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## Conjugate prior in the Mallows model with Spearman distance

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## Introduction

► **Ranking and comparing items:** crucial for collecting information about preferences in many areas: e.g. marketing, business, politics, genetics.

► **Mallows model:** powerful model for rankings

► **Question:** How to include **expert opinions** into the analysis? How to elicit a meaningful **prior on the consensus** ranking of a population?

## 1. The Mallows model with Spearman distance

The **Mallows model** (Mallows 1957) is a class of non-uniform distributions for  $\mathbf{R} \in \mathcal{P}_n$ , the set of  $n$ -dimensional permutations, of the form

$$P(\mathbf{R}|\theta, \boldsymbol{\rho}) = \frac{e^{-\theta d(\mathbf{R}, \boldsymbol{\rho})}}{Z(\theta)}$$

►  $\boldsymbol{\rho} \in \mathcal{P}_n$ : consensus ranking, location parameter

►  $\theta \geq 0$ : precision parameter

►  $d(\cdot, \cdot)$ : right-invariant (Diaconis 1988) distance

►  $Z(\theta) = \sum_{\mathbf{r} \in \mathcal{P}_n} e^{-\theta d(\mathbf{r}, \mathbf{1}_n)}$ : partition function (independent of  $\boldsymbol{\rho}$  because of right-invariance of  $d(\cdot, \cdot)$ )

**Spearman distance:** the squared  $l_2$  norm on  $\mathcal{P}_n$

$$d_S(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \sum_{i=1}^n (\rho_i - \sigma_i)^2, \quad \boldsymbol{\rho}, \boldsymbol{\sigma} \in \mathcal{P}_n$$

Consider  $n$  items, ranked by  $N$  assessors.

Denote by  $\mathbf{R}_j = (R_{1j}, R_{2j}, \dots, R_{nj}) \in \mathcal{P}_n$ , the ranking of user  $j$ ,  $j = 1, \dots, N$ .

Under the Mallows model with Spearman distance (MMS), given  $\theta$ , the likelihood can be written as

$$P(\mathbf{R}_1, \dots, \mathbf{R}_N | \boldsymbol{\rho}) \propto \exp\left(2\theta N \sum_{i=1}^n \rho_i \bar{R}_i\right)$$

where  $\bar{R}_i = \frac{1}{N} \sum_{j=1}^N R_{ij}$ ,  $i = 1, \dots, n$ , is the sample average of the  $i$ -th rank.

