sum_{j=1}^d \partial_{v_j} \partial_{x_j} \, .$$

III. **Proof of the uniform boundedness of $\|\nabla_v \, h_\delta\|_{L^2(\mathbb{R}^{2d})}$.**
In order to show (43), we multiply equation (51) by h_δ and integrate, we obtain

$$
\begin{aligned}
(56) \quad \langle P_0 \, h_\delta, h_\delta \rangle = & \sum_{j=1}^d \langle \chi_2 \, (1 - \delta^2 \, \Delta_{x,v})^{-1} \, \chi_1 \, c_j \, \partial_{v_j} \, h_j, h_\delta \rangle + \langle B_0 \, \tilde{g}, h_\delta \rangle \\
& + \langle B_1 \, h, h_\delta \rangle + \langle 2 \, \chi_2 \, (1 - \delta^2 \, \Delta_{x,v})^{-1} \, \nabla_v \, \chi_1 \cdot \nabla_v \, h, h_\delta \rangle \\
& + \langle B_2 \, h, h_\delta \rangle + \langle 2 \, \nabla_v \chi_2 \cdot \nabla_v \, \left((1 - \delta^2 \, \Delta_{x,v})^{-1} \, \chi_1 \, h \right), h_\delta \rangle \, ,
\end{aligned}
$$

and we estimate term by term of the right-hand side of the previous equality.

For the first term, we use relation (48) to switch ∂_{v_j} and χ_k for $k = 1, 2$, and obtain:

$$\langle \chi_2 \, (1 - \delta^2 \, \Delta_{x,v})^{-1} \, \chi_1 \, c_j \, \partial_{v_j} \, h_j, h_\delta \rangle = I_0 + I_1 + I_2 \, ,$$

where the I_j for $j = 0, 1, 2$ are defined by

$$(57) \qquad I_0 = \langle \partial_{v_j} \left(\chi_2 \, (1 - \delta^2 \, \Delta_{x,v})^{-1} \, \chi_1 \, c_j \, h_j \right), h_\delta \rangle \, ,$$

$$(58) \qquad I_1 = \langle \chi_2 \, (1 - \delta^2 \, \Delta_{x,v})^{-1} \, (\partial_{v_j} \chi_1) \, c_j \, h_j, h_\delta \rangle \, ,$$

$$(59) \qquad I_2 = \langle (\partial_{v_j} \, \chi_2) \, (1 - \delta^2 \, \Delta_{x,v})^{-1} \, \chi_1 \, c_j \, h_j, h_\delta \rangle \, .$$

To estimate I_0, we integrate by parts with respect to v_j

$$I_0 = - \left\langle \chi_2 \left(1 - \delta^2 \, \Delta_{x,v}\right)^{-1} \chi_1 \, c_j \, h_j, \partial_{v_j} \, h_\delta \right\rangle,$$

and then, applying the Cauchy-Schwarz inequality, we get

$$|I_0| \leq \|B_0 \, (c_j \, h_j)\|_{L^2(\mathbb{R}^{2d})} \, \|\partial_{v_j} \, h_\delta\|_{L^2(\mathbb{R}^{2d})} \, .$$

Then using the uniform boundedness of B_0 we obtain that for all $\eta' > 0$, there exists $C_{\eta'} > 0$ such that

$$(60) \qquad |I_0| \leq C_{\eta'} \, \|c_j \, h_j\|^2_{L^2(B((x_0,v_0),r_0))} + \eta' \, \|\partial_{v_j} \, h_\delta\|^2_{L^2(\mathbb{R}^{2d})} \, .$$

The two terms $|I_1|$ and $|I_2|$ satisfy the following estimates:

$$|I_1| \leq \|\chi_2 \left(1 - \delta^2 \, \Delta_{x,v}\right)^{-1} \left(\partial_{v_j} \chi_1\right) c_j \, h_j\|_{L^2(\mathbb{R}^{2d})} \, \|h_\delta\|_{L^2(\mathbb{R}^{2d})}$$
$$(61) \qquad \leq C \, \|c_j \, h_j\|_{L^2(B((x_0,v_0),r_0))} \, \|h_\delta\|_{L^2(\mathbb{R}^{2d})}$$

$$|I_2| \leq \|\left(\partial_{v_j} \, \chi_2\right) \left(1 - \delta^2 \, \Delta_{x,v}\right)^{-1} \chi_1 \, c_j \, h_j\|_{L^2(\mathbb{R}^{2d})} \, \|h_\delta\|_{L^2(\mathbb{R}^{2d})}$$
$$(62) \qquad \leq C \, \|c_j \, h_j\|_{L^2(B((x_0,v_0),r_0))} \, \|h_\delta\|_{L^2(\mathbb{R}^{2d})} \, .$$

