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# TREND TO THE EQUILIBRIUM FOR THE FOKKER-PLANCK SYSTEM WITH A STRONG EXTERNAL MAGNETIC FIELD

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ABSTRACT. We consider the Fokker-Planck equation with a strong external magnetic field. Global-in-time solutions are built near the Maxwellian, the global equilibrium state for the system. Moreover, we prove the convergence to equilibrium at exponential rate. The results are first obtained on spaces with an exponential weight. Then they are extended to larger functional spaces, like the Lebesgue space and the Sobolev space with polynomial weight, by the method of factorization and enlargement of the functional space developed in [Galdani, Mischler, Mouhot, 2017].

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## 1. INTRODUCTION AND MAIN RESULTS

**1.1. Introduction.** In this article, we are interested in inhomogeneous kinetic equations. These equations model the dynamics of a charged particle system described by a probability density  $F(t, x, v)$  representing at time  $t \geq 0$  the density of

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particles at position  $x \in \mathbb{T}^3$  and at velocity  $v \in \mathbb{R}^3$ .

In the absence of force and collision, the particles move in a straight line at constant speed according to the principle of Newton, and  $F$  is the solution of the Vlasov equation

$$\partial_t F + v \cdot \nabla_x F = 0,$$

where  $\nabla_x$  is the gradient operator with respect to the variable  $x$ , and the symbol « $\cdot$ » represents the scalar product in the Euclidean space  $\mathbb{R}^3$ . When there are forces and shocks, this equation must be corrected. This leads to various kinetic equations, the most famous being those of Boltzmann, Landau and Fokker-Planck. The general model for the dynamics of the charged particles, assuming that they undergo shocks modulated by a collision kernel  $Q$  and under the action of an external force  $\mathcal{F} \in \mathbb{R}^3$ , is written by the following kinetic equation:

$$(1) \quad \partial_t F + v \cdot \nabla_x F + \mathcal{F}(t, x) \cdot \nabla_v F = Q(F),$$

where  $Q$ , possibly non-linear, acts only in velocity and where  $\mathcal{F}$  can even depend on  $F$  via Poisson or Maxwell equations.

According to the  $H$ -theorem of Boltzmann in 1872, there exists a quantity  $H(t)$  called entropy which varies monotonous over time, while the gas relaxes towards the thermodynamic equilibrium characterized by the Maxwellian: it is a solution time independent of equation (1) having the same mass as the initial system. The effect of the collisions will bring the distribution  $F(t)$  to the Maxwellian with time. A crucial question is then to know the rate of convergence and this question has been widely studied over the past 15 years, in particular with the so called hypocoercive strategy (see [21] or [10] for an introductive papers).

## 1.2. Fokker-Planck equation with a given external magnetic field.

1.2.1. *Presentation of the equation.* We are interested in the Fokker-Planck inhomogeneous linear kinetic equation with a fixed external magnetic field  $x \mapsto B_e(x) \in \mathbb{R}^3$  which depends only on the spatial variables  $x \in \mathbb{T}^3 \equiv [0, 2\pi]^3$ . The Cauchy problem is the following:

$$(2) \quad \begin{cases} \partial_t F + v \cdot \nabla_x F - (v \wedge B_e) \cdot \nabla_v F = \nabla_v \cdot (\nabla_v + v)F \\ F(0, x, v) = F_0(x, v), \end{cases}$$

Here  $F$  is the distribution function of the particles, and represents the density of probability of presence of particles at time  $t \geq 0$  at the position  $x \in \mathbb{T}^3$  and with a speed  $v \in \mathbb{R}^3$ . (Where « $\wedge$ » indicates the vector (cross) product.)

We define the Maxwellian

$$\mu(v) := \frac{1}{(2\pi)^{3/2}} e^{-v^2/2}.$$

It is the (only) time independent solution of the system (2), since

$$\partial_t \mu + v \cdot \nabla_x \mu = 0, \quad \nabla_v \cdot (-\nabla_v + v)\mu = 0 \quad \text{and} \quad (v \wedge B_e) \cdot \nabla_v \mu = 0.$$

Concerning (2), we are interested in the return to the global equilibrium  $\mu$  and the convergence of  $F$  to  $\mu$  in norms  $L^2(dx d\mu)$  and  $H^1(dx d\mu)$  defined by

$$\begin{aligned} \forall h \in L^2(dx d\mu), \quad \|h\|_{L^2(dx d\mu)}^2 &= \iint_{\mathbb{T}^3 \times \mathbb{R}^3} |h(x, v)|^2 dx d\mu, \\ \forall h \in H^1(dx d\mu), \quad \|h\|_{H^1(dx d\mu)}^2 &= \|h\|_{L^2(dx d\mu)}^2 + \|\nabla_x h\|_{L^2(dx d\mu)}^2 + \|\nabla_v h\|_{L^2(dx d\mu)}^2, \end{aligned}$$

where  $d\mu := \mu(v)dv$  and the (real) Hilbertian scalar product  $\langle \cdot, \cdot \rangle$  on the space  $L^2(dx d\mu)$  defined by

$$\forall g, h \in L^2(dx d\mu), \quad \langle h, g \rangle = \iint hg dx d\mu.$$

To answer such questions, when  $F$  is close to the equilibrium  $\mu$ , we define  $f$  to be the standard perturbation of  $F$  defined by

$$F = \mu + \mu f.$$

We then rewrite equation (2) in the following form:

$$(3) \quad \begin{cases} \partial_t f + v \cdot \nabla_x f - (v \wedge B_e) \cdot \nabla_v f = -(-\nabla_v + v) \cdot \nabla_v f \\ f(0, x, v) = f_0(x, v) \end{cases}$$

We now introduce the main assumption on  $B_e$ .

**Hypothesis 1.1.** *The magnetic field  $B_e$  is indefinitely derivable on  $\mathbb{T}^3$ .*

1.2.2. *The main results.* First we will show that the problem (3) is well-posed in the  $L^2(dx d\mu)$  space, in the sense of the associated semi-group (See [20]). We associate with the problem (3) the operator  $P_1$  defined by

$$(4) \quad P_1 := X_0 - L,$$

$$(5) \quad \text{where } X_0 = v \cdot \nabla_x - (v \wedge B_e) \cdot \nabla_v \text{ and } L = (-\nabla_v + v) \cdot \nabla_v$$

The problem (3) is then written

$$(6) \quad \begin{cases} \partial_t f + P_1 f = 0 \\ f(0, x, v) = f_0(x, v) \end{cases}$$

**Theorem 1.2.** *Under Hypothesis 1.1 and with  $f_0 \in C_0^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$ , the problem (6) admits a unique solution  $f \in C^1([0, +\infty[, L^2(dx d\mu)) \cap C([0, +\infty[, C_0^\infty(\mathbb{T}^3 \times \mathbb{R}^3))$ .*

We also show the exponential convergence towards equilibrium in the norm  $L^2(dx d\mu)$ .

**Theorem 1.3.** *Let  $f_0 \in L^2(dx d\mu)$  such that  $\langle f_0 \rangle = \iint f_0(t, x, v) dx d\mu = 0$ . If  $B_e$  satisfies Hypothesis 1.1, then there exist  $\kappa > 0$  and  $c > 0$  (two explicit constants independent of  $f_0$ ) such that*

$$\forall t \geq 0, \quad \|f(t)\|_{L^2(dx d\mu)} \leq ce^{-\kappa t} \|f_0\|_{L^2(dx d\mu)}.$$

Note that in the preceding statement the mean  $\langle f_0 \rangle$  is preserved over time.

We give a result about the return to the global equilibrium  $\mu$  with an exponential decay rate in the space  $H^1(dx d\mu)$ .

**Theorem 1.4.** *There exist  $c, \kappa > 0$  such that  $\forall f_0 \in H^1(dx d\mu)$  with  $\langle f_0 \rangle = 0$ , the solution  $f$  of the system (3) satisfies*

$$\forall t \geq 0, \quad \|f(t)\|_{H^1(dx d\mu)} \leq ce^{-\kappa t} \|f_0\|_{H^1(dx d\mu)}.$$

We are interested in extending the results about the exponential decay of the semi-group to much larger spaces, following the work of Gualdani-Mischler-Mouhot in [6]. The following result gives convergence in  $L^p(m)$  norms of  $F$  to  $\mu$ , where the space  $L^p(m)$  for  $p \in [1, 2]$  is the weighted Lebesgue space associated with the norm

$$\|F\|_{L^p(m)} := \|Fm\|_{L^p} = \left( \int_{\mathbb{R}^3 \times \mathbb{T}^3} F^p(x, v) m^p(v) dv dx \right)^{\frac{1}{p}},$$

for some given weight function  $m = m(v) > 0$  on  $\mathbb{R}^3$ . Since there is no ambiguity we again denote

$$\langle H \rangle = \iint H dx dv,$$

the mean with respect to the usual  $L^1$  norm. The main result of this paper in this direction is the following.

**Theorem 1.5.** *Let  $p \in [1, 2]$ , let  $m = \langle v \rangle^k := (1 + |v|^2)^{k/2}$ ,  $k > 3(1 - \frac{1}{p})$ , and assume Hypothesis 1.1. Then for all  $0 > a > 3(1 - \frac{1}{p}) - k$  and for all  $F_0 \in L^p(m)$ , there exists  $c_{k,p} > 0$  such that the solution  $F$  of the problem (2) satisfies the decay estimate*

$$(7) \quad \forall t \geq 0, \quad \|F(t) - \mu \langle F_0 \rangle\|_{L^p(m)} \leq c_{k,p} e^{at} \|F_0 - \mu \langle F_0 \rangle\|_{L^p(m)}.$$

It is also possible to obtain the same type of results in weighted Sobolev space  $\tilde{W}^{1,p}(m)$  which is defined by

$$\tilde{W}^{1,p}(m) = \{h \in L^p(m) \text{ such that } \langle v \rangle h, \nabla_v h \text{ and } \nabla_x h \in L^p(m)\}.$$

We equip the previous space with the following standard norm:

$$(8) \quad \|h\|_{\tilde{W}^{1,p}(m)} = \left( \|h\|_{L^p(m \langle v \rangle)}^p + \|\nabla_v h\|_{L^p(m)}^p + \|\nabla_x h\|_{L^p(m)}^p \right)^{\frac{1}{p}}.$$

**Hypothesis 1.6.** *Let  $p \in [1, 2]$ , the polynomial weight  $m(v) = \langle v \rangle^k$  is such that*

$$(9) \quad k > 3\left(1 - \frac{1}{p}\right) + \frac{7}{2} + \max \left( \|B_e\|_{L^\infty(\mathbb{T}^3)}, \frac{1}{2} \|\nabla_x B_e\|_{L^\infty(\mathbb{T}^3)} \right).$$

The second main result of this paper is the following.

**Theorem 1.7.** *Let  $m$  be a weight that satisfies Hypothesis 1.6 with  $p \in [1, 2]$  and assume Hypothesis 1.1. If  $F_0 \in \tilde{W}^{1,p}(m)$ , then there is a solution  $F$  of the problem (2), such that  $F(t) \in \tilde{W}^{1,p}(m)$  for all  $t \geq 0$ , and it satisfies the following decay estimate:*

$$(10) \quad \forall t \geq 0, \quad \|F(t) - \mu \langle F_0 \rangle\|_{\tilde{W}^{1,p}(m)} \leq Ce^{at} \|F_0 - \mu \langle F_0 \rangle\|_{\tilde{W}^{1,p}(m)}$$

with  $0 > a > \max(a_{m,1}^i, a_{m,2}^i, -\kappa)$ ,  $i \in \{1, 2, 3\}$ , where  $a_{m,1}^i$  and  $a_{m,2}^i$  are functions defined afterwards in (45)-(47) and (54)-(56) and  $\kappa$  is defined in Theorem 1.4.

We will end this part by a brief review of the literature related to the analysis of kinetic PDEs using hypocoercivity methods. In some studies [11, 19, 21, 22], the treated hypocoercivity method is very close to that of hypoellipticity following the method of Kohn, which deals simultaneously with regularity properties and trend to the equilibrium.

The hypocoercive results were developed for simple models in [5, 9, 17, 22], the methods used were close in spirit to the ones developed in Guo [7, 8] in functional spaces with exponential weights.

In recent years, the theory of factorization and enlargement of Banach spaces was introduced in [6] and [16]. This theory allows us to extend hypocoercivity results into much larger spaces with polynomial weights. We refer for example to [3] and [16], where the authors show, using a factorization argument, the return to equilibrium with an exponential decay rate for the Fokker-Planck equation with an external electrical potential, or [13] for the inhomogeneous Boltzmann equation without angular cutoff case.

We conclude this section with some comments on our result. For the proof of Theorem 1.3, we follow the micro-macro method proposed in [10]. Note that for the proof of Theorem 1.3, the black box method proposed in [5] (see also [3]) could perhaps be employed, anyway the presence of the Magnetic field induces same difficulties. To prove Theorem 1.5 and 1.7, we apply the abstract theorem of enlargement from [6, 16] to our Fokker-Planck-Magnetic linear operator. We deduce the semi-group estimates of Theorem 1.3 on large spaces like  $L^p(\langle v \rangle^k)$  and  $\tilde{W}^{1,p}(\langle v \rangle^k)$  with  $p \in [1, 2]$ .

We hope that this first work will help in future investigations of non-linear problems like the Vlasov-Poisson-Fokker-Planck or Vlasov-Maxwell-Fokker-Planck equations (see [7, 12] and [8, 23]).

**Plan of the paper:** This article is organized as follows. In Section 2, we prove that the Fokker-Planck-magnetic operator  $P_1$  is a generator of a strongly continuous semi-group over the space  $L^2(dx d\mu)$ . In section 3, we show hypocoercivity in the weighted spaces  $L^2$  and  $H^1$  with an exponential weight. Finally, section 4 is devoted to the proofs of Theorems 1.5 and 1.7 with factorization and enlargement of the functional space arguments.

## 2. STUDY OF THE OPERATOR $P_1$

In this part, we show that the problem (6) is well-posed in the space  $L^2(dx d\mu)$  in the sense of semi-groups. By the Hille-Yosida Theorem, it is sufficient to show that  $P_1$  is maximal accretive in the space  $L^2(dx d\mu)$ .

**Notation 2.1.** We define  $P_0$  by

$$P_0 = v \cdot \nabla_x - (v \wedge B_e) \cdot \nabla_v - \nabla_v \cdot (\nabla_v + v).$$

The perturbation of the Cauchy problem (2) reduces the study of the operator  $P_1$  defined in (4) to the study of  $P_0$ , since  $P_1$  is obtained via a conjugation of the operator  $P_0$  by the function  $\mu$ , that is to say

$$P_1 u = (\mu^{-1} P_0 \mu) u \quad \forall u \in D(P_1).$$

Similarly, we can define the operator  $P_\theta$  as the conjugation of the operator  $P_1$  by the function  $\mu^\theta$  with  $\theta \in ]0, 1]$ . Note that any result on the operator  $P_\theta$  is also true on the operator  $P_1$  in the corresponding conjugated space.