## 2. Sufficient statistics and mle

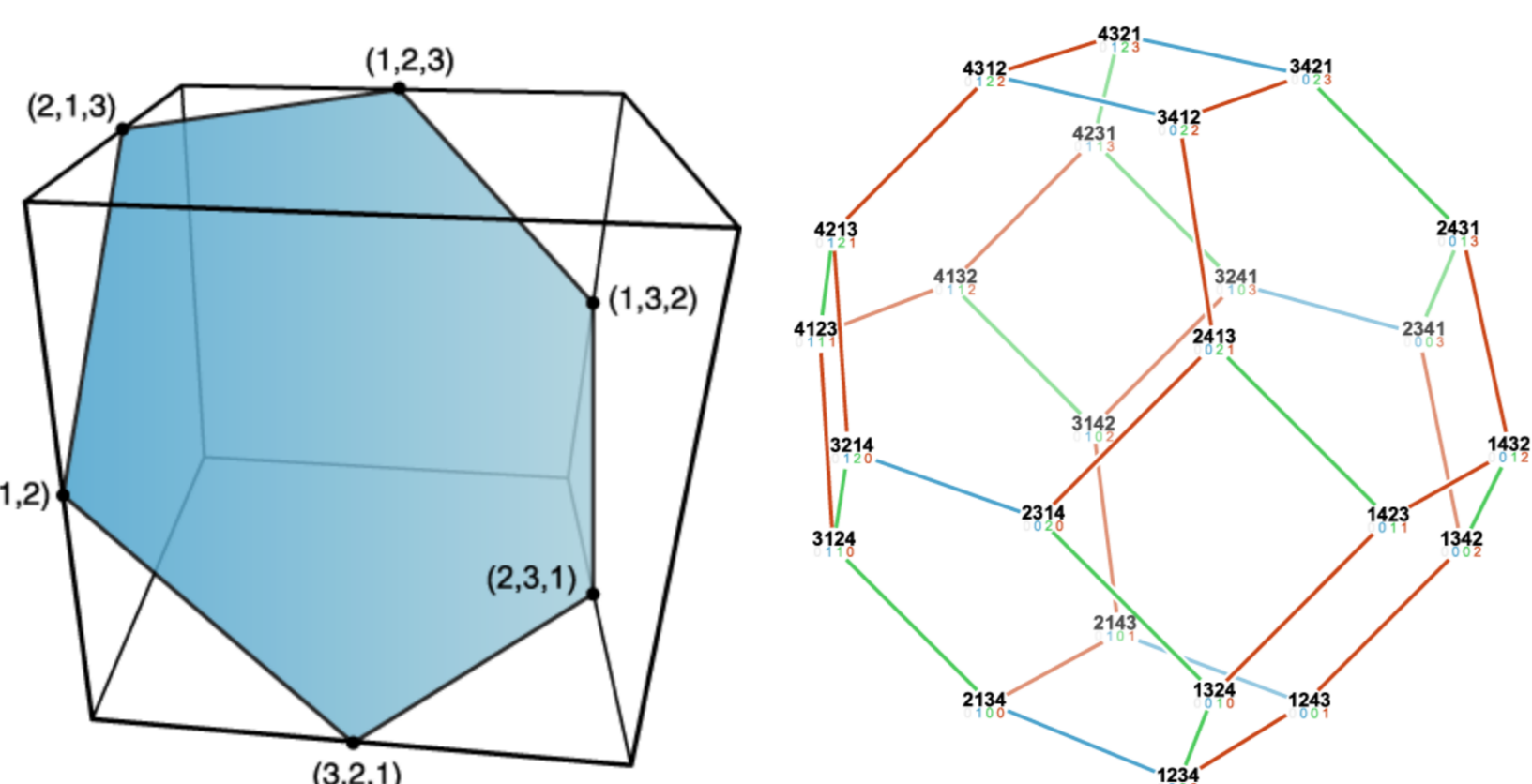
The sufficient statistic for  $\boldsymbol{\rho} = (\rho_1, \dots, \rho_n)$ , when  $\theta$  is known, is  $\bar{\mathbf{R}} = (\bar{R}_1, \dots, \bar{R}_n)$ .

**Proposition 1.** Let  $\mathbf{R}_1, \dots, \mathbf{R}_N | \boldsymbol{\rho}, \theta \sim \text{Mall}(\theta, \boldsymbol{\rho})$ , and define the vector of sample ranks  $\bar{\mathbf{R}}$  as above. Assume  $\bar{R}_i \neq \bar{R}_j$ , for each  $i \neq j$ , and denote by  $\mathbf{Y}(\bar{\mathbf{R}}) = [Y_1(\bar{\mathbf{R}}), \dots, Y_n(\bar{\mathbf{R}})] \in \mathcal{P}_n$  the rank vector of  $\bar{\mathbf{R}}$ , i.e.  $Y_i(\bar{\mathbf{R}}) = Y_i = \sum_{h=1}^n \mathbb{1}(\bar{R}_h \leq \bar{R}_i)$ ,  $i = 1, \dots, n$ . Then the unique mle of  $\boldsymbol{\rho}$  is

$$\boldsymbol{\rho}_{mle} = \operatorname{argmax}_{\boldsymbol{\rho} \in \mathcal{P}_n} \sum_{i=1}^n \rho_i \bar{R}_i = \mathbf{Y}(\bar{\mathbf{R}}).$$

Notice that, in general,  $\bar{\mathbf{R}} \notin \mathcal{P}_n$ . However  $\bar{\mathbf{R}}$  lives in the permutohedron of order  $n$ .

**Definition 1.** The **permutohedron** of order  $n$ ,  $\mathbb{P}\mathcal{P}_n$ , is an  $(n-1)$ -dimensional polytope embedded in an  $n$ -dimensional space, the vertices of which are formed by permuting the coordinates of the vector  $(1, 2, 3, \dots, n)$ . Equivalently, it is the convex hull of the points  $\boldsymbol{\rho} \in \mathcal{P}_n$ , the set of  $n$ -dim permutations.