Using (60)-(62), we get that for all $\eta' > 0$, there exists $C_{\eta'} > 0$ such that

$$(63) \qquad \left| \sum_{j=1}^d \left\langle \chi_2 \left(1 - \delta^2 \, \Delta_{x,v}\right)^{-1} \chi_1 \, c_j \, \partial_{v_j} \, h_j, h_\delta \right\rangle \right|$$

$$\leq C_{\eta'} \sum_{j=1}^d \|c_j \, h_j\|^2_{L^2(B((x_0,v_0),r_0))} + \eta' \, \|\nabla_v \, h_\delta\|^2_{L^2(\mathbb{R}^{2d},\mathbb{R}^{2d})} + C_{\eta'} \, \|h_\delta\|^2_{L^2(\mathbb{R}^{2d})} \, ,$$

We show for the fourth and the sixth term of (56) that for all η'' and $\eta''' > 0$, there exists $C_{\eta''}$ and $C_{\eta'''} > 0$ such that

$$(64)$$
$$\left\langle 2 \, \chi_2 \left(1 - \delta^2 \, \Delta_{x,v}\right)^{-1} \nabla_v \, \chi_1 \cdot \nabla_v \, h, h_\delta \right\rangle \leq C_{\eta''} \|h\|^2_{L^2(B((x_0,v_0),r_0))} + \eta'' \, \|\nabla_v \, h_\delta\|^2_{L^2(\mathbb{R}^{2d})}$$
$$+ C_{\eta''} \, \|h_\delta\|^2_{L^2(\mathbb{R}^{2d})}$$

$$(65)$$
$$\left\langle 2 \, \nabla_v \chi_2 \cdot \nabla_v \left(\left(1 - \delta^2 \, \Delta_{x,v}\right)^{-1} \chi_1 \, h\right), h_\delta \right\rangle \leq C_{\eta'''} \|h\|^2_{L^2(B((x_0,v_0),r_0))} + \eta''' \, \|\nabla_v \, h_\delta\|^2_{L^2(\mathbb{R}^{2d},\mathbb{R}^{2d})}$$
$$+ C_{\eta'''} \, \|h_\delta\|^2_{L^2(\mathbb{R}^{2d})} \, .$$

It remains to estimate the three remaining terms corresponding to the following scalar products:

$$|\langle B_0 \, \tilde{g}, h_\delta \rangle| \leq \|B_0 \, \tilde{g}\|_{L^2(\mathbb{R}^{2d})} \, \|h_\delta\|_{L^2(\mathbb{R}^{2d})} \, ,$$
$$|\langle B_1 \, h, h_\delta \rangle| \leq \|B_1 \, h\|_{L^2(\mathbb{R}^{2d})} \, \|h_\delta\|_{L^2(\mathbb{R}^{2d})} \, ,$$
$$|\langle B_2 \, h, h_\delta \rangle| \leq \|B_2 \, h\|_{L^2(\mathbb{R}^{2d})} \, \|h_\delta\|_{L^2(\mathbb{R}^{2d})} \, .$$

Applying (44), we obtain the existence of constants $C_1 > 0$ and $C_2 > 0$ such that

$$(66) \qquad \langle B_0 \, \tilde{g}, h_\delta \rangle \leq C_1 \, \|\tilde{g}\|_{L^2(B((x_0,v_0),r_0))} \, \|h_\delta\|_{L^2(\mathbb{R}^{2d})} \, ,$$
$$(67) \qquad \langle B_1 \, h, h_\delta \rangle \leq C_2 \, \|h\|_{L^2(B((x_0,v_0),r_0))} \, \|h_\delta\|_{L^2(\mathbb{R}^{2d})} \, .$$