We will work in this section on operator  $P_{1/2}$  which is defined by

$$P_{1/2} := v \cdot \nabla_x - (v \wedge B_e) \cdot \nabla_v + \left(-\nabla_v + \frac{v}{2}\right) \cdot \left(\nabla_v + \frac{v}{2}\right) = X_0 + b^*b,$$

here  $b = \left(\nabla_v + \frac{v}{2}\right)$  and  $X_0$  is defined in (5). We now show that operator  $P_{1/2}$  is maximal accretive in the space  $L^2(dx dv)$  and note that this gives the same result for  $P_1$  in the space  $L^2(dx d\mu)$ . We study the following problem:

$$(11) \quad \begin{cases} \partial_t u + P_{1/2} u = 0 \\ u(0, x, v) = u_0(x, v). \end{cases}$$

**Proposition 2.2.** *Suppose that  $B_e$  satisfies the Hypothesis 1.1. Then the closure with respect to the norm  $L^2(\mathbb{T}^3 \times \mathbb{R}^3)$  of the magnetic-Fokker-Planck operator  $\overline{P}_{1/2}$  on the space  $C_0^\infty(\mathbb{R}^3 \times \mathbb{T}^3)$  is maximally accretive.*

*Proof.* We adapt here the proof given in [19, page 44]. We apply the abstract criterion by taking  $H = L^2(\mathbb{T}^3 \times \mathbb{R}^3)$  and the domain of  $P_{1/2}$  defined by  $D(P_{1/2}) = C_0^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$ . First, we show the accretivity of the operator  $P_{1/2}$ . When  $u \in D(P_{1/2})$ , we have to show that  $\langle P_{1/2} u, u \rangle \geq 0$ . Indeed,

$$\begin{aligned} \langle P_{1/2} u, u \rangle &= \langle v \cdot \nabla_x u - (v \wedge B_e) \cdot \nabla_v u + \left(-\nabla_v + \frac{v}{2}\right) \left(\nabla_v + \frac{v}{2}\right) u, u \rangle \\ &= \underbrace{\iint v \cdot \nabla_x u \times u \, dx dv}_{=0} - \underbrace{\iint (v \wedge B_e) \cdot \nabla_v u \times u \, dx dv}_{=0} + \langle b^* b u, u \rangle \\ &= \|b u\|^2 \\ &\geq 0, \end{aligned}$$

since operators  $(v \wedge B_e) \cdot \nabla_v$  and  $v \cdot \nabla_x$  are skew-adjoint see Lemma A.1.

Let us now show that there exists  $\lambda_0 > 0$  such that the operator

$$T = P_{1/2} + \lambda_0 Id$$

has dense image in  $H$ . We take  $\lambda_0 = \frac{3}{2} + 1$  (following [19]). Let  $u \in H$  satisfy

$$(12) \quad \langle u, (P_{1/2} + \lambda_0 Id) h \rangle = 0, \quad \forall h \in D(P_{1/2}).$$

We have to show that  $u = 0$ .

First, we observe that equality (12) implies that

$$\left(-\Delta_v + \frac{v^2}{4} + 1 - X_0\right) u = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^3 \times \mathbb{T}^3).$$

Under Hypothesis 1.1, and following Hormander [14, 15] or Helffer-Nier [19, Chapter 8], operator  $-\Delta_v + \frac{v^2}{4} + 1 - X_0$  is hypoelliptic, so  $u \in C^\infty(\mathbb{R}^3 \times \mathbb{T}^3)$ .

Now we introduce the family of truncation functions  $\xi_k$  indexed by  $k \in \mathbb{N}^*$  and defined by

$$\xi_k(v) := \xi\left(\frac{v}{k}\right) \quad \forall k \in \mathbb{N}^*,$$

where  $\xi$  is a  $C^\infty$  function satisfying  $0 \leq \xi \leq 1$ ,  $\xi = 1$  on  $B(0, 1)$ , and  $\text{Supp } \xi \subset B(0, 2)$ .

For all  $u, w \in C^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$ , we have

$$\begin{aligned} & \iint \nabla_v(\xi_k u) \cdot \nabla_v(\xi_k w) dx dv + \iint \xi_k^2 \left(\frac{v^2}{4} + 1\right) w u dx dv + \iint u X_0(\xi_k^2 w) dx dv \\ &= \iint |\nabla_v \xi_k|^2 w u dx dv + \iint (u \nabla_v w - w \nabla_v u) \cdot \xi_k \nabla_v \xi_k dx dv + \langle u, T(\xi_k^2 w) \rangle. \end{aligned}$$

When  $u$  satisfies (12) in particular, when  $h = \xi_k^2 w$ , we get for all  $w \in C^\infty$

$$\begin{aligned} & \iint \nabla_v(\xi_k u) \cdot \nabla_v(\xi_k w) dx dv + \iint \xi_k^2 \left(\frac{v^2}{4} + 1\right) w u dx dv + \iint u X_0(\xi_k^2 w) dx dv \\ &= \iint |\nabla_v \xi_k|^2 w u dx dv + \iint (u \nabla_v w - w \nabla_v u) \cdot \xi_k \nabla_v \xi_k dx dv. \end{aligned}$$

In particular, we take the test function  $w = u$ , so

$$\begin{aligned} & \langle \nabla_v(\xi_k u), \nabla_v(\xi_k u) \rangle + \iint \xi_k^2 \left(\frac{v^2}{4} + 1\right) u^2 dx dv + \iint u X_0(\xi_k^2 u) dx dv \\ &= \iint |\nabla_v \xi_k|^2 u^2 dx dv. \end{aligned}$$

By an integration by parts, we obtain

$$\begin{aligned} & \langle \nabla_v(\xi_k u), \nabla_v(\xi_k u) \rangle + \iint \xi_k^2 \left(\frac{v^2}{4} + 1\right) u^2 dx dv + \iint \xi_k u^2 X_0(\xi_k) dx dv \\ &= \iint |\nabla_v \xi_k|^2 u^2 dx dv. \end{aligned}$$

Which gives the existence of a constant  $c > 0$  such that, for all  $k \in \mathbb{N}^2$ ,

$$\begin{aligned} \|\xi_k u\|^2 + \frac{1}{4} \|\xi_k v u\|^2 \\ \leq \frac{c}{k^2} \|u\|^2 + \frac{c}{k} \|(v \wedge B_e) \xi_k u\| \|u\|. \end{aligned}$$

This leads to

$$\|\xi_k u\|^2 + \frac{1}{8} \|\xi_k v u\|^2 \leq c \left( \frac{1}{k^2} + \frac{c_\eta}{k} \|B_e\|_\infty^2 \right) \|u\|^2 + \eta \|\xi_k v u\|^2.$$

Choosing  $\eta \leq \frac{1}{8}$ , we get

$$(13) \quad \|\xi_k u\|^2 \leq c \left( \frac{1}{k^2} + \frac{c_\eta}{k} \|B_e\|_\infty^2 \right) \|u\|^2.$$

Taking  $k \rightarrow +\infty$  in (13), leads to  $u = 0$ .  $\square$

*Proof of Theorem 1.2.* According to Remark 2.1, the operator  $P_1$  has a closure  $\overline{P_1}$  from  $C_0^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$ . This gives Theorem 1.2, by a direct application of Hille-Yosida's theorem (cf. [20] for more details for the semi-group theory) to the problem (3), with  $D(P_1) = C_0^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$  and  $H = L^2(dx d\mu)$ .  $\square$

From now on, we write  $P_\theta$  for the closure of the operator  $P_\theta$  from the space  $C_0^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$  with respect the norm  $L^2(\mathbb{T}^3 \times \mathbb{R}^3)$ .

## 3. TREND TO THE EQUILIBRIUM

**3.1. Hypocoercivity in the space  $L^2(dx d\mu)$ .** The purpose of this subsection is to show the exponential time decay of the  $L^2(dx d\mu)$  entropy for  $P_1$ , based on macroscopic equations. First, we try to find the macroscopic equations associated with system (3). We write  $f$  in the following form:

$$(14) \quad f(x, v) = r(x) + h(x, v),$$

where  $r(f)(x) = \int f(x, v) d\mu(v)$  and  $m(f)(x) = \int v f(x, v) d\mu(v)$  will be use later.

**Definition 3.1.** *In the following, we define*

$$\Lambda_x = (1 - \Delta_x)^{1/2},$$

and introduce a class of Hilbert spaces

$$\mathbb{H}^\alpha := \{u \in \mathcal{S}', \Lambda_x^\alpha \in L^2(dx d\mu)\} \quad \text{with } \alpha \in \mathbb{R},$$

where  $\mathcal{S}'$  is the space of temperate distributions.

We recall that the operator  $\Lambda_x^2$  is an elliptic, self-adjoint, invertible operator from  $\mathbb{H}^2(dx d\mu)$  to  $L^2(dx d\mu)$  and  $\Lambda_x \geq Id$ . (cf [11, section 6] for a proof of these properties).

**Lemma 3.2.** *Let  $f$  be the solution of the system (3), with the decomposition given in (14). Then we have*

$$(15) \quad \partial_t r = Op_1(h),$$

$$(16) \quad \partial_t m = -\nabla_x r - m \wedge B_e + Op_1(h).$$

Where  $Op_1$  denotes a bounded generic operator of  $L^2$  to  $\mathbb{H}^{-1}$ .

*Proof.* We suppose is  $f$  is a Schwarz function. In order to show equation (14), we integrate equation (3) with respect to the measure  $d\mu$ . We get

$$\begin{aligned} \partial_t \int f d\mu + \int v \cdot \nabla_x f d\mu - \int (v \wedge B_e) \cdot \nabla_v f d\mu &= - \int (-\nabla_v + v) \cdot \nabla_v f d\mu \\ &= \langle Lf, 1 \rangle = \langle f, L1 \rangle = 0, \end{aligned}$$

since,  $L1 = 0$ ,  $L$  is a self-adjoint operator and

$$(v \wedge B_e) \cdot \nabla_v f = \nabla_v \cdot (v \wedge B_e) f.$$

Then, we obtain

$$\partial_t r = \nabla_x \cdot \int v h d\mu = Op_1(h),$$

hence equality (14).

To show (15), we multiply equation (3) by  $v$  before performing an integration with respect to the measure  $d\mu$ , we obtainning

$$(17) \quad \partial_t \int v f d\mu + \nabla_x \cdot \int (v \otimes v) f d\mu - \int v((v \wedge B_e) \cdot \nabla_v f) d\mu = \langle Lf, v \rangle,$$

where  $\nabla_x \cdot \int (v \otimes v) f d\mu = \sum_{i=1}^3 \sum_{j=1}^3 \int v_j v_i \partial_{x_i} f d\mu$ . Now, we will calculate term by term the left-hand side of the equality (17). We first observe that

$$\nabla_x \cdot \int (v \otimes v) f d\mu = Op_1(h) + \nabla_x r.$$

Furthermore,

$$\langle Lf, v \rangle = \langle f, Lv \rangle = \langle f, v \rangle = \int hvd\mu.$$

It remains to compute component by component  $\int v((v \wedge B_e \cdot \nabla_v f)) d\mu$ . We have for all  $1 \leq j \leq 3$ ,

$$\begin{aligned} \int v_j((v \wedge B_e) \cdot \nabla_v f) d\mu &= \int v_j \nabla_v \cdot ((v \wedge B_e) f) d\mu \\ &= \int (-\nabla_v + v)(v_j) \cdot (v \wedge B_e) f d\mu \\ &= -\delta_j \cdot \int (v \wedge B_e) f d\mu \\ &= -\delta_j \cdot \left( \int v f d\mu \right) \wedge B_e \\ &= -\delta_j \cdot (m \wedge B_e) \\ &= -(m \wedge B_e)_j. \end{aligned}$$

Therefore  $\int v((v \wedge B_e \cdot \nabla_v f)) d\mu = -m \wedge B_e$ , where  $m$  is defined in (14). By combining all the previous equalities in (17), we obtain

$$\partial_t m = -\nabla_x r - m \wedge B_e + Op_1(h).$$

□

**Remark 3.3.** Under Hypothesis 1.1, since  $B_e \in L^\infty(\mathbb{T}^3)$ ,

$$m \wedge B_e = Op_1(h),$$

so the macroscopic equation (14) takes the following form:

$$\partial_t m = -\nabla_x r + Op_1(h).$$

Now we are ready to build a new entropy, defined for any  $u \in L^2(dx d\mu)$  by

$$\mathcal{F}_\varepsilon(u) = \|u\|^2 + \varepsilon \langle \Lambda_x^{-2} \nabla_x r(u), m(u) \rangle, \quad r(u) := \int u d\mu \text{ and } m(u) := \int v u d\mu.$$

Using the Cauchy-Schwarz inequality gives us directly that

**Lemma 3.4.** If  $\varepsilon \leq \frac{1}{2}$ , then

$$(18) \quad \frac{1}{2} \|u\|^2 \leq \mathcal{F}_\varepsilon(u) \leq 2 \|u\|^2$$

Now, we can prove the main result of hypocoercivity leading to the proof of Theorem 1.3.

