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Refined forms of Oppenheim and Cusa-Huygens type inequalities

Yogesh J. Bagul¹, Christophe Chesneau²,

¹Department of Mathematics, K. K. M. College Manwath,

Dist : Parbhani(M.S.) - 431505, India.

Email : yjbagul@gmail.com

²LMNO, University of Caen Normandie, France

Email : christophe.chesneau@unicaen.fr

Abstract : In this article we refine Oppenheim's inequality as well as generalized Cusa-Huygens type inequalities established recently by some researchers. One of the results where the bounds of $\sin x/x$ are tractable will be used to obtain sharp version of Yang Le's inequality.

1 Introduction

The famous Cusa-Huygens inequality [4, 6, 7, 9, 11, 12, 16–18] is known as

$$\frac{\sin x}{x} < \frac{2 + \cos x}{3}; x \in (0, \pi/2) \quad (1.1)$$

and its hyperbolic version, sometimes called hyperbolic Cusa-Huygens inequality [12] is stated as follows:

$$\frac{\sinh x}{x} < \frac{2 + \cosh x}{3}; x \neq 0. \quad (1.2)$$

Obtaining extended and generalized sharp versions of the above inequalities have been the interest among many researchers in recent years. In [6, 16], the following two sided inequality has been obtained:

$$\left(\frac{2 + \cos x}{3}\right)^a < \frac{\sin x}{x} < \frac{2 + \cos x}{3}; x \in (0, \pi/2) \quad (1.3)$$

with the best positive constants $a \approx 1.11374$ and 1.

J. Sándor and R. Oláh-Gál [18, Theorems 1 and 2] proved the following inequalities:

$$\frac{2 + \cos x}{\pi} < \frac{\sin x}{x} < \frac{2 + (\pi/2) \cos x}{\pi}; x \in (0, \pi/2). \quad (1.4)$$

*Corresponding author

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For $x \in (-\pi/2, \pi/2)$, the double inequality

$$\frac{p-1+\cos x}{p} < \frac{\sin x}{x} < \frac{2+\cos x}{3}, \quad (1.5)$$

where $p \approx 2.75194$ is due to B. A. Bhayo, R. Klén and J. Sándor [4]. Left inequality in (1.5) is sharper than the corresponding inequalities in (1.3) and (1.4). For $x \in (0, \pi/2)$ the two sided inequality

$$\frac{p-1+\cos x}{p} < \frac{\sin x}{x} < \frac{2+(4/\pi)\cos x}{\pi}, \quad (1.6)$$

where $p \approx 2.75194$ refines (1.4). It gives an optimal answer to the Oppenheim's problem [13]. It is already discussed in [3] and proved by F. Qi et al. [15] with more general case.

On the other side, as a natural approach C. Mortici established trigonometric-polynomial bounds for $\sin x/x$ as follows:

$$\frac{2+\cos x}{3} - \frac{x^4}{180} < \frac{\sin x}{x} < \frac{2+\cos x}{3} - \frac{x^4}{180} + \frac{x^6}{3780}; \quad x \in (0, \pi/2). \quad (1.7)$$

Recently, B. Malešević et al. [9] extended and generalized (1.7) as follows: If $x \in (0, \pi/2)$ and n is a natural number, then

$$\begin{aligned} \frac{2+\cos x}{3} + \sum_{k=2}^{2n} (-1)^{k+1} B(k) x^{2k} &< \frac{\sin x}{x} \\ &< \frac{2+\cos x}{3} + \sum_{k=2}^{2n+1} (-1)^{k+1} B(k) x^{2k}, \end{aligned} \quad (1.8)$$

where $B(k) = 2(k-1)/[3(2k+1)!]$.

Putting $n = 1$ in (1.8), the authors of the paper [9] obtain (1.7). Putting $n = 2, 3$ they obtain respectively the inequalities listed below for $x \in (0, \pi/2)$.

$$\begin{aligned} \frac{2+\cos x}{3} - \frac{x^4}{180} + \frac{x^6}{3780} - \frac{x^8}{181440} &< \frac{\sin x}{x} \\ &< \frac{2+\cos x}{3} - \frac{x^4}{180} + \frac{x^6}{3780} - \frac{x^8}{181440} + \frac{x^{10}}{14968800} \end{aligned} \quad (1.9)$$

and

$$\begin{aligned}
& \frac{2 + \cos x}{3} - \frac{x^4}{180} + \frac{x^6}{3780} - \frac{x^8}{181440} + \frac{x^{10}}{14968800} - \frac{x^{12}}{1868106240} < \frac{\sin x}{x} \\
& < \frac{2 + \cos x}{3} - \frac{x^4}{180} + \frac{x^6}{3780} - \frac{x^8}{181440} + \frac{x^{10}}{14968800} - \frac{x^{12}}{1868106240} \\
& + \frac{x^{14}}{326918592000}. \tag{1.10}
\end{aligned}$$

In this paper, we give simpler alternative proofs of (1.5) and (1.6) as well as establish new extended refined forms of the inequalities listed above. It will be interested to refine the inequalities in (1.8). We also aim to improve Yang Le's inequality [19] in the last section.