**Figure 1:** Left: The permutohedron of order 3 is a regular hexagon, filling a cross-section of a  $2 \times 2 \times 2$  cube; Right: The permutohedron of order 4 is a truncated octahedron.

## 3. The Bayesian Mallows

Vitelli et al. (2018) proposed a framework for performing **Bayesian inference on the Mallows** model.

They assume that  $\boldsymbol{\rho}$  is a priori uniformly distributed,  $\boldsymbol{\rho} \sim \text{Unif}(\mathcal{P}_n)$ .

**Contribution:** the objective prior in the sense of Villa & Walker (2015) is the uniform prior density on the space of permutations  $\boldsymbol{\rho} \sim \text{Unif}(\mathcal{P}_n)$ .

Can we go further?

How to put an informative prior on  $\boldsymbol{\rho}$ ?

A:  $\theta$  given: conjugate prior for  $\boldsymbol{\rho}$

B:  $\theta$  not given: conjugate conditioned on  $\theta$  + clever prior on  $\theta \rightarrow$  MH sampling scheme to approximate the posterior

### 3.A Conjugate prior for $\boldsymbol{\rho}$ (given $\theta$ )

**Proposition 2.** Keeping  $\theta$  fixed, the conjugate prior for  $\boldsymbol{\rho} \in \mathcal{P}_n$  is

$$\pi(\boldsymbol{\rho} | \boldsymbol{\rho}_0, \theta N_0) = \frac{\exp\left[-\theta N_0 \sum_{i=1}^n (\rho_i - \rho_{0i})^2\right]}{Z^*(\theta N_0, \boldsymbol{\rho}_0)},$$

where  $\boldsymbol{\rho}_0 \in \mathbb{P}\mathcal{P}_n$ , and  $N_0 \in \mathbb{N}$ . We call  $\pi(\cdot | \boldsymbol{\rho}_0, \theta N_0)$  the extended Mallows density: it is a Mallows model where the consensus  $\boldsymbol{\rho}_0$  belongs to  $\mathbb{P}\mathcal{P}_n$ .

Posterior consensus: weighted average of prior parameter and observed mean (recall Diaconis et al. 1979):

$$\boldsymbol{\rho}_N = \frac{N}{N_0 + N} \bar{\mathbf{R}} + \frac{N_0}{N_0 + N} \boldsymbol{\rho}_0 \in \mathbb{P}\mathcal{P}_n$$