**Proposition 3.5.** There exists  $\kappa > 0$  such that, if  $f_0 \in L^2(dx d\mu)$  and  $\langle f_0 \rangle = 0$ , then the solution of system (3) satisfies

$$\forall t \geq 0, \quad \mathcal{F}_\varepsilon(f(t)) \leq e^{-\kappa t} \mathcal{F}_\varepsilon(f_0).$$

*Proof.* We write

$$\frac{d}{dt} \mathcal{F}_\varepsilon(f(t)) = \frac{d}{dt} \|f\|^2 + \varepsilon \frac{d}{dt} \langle \Lambda_x^{-2} r, m \rangle.$$

We will omit the dependence of  $f$  with respect to  $t$ . For the first term, we notice that

$$(19) \quad \frac{d}{dt} \|f\|^2 = 2\langle Lf, f \rangle = -2\|\nabla_v f\|^2 \leq -2\|h\|^2,$$

by the spectral property of the operator  $L$ . For the second term, using the macroscopic equations, we get

$$\begin{aligned} \frac{d}{dt} \langle \Lambda_x^{-2} \nabla_x r, m \rangle &= \langle \Lambda_x^{-2} \nabla_x \partial_t r, m \rangle + \langle \Lambda_x^{-2} \nabla_x r, \partial_t m \rangle \\ &= -\langle \Lambda_x^{-2} \nabla_x r, \nabla_x r \rangle + \langle \Lambda_x^{-2} \nabla_x Op_1(h), m \rangle + \langle \Lambda_x^{-2} \nabla_x r, Op_1(h) \rangle \\ &\leq \|\Lambda_x^{-1} \nabla_x r\|^2 + C \|\Lambda_x^{-1} Op_1(h)\| (\|\Lambda_x^{-1} \nabla_x r\| + \|\Lambda_x^{-1} \nabla_x m\|). \end{aligned}$$

Now, using  $\|m\| \leq \|h\|$ , the Cauchy-Schwarz inequality and the following estimate:

$$\|\Lambda_x^{-1} \nabla_x \phi\| \leq \|\phi\|, \quad \forall \phi \in L^2(dx d\mu),$$

we obtain

$$\frac{d}{dt} \langle \Lambda_x^{-2} r, m \rangle \leq -\frac{1}{2} \|\Lambda_x^{-1} \nabla_x r\|^2 + C \|h\|^2.$$

Poincaré's inequality on  $L^2(dx)$  takes the form

$$\forall \phi \in L^2(dx), \quad \text{such that } \langle \phi \rangle = 0, \quad \|\Lambda_x^{-1} \nabla_x \phi\|^2 \geq \frac{c_P}{c_P + 1} \|\phi\|^2,$$

where  $\langle \phi \rangle = \int \phi(x) dx$  and  $c_P > 0$  is the spectral gap of  $-\Delta_x$  on the torus (see [10, Lemma 2.6] for the proof of the previous inequality). Using this, we obtain, by applying the previous estimate to  $r$  (since  $\langle r \rangle = \langle f \rangle = \langle f_0 \rangle = 0$ ),

$$(20) \quad \frac{d}{dt} \langle \Lambda_x^{-2} \nabla_x r, m \rangle \leq -\frac{1}{2} \frac{c_P}{c_P + 1} \|r\|^2 + C \|h\|^2.$$

gathering (19) and (20), we get

$$\frac{d}{dt} \mathcal{F}_\varepsilon(f) \leq -\|h\|^2 - \frac{\varepsilon}{2} \frac{c_P}{c_P + 1} \|r\|^2 + C\varepsilon \|h\|^2$$

Now we choose  $\varepsilon$  such that  $C\varepsilon \leq \frac{1}{2}$ , we get

$$\frac{d}{dt} \mathcal{F}_\varepsilon(f) \leq -\frac{1}{2} \|h\|^2 - \frac{\varepsilon}{2} \frac{c_P}{c_P + 1} \|r\|^2 \leq -\frac{\varepsilon}{2} \frac{c_P}{c_P + 1} \|f\|^2 \leq -\frac{\varepsilon}{4} \frac{c_P}{c_P + 1} \mathcal{F}_\varepsilon(f).$$

Which gives the result with  $\kappa = \frac{\varepsilon}{4} \frac{c_P}{c_P + 1} > 0$ .  $\square$

We can deduce the proof of Theorem 1.3.

*Proof of Theorem 1.3.* Starting from Lemma 3.4 and Proposition 3.5, we have, for  $f$  the solution of the system (3),

$$\|f\|^2 \leq 2\mathcal{F}_\varepsilon(f) \leq 2e^{-\kappa t} \mathcal{F}_\varepsilon(f_0) \leq 4e^{-\kappa t} \|f_0\|^2.$$

This completes the proof of Theorem 1.3.  $\square$

**3.2. Hypocoercivity in the space  $H^1(dx d\mu)$ .** We will establish some technical lemmas, which will help us to deduce the exponential time decay of the norm  $H^1(dx d\mu)$ , noting that we work in 3 dimensions.

The following lemma gives the exact values of some commutators will be used later.

**Lemma 3.6.** *The following equalities*

- (1)  $[\partial_{v_i}, v \cdot \nabla_x] = \partial_{x_i} \quad \forall i \in \{1, 2, 3\}$ .
- (2)  $[\partial_{v_i}, (-\partial_{v_j} + v_j)] = \delta_{ij} \quad \forall i, j \in \{1, 2, 3\}$ .
- (3)  $[\nabla_v, (v \wedge B_e) \cdot \nabla_v] = B_e \wedge \nabla_v$ .
- (4)  $[\nabla_x, (v \wedge B_e) \cdot \nabla_v] = (v \wedge \nabla_x B_e) \cdot \nabla_v$ .

*Proof.* Let  $f \in C_0^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$ . The first two equalities are obvious. We directly go to the proof of (3) in component. Writting  $B_e = (B_1, B_2, B_3)$ ,

$$\begin{aligned} [\partial_{v_1}, (v \wedge B_e) \cdot \nabla_v] f &= \partial_{v_1}((v \wedge B_e) \cdot \nabla_v) f - ((v \wedge B_e) \cdot \nabla_v) \partial_{v_1} f \\ &= \partial_{v_1}[(v_2 B_3 - v_3 B_2) \partial_{v_1} f + (v_3 B_1 - v_1 B_3) \partial_{v_2} f \\ &\quad + (v_1 B_2 - v_2 B_1) \partial_{v_3} f] - ((v \wedge B_e) \cdot \nabla_v) \partial_{v_1} f \\ &= (B_2 \partial_{v_3} f - B_3 \partial_{v_2} f) \\ &= (B_e \wedge \nabla_v)_1 f. \end{aligned}$$

Similarly we can show that, for all  $1 \leq i \leq 3$ ,

$$[\partial_{v_i}, (v \wedge B_e) \cdot \nabla_v] = (B_e \wedge \nabla_v)_i.$$

This proves the equality (3). Now, we will show (4),

$$\begin{aligned} [\nabla_x, (v \wedge B_e) \cdot \nabla_v] f &= \nabla_x((v \wedge B_e) \cdot \nabla_v) f - ((v \wedge B_e) \cdot \nabla_v) \nabla_x f \\ &= (v \wedge \nabla_x B_e) \cdot \nabla_v f. \end{aligned}$$

□

Now, we are ready to build a new entropy that will allow us to show the exponential decay of the norm  $H^1(dx d\mu)$ . We define this modified entropy by

$$\mathcal{E}(u) = C\|u\|^2 + D\|\nabla_v u\|^2 + E\langle \nabla_x u, \nabla_v u \rangle + \|\nabla_x u\|^2, \quad \forall u \in H^1(dx d\mu),$$

where  $C > D > E > 1$  are constants fixed below. We first show that  $\mathcal{E}(u)$  is equivalent to the norm  $H^1(dx d\mu)$  of  $u$ .

**Lemma 3.7.** *If  $E^2 < D$ , then  $\forall u \in H^1(dx d\mu)$*

$$(21) \quad \frac{1}{2}\|u\|_{H^1(dx d\mu)}^2 \leq \mathcal{E}(u) \leq 2C\|u\|_{H^1(dx d\mu)}^2.$$

*Proof.* Let  $u \in H^1(dx d\mu)$ . Using the Cauchy-Schwarz inequality, we get

$$|E\langle \nabla_x u, \nabla_v u \rangle| \leq \frac{E^2}{2}\|\nabla_v u\|^2 + \frac{1}{2}\|\nabla_x u\|^2,$$

which implies,

$$\begin{aligned} C\|u\|^2 + (D - \frac{E^2}{2})\|\nabla_v u\|^2 + (1 - \frac{1}{2})\|\nabla_x u\|^2 &\leq \mathcal{E}(u) \\ &\leq C\|u\|^2 + (D + \frac{E^2}{2})\|\nabla_v u\|^2 + (1 + \frac{1}{2})\|\nabla_x u\|^2. \end{aligned}$$

This implies (21) if  $E^2 < D$ . □

Note that using the same approach as in Section 3, we can show the existence of a solution of the problem (3), which will be denoted as  $f$ , in the space  $H^1(dx d\mu)$  in the sense of an associated semi-group. Using the preceding results, we are able to study the decrease of the modified entropy  $\mathcal{E}(f(t))$ .

**Proposition 3.8.** *Suppose that  $B_e$  satisfies the Hypothesis 1.1, then there exist  $C, D, E$  and  $\kappa > 0$ , such that for all  $f_0 \in H^1(dx d\mu)$  with  $\langle f_0 \rangle = 0$ , the solution  $f$  of the system (3) satisfies*

$$\forall t > 0, \quad \mathcal{E}(f(t)) \leq \mathcal{E}(f_0) e^{-\kappa t}.$$

*Proof.* The time derivatives of the four terms defining  $\mathcal{E}(f(t))$  will be calculated separately. For the first term we have

$$\begin{aligned} \frac{d}{dt} \|f\|^2 &= -2\langle \partial_t f, f \rangle \\ &= -2 \underbrace{\langle v \cdot \nabla_x f, f \rangle}_{=0} + 2 \underbrace{\langle (v \wedge B_e) \cdot \nabla_v f, f \rangle}_{=0} - 2\langle (-\nabla_v + v) \cdot \nabla_v f, f \rangle \\ &= -2\|\nabla_v f\|^2. \end{aligned}$$

The second term writes

$$\begin{aligned} \frac{d}{dt} \|\nabla_v f\|^2 &= 2\langle \nabla_v \nabla_t f, \nabla_v f \rangle \\ &= -2\langle \nabla_v (v \cdot \nabla_x f), \nabla_v f \rangle + 2\langle \nabla_v ((v \wedge B_e) \cdot \nabla_v f), \nabla_v f \rangle \\ &\quad - 2\langle \nabla_v (-\nabla_v + v) \cdot \nabla_v f, \nabla_v f \rangle \\ &= -2 \underbrace{\langle v \nabla_x \nabla_v f, \nabla_v f \rangle}_{=0} - 2\langle [\nabla_v, v \cdot \nabla_x] f, \nabla_v f \rangle + 2\langle [\nabla_v, (v \wedge B_e) \cdot \nabla_v] f, \nabla_v f \rangle \\ &\quad + 2 \underbrace{\langle ((v \wedge B_e) \cdot \nabla_v) \nabla_v f, \nabla_v f \rangle}_{=0} - 2\|(-\nabla_v + v) \cdot \nabla_v f\|^2. \end{aligned}$$

We used the fact that the operators  $v \cdot \nabla_x$  and  $(v \wedge B_e) \cdot \nabla_v$  are skew-adjoint in  $L^2(dx d\mu)$  by Lemma A.1. According to equalities (1) and (3) of Lemma 3.6, we then obtain

$$\frac{d}{dt} \|\nabla_v f\|^2 = -2\langle \nabla_x f, \nabla_v f \rangle + 2\langle (B_e \wedge \nabla_v) f, \nabla_v f \rangle - 2\|(-\nabla_v + v) \cdot \nabla_v f\|^2.$$

The time derivative of the third term can be calculated as follows:

$$(22) \quad \frac{d}{dt} \langle \partial_v f, \nabla_x f \rangle = \langle \nabla_v \partial_t f, \nabla_x f \rangle + \langle \nabla_v f, \nabla_x \partial_t f \rangle.$$

We calculate each term of equality (22). For the first term, using equalities (1), (2) and (3) of Lemma 3.6, we obtain

$$\begin{aligned} \langle \nabla_v \partial_t f, \nabla_x f \rangle &= -\langle \nabla_v (v \cdot \nabla_x f - (v \wedge B_e) \cdot \nabla_v f + (-\nabla_v + v) \cdot \nabla_v f), \nabla_x f \rangle \\ &= -\|\nabla_x f\|^2 - \langle v \cdot \nabla_x (\nabla_v f), \nabla_x f \rangle - \langle \nabla_v f, \nabla_x f \rangle - \langle \Delta_v f, \nabla_v \cdot \nabla_x f \rangle \\ &\quad + \langle (B_e \wedge \nabla_v) f, \nabla_x f \rangle + \langle ((v \wedge B_e) \cdot \nabla_v) \nabla_v f, \nabla_x f \rangle. \end{aligned}$$

For the second term of equality (22), using equality (4) of Lemma 3.6, we have

$$\begin{aligned} \langle \nabla_v f, \nabla_x \partial_t f \rangle &= -\langle \nabla_v f, \nabla_x (v \cdot \nabla_x f - (v \wedge B_e) \cdot \nabla_v f + (-\nabla_v + v) \cdot \nabla_v f) \rangle \\ &= -\langle \nabla_v f, v \cdot \nabla_x (\nabla_x f) \rangle + \langle \nabla_v f, (v \wedge \nabla_x B_e) \cdot \nabla_v f \rangle \\ &\quad + \langle \nabla_v f, ((v \wedge B_e) \cdot \nabla_v) \nabla_x f \rangle - \langle \nabla_v f, \nabla_x f \rangle - \langle \Delta_v f, \nabla_v \cdot \nabla_x f \rangle. \end{aligned}$$

Combining the proceeding equalities of the two terms in (22), we get

$$\begin{aligned} \frac{d}{dt} \langle \nabla_v f, \nabla_x f \rangle &= - \|\nabla_x f\|^2 - \langle \nabla_v f, \nabla_x f \rangle + 2 \langle (-\nabla_v + v) \cdot \nabla_v f, \nabla_v \cdot (\nabla_x f) \rangle \\ &\quad - [\langle v \cdot \nabla_x (\nabla_v f), \nabla_x f \rangle + \langle \nabla_v f, v \cdot \nabla_x (\nabla_x f) \rangle] \\ &\quad + \langle (B_e \wedge \nabla_v) f, \nabla_x f \rangle + \langle \nabla_v f, (v \wedge \nabla_x B_e) \cdot \nabla_v f \rangle \\ &\quad + [\langle (v \wedge B_e) \cdot \nabla_v \nabla_v f, \nabla_x f \rangle + \langle \nabla_v f, ((v \wedge B_e) \cdot \nabla_v) \nabla_x f \rangle]. \end{aligned}$$

According to Lemma A.1, the operators  $v \cdot \nabla_x$  and  $(v \wedge B_e) \cdot \nabla_v$  are skew-adjoint in  $L^2(dx d\mu)$ , we have

$$(23) \quad \langle v \cdot \nabla_x (\nabla_v f), \nabla_x f \rangle + \langle \nabla_v f, v \cdot \nabla_x (\nabla_x f) \rangle = 0$$

$$(24) \quad \langle ((v \wedge B_e) \cdot \nabla_v) \nabla_v f, \nabla_x f \rangle + \langle \nabla_v f, ((v \wedge B_e) \cdot \nabla_v) \nabla_x f \rangle = 0.$$

Using equality (23)-(24), we obtain

$$\begin{aligned} \frac{d}{dt} \langle \nabla_v f, \nabla_x f \rangle &= - \|\nabla_x f\|^2 - \langle \nabla_v f, \nabla_x f \rangle + 2 \langle (-\nabla_v + v) \nabla_v f, \nabla_v \cdot (\nabla_x f) \rangle \\ &\quad + \langle (B_e \wedge \nabla_v) f, \nabla_x f \rangle + \langle \nabla_v f, (v \wedge \nabla_x B_e) \cdot \nabla_v f \rangle. \end{aligned}$$

Eventually, the time derivative of the last term takes the following form

$$\begin{aligned} \frac{d}{dt} \|\nabla_x f\|^2 &= 2 \langle \nabla_x \partial_t f, \nabla_x f \rangle \\ &= -2 \underbrace{\langle \nabla_x (v \cdot \nabla_x f), \nabla_x f \rangle}_{=0} + 2 \langle \nabla_x ((v \wedge B_e) \cdot \nabla_v f), \nabla_x f \rangle \\ &\quad - 2 \langle \nabla_x (-\nabla_v + v) \cdot \nabla_v f, \nabla_x f \rangle \\ &= -2 \|\nabla_x \nabla_v f\|^2 + 2 \langle (v \wedge \nabla_x B_e) \cdot \nabla_v f, \nabla_x f \rangle + 2 \underbrace{\langle ((v \wedge B_e) \cdot \nabla_v) \nabla_x f, \nabla_x f \rangle}_{=0} \\ &= -2 \|\nabla_x \nabla_v f\|^2 + 2 \langle (v \wedge \nabla_x B_e) \cdot \nabla_v f, \nabla_x f \rangle, \end{aligned}$$

By collecting all the ties, we get

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(f) &= -2C \|\nabla_v f\|^2 - 2D \|(-\nabla_v + v) \cdot \nabla_v f\|^2 - E \|\nabla_x f\|^2 - 2 \|\nabla_x \nabla_v f\|^2 \\ &\quad - (2D + E) \langle \nabla_x f, \nabla_v f \rangle + 2E \langle (-\nabla_v + v) \cdot \nabla_v f, \nabla_v \cdot (\nabla_x f) \rangle \\ &\quad + 2D \langle (B_e \wedge \nabla_v) f, \nabla_v f \rangle + E \langle (B_e \wedge \nabla_v) f, \nabla_x f \rangle \\ &\quad + E \langle (v \wedge \nabla_x B_e) \cdot \nabla_v f, \nabla_v f \rangle + 2 \langle (v \wedge \nabla_x B_e) \cdot \nabla_v f, \nabla_x f \rangle. \end{aligned}$$

Now, we need the following technical lemma.