To estimate the norm $\|B_2 \, h\|_{L^2(\mathbb{R}^{2d})}$, we apply (44) (with $Q_1 = \left(1 - \delta^2 \, \Delta_{x,v}\right)^{-1}$ and $Q_2 = \delta^2 \, R$ where R is defined in (55)). We also obtain the existence of C_3 and $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0]$

$$(68) \qquad |\langle B_2 \, h, h_\delta \rangle| \leq C_3 \, \|h\|_{L^2(B((x_0,v_0),r_0))} \, \|h_\delta\|_{L^2(\mathbb{R}^{2d})} \, .$$

Finally, using the estimates (63)-(68), and choosing η', η'' and $\eta''' > 0$ such that $\eta' + \eta'' + \eta''' < 1$, we get the existence of a constant $\tilde{C}$ and $\delta_0 > 0$ such that $\forall \delta \in (0, \delta_0]$

(69)

$$\|\nabla_v\, h_\delta\|^2_{L^2(\mathbb{R}^{2d}, \mathbb{R}^d)} \le \tilde{C}\, \left(\sum_{j=1}^d \|c_j\, h_j\|^2_{L^2(B((x_0, v_0), r_0))} + \|\tilde{g}\|^2_{L^2(B((x_0, v_0), r_0))} + \|h\|^2_{L^2(B((x_0, v_0), r_0))} \right)\, .$$

This completes the proof of (43) .

IV. **Deduce the uniform local boundedness of $\nabla_v\, h$.**

Let $h \in L^2(\mathbb{R}^{2d})$ satisfy (38). On the one hand, according to (42) we know that

$$h_\delta \to \chi_1\, h \text{ in } L^2(\mathbb{R}^{2d}) \text{ when } \delta \to 0\, .$$

This implies that

$$h_\delta \to \chi_1\, h \text{ and } \nabla_v\, h_\delta \to \nabla_v(\chi_1\, h) \text{ in } \mathcal{D}'(\mathbb{R}^{2d}) \text{ when } \delta \to 0\, .$$

On the other hand, according to inequality (69), we obtain that $(\nabla_v\, h_\delta)_\delta$ is bounded in $L^2(\mathbb{R}^{2d})$ for $\delta \in (0, \delta_0]$. By weak compacity, there is a subsequence $(\delta_k)_{k \in \mathbb{N}}$ tending to 0 and a function $u \in L^2(\mathbb{R}^{2d}, \mathbb{R}^d)$ such that

$$\nabla_v\, h_{\delta_k} \to u \text{ in } \mathcal{D}'(\mathbb{R}^{2d}) \text{ when } k \to +\infty\, .$$

Hence $\nabla_v(\chi_1\, h) = u$ in $\mathcal{D}'(\mathbb{R}^{2d})$. Then $\nabla_v\, h \in L^2(B((x_0, v_0), r_0/2), \mathbb{R}^d)$.

Step 3. Conclusion.

Taking the information obtained near each point (x_0, v_0), we deduce that $\nabla_v\, h \in L^2_{loc}(\mathbb{R}^{2d}, \mathbb{R}^d)$, which finishes the proof of Proposition A.3 . $\qquad\square$

We now give the proof of Theorem A.2.

Proof of Theorem A.2. The key idea of the proof is to decompose the operator K^* defined in (35) as follows:

(70) $$K^* = P_0 + (v \wedge B_e) \cdot \nabla_v + v^2/4 - d/2\, .$$

Let $f \in L^2_{loc}(\mathbb{R}^{2d})$ and $B_e \in L^\infty(\mathbb{R}^d, \mathbb{R}^{d(d-1)/2})$ such that

$$K^*\, f = g \text{ with } g \in L^2_{loc}(\mathbb{R}^{2d})\, ,$$

the goal is to show that $f \in \mathcal{H}^2_{loc}(\mathbb{R}^{2d})$.

Step 1. Reformulation of the problem.

By following decomposition given in (70) of the operator K^*, we can consider our problem as a special case of the following generalized problem:

(71) $$\begin{cases} P_0\, f = \sum\limits_{j=1}^d c_j(x, v)\, \partial_{v_j}\, h_j + \tilde{g} \\ h_j = f \in L^2_{loc}(\mathbb{R}^{2d})\, , \\ \tilde{g} = g - \frac{v^2}{4}\, f + \frac{d}{2}\, f \in L^2_{loc}(\mathbb{R}^{2d}), \forall j = 1, ..., d\, , \end{cases}$$

where P_0 the Kolomogrov operator defined in (37) and

$$c_j(x, v) = -(v \wedge B_e)_j \in L^\infty(\mathbb{R}^d, L^2_{loc}(\mathbb{R}^d))\, , \forall j = 1, ..., d\, ,$$

and where we denote by $(L)_j$ the jth component of the vector $L \in \mathbb{R}^d$.