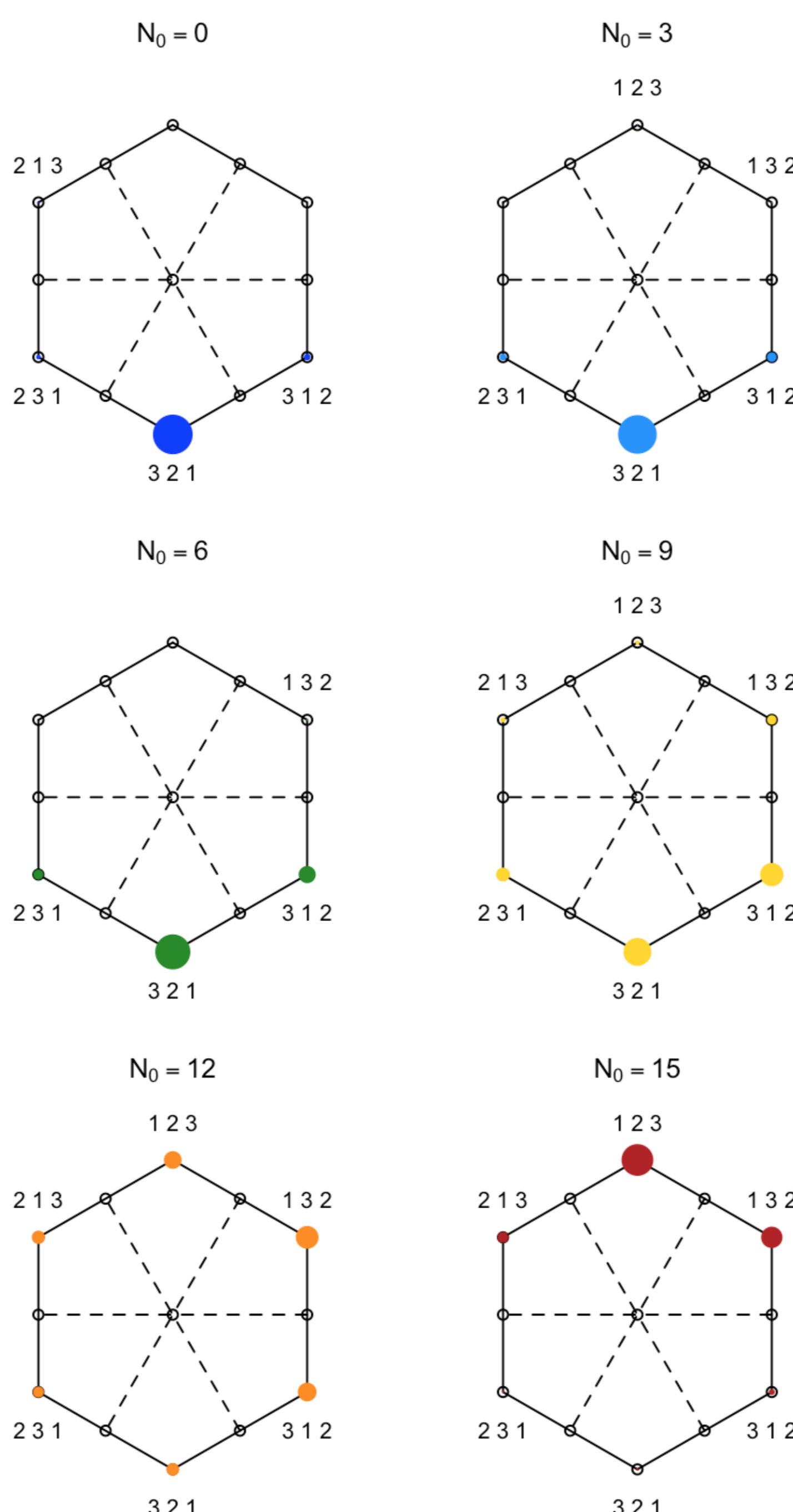
$$\theta_N = \theta(N_0 + N) \in [0, \infty)$$

Then  $N_0$  can be interpreted as an equivalent sample size (recall the Gaussian).

**Remark 1.** When  $N_0 = 0$ , the conjugate prior reduces to the uniform, for all  $\boldsymbol{\rho}_0$ . When  $\boldsymbol{\rho}_0 = (\frac{n+1}{2}, \dots, \frac{n+1}{2})$  the conjugate prior reduces to the uniform, for all  $\theta_0$ .