**Lemma 3.9.** *We have the following equalities in  $L^2(dx d\mu)$ :*

- i.  $\langle (v \wedge \nabla_x B_e) \cdot \nabla_v f, \nabla_v f \rangle = - \langle \nabla_v \wedge (\nabla_x B_e \cdot \nabla_v f), \nabla_v f \rangle.$
- ii.  $\langle (v \wedge \nabla_x B_e) \cdot \nabla_v f, \nabla_x f \rangle = - \langle \nabla_v \wedge (\nabla_x B_e \cdot \nabla_x f), \nabla_v f \rangle.$

where  $\nabla_x B_e$  is the Jacobian matrix of the function

$$x \rightarrow B_e(x) = (B_1(x), B_2(x), B_3(x)),$$

and

$$\begin{aligned} \langle (v \wedge \nabla_x B_e) \cdot \nabla_v f, \nabla_v f \rangle &= \sum_{i=1}^3 \left[ \int \partial_{v_1} \mu(v) ((\nabla_x B_e)_{i2} \partial_{v_3} f - (\nabla_x B_e)_{i3} \partial_{v_2} f) \partial_{v_i} f \, dx dv \right] \\ &+ \sum_{i=1}^3 \left[ \int \partial_{v_2} \mu(v) ((\nabla_x B_e)_{i3} \partial_{v_1} f - (\nabla_x B_e)_{i1} \partial_{v_3} f) \partial_{v_i} f \, dx dv \right] \\ &+ \sum_{i=1}^3 \left[ \int \partial_{v_3} \mu(v) ((\nabla_x B_e)_{i1} \partial_{v_2} f - (\nabla_x B_e)_{i2} \partial_{v_1} f) \partial_{v_i} f \, dx dv \right]. \end{aligned}$$

*Proof.* Using the fact that  $v \mu(v) = \nabla_v(\mu(v))$  and integrations by part, we obtain the result by simple computations.  $\square$

Let's go back to the proof of Proposition 3.8. Using Lemma 3.9, the time derivative of  $\mathcal{E}(f(t))$  takes the following form:

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(f(t)) &= -2C \|\nabla_v f\|^2 - 2D \|(-\nabla_v + v) \cdot \nabla_v f\|^2 - E \|\nabla_x f\|^2 - 2 \|\nabla_x \nabla_v f\|^2 \\ &- (2D + E) \langle \nabla_x f, \nabla_v f \rangle + 2E \langle (-\nabla_v + v) \cdot \nabla_v f, \nabla_v \cdot (\nabla_x f) \rangle \\ &+ 2D \langle (B_e \wedge \nabla_v) f, \nabla_v f \rangle + E \langle (B_e \wedge \nabla_v) f, \nabla_x f \rangle \\ &- E \langle \nabla_v \wedge (\nabla_x B_e \cdot \nabla_v f), \nabla_v f \rangle - 2 \langle \nabla_v \wedge (\nabla_x B_e \cdot \nabla_x f), \nabla_v f \rangle. \end{aligned}$$

Now we estimate the scalar products in the previous equality in  $L^2(dx d\mu)$ . For all  $\eta, \eta'$  and  $\eta'' > 0$ , we have

$$\begin{aligned} |(2D + E) \langle \nabla_x f, \nabla_v f \rangle| &\leq \frac{1}{2} \|\nabla_x f\|^2 + \frac{1}{2} (2D + E)^2 \|\nabla_v f\|^2, \\ |2E \langle (-\nabla_v + v) \cdot \nabla_v f, \nabla_v \cdot (\nabla_x f) \rangle| &\leq \|\nabla_v \cdot (\nabla_x f)\|^2 + E^2 \|(-\nabla_v + v) \cdot \nabla_v f\|^2, \\ |2D \langle (B_e \wedge \nabla_v) f, \nabla_v f \rangle| &\leq 2D \|B_e\|_\infty \|\nabla_v f\|^2, \\ |E \langle (B_e \wedge \nabla_v) f, \nabla_x f \rangle| &\leq E \frac{\eta}{2} \|\nabla_x f\|^2 + E \frac{1}{2\eta} \|B_e\|_\infty^2 \|\nabla_v f\|^2, \end{aligned}$$

and using that  $\|\nabla_v^2 f\|^2 \leq \|(-\nabla_v + v) \cdot \nabla_v f\|^2 + \|\nabla_v f\|^2$ , we obtain

$$\begin{aligned} |E \langle (\nabla_v \wedge (\nabla_x B_e \cdot \nabla_v)) f, \nabla_v f \rangle| &\leq \frac{\eta'}{2} \|\nabla_v^2 f\|^2 + \frac{E^2}{2\eta'} \|\nabla_x B_e\|_\infty^2 \|\nabla_v f\|^2 \\ &\leq \frac{\eta'}{2} \|(-\nabla_v + v) \cdot \nabla_v f\|^2 + \left( \frac{E^2}{2\eta'} \|\nabla_x B_e\|_\infty^2 \right. \\ &\quad \left. + \frac{\eta'}{2} \right) \|\nabla_v f\|^2, \end{aligned}$$

The last scalar product is bounded by

$$|2 \langle (\nabla_v \wedge (\nabla_x B_e \cdot \nabla_x)) f, \nabla_v f \rangle| \leq \eta'' \|\nabla_x \nabla_v f\|^2 + \frac{1}{\eta''} \|\nabla_x B_e\|_\infty^2 \|\nabla_v f\|^2.$$

Combining all the previous estimates, we have

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(f) &\leq (-2C + \frac{1}{2}(2D + E)^2 + 2D\|B_e\|_\infty + \frac{E}{2\eta}\|B_e\|_\infty^2 \\ &\quad + \frac{E^2}{2\eta'}\|\nabla_x B_e\|_\infty^2 + \frac{\eta'}{2} + \frac{1}{\eta''}\|\nabla_x B_e\|_\infty^2)\|\nabla_v f\|^2 \\ &\quad + (-2D + E^2 + \frac{\eta'}{2})\|(-\nabla_v + v) \cdot \nabla_v f\|^2 \\ &\quad + (-E + \frac{1}{2} + \frac{\eta}{2}E)\|\nabla_x f\|^2 \\ &\quad + (-2 + 1 + \eta'')\|\nabla_x \nabla_v f\|^2. \end{aligned}$$

We notice that

$$\begin{aligned} A &= \frac{1}{2}(2D + E)^2 + 2D\|B_e\|_\infty + \frac{E}{2\eta}\|B_e\|_\infty^2 \\ &\quad + \frac{E^2}{2\eta'}\|\nabla_x B_e\|_\infty^2 + \frac{\eta'}{2} + \frac{1}{\eta''}\|\nabla_x B_e\|_\infty^2. \end{aligned}$$

We choose  $\eta$ ,  $\eta''$ ,  $E$ ,  $D$  and  $C$  such that

- (1)  $\eta \leq 1$  and  $\eta'' \leq 1$ .
- (2)  $E \geq 2$ .
- (3)  $D \geq \frac{1}{2}(E^2 + \frac{\eta'}{2})$ .
- (4)  $C \geq A$ .

Under the previous conditions, we get

$$\frac{d}{dt}\mathcal{E}(f) \leq -C\|\nabla_v f\|^2 - \frac{E}{4}\|\nabla_x f\|^2 \leq -\frac{E}{4}(\|\nabla_v f\|^2 + \|\nabla_x f\|^2).$$

Using the Poincaré inequality in space and velocity variables, we then obtain

$$\frac{d}{dt}\mathcal{E}(f) \leq -\frac{E}{8}(\|\nabla_v f\|^2 + \|\nabla_x f\|^2) - \frac{E}{8}c_p\|f\|^2 \leq -\frac{E}{8}\frac{c_p}{2C}\mathcal{E}(f).$$

Which completes Proposition 3.5 with  $\kappa = \frac{E}{8}\frac{c_p}{2C} > 0$ .  $\square$

*Proof of Theorem 1.4.* Using Lemma 3.7 and Proposition 3.8, we get  $\kappa > 0$  and  $1 < E < D < C$  such that

$$\begin{aligned} \|f\|_{H^1(dx d\mu)}^2 &\leq 2\mathcal{E}(f) \leq 2Ce^{-\kappa t}\mathcal{E}(f_0) \\ &\leq 4Ce^{-\kappa t}\|f_0\|_{H^1(dx d\mu)}^2. \end{aligned}$$

This completes the proof of Theorem 1.4.  $\square$

#### 4. ENLARGEMENT OF THE FUNCTIONAL SPACE

**4.1. Intermediate results.** In this section, we extend the results of exponential time decay of the semi-group to enlarged spaces (which we will define later), following the recent work of Gualdani, Mischler, Mouhot in [6].

**Notation:** Let  $E$  be a Banach space.

- We denote by  $\mathcal{C}(E)$  the space of unbounded, closed operators with dense domains in  $E$ .
- We denote by  $B(E)$  the space of bounded operators in  $E$ .

- Let  $a \in \mathbb{R}$ . We define the complex half-plane

$$\Delta_a = \{z \in \mathbb{C}, \operatorname{Re} z > a\}.$$

- Let  $L \in \mathcal{C}(E)$ .  $\Sigma(L)$  denote the spectrum of the operator  $L$  and  $\sigma_d(L)$  its discrete spectrum.

- Let  $\xi \in \Sigma_d(L)$ , for  $r$  sufficiently small we define the spectral projection associated with  $\xi$  by

$$\Pi_{L,\xi} := \frac{1}{2i\pi} \int_{|z-\xi|=r} (L-z)^{-1} dz.$$

- Let  $a \in \mathbb{R}$  be such that  $\Delta_a \cap \Sigma(L) = \{\xi_1, \xi_2, \dots, \xi_k\} \subset \Sigma_d(L)$ . We define  $\Pi_{L,a}$  as the operator

$$\Pi_{L,a} = \sum_{j=1}^k \Pi_{L,\xi_j}.$$

We need the following definition on the convolution of semigroup (corresponding to composition at the level of the resolvent operators).

**Definition 4.1** (Convolution of time dependent operators). *Let  $X_1, X_2$  and  $X_3$  be Banach spaces. For two given functions*

$$\mathcal{S}_1 \in L^1(\mathbb{R}^+; B(X_1, X_2)) \text{ and } \mathcal{S}_2 \in L^1(\mathbb{R}^+; B(X_2, X_3)),$$

*we define the convolution  $\mathcal{S}_2 * \mathcal{S}_1 \in L^1(\mathbb{R}^+; B(X_1, X_3))$  by*

$$\forall t \geq 0, (\mathcal{S}_2 * \mathcal{S}_1)(t) := \int_0^t \mathcal{S}_2(s) \mathcal{S}_1(t-s) ds.$$

When  $\mathcal{S} = \mathcal{S}_1 = \mathcal{S}_2$  and  $X_1 = X_2 = X_3$ , we define inductively  $\mathcal{S}^{(*1)} = \mathcal{S}$  and  $\mathcal{S}^{(*\ell)} = \mathcal{S} * \mathcal{S}^{(*(\ell-1)})$  for any  $\ell \geq 2$ .

We say that  $L \in \mathcal{C}(E)$  is hypodissipative if it is dissipative for some norm equivalent to the canonical norm of  $E$  and we say that  $L$  is dissipative for the norm  $\|\cdot\|_E$  on  $E$  if

$$\forall f \in D(L), \forall f^* \in E^* \text{ such that } \langle f, f^* \rangle = \|f\|_E = \|f^*\|_{E^*}, \operatorname{Re} \langle Lf, f^* \rangle \leq 0.$$

We refer to the paper [6, Section 2.3] for an introduction to this subject. Now, we recall the crucial Theorem of enlargement of the functional space.

**Theorem 4.2** (Theorem 2.13 in [6]). *Let  $E$  and  $\mathcal{E}$  be two Banach spaces such that  $E \subset \mathcal{E}$ ,  $L \in \mathcal{C}(E)$  and  $\mathcal{L} \in \mathcal{C}(\mathcal{E})$  such that  $\mathcal{L}|_E = L$ . We suppose that there exist  $\mathcal{A}$  and  $\mathcal{B} \in \mathcal{C}(\mathcal{E})$  such that  $\mathcal{L} = \mathcal{A} + \mathcal{B}$  (with corresponding restrictions  $A, B$  on  $E$ ). Suppose there exists  $a \in \mathbb{R}$  and  $n \in \mathbb{N}$  such that*

(H<sub>1</sub>) **Locating the spectrum of  $L$ :**

$$\Sigma(L) \cap \Delta_a = \{0\} \subset \Sigma_d(L),$$

*and  $L - a$  is dissipative on  $\operatorname{Im}(Id_E - \Pi_{L,0})$*

(H<sub>2</sub>) **Dissipativity of  $\mathcal{B}$  and bounded character of  $\mathcal{A}$ :** *( $\mathcal{B} - a$ ) is hypodissipative on  $\mathcal{E}$  and  $\mathcal{A} \in B(\mathcal{E})$  and  $A \in B(E)$ .*

(H<sub>3</sub>) **Regularization properties of  $T_n(t) = (\mathcal{A}\mathcal{S}_{\mathcal{B}}(t))^{(*n)}$ :**

$$\|T_n(t)\|_{B(\mathcal{E}, E)} \leq C_{a,n} e^{at}.$$

Then for all  $a' > a$ , we have the following estimate:

$$\forall t \geq 0, \quad \|S_{\mathcal{L}}(t) - S_{\mathcal{L}}(t)\Pi_{\mathcal{L},0}\|_{B(\mathcal{E})} \leq C_{a'} e^{a't}.$$

To finish this subsection, we give a lemma providing a practical criterion to prove hypothesis  $(H_3)$  in the previous theorem.