Note that the coefficients c_j verify the condition (36) of Proposition A.3 because

$$\partial_{v_j}(v \wedge B_e)_j = 0\, , \forall j = 1, .., d\, .$$

then Proposition A.3 gives $\nabla_v f \in L^2_{loc}(\mathbb{R}^{2d}, \mathbb{R}^d)$.

Step 2. Application to Theorem A.1.
Our operator P_0 can be written as follows:

$$(72) \qquad P_0 = -\left(\sum_{j=1}^d X_j^2 + X_0 \right) ,$$

where X_j for $j = 0, 1, .., d$ are defined by

$$X_j = \begin{cases} \partial_{v_j} & \text{if } j \neq 0 \,, \\ v \cdot \nabla_x & \text{if } j = 0 \,. \end{cases}$$

According to step 1, Problem (71) is a special case of the problem of regularity of type $P_0\, f = \tilde{h}$, with f and $g \in L^2_{loc}(\mathbb{R}^{2d})$ and $\tilde{h}$ is given by

$$\tilde{h} = \sum_{j=1}^d c_j(x, v)\, \partial_{v_j} f + g - \frac{v^2}{4}\, f + \frac{d}{2}\, f \,.$$

According to step 1, we have shown that $\nabla_v f \in L^2_{loc}(\mathbb{R}^{2d}, \mathbb{R}^d)$, which implies that $\tilde{h} \in L^2_{loc}(\mathbb{R}^{2d})$. Hence, by applying Theorem A.1, we obtain $f \in \mathcal{H}^2_{loc}(\mathbb{R}^{2d})$, which completes the proof of Theorem A.2 .

$\square$

Remark A.5. *In this part, we have shown a local Sobolev regularity on $\mathbb{R}^{2d}$, we actually need it only in $\Omega \times \mathbb{R}^d$ where Ω is one of the open set of $\mathbb{R}^d$ appearing when choosing local coordinates for $\mathbb{T}^d$.*

A.4. **Proof of Theorem 1.1.** Finally, we are ready to give the proof of Theorem 1.1. The accretivity of the operator K is clear. To show that the operator is maximal, it suffices to show that there exists $\lambda_0 > 0$ such that the operator $T = K + \lambda_0\, Id$ is of dense image in $L^2(\mathbb{T}^d \times \mathbb{R}^d)$. As in [5], we take $\lambda_0 = \frac{d}{2} + 1$. Let $f \in L^2(\mathbb{T}^d \times \mathbb{R}^d)$ such that

$$(73) \qquad \langle u, (K + \lambda_0 Id)\, w \rangle = 0, \quad \forall w \in D(K) \,,$$

we have to show that $u = 0$.

For this we observe that equation (73) implies that

$$\left(K^* + \frac{d}{2} + 1 \right) u = 0, \ \text{in } \mathcal{D}'(\mathbb{T}^d \times \mathbb{R}^d) \Longleftrightarrow K^*\, u = -(\frac{d}{2} + 1)u, \ \text{in } \mathcal{D}'(\mathbb{T}^d \times \mathbb{R}^d) \,,$$

where K^* is the operator defined in (35).

Under the assumption that $B_e \in L^\infty(\mathbb{T}^d, \mathbb{R}^{d(d-1)/2})$ and $u \in D(K) \subset L^2_{loc}(\mathbb{T}^d \times \mathbb{R}^d)$, Theorem A.2 and Remark A.5 show that $f \in \mathcal{H}^2_{loc}(\mathbb{T}^d \times \mathbb{R}^d)$. More precisely, using the compactness of $\mathbb{T}^d$, we have $\chi(v)f \in \mathcal{H}^2(\mathbb{T}^d \times \mathbb{R}^d)$ for any $\chi \in C_0^\infty(\mathbb{R}^d)$. The rest of the proof is standard. The regularity obtained for f allows us to justify the integrations by parts and the cut-off argument given in [5, Proposition 5.5]. We note that in [5], a cut-off in x and v was necessary to develop the argument whereas here it suffices to perform a cut-off in v. Here we refer to [12, Proposition 3.1].

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