### 3.A.bis Toy example 1

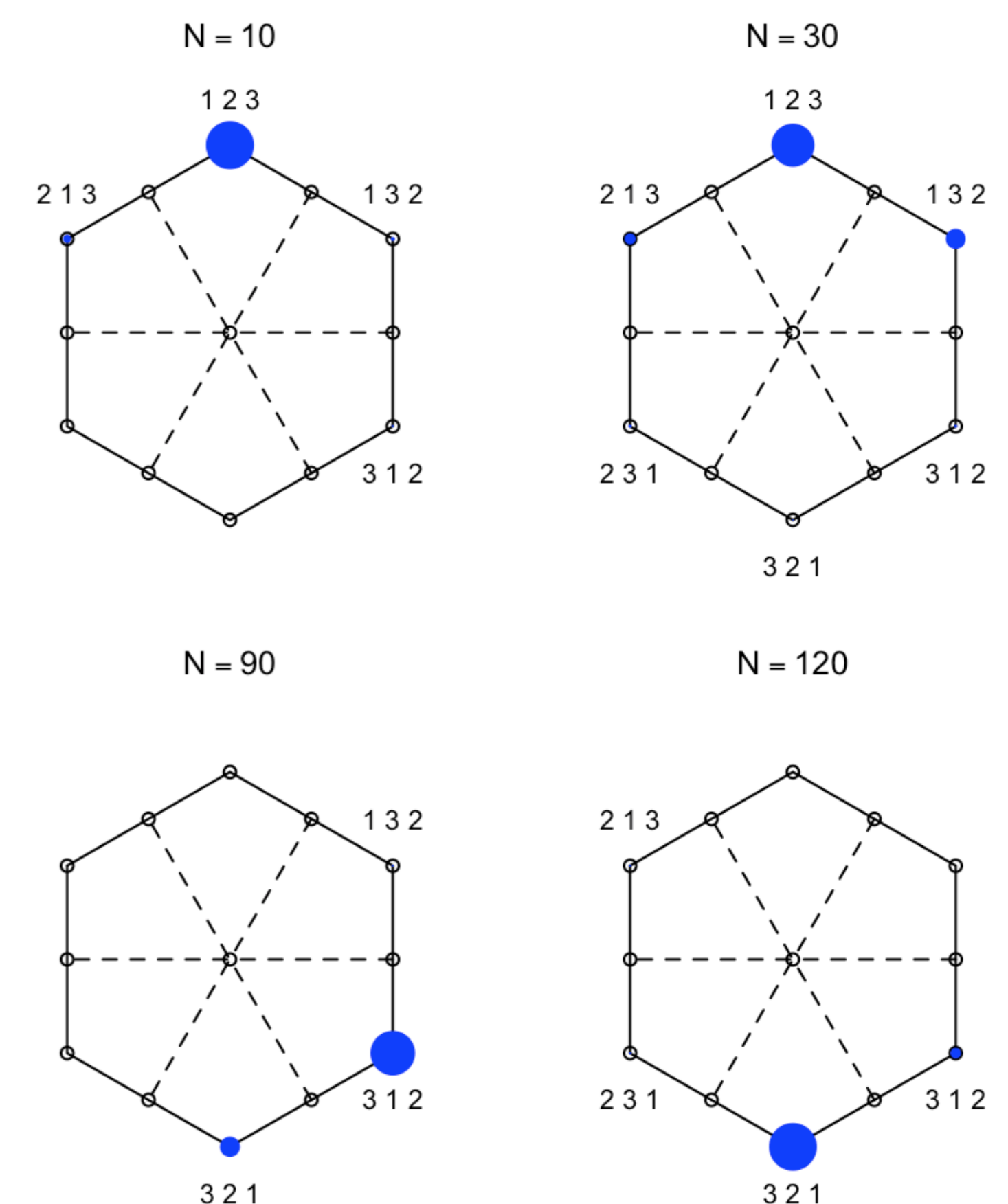
Sample  $N = 30$  rankings from the MMS with  $\boldsymbol{\rho} = (3, 1, 2)$  and  $\theta = 0.18$ . Conjugate prior with  $\boldsymbol{\rho}_0 = (1, 2, 3)$ , and varying  $\theta_0 = \theta N_0$ .



**Figure 2:** The balls have radius proportional to the frequency of rankings in the posterior sample.

### 3.A.bis Toy example 2

Same model. Increasing sample size ( $N$ ). Conjugate prior with  $\boldsymbol{\rho}_0 = (1, 2, 3)$  and  $\theta N_0 = 3$  (i.e.  $N_0 \approx 16$ ).