**Lemma 4.3** (Lemma 2.4 in [16]). *Let  $E$  and  $\mathcal{E}$  be two Banach spaces with  $E \subset \mathcal{E}$  dense with continuous embedding, and consider  $L \in \mathcal{C}(E)$  and  $\mathcal{L} \in \mathcal{C}(\mathcal{E})$  with  $\mathcal{L}|_E = L$  and  $a \in \mathbb{R}$ . Let us assume that:*

- a)  $\mathcal{B} - a$  is hypodissipative on  $\mathcal{E}$  and  $B - a$  on  $E$ .
- b)  $\mathcal{A} \in B(\mathcal{E})$  and  $A \in B(E)$ .
- c) There are constants  $b \in \mathbb{R}$  and  $\Theta \geq 0$  such that

$$\|S_{\mathcal{B}}(t)\mathcal{A}\|_{B(\mathcal{E},E)} \leq C e^{bt} t^{-\Theta} \text{ et } \|\mathcal{A}S_{\mathcal{B}}(t)\|_{B(\mathcal{E},E)} \leq C e^{bt} t^{-\Theta}.$$

Then for all  $a' > a$ , there exist some explicit constants  $n \in \mathbb{N}$  and  $C_{a'} \geq 1$ , such that

$$\forall t \geq 0, \quad \|T_n\|_{B(\mathcal{E},E)} \leq C_{a'} e^{a't}.$$

**4.2. Study of the magnetic-Fokker-Planck operator on the spaces  $L^p(m)$  and  $\tilde{W}^{1,p}(m)$ :** This part consists in building the general framework of the problem.

Recall first the equation of Fokker-Planck (2) written in original variable:

$$(25) \quad \begin{aligned} \partial_t F &= -P_0 F, \quad F(0, x, v) = F_0(x, v), \\ \text{where } -P_0 F &= \nabla_v \cdot (\nabla_v F + K F) - v \cdot \nabla_x F, \end{aligned}$$

and where we recall that  $P_0$  was introduced in Section 2 and with

$$K(x, v) = v + v \wedge B_e(x) = \nabla_v \Phi + U, \text{ where } \Phi(v) = \frac{|v|^2}{2}$$

and  $B_e$  is the external magnetic field satisfying Hypothesis 1.1. As mentioned in Section 2, the Maxwellian  $\mu$  is a solution of the system (2). We will need the following modified Poincaré inequality:

$$(26) \quad \begin{aligned} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \left| \nabla_v \left( \frac{F}{\mu} \right) \right|^2 \mu(v) dx dv \\ \geq 2\lambda_p \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \left( F - \int_{\mathbb{R}^3} F(v') dv' \right)^2 (1 + |\nabla_v \Phi|^2) \mu^{-1}(v) dx dv, \end{aligned}$$

where  $\lambda_p > 0$  which depends on the dimension (see [16, Lemma 3.6]). See also [17], [2] and [1].

Now we will define the expanded functional space.

**Definition 4.4.** *Let  $m = m(v) > 0$  on  $\mathbb{R}^3$  be a weight of class  $C^\infty$  and recall that*

- The space  $L^p(m)$  for  $p \in [1, 2]$ , is the Lebesgue space with weight associated with the norm

$$\|F\|_{L^p(m)} := \|Fm\|_{L^p} = \left( \int_{\mathbb{R}^3 \times \mathbb{T}^3} F^p(x, v) m^p(v) dv dx \right)^{\frac{1}{p}}.$$

- We define the technical function  $\Psi_{m,p}$  by

$$\Psi_{m,p} := (p-1) \frac{|\nabla_v m|^2}{m^2} + \frac{\Delta_v m}{m} + \left(1 - \frac{1}{p}\right) \nabla_v \cdot K + K \cdot \frac{\nabla_v m}{m},$$

where  $K(x, v) = v \wedge B_e(x) + v$ .

We will show the decay of the semi-group associated with the problem (2) in the spaces  $L^p(m)$  where  $p \in [1, 2]$ , when  $m$  verifies the following hypothesis:

(W<sub>p</sub>) The weight  $m$  satisfies  $L^2(\mu^{-\frac{1}{2}}) \subset L^p(m)$  with continuous injection and

$$\limsup_{|v| \rightarrow +\infty} \Psi_{m,p} := a_{m,p} < 0.$$

**Remark 4.5.** In the following, we note  $m_0 = \mu^{-1/2}$  the exponential weight. By direct computation,  $L^2(\mu^{-1/2}) \subset L^q(m_0)$  for any  $q \in [1, 2]$  with continuous injection and there exists  $b \in \mathbb{R}$  such that

$$(27) \quad \begin{cases} \sup_{q \in [1, 2], v \in \mathbb{R}^3} \Psi_{m_0, q} \leq b \\ \sup_{v \in \mathbb{R}^3} \left( \frac{\Delta_v m_0}{m_0} - \frac{|\nabla_v m_0|^2}{m_0^2} \right) \leq b. \end{cases}$$

(See Lemma 3.7 in [6] for a proof of the previous property). Under the previous hypothesis, by direct computation we obtain that the semi-group  $\mathcal{S}_{L_0}$  is bounded from  $L^p(m_0)$  to  $L^p(m_0)$ .

We work now in  $L^p(m)$  with a polynomial weight  $m$  satisfying Hypothesis (W<sub>p</sub>).

**Lemma 4.6.** Let  $m = \langle v \rangle^k := (1 + |v|^2)^{k/2}$  and  $p \in [1, 2]$ . Then hypothesis (W<sub>p</sub>) is true when  $k$  satisfies the following estimate:

$$k > 3\left(1 - \frac{1}{p}\right).$$

*Proof.* For the proof, see Lemma 3.7 in [6]. □

4.2.1. *Proof of Theorem 1.5.* From now on, we write  $L_0$  for the operator  $-P_0$ , the Fokker-Planck operator considered on the space  $L^2(m_0)$  defined in (25) (respectively  $\mathcal{L}_0$  for  $-P_0$  the Fokker-Planck operator considered on the space  $\mathcal{E} = L^p(m)$ , with  $m = \langle v \rangle^k$ , where  $k > 3(1 - \frac{1}{p})$  and  $p \in [1, 2]$ ). We will prove Theorem 1.5 by applying Theorem 4.2 to  $\mathcal{L}_0$ . To verify Hypotheses (H<sub>2</sub>) and (H<sub>3</sub>) of Theorem 4.2, we need two lemmas about the dissipativity and regularization properties of  $\mathcal{L}_0$  following [6].

**Definition 4.7.** We split operator  $\mathcal{L}_0$  into two pieces: for  $M, R > 1$ , we define the operator  $\mathcal{B}$  by

$$(28) \quad \mathcal{B} = \mathcal{L}_0 - \mathcal{A} \text{ with } \mathcal{A}f = M\chi_{Rf},$$

where  $\chi_R(v) = \chi(v/R)$ , and  $0 \leq \chi \in C_0^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$  is such that  $\chi(v) = 1$  when  $|v| \leq 1$ . We also denote by  $A$  and  $B$  the restriction of the operators  $\mathcal{A}$  and  $\mathcal{B}$  to the space  $E$ .

**Lemma 4.8** (Dissipativity of  $\mathcal{B}$ ). Under Assumption (W<sub>p</sub>), for all  $0 > a > a_{m,p}$ , we can choose  $R, M > 1$  such that the operator  $\mathcal{B} - a$  satisfies the dissipativity estimate for some  $C > 0$

$$\forall t \geq 0, \quad \|S_{\mathcal{B}}(t)f\|_{L^p(m)} \leq Ce^{at} \|f\|_{L^p(m)}.$$

*Proof.* The proof follows the one given in Lemma 3.8 in [6]. Let  $F$  be smooth, rapidly decaying and positive function  $F$ . Since of  $\Psi_{m,p}$  is independent of the

magnetic field, by integration by parts with respect to  $v$  and using Remark A.2, we have

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|F\|_{L^p(m)}^p &= \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (\mathcal{L}_0 F - M \chi_R(v) F) |F|^{p-2} F m^p(v) dx dv \\ &= -(p-1) \iint_{\mathbb{T}^3 \times \mathbb{R}^3} |\nabla_v F|^2 |F|^{p-2} m^p(v) dx dv \\ &\quad + \iint_{\mathbb{T}^3 \times \mathbb{R}^3} |F|^p \Psi_{m,p} m^p(v) dx dv - \iint_{\mathbb{T}^3 \times \mathbb{R}^3} M \chi_R(v) |F|^p m^p(v) dx dv \\ &\leq \iint_{\mathbb{T}^3 \times \mathbb{R}^3} |F|^p (\Psi_{m,p} - M \chi_R) m^p(v) dx dv. \end{aligned}$$

Let now take  $a > a_{m,p}$ . As  $m$  satisfies the hypothesis  $(W_p)$ , there exist  $M$  and  $R$  two large constants such that

$$\forall v \in \mathbb{R}^3, \quad \Psi_{m,p} - M \chi_R \leq a,$$

and we obtain

$$\frac{1}{p} \frac{d}{dt} \|F\|_{L^p(m)}^p \leq a \int_{\mathbb{T}^3 \times \mathbb{R}^3} |F|^p m^p(v) dx dv.$$

This completes the proof of Lemma 4.8.  $\square$

From now on,  $a$ ,  $M$  and  $R$  are fixed. We note that  $\mathcal{B}^*$  is the dual operator of  $\mathcal{B}$  relative to the pivot space  $L^2(\mathbb{T}^3 \times \mathbb{R}^3)$ , which is defined as follows:

$$\mathcal{B}^* F := \nabla_v \cdot (\nabla_v F - K F) + v \cdot \nabla_x F - M \chi_R F.$$

**Lemma 4.9** (Regularization properties). *There exists  $b \in \mathbb{R}$  and  $C > 0$  such that, for all  $t \geq 0$ ,*

$$\begin{aligned} \forall 1 \leq p \leq q \leq 2, \quad &\|S_{\mathcal{B}}(t) F_0\|_{L^q(m_0)} \leq C e^{bt} t^{-(3d+1)(\frac{1}{p}-\frac{1}{q})} \|F_0\|_{L^p(m_0)}, \\ \forall 2 \leq q' \leq p' \leq +\infty, \quad &\|S_{\mathcal{B}^*}(t) F_0\|_{L^{p'}(m_0)} \leq C e^{bt} t^{-(3d+1)(\frac{1}{p}-\frac{1}{q})} \|F_0\|_{L^{q'}(m_0)}, \end{aligned}$$

where  $p'$  and  $q'$  are the conjugates of  $p$  and  $q$  respectively

*Proof.* We consider  $F(t)$  the solution of the evolution equation

$$\partial_t F(t) = \mathcal{B} F(t), \quad F|_{t=0} = F_0.$$

We introduce the following entropy defined for all  $t \in [0, T]$ , with  $T \ll 1$  and  $r > 1$  to be fixed later:

$$\mathcal{H}(t, h) = B \|h\|_{L^1(m_0)}^2 + t^r \mathcal{G}(t, h),$$

with

$$\mathcal{G}(t, h) = \alpha \|h\|_{L^2(m_0)}^2 + D t \|\nabla_v h\|_{L^2(m_0)}^2 + E t^2 \langle \nabla_x h, \nabla_v h \rangle_{L^2(m_0)} + \beta t^3 \|\nabla_x h\|_{L^2(m_0)}^2,$$

where  $B > \alpha > D, \beta, E < \sqrt{\beta D}$  and  $r$  is an integer that will be determined later. We will omit the dependence of  $F$  on  $t$ . Using the methods and computations of the proof of Proposition 3.8 and adapting the techniques used in [10], we choose the constants  $\alpha, D$  and  $E > 0$  large enough such that there exist a constant  $C_{\mathcal{G}} > 0$

(depending on  $\|B_e\|_{L^\infty(\mathbb{T}^3)}$  and  $\|\nabla_x B_e\|_{L^\infty(\mathbb{T}^3)}$ ) such that

$$\begin{aligned} \frac{d}{dt}\mathcal{G}(t, F) &\leq -C_{\mathcal{G}}(\|\nabla_v F\|_{L^2(m_0)}^2 + t^2\|\nabla_x F\|_{L^2(m_0)}^2) \\ &\quad + \left( \frac{M}{2}\|\Delta_v \chi_R\|_{L^\infty(\mathbb{T}^3)}^2 + \frac{M}{2}\|\chi_R\|_{L^\infty(\mathbb{T}^3)}^2 + M\|\nabla_v \chi_R\|_{L^\infty(\mathbb{T}^3)}^2 \right) \|F\|_{L^2(m_0)}^2 \\ &\leq -C_{\mathcal{G}}(\|\nabla_v F\|_{L^2(m_0)}^2 + t^2\|\nabla_x F\|_{L^2(m_0)}^2) + C_\chi \|F\|_{L^2(m_0)}^2. \end{aligned}$$

Here,  $C_\chi > 0$  is a uniform constant in  $R > 1$  but depends on  $M$ .

$$\begin{aligned} \frac{d}{dt}\mathcal{H}(t, F) &= B \frac{d}{dt}\|F\|_{L^1(m_0)}^2 + r t^{r-1}\mathcal{G}(t, F) + t^r \frac{d}{dt}\mathcal{G}(t, F) \\ &\leq B \frac{d}{dt}\|F\|_{L^1(m_0)}^2 + r t^{r-1}\mathcal{G}(t, F) \\ &\quad - C_{\mathcal{G}} t^r (\|\nabla_v F\|_{L^2(m_0)}^2 + t^2\|\nabla_x F\|_{L^2(m_0)}^2) + C_\chi t^r \|F\|_{L^2(m_0)}^2. \end{aligned}$$

We choose the constants  $\beta$  and  $T > 0$  such that

$$\beta < \frac{C_{\mathcal{G}}}{2r} \text{ and } T \leq \frac{C_{\mathcal{G}}}{2r} \left( \frac{1}{D} + \frac{1}{\beta} \right).$$

We deduce that

$$\begin{aligned} \frac{d}{dt}\mathcal{H}(t, F) &\leq B \frac{d}{dt}\|F\|_{L^1(m_0)}^2 - \frac{C_{\mathcal{G}}}{2} t^r (\|\nabla_v F\|_{L^2(m_0)}^2 + t^2\|\nabla_x F\|_{L^2(m_0)}^2) \\ (29) \quad &\quad + \frac{C_\chi}{2} t^{r-1} \|F\|_{L^2(m_0)}^2. \end{aligned}$$

Now, the Nash inequality [18] implies that there exists  $C_d > 0$  such that

$$\begin{aligned} \iint_{\mathbb{T}^d \times \mathbb{R}^d} |F(x, v)|^2 m_0^2 dx dv &\leq C_d \left( \iint_{\mathbb{T}^d \times \mathbb{R}^d} |\nabla_{x,v}(F m_0)|^2 dx dv \right)^{\frac{d}{d+1}} \\ (30) \quad &\quad \times \left( \iint_{\mathbb{T}^d \times \mathbb{R}^d} |F| m_0 dx dv \right)^{\frac{2}{d+1}}. \end{aligned}$$

We need to have an estimate based on  $\|\nabla_{x,v} F\|_{L^2(m_0)}$ . Firstly,

$$\begin{aligned} \iint |\nabla_v(F m_0)|^2 dx dv &= \iint |\nabla_v F + \frac{v}{2} F|^2 m_0^2 dx dv \\ &\leq 2 \left( \iint |\nabla_v F|^2 m_0^2 dx dv + \iint |F|^2 |v|^2 m_0^2 dx dv \right) \\ (31) \quad &\leq 2 \left( \|\nabla_v F\|_{L^2(m_0)}^2 + \|v F\|_{L^2(m_0)}^2 \right). \end{aligned}$$

On the other hand, we use the fact that  $v m_0^2 = \nabla_v(m_0^2)$  to estimate  $\|v F\|_{L^2(m_0)}$ . We get

$$\begin{aligned} \iint |F|^2 |v|^2 m_0^2 dx dv &= \iint v |F|^2 \cdot v m_0^2 dx dv \\ &= \iint v |F|^2 \cdot \nabla_v(m_0^2) dx dv, \end{aligned}$$

and integrating by parts in  $v$  in the previous estimate, we obtain

$$\begin{aligned} \iint |F|^2 |v|^2 m_0^2 dx dv &\leq - \iint \nabla_v \cdot (v |F|^2) m_0^2 dx dv \\ &= -3 \iint |F|^2 m_0^2 dx dv - 2 \iint v \cdot F \nabla_v F m_0^2 dx dv \\ &\leq -2 \iint v \cdot F \nabla_v F m_0^2 dx dv. \end{aligned}$$

Applying Cauchy-Schwarz inequality, we get

$$\begin{aligned} \iint |F|^2 |v|^2 m_0^2 dx dv &\leq 2 \left( \iint |v|^2 |F|^2 m_0^2 dx dv \right)^{1/2} \times \left( \iint |\nabla_v F|^2 m_0^2 dx dv \right)^{1/2} \\ &\leq 8 \iint |\nabla_v F|^2 m_0^2 dx dv + \frac{1}{2} \iint |v|^2 |F|^2 m_0^2 dx dv. \end{aligned}$$

Therefore

$$(32) \quad \iint |F|^2 |v|^2 m_0^2 dx dv \leq 16 \iint |\nabla_v F|^2 m_0^2 dx dv.$$

Using the previous estimate and inequality (31), we have

$$\iint |\nabla_v (F m_0)|^2 dx dv \leq 34 \iint |\nabla_v F|^2 m_0^2 dx dv.$$

Using the previous inequality and the fact that  $\nabla_x (F m_0) = m_0 \nabla_x F$  (since  $m_0$  does not depend on  $x$ ), there exists  $C'_d > 0$  such that the estimate (30) becomes

$$\begin{aligned} \iint_{\mathbb{T}^d \times \mathbb{R}^d} |F(x, v)|^2 m_0^2 dx dv &\leq C'_d \left( \iint_{\mathbb{T}^d \times \mathbb{R}^d} |\nabla_{x,v} (F)|^2 m_0^2 dx dv \right)^{\frac{d}{d+1}} \\ &\quad \times \left( \iint_{\mathbb{T}^d \times \mathbb{R}^d} |F| m_0 dx dv \right)^{\frac{2}{d+1}}. \end{aligned}$$

Using Young's inequality with  $p = (d+1)$  and  $q = (d+1)/d$ , we get, for all  $\varepsilon > 0$ ,

$$\begin{aligned} \iint_{\mathbb{T}^d \times \mathbb{R}^d} |F(x, v)|^2 m_0^2 dx dv &\leq C'_d t^{-3d/d+1} \left( \iint_{\mathbb{T}^d \times \mathbb{R}^d} |F| m_0 dx dv \right)^{\frac{2}{d+1}} \\ &\quad \times t^{3d/d+1} \left( \iint_{\mathbb{T}^d \times \mathbb{R}^d} |\nabla_{x,v} F|^2 m_0^2 dx dv \right)^{\frac{d}{d+1}} \\ &\leq C_{\varepsilon, d} t^{-3d} \|F\|_{L^1(m_0)}^2 + \varepsilon t^3 \|\nabla_{x,v} F\|_{L^2(m_0)}^2. \end{aligned}$$

Using the previous estimate, we choose  $\varepsilon > 0$  small enough that there is a  $C'' > 0$

$$\frac{d}{dt} \mathcal{H}(t, F) \leq B \frac{d}{dt} \|F\|_{L^1(m_0)}^2 + C'' t^{r-1-3d} \|F\|_{L^1(m_0)}^2.$$

According to Remark 4.5 there exists  $b \in \mathbb{R}$  such that  $\forall p \in [1, 2]$

$$\frac{d}{dt} \|F\|_{L^p(m_0)} \leq b \|F\|_{L^p(m_0)}, \quad \forall t \geq 0,$$

Finally, using the previous estimate when  $p = 1$  and choosing  $r = 3d + 1$ , we deduce that there exists  $B'' > 0$  such

$$\frac{d}{dt} \mathcal{H}(t, F) \leq B'' \|F\|_{L^1(m_0)}^2 \leq \frac{B''}{B} \mathcal{H}(t, F).$$

Thanks to Gronwall's Lemma, there exists  $B''' > 0$  such that

$$\forall t \in [0, T], \quad \mathcal{H}(t, F) \leq B''' \mathcal{H}(0, F_0) \leq C \|F_0\|_{L^1(m_0)}^2.$$

Then,

$$\forall t \in (0, T], \quad \|F\|_{L^2(m_0)}^2 \leq \frac{\alpha}{t^r} \mathcal{H}(t, F) \leq \frac{C}{t^{3d+1}} \|F_0\|_{L^1(m_0)}^2.$$

As a consequence, using the continuity of  $S_{\mathcal{B}}(t)$  on  $L^p(m_0)$  with  $p = 2$ ,

$$\forall t \in (T, +\infty), \quad \|F\|_{L^2(m_0)}^2 = \|S_{\mathcal{B}}(t - T + T)F_0\|_{L^2(m_0)}^2 \leq C e^{(t-T)b} \|S_{\mathcal{B}}(T)F_0\|_{L^2(m_0)}^2,$$

and eventually for all  $t \in (0, +\infty)$

$$\|F\|_{L^2(m_0)}^2 \leq \frac{C}{t^{3d+1}} \|F_0\|_{L^1(m_0)}^2$$

Let us now consider  $p$  and  $q$  satisfying  $1 \leq p \leq q \leq 2$ .  $S_{\mathcal{B}}(t)$  is continuous from  $L^p(m_0)$  into  $L^q(m_0)$  using the Riesz-Thorin Interpolation Theorem. Moreover, if we denote by  $C_{p,q}(t)$  the norm of  $S_{\mathcal{B}}(t) : L^p(m_0) \rightarrow L^q(m_0)$ , we get the following estimate:

$$C_{p,q}(t) \leq C_{2,2}^{2-\frac{2}{p}}(t) C_{1,1}^{\frac{2}{q}-1}(t) C_{1,2}^{\frac{2}{p}-\frac{2}{q}}(t) \leq C \frac{e^{bt}}{t^{(3d+1)(1/p-1/q)}}.$$

This shows the first estimate.

Now we will show the second estimate. According to the first estimate, we have

$$\forall 1 \leq p \leq q \leq 2, \quad \|S_{\mathcal{B}}(t)F_0\|_{L^q(m_0)} \leq C e^{bt} t^{-(3d+1)(\frac{1}{p}-\frac{1}{q})} \|F_0\|_{L^p(m_0)},$$

which means

$$\|S_{m_0 \mathcal{B} m_0^{-1}}(t)h\|_{L^q} \leq C e^{bt} t^{-(3d+1)(\frac{1}{p}-\frac{1}{q})} \|h\|_{L^p},$$

where  $h = m_0 F_0$ . Then by duality, we get

$$\|S_{m_0 \mathcal{B}^* m_0^{-1}}(t)h\|_{L^{p'}} \leq C e^{bt} t^{-(3d+1)(\frac{1}{p}-\frac{1}{q})} \|h\|_{L^{q'}},$$

where  $p'$  and  $q'$  are the conjugates of  $p$  and  $q$  respectively. Which gives the result by reusing the definition of weighted dual spaces

$$\|S_{\mathcal{B}^*}(t)F_0\|_{L^{p'}(m_0)} \leq C e^{bt} t^{-(3d+1)(\frac{1}{p}-\frac{1}{q})} \|F_0\|_{L^{q'}(m_0)}.$$

This completes the proof.  $\square$

**Corollary 4.10.** *Let  $m$  be a weight that satisfies Hypothesis 1.6, then there exists  $\Theta \geq 0$  such that for all  $F_0 \in L^p(m)$  with  $p \in [1, 2]$ , we have the following estimate*

$$\begin{aligned} \forall t \geq 0, \quad \|\mathcal{A}S_{\mathcal{B}}(t)F_0\|_{L^2(m_0)} &\leq C e^{bt} t^{-\Theta} \|F_0\|_{L^p(m)}, \\ \forall t \geq 0, \quad \|S_{\mathcal{B}}(t)\mathcal{A}F_0\|_{L^2(m_0)} &\leq C e^{bt} t^{-\Theta} \|F_0\|_{L^p(m)}. \end{aligned}$$

*Proof.* We first prove the second inequality. Let  $F_0 \in L^p(m)$  with  $m$  a polynomial weight satisfying Hypothesis 1.6. For all  $1 \leq p \leq 2$  and for all  $t \in ]0, 1]$  and  $v \in \mathbb{R}^3$ ,

using Lemma 4.9 with  $q = 2$ , we get

$$\begin{aligned} \|S_{\mathcal{B}}(t)\mathcal{A}F_0\|_{L^2(m_0)} &\leq C e^{bt} t^{-(3d+1)(\frac{1}{p}-\frac{1}{2})} \|\mathcal{A}F_0\|_{L^p(m_0)} \\ &\leq C e^{bt} t^{-(3d+1)(\frac{1}{p}-\frac{1}{2})} \left\| \mathcal{A}F_0 \times \frac{m_0}{m} \right\|_{L^p(m)} \\ &\leq C M e^{bt} t^{-(3d+1)(\frac{1}{p}-\frac{1}{2})} \times \left( \sup_{v \in B(0,R)} \frac{m_0(v)}{m(v)} \right) \|F_0\|_{L^p(m)} \\ &\leq C' e^{bt} t^{-(3d+1)(\frac{1}{p}-\frac{1}{2})} \|F_0\|_{L^p(m)} \leq C' e^{bt} t^{-\Theta} \|F_0\|_{L^p(m)}, \end{aligned}$$

where  $\Theta = (3d+1)(1/p - 1/2) > 0$ .

To show the first estimate, we proceed step by step.

**Step 1:** First, we will show the following estimate:

$$(33) \quad \|S_{\mathcal{B}^*}(t)g\|_{L^{p'}(m)} \leq C e^{bt} t^{-\Theta} \|g\|_{L^2(m_0)}, \quad \forall t \geq 0.$$

Indeed, using the continuous and dense injection  $L^{p'}(m_0) \subset L^{p'}(m)$ , we obtain

$$\|S_{\mathcal{B}^*}(t)g\|_{L^{p'}(m)} \leq \|S_{\mathcal{B}^*}(t)g\|_{L^{p'}(m_0)},$$

then using Lemma 4.9 with  $q' = 2$ , we obtain

$$(34) \quad \|S_{\mathcal{B}^*}(t)g\|_{L^{p'}(m_0)} \leq C e^{bt} t^{-\Theta} \|g\|_{L^2(m_0)}, \quad \forall t \geq 0,$$

where  $\Theta = (3d+1)(1/p - 1/2)$ .

**Step 2:** Of the inequality (34), it follows that for  $g = \mathcal{A}F_0$ , we get

$$\|S_{\mathcal{B}^*} \mathcal{A}F_0\|_{L^{p'}(m)} \leq C e^{bt} t^{-\Theta} \|\mathcal{A}F_0\|_{L^2(m_0)},$$

which means, denoting  $h = mF_0$

$$\|S_{m\mathcal{B}^* m^{-1}} \mathcal{A}h\|_{L^{p'}} \leq C e^{bt} t^{-\Theta} \left\| \mathcal{A}h \times \frac{m_0}{m} \right\|_{L^2} \leq C' e^{bt} t^{-\Theta} \|h\|_{L^2},$$

by a duality argument and noting that  $\mathcal{A}^* = \mathcal{A}$ , we get

$$\|\mathcal{A} S_{m\mathcal{B}^* m^{-1}} h\|_{L^2} \leq C e^{bt} t^{-\Theta} \|h\|_{L^p}.$$

Finally, according to our definition of weighted dual spaces and replacing  $h$  by  $mF_0$ , we obtain

$$(35) \quad \|\mathcal{A} S_{\mathcal{B}}(t) F_0\|_{L^2(m)} \leq C e^{bt} t^{-\Theta} \|F_0\|_{L^p(m)}.$$

To obtain the result, we notice that

$$\|\mathcal{A} S_{\mathcal{B}}(t) F_0\|_{L^2(m_0)} \leq \|\mathcal{A} S_{\mathcal{B}}(t) F_0\|_{L^2(m)},$$

and we combine the previous estimate with the estimate (35), which completes the proof of the first estimate.  $\square$

Now we prove Theorem 1.5.

*Proof of Theorem 1.5.* For  $p \in [1, 2]$ . We consider  $\mathcal{E} = L^p(m)$ ,  $E = L^2(m_0)$ , and denote  $\mathcal{L}_0$  and  $L_0$  the Fokker-planck operator considered respectively on  $\mathcal{E}$  and  $E$  (defined in (25)). We split the operator as  $\mathcal{L}_0 = \mathcal{A} + \mathcal{B}$  as in (28). Let us proceed step by step:

• **Step 1: Verification of condition  $(H_1)$  of Theorem 4.2**

Theorem 1.3 shows us the existence of the semi-group  $S_{L_0}(t)$ , associated with the

Fokker-Planck operator defined in (25) on the space  $L^2(m_0)$  and the constants  $\kappa$  and  $c > 0$ , for which, for all  $F_0 \in L^2(m)$  such that  $\langle F_0 \rangle = 0$ ,

$$(36) \quad \forall t \geq 0, \quad \|F(t)\|_{L^2(m_0)} \leq ce^{-\kappa t} \|F_0\|_{L^2(m_0)}.$$

Which implies the dissipativity of the operator  $L_0 - a$  on  $E$ , for all  $0 > a > -\kappa$ .