**Figure 3:** The balls have radius proportional to the frequency of rankings in the posterior sample.

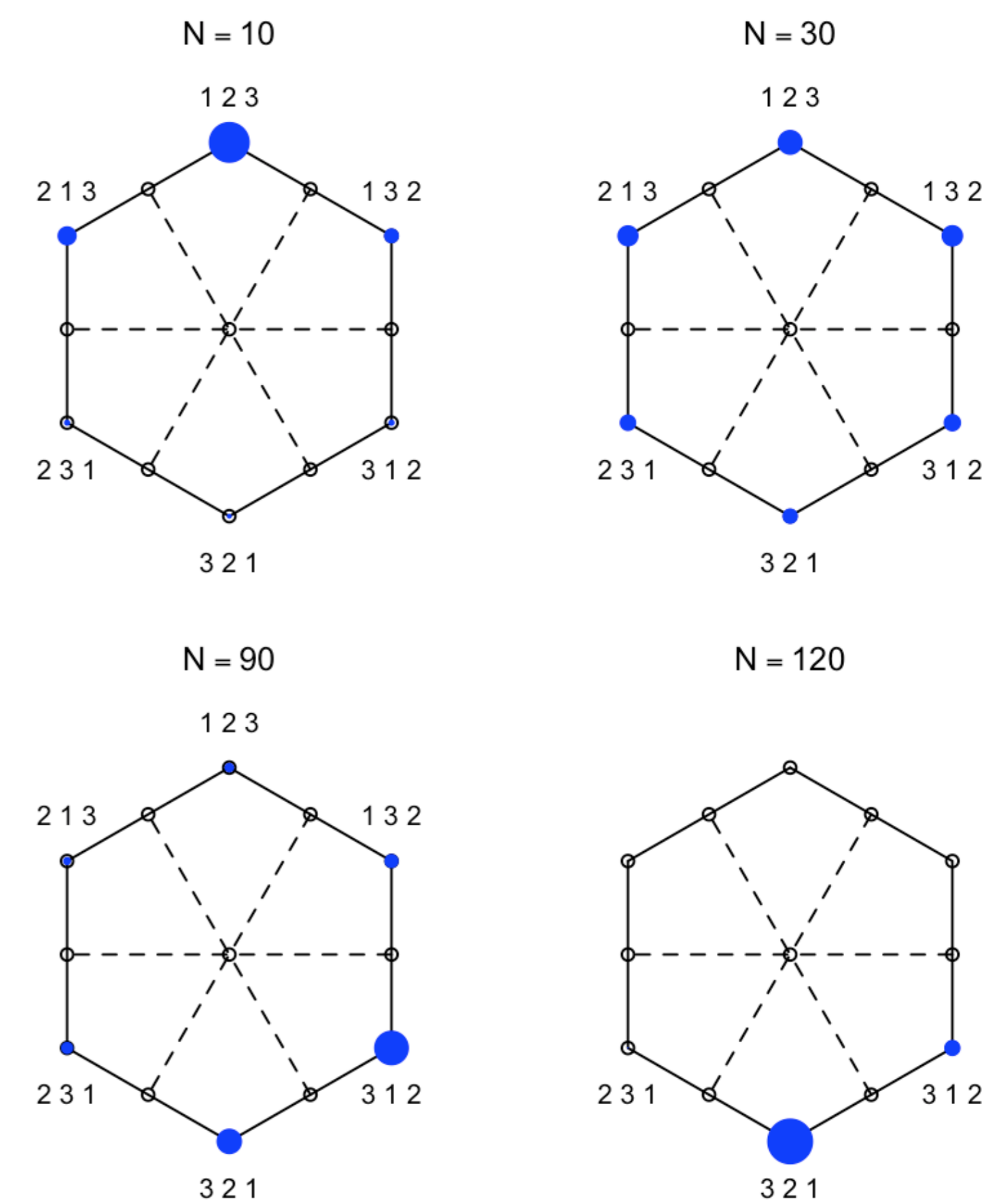
### 3.B Prior when $\theta$ not given

When  $\theta$  is unknown, the partition function of  $\pi(\cdot | \boldsymbol{\rho}_0, \theta N_0)$  depends on the model parameters and cannot be avoided.

**Solution:** Let  $\pi(\theta) \propto Z^*(\theta N_0, \boldsymbol{\rho}_0)$ , so that the posterior full conditional is treatable.

### 3.B Prior when $\theta$ not given

Same model. Increasing sample size ( $N$ ).  $\boldsymbol{\rho}_0 = (1, 2, 3)$  and  $N_0 = 16$ .



**Figure 4:** The balls have radius proportional to the frequency of rankings in the posterior sample.

## Ongoing work

- Can we say something about the convergence rate?  
→ Can we say something about  $Z^*(\cdot, \cdot)$ ?
- Applications?

**Interesting data to test our methods on? Please take contact!**

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