• **Step 2: Verification of condition  $(H_2)$  of Theorem 4.2.**

According to Lemma 4.8, the operator  $\mathcal{B} - a$  is dissipative on  $\mathcal{E}$ , for all  $0 > a > a_{m,p}$ , and by definition of the operator  $\mathcal{A}$  and  $A$ , we have  $\mathcal{A} \in B(\mathcal{E})$  and  $A \in B(E)$ .

• **Step 3: Verification of condition  $(H_3)$  of Theorem 4.2.**

According to Corollary 4.10, the operators  $\mathcal{A}S_{\mathcal{B}}$  and  $S_{\mathcal{B}}\mathcal{A}$  satisfy the property  $c)$  of Lemma 4.3. By applying Lemma 4.3,

$$\|S_{\mathcal{B}}(t)\mathcal{A}\|_{B(\mathcal{E},E)} \leq Ce^{bt} t^{-\Theta} \quad \text{and} \quad \|\mathcal{A}S_{\mathcal{B}}(t)\|_{B(\mathcal{E},E)} \leq Ce^{bt} t^{-\Theta}.$$

Then for all  $a' > a$ , there exist constructible constants  $n \in \mathbb{N}$  and  $C_{a'} \geq 1$ , such that

$$\forall t \geq 0, \quad \|T_n(t)\|_{B(\mathcal{E},E)} \leq C_{a'} e^{a't}.$$

• **Step 4: End of the Proof**

All the hypotheses of Theorem 4.2 are satisfied. We deduce that  $\mathcal{L}_0 - a$  is a dissipative operator on  $\mathcal{E}$  for all  $a > \max(a_{m,p}, -\kappa)$ , with the semi-group  $S_{\mathcal{L}_0}(t)$  satisfying estimate (7). □

4.2.2. *Proof of Theorem 1.7.* This part is dedicated to the proof of the exponential time decay estimates of the semi-group associated with the Cauchy problem (2) with an external magnetic field  $B_e$ , with an initial datum in  $\tilde{W}^{1,p}(m)$  defined in (8).

For the proof of Theorem 1.7, we consider the space  $\mathcal{E} = \tilde{W}^{1,p}(m)$  and  $E = H^1(m_0)$ .

**Definition 4.11.** *We split operator  $\mathcal{L}_0$  into two pieces and define for all  $R, M > 0$*

$$(37) \quad \mathcal{B}u = \mathcal{L}_0 u - \mathcal{A}u \quad \text{with} \quad \mathcal{A}u = M\chi_R u,$$

where  $M > 0$ ,  $\chi_R(v) = \chi(v/R)$   $R > 1$ , and  $\chi \in C_0^\infty(\mathbb{R}^3)$  such that  $\chi(v) = 1$   $|v| \leq 1$ . We also denote  $A$  and  $B$  the restriction of operators  $\mathcal{A}$  and  $\mathcal{B}$  on the space  $E$  respectively.

**Lemma 4.12** (Dissipativity of  $\mathcal{B}$ ). *Under Assumptions 1.1 and 1.6, there exists  $M$  and  $R > 0$  such that for all  $0 > a > \max(a_{m,1}^i, a_{m,2}^i)$  (defined in (45)-(47) and (54)-(56)) such that operator  $\mathcal{B} - a$  is dissipative in  $\tilde{W}^{1,p}(m)$  where  $p \in [1, 2]$ . In other words, the semi-group  $\mathcal{S}_{\mathcal{B}}$  satisfies the following estimate:*

$$\forall t \geq 0, \quad \|\mathcal{S}_{\mathcal{B}}(t)F_0\|_{\tilde{W}^{1,p}(m)} \leq e^{at} \|F_0\|_{\tilde{W}^{1,p}(m)}, \quad \forall F_0 \in \tilde{W}^{1,p}(m).$$

*Proof.* Let  $F_0 \in \tilde{W}^{1,p}(m)$ . We consider  $F$  the solution of the evolution equation

$$(38) \quad \partial_t F = \mathcal{B}F, \quad F|_{t=0} = F_0.$$

Recall that the norm on the space  $\tilde{W}^{1,p}(m)$  is given by

$$\|F\|_{\tilde{W}^{1,p}(m)}^p = \|F\|_{L^p(\tilde{m})}^p + \|\nabla_v F\|_{L^p(m)}^p + \|\nabla_x F\|_{L^p(m)}^p,$$

where  $\tilde{m} = m\langle v \rangle$ . Differentiating the previous equality with respect to  $t$ , we get

$$(39) \quad \frac{d}{dt} \frac{1}{p} \|F\|_{\tilde{W}^{1,p}(m)}^p = \frac{d}{dt} \frac{1}{p} \|F\|_{L^p(\tilde{m})}^p + \frac{d}{dt} \frac{1}{p} \|\nabla_v F\|_{L^p(m)}^p + \frac{d}{dt} \frac{1}{p} \|\nabla_x F\|_{L^p(m)}^p.$$

We now estimate each term of the equality (39).

For the first term in (39), we apply Lemma 4.8 and get

$$\frac{1}{p} \frac{d}{dt} \|F\|_{L^p(\tilde{m})}^p \leq \iint_{\mathbb{T}^3 \times \mathbb{R}^3} |F|^p (\Psi_{\tilde{m},p} - M\chi_R) \tilde{m}^p(v) dx dv,$$

Secondly, we differentiate the equation (38) with respect to  $v$ , and then we use the equalities of Lemma 3.6. We get the following equation (recall  $d = 3$ ):

$$(40) \quad \partial_t \nabla_v F = \mathcal{B}(\nabla_v F) + 3\nabla_v F + (B_e \wedge \nabla_v)F - \nabla_x F - M(\nabla_v \cdot \chi_R)F_t.$$

This gives

$$\begin{aligned} \frac{d}{dt} \frac{1}{p} \|\nabla_v F\|_{L^p(m)}^p &= \iint \partial_t \nabla_v F |\nabla_v F|^{p-2} \cdot \nabla_v F m^p dx dv \\ &= \iint \mathcal{B}(\nabla_v F) |\nabla_v F|^{p-2} \cdot \nabla_v F m^p dx dv + 3 \|\nabla_v F\|_{L^p(m)}^p \\ &\quad - \iint \nabla_x F |\nabla_v F|^{p-2} \cdot \nabla_v F m^p dx dv \\ &\quad + \iint (B_e \wedge \nabla_v)F |\nabla_v F|^{p-2} \cdot \nabla_v F m^p dx dv \\ &\quad - M \iint (\nabla_v \chi_R)F |\nabla_v F|^{p-2} \cdot \nabla_v F m^p dx dv. \end{aligned}$$

Then, proceeding exactly as in the proof of Lemma 4.8 and applying Young's inequality, we obtain for all  $\eta_1 > 0$

$$\begin{aligned} &\frac{d}{dt} \frac{1}{p} \|\nabla_v F\|_{L^p(m)}^p \\ &\leq \iint |\nabla_v F|^p (\Psi_{m,p} - M\chi_R) m^p dx dv + 3 \|\nabla_v F\|_{L^p(m)}^p \\ &\quad + \frac{1}{2} \|\nabla_x F\|_{L^p(m)}^p + \frac{1}{2} \|\nabla_v F\|_{L^p(m)}^p + \frac{M}{R} C_{\eta_1} \|\nabla_v \chi\|_{L^\infty(\mathbb{R}^3)} \|F\|_{L^p(m)}^p \\ &\quad + \frac{M}{R} \eta_1 \|\nabla_v \chi\|_{L^\infty(\mathbb{R}^3)} \|\nabla_v F\|_{L^p(m)} + \|B_e\|_{L^\infty(\mathbb{T}^3)} \|\nabla_v F\|_{L^p(m)}^p \\ &\leq \iint \left( |\nabla_v F|^p (\Psi_{m,p} - M\chi_R + 3 + \frac{1}{2} + \frac{M}{R} \|\nabla_v \chi\|_{L^\infty(\mathbb{R}^3)} \eta_1 + \|B_e\|_{L^\infty(\mathbb{T}^3)}) m^p dx dv \right. \\ &\quad \left. + \frac{1}{2} \|\nabla_x F\|_{L^p(m)}^p + \frac{M}{R} C_{\eta_1} \|\nabla_v \chi\|_{L^\infty(\mathbb{R}^3)} \|F\|_{L^p(m)}^p \right). \end{aligned}$$

Finally, we estimate the last term of the equality (39). We treat two cases, and then we use an interpolation argument to complete the proof.

• **Case 1:**  $p = 1$ .

We differentiate the equation (38) with respect to  $x_i$  for all  $i = 1, 2, 3$ , then we use the equalities of Lemma 3.6. We will have the following equation:

$$(41) \quad \partial_i \partial_{x_i} F = \mathcal{B}(\partial_{x_i} F) + (v \wedge \partial_{x_i} B_e) \cdot \nabla_v F.$$

Using the previous equation, we obtain

$$\begin{aligned}
\frac{d}{dt} \|\partial_{x_i} F\|_{L^1(m)} &= \iint \partial_t |\partial_{x_i} F| m \, dx dv \\
&= \iint (\partial_{x_i} \partial_t F) \partial_{x_i} F |\partial_{x_i} F|^{-1} m \, dx dv \\
&= \iint \mathcal{B}(\partial_{x_i} F) \partial_{x_i} F |\partial_{x_i} F| m \, dx dv \\
&+ \iint (v \wedge \partial_{x_i} B_e) \cdot \nabla_v F \partial_{x_i} F |\partial_{x_i} F|^{-1} m \, dx dv.
\end{aligned}$$

Using the computations made in Lemma 4.8 for  $p = 1$ , using Lemma B.1 in the appendix B, and performing an integration by parts with respect to  $v$ , we get

$$\begin{aligned}
&\frac{d}{dt} \|\partial_{x_i} F\|_{L^1(m)} \\
&\leq \iint (\Psi_{m,1} - M\chi_R) |\partial_{x_i} F| m \, dx dv - \underbrace{\iint (v \wedge \partial_{x_i} B_e) F \partial_{x_i} F |\partial_{x_i} F|^{-1} \nabla_v m \, dx dv}_{=0},
\end{aligned}$$

where, we used the fact that  $(v \wedge \partial_{x_i} B_e) \cdot \nabla_v m = 0$ . Then, defining the norm

$$\|\nabla_x F\|_{L^p(m)} := \sum_{i=1}^3 \|\partial_{x_i} F\|_{L^p(m)},$$

and using the previous definition, we have

$$\frac{d}{dt} \|\nabla_x F\|_{L^1(m)} \leq \iint (\Psi_{m,1} - M\chi_R) |\nabla_x F| m \, dx dv.$$

Collecting all the estimates, we obtain

$$\begin{aligned}
&\frac{d}{dt} \|F\|_{\tilde{W}^{1,1}(m)} \\
&\leq \iint \left( \Psi_{\tilde{m},1} - M\chi_R + \frac{M}{R} C_{\eta_1} \|\nabla_v \chi\|_{L^\infty(\mathbb{R}^3)} \right) |F| \tilde{m} \, dx dv \\
&+ \iint \left( \Psi_{m,1} - M\chi_R + 3 + \frac{1}{2} + \frac{M}{R} \eta_1 \|\nabla_v \chi\|_{L^\infty(\mathbb{R}^3)} + \|B_e\|_{L^\infty(\mathbb{T}^3)} \right) |\nabla_v F| m \, dx dv \\
&+ \iint \left( \Psi_{m,1} - M\chi_R + \frac{1}{2} \right) |\nabla_x F| m \, dx dv.
\end{aligned}$$

We define then (for  $M$  and  $R$  to be fixed below).

$$(42) \quad \Psi_{m,1}^1 := \Psi_{\tilde{m},1} - M\chi_R + \frac{M}{R} C_{\eta_1} \|\nabla_v \chi\|_{L^\infty(\mathbb{R}^3)},$$

$$(43) \quad \Psi_{m,1}^2 := \Psi_{m,1} - M\chi_R + 3 + \frac{1}{2} + \frac{M}{R} \eta_1 \|\nabla_v \chi\|_{L^\infty(\mathbb{R}^3)} + \|B_e\|_{L^\infty(\mathbb{T}^3)},$$

$$(44) \quad \Psi_{m,1}^3 := \Psi_{m,1} - M\chi_R + \frac{1}{2}.$$

(Recall that  $\limsup_{|v| \rightarrow +\infty} \Psi_{m,1} = -k$ ). We denote then

$$(45) \quad a_{m,1}^1 = -k - 1,$$

$$(46) \quad a_{m,1}^2 = -k + \frac{7}{2} + \|B_e\|_{L^\infty(\mathbb{T}^3)},$$

$$(47) \quad a_{m,1}^3 = -k + \frac{1}{2}.$$

We now assume that  $k$  satisfies

$$(48) \quad k > \frac{7}{2} + \|B_e\|_{L^\infty(\mathbb{T}^3)}.$$

Hypothesis (48) implies that  $a_{m,1}^i < 0$ , for all  $i = 1, 2, 3$ . Consequently, for  $\eta_1$  sufficiently small, we may then find  $M$  and  $R > 0$  large enough so that, for all  $0 > a > \max(a_{m,1}^1, a_{m,1}^2, a_{m,1}^3)$ , we have

$$(49) \quad \frac{d}{dt} \|F(t)\|_{\tilde{W}^{1,1}(m)} \leq a \|F(t)\|_{\tilde{W}^{1,1}(m)}.$$

Hence the operator  $\mathcal{B} - a$  is dissipative on  $\tilde{W}^{1,1}(m)$ .

• **Case 2:**  $p = 2$ .

Again, we differentiate the equation (38) with respect to  $x$ , and we use the equalities of Lemma 3.6 to obtain the following equation:

$$(50) \quad \partial_t \nabla_x F = \mathcal{B}(\nabla_x F) + (v \wedge \nabla_x B_e) \cdot \nabla_v F.$$

Using the calculations made in Lemma 4.8 and the previous equation, we obtain

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|\nabla_x F\|_{L^2(m)}^2 &= - \iint |\nabla_v \nabla_x F|^2 m^2 dx dv \\ &\quad + \iint (\Psi_{m,2} - M\chi_R) |\nabla_x F|^2 m^2 dx dv \\ &\quad + \iint (v \wedge \nabla_x B_e) \cdot \nabla_v F \nabla_x F m^2 dx dv. \end{aligned}$$

Then, by integration by parts with respect to  $v$ , we get

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|\nabla_x F\|_{L^2(m)}^2 &\leq - \iint |\nabla_v \nabla_x F|^2 m^2 dx dv \\ &\quad + \iint (\Psi_{m,2} - M\chi_R) |\nabla_x F|^2 m^2 dx dv \\ &\quad + \iint |v \wedge \nabla_x B_e| |F| |\nabla_x \nabla_v F| m^2 dx dv \end{aligned}$$

According to the Cauchy-Schwarz inequality, for every  $\varepsilon > 0$ , there is a  $C_\varepsilon > 0$  such that

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|\nabla_x F\|_{L^2(m)}^2 &\leq - \iint |\nabla_v \nabla_x F|^2 m^2 dx dv + \iint (\Psi_{m,2} - M\chi_R) |\nabla_x F|^2 m^2 dx dv \\ &\quad + \varepsilon \iint |\nabla_v \nabla_x F|^2 m^2 dx dv + C_\varepsilon \iint |v \wedge B_e|^2 |F|^2 m^2 dx dv. \end{aligned}$$

We choose  $\varepsilon = \frac{1}{4}$ , and we finally get

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|\nabla_x F\|_{L^2(m)}^2 &\leq \iint (\Psi_{m,2} - M\chi_R) |\nabla_x F|^2 m^2 dx dv \\ &\quad + \frac{1}{2} \|\nabla_x B_e\|_{L^\infty(\mathbb{T}^3)} \|F\|_{L^2(\tilde{m})}^2 + \frac{1}{2} \|\nabla_x F\|_{L^2(m)}^2. \end{aligned}$$

Collecting all the estimates, we thus obtain

$$\begin{aligned} &\frac{d}{dt} \frac{1}{2} \|F\|_{\tilde{W}^{1,2}(m)}^2 \\ &\leq \iint \left( \Psi_{\tilde{m},2} - M\chi_R + \frac{M}{R} C_{\eta_1} \|\nabla_v \chi\|_{L^\infty(\mathbb{R}^3)} + \frac{1}{2} \|\nabla_x B_e\|_{L^\infty(\mathbb{T}^3)} \right) |F|^2 \tilde{m}^2 dx dv \\ &\quad + \iint \left( \Psi_{m,2} - M\chi_R + 3 + \frac{1}{2} + \frac{M}{R} \eta_1 \|\nabla_v \chi\|_{L^\infty(\mathbb{R}^3)} + \|B_e\|_{L^\infty(\mathbb{T}^3)} \right) |\nabla_v F|^2 m^2 dx dv \\ &\quad + \iint \left( \Psi_{m,2} - M\chi_R + \frac{1}{2} + \frac{1}{2} \right) |\nabla_x F|^2 m^2 dx dv \end{aligned}$$

Again, we define then, for  $M$  and  $R$  to be fixed in the next paragraph

$$(51) \quad \Psi_{m,2}^1 = \Psi_{\tilde{m},2} - M\chi_R + \frac{M}{R} C_{\eta_1} \|\nabla_v \chi\|_{L^\infty(\mathbb{R}^3)} + \frac{1}{2} \|\nabla_x B_e\|_{L^\infty(\mathbb{T}^3)},$$

$$(52) \quad \Psi_{m,2}^2 = \Psi_{m,2} - M\chi_R + 3 + \frac{1}{2} + \frac{M}{R} \eta_1 \|\nabla_v \chi\|_{L^\infty(\mathbb{R}^3)} + \|B_e\|_{L^\infty(\mathbb{T}^3)},$$

$$(53) \quad \Psi_{m,2}^3 = \Psi_{m,2} - M\chi_R + 1.$$

Again, we denote

$$(54) \quad a_{m,2}^1 = \frac{3}{2} - k - 1 + \frac{1}{2} \|\nabla_x B_e\|_{L^\infty(\mathbb{T}^3)}$$

$$(55) \quad a_{m,2}^2 = \frac{3}{2} - k + \frac{7}{2} + \|B_e\|_{L^\infty(\mathbb{T}^3)}$$

$$(56) \quad a_{m,2}^3 = \frac{3}{2} - k + 1,$$

Assuming  $k$  satisfies

$$(57) \quad k > 5 + \max \left( \|B_e\|_{L^\infty(\mathbb{T}^3)}, \frac{1}{2} \|\nabla_x B_e\|_{L^\infty(\mathbb{T}^3)} \right),$$

we obtain that  $a_{m,2}^i < 0$  for all  $i = 1, 2, 3$ . Consequently, we may find  $M$  and  $R > 0$  large enough so that for all  $0 > a > \max(a_{m,2}^1, a_{m,2}^2, a_{m,2}^3)$

$$\frac{d}{dt} \frac{1}{2} \|F(t)\|_{\tilde{W}^{1,2}(m)}^p \leq a \|F(t)\|_{\tilde{W}^{1,2}(m)}^2.$$

Hence the operator  $\mathcal{B} - a$  is dissipative on  $\tilde{W}^{1,2}(m)$  for such  $M$  and  $R$ .

**For the general case  $1 \leq p \leq 2$ :** The cases 1 and 2 show us that the operator  $S_{\mathcal{B}}(t)$  is continuous on  $\tilde{W}^{1,1}(m)$  (on  $\tilde{W}^{1,2}(m)$ ) with the operator  $\mathcal{B}$  is given by

$$\mathcal{B} = \mathcal{L}_0 - M\chi_R,$$

where  $M$  and  $R > 0$  agree with the conditions given in case 1 and case 2. Applying the Riesz-Thorin interpolation Theorem and using Hypothesis 1.6, we obtain that the operator  $S_{\mathcal{B}}(t)$  is continuous on  $\tilde{W}^{1,p}(m)$  for all  $1 \leq p \leq 2$ , with the following dissipative estimate:

$$\forall 0 > a > \max(a_{m,1}^i, a_{m,2}^i, i = 1, 2, 3), \quad \|S_{\mathcal{B}}(t) F_0\|_{\tilde{W}^{1,p}(m)} \leq C e^{at} \|F_0\|_{\tilde{W}^{1,p}(m)}.$$

This completes the proof.  $\square$

From now on,  $M$  and  $R$  are fixed.

**Lemma 4.13** (Property of regularization). *There exist  $b$  and  $C > 0$  such that, for all  $p, q$  with  $1 \leq p \leq q \leq 2$ , we have*

$$(58) \quad \forall t \geq 0, \quad \|S_{\mathcal{B}}(t)F_0\|_{\tilde{W}^{1,q}(m_0)} \leq C e^{bt} t^{-(3d+4)(\frac{1}{p}-\frac{1}{q})} \|F_0\|_{\tilde{W}^{1,p}(m_0)}.$$

$$(59) \quad \forall t \geq 0, \quad \|S_{\mathcal{B}^*}(t)F_0\|_{\tilde{W}^{-1,p'}(m_0)} \leq C e^{bt} t^{-(3d+4)(\frac{1}{p}-\frac{1}{q})} \|F_0\|_{\tilde{W}^{-1,q'}(m_0)}.$$

Here  $2 \leq q' \leq p' \leq +\infty$  are the conjugates of  $p$  and  $q$  respectively.

*Proof.* Let  $F$  be the solution of the evolution equation

$$\partial_t F = \mathcal{B}F, \quad F|_{t=0} = F_0.$$

In to the proof of Lemma 4.9, the following relative entropy has been introduced

$$\mathcal{H}(t, h) = B \|h\|_{L^1(m_0)}^2 + t^r \mathcal{G}(t, h),$$

with  $\mathcal{G}$  defined by

$$\mathcal{G}(t, h) = C \|h\|_{L^2(m_0)}^2 + D t \|\nabla_v h\|_{L^2(m_0)}^2 + E t^2 \langle \nabla_x h, \nabla_v h \rangle + a t^3 \|\nabla_x h\|_{L^2(m_0)}^2.$$

We have shown, for constants  $\alpha, D, E$  and  $\beta > 0$  well chosen, that there exist  $C > 0$  and  $r = 3d + 1$  such that

$$\forall t \geq 0, \quad \mathcal{H}(t, F) \leq B''' \mathcal{H}(0, F_0) \leq C \|F_0\|_{L^1(m_0)}^2.$$

Using the previous estimate and the definition of  $\mathcal{H}$  and  $\mathcal{G}$ , we get

$$\begin{aligned} \|S_{\mathcal{B}}(t)F_0\|_{L^2(m_0)}^2 &\leq \frac{\alpha}{t^{3d+1}} \mathcal{H}(t, F) \leq \frac{C'}{t^{3d+1}} e^{bt} \|F_0\|_{L^1(m_0)}^2, \\ \|\nabla_v S_{\mathcal{B}}(t)F_0\|_{L^2(m_0)}^2 &\leq \frac{D}{t^{3d+2}} \mathcal{H}(t, F) \leq \frac{C''}{t^{3d+2}} e^{bt} \|F_0\|_{L^1(m_0)}^2, \\ \|\nabla_x S_{\mathcal{B}}(t)F_0\|_{L^2(m_0)}^2 &\leq \frac{\beta}{t^{3d+4}} \mathcal{H}(t, F) \leq \frac{C'''}{t^{3d+4}} e^{bt} \|F_0\|_{L^1(m_0)}^2. \end{aligned}$$

Therefore,

$$\forall t \in [0, +\infty), \quad \|S_{\mathcal{B}}(t)F_0\|_{H^1(m_0)}^2 \leq \frac{\tilde{C}}{t^{3d+4}} e^{bt} \|F_0\|_{\tilde{W}^{1,1}(m_0)}^2.$$

Finally, to complete the proof, we use the Riesz-Thorin Interpolation Theorem in the real case on the operator  $S_{\mathcal{B}}(t)$ . We obtain the continuity of  $S_{\mathcal{B}}(t)$  from  $\tilde{W}^{1,p}(m_0)$  to  $\tilde{W}^{1,q}(m_0)$ , with  $S_{\mathcal{B}}$  satisfying the estimate (58).

The estimate (59) follows from (58) by duality.  $\square$

**Corollary 4.14.** *Let  $m$  be a weight that satisfies Hypothesis 1.6. Then there exists  $\Theta \geq 0$  such that for all  $F_0 \in \tilde{W}^{1,p}(m)$  with  $p \in [1, 2]$*

$$\begin{aligned} \forall t \geq 0, \quad \|\mathcal{A}S_{\mathcal{B}}(t)F_0\|_{H^1(m_0)} &\leq C e^{bt} t^{-\Theta} \|F_0\|_{\tilde{W}^{1,p}(m)}, \\ \forall t \geq 0, \quad \|S_{\mathcal{B}}\mathcal{A}(t)F_0\|_{H^1(m_0)} &\leq C e^{bt} t^{-\Theta} \|F_0\|_{\tilde{W}^{1,p}(m)}. \end{aligned}$$

*Proof.* The proof is similar to that of Corollary 4.10.  $\square$

*Proof of Theorem 1.7.* The estimate (10) is an immediate consequence of Theorem 4.2 together with Theorem 1.4, Lemma 4.12, Lemma 4.13 and Corollary 4.14.  $\square$

APPENDIX A. PROPERTY OF THE OPERATOR  $(v \wedge B_e) \cdot \nabla_v$ 

In the following Lemma we show that operator  $(v \wedge B_e) \cdot \nabla_v$  and  $v \cdot \nabla_x$  are formally skew-adjoint operators in the space  $L^2(dx d\mu)$ .

**Lemma A.1.** *Let  $B_e$  be the external magnetic field, then, with adjoints in the space  $L^2(dx d\mu)$ ,*

$$((v \wedge B_e) \cdot \nabla_v)^* = -(v \wedge B_e) \cdot \nabla_v,$$

and

$$(v \cdot \nabla_x)^* = -v \cdot \nabla_x.$$

*Proof.* Let  $f$  and  $g \in C_0^\infty(\mathbb{R}^3 \times \mathbb{T}^3)$ . We have

$$\begin{aligned} \langle ((v \wedge B_e) \cdot \nabla_v)^* f, g \rangle &= \langle f, ((v \wedge B_e) \cdot \nabla_v) g \rangle \\ &= \iint f ((v \wedge B_e) \cdot \nabla_v) g dx d\mu. \end{aligned}$$

Using the fact

$$(v \wedge B_e) \cdot \nabla_v f = \nabla_v \cdot (v \wedge B_e) f,$$

we obtain

$$\begin{aligned} \langle ((v \wedge B_e) \cdot \nabla_v)^* f, g \rangle &= \iint f \nabla_v \cdot (v \wedge B_e) g dx d\mu \\ &= - \iint f (-\nabla_v + v) \cdot (v \wedge B_e) g dx d\mu, \end{aligned}$$

since  $(v \wedge B_e) \cdot v = 0$ . By integration by parts, we have then

$$\begin{aligned} \langle ((v \wedge B_e) \cdot \nabla_v)^* f, g \rangle &= - \iint \nabla_v f \cdot (v \wedge B_e) g dx d\mu \\ &= - \iint g (v \wedge B_e) \cdot \nabla_v f dx d\mu \\ &= - \langle ((v \wedge B_e) \cdot \nabla_v) f, g \rangle. \end{aligned}$$

For the second equality, we obtain

$$\langle (v \cdot \nabla_x)^* f, g \rangle = \langle f, v \cdot \nabla_x g \rangle = \iint f (v \cdot \nabla_x g) dx d\mu,$$

by integration by parts with respect to  $x$ . Since  $\mu$  is independent of  $x$ , we have then

$$\langle (v \cdot \nabla_x)^* f, g \rangle = - \iint (v \cdot \nabla_x f) g dx d\mu = - \langle v \cdot \nabla_x f, g \rangle.$$

This completes the proof.  $\square$

**Remark A.2.** *We can generalize the results of the preceding Lemma for all function  $m$  which are radial in  $v$  and independant of  $x$ . We obtain that  $v \cdot \nabla_x$  and  $(v \wedge B_e) \cdot \nabla_v$  are formally skew-adjoint operators in the space  $L^2(m)$ .*

## APPENDIX B. NON POSITIVITY OF A CERTAIN INTEGRAL

The following well-know lemma is used in the proof of the dissipativity of the operator  $\mathcal{B} - a$  in the spaces  $L^p(m)$  and  $\tilde{W}^{1,p}(m)$  in Section 4. This lemma is a special case of the general study done in the article [4].

**Lemma B.1.** *Let  $g$  be a smooth function and let  $p \geq 1$ . Then the following integral is well-posed and satisfy the following estimate*

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} (\Delta_v g) |g|^{p-2} g \, dx dv \leq 0.$$

*Proof.* Formal integration by parts with respect to  $v$  justifies the property for all  $p > 1$ . For  $p = 1$ , we regularize and use convexity of the function  $\Psi : s \rightarrow |s|$ .  $\square$

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