2 Preliminaries and Lemmas

We start by recalling the following power series expansions:

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}; \quad x \in \mathbb{R} \tag{2.1}$$

and

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}; \quad x \in \mathbb{R}. \tag{2.2}$$

The following lemmas will be applied to prove the main results of this paper.

Lemma 1. *(The l'Hôpital's monotonicity rule [1]) : Let $f, g : [p, q] \rightarrow \mathbb{R}$ be two continuous functions which are derivable on (p, q) and $g'(x) \neq 0$ for any $x \in (p, q)$. Let,*

$$A(x) = \frac{f(x) - f(p)}{g(x) - g(p)}$$

and

$$B(x) = \frac{f(x) - f(q)}{g(x) - g(q)}.$$

If f'/g' is increasing (or decreasing) on (p, q) , then the functions $A(x)$ and $B(x)$ are also increasing (or decreasing) on (p, q) . The strictness of the monotonicity in the conclusion depends on the strictness of the monotonicity of f'/g' .

Lemma 2. [2, Lemma 2] The function $H(x) = \frac{\sin x - x \cos x}{x^2 \sin x}$ is strictly positive increasing in $(0, \pi/2)$.

Lemma 3. The function $\phi(x) = \frac{\sinh x - x \cosh x}{x^2 \sinh x}$ is strictly negative increasing for $x > 0$.

Proof. The function $\phi(x)$ is clearly negative as $\sinh x/x < \cosh x$ for $x > 0$ (see [10]). Let us now set

$$\phi(x) = \frac{\sinh x - x \cosh x}{x^2 \sinh x} = \frac{\phi_1(x)}{\phi_2(x)},$$

where $\phi_1(x) = \sinh x - x \cosh x$ and $\phi_2(x) = x^2 \sinh x$ are such that $\phi_1(0) = 0$ and $\phi_2(0) = 0$. By differentiating, we obtain

$$\frac{\phi_1'(x)}{\phi_2'(x)} = \frac{-\sinh x}{x \cosh x + 2 \sinh x} = \frac{\phi_3(x)}{\phi_4(x)},$$

where $\phi_3(x) = -\sinh x$ and $\phi_4(x) = x \cosh x + 2 \sinh x$ with $\phi_3(0) = 0$ and $\phi_4(0) = 0$. By differentiating, we get

$$\frac{\phi_3'(x)}{\phi_4'(x)} = \frac{-\cosh x}{x \sinh x + 3 \cosh x} = \frac{-1}{x \tanh x + 3}.$$

Now it is well known that $x \tanh x$ is increasing, implying that $x \tanh x + 3$ is increasing. By Lemma 1, $\phi(x)$ is strictly increasing function for $x > 0$. \square

3 Main Results and Proofs

We first give a simpler alternative proof of (1.5) in the following theorem.

Theorem 1. [4, Theorem 1] If $x \in (-\pi/2, \pi/2)$ then it holds that

$$\frac{2 + (\pi - 2) \cos x}{\pi} < \frac{\sin x}{x} < \frac{2 + \cos x}{3}. \quad (3.1)$$

Proof. It is enough to prove the result for $x \in (0, \pi/2)$. Let us set

$$f(x) = \frac{\frac{\sin x}{x} - \cos x}{\cos x - 1} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = \sin x/x - \cos x$ and $f_2(x) = \cos x - 1$ satisfy $f_1(0+) = 0$ and $f_2(0) = 0$. Then

$$\frac{f_1'(x)}{f_2'(x)} = \frac{\frac{x \cos x - \sin x}{x^2} + \sin x}{-\sin x} = \frac{\sin x - x \cos x}{x^2 \sin x} - 1.$$

By Lemmas 1 and 2, $f(x)$ is strictly increasing in $(0, \pi/2)$. We end the proof by applying l'Hôpital's rule with $f(0+) = -2/3$ and $f(\pi/2) = -2/\pi$. \square

In the next result, we extend the inequality (1.2).

Theorem 2. *If $x \in (0, \lambda)$ where $\lambda > 0$ then*

$$-\delta + (1 + \delta) \cosh x < \frac{\sinh x}{x} < \frac{2 + \cosh x}{3}, \quad (3.2)$$

where $\delta = (\sinh \lambda / \lambda - \cosh \lambda) / (\cosh \lambda - 1)$.

Proof. Let us consider

$$f(x) = \frac{\frac{\sinh x}{x} - \cosh x}{\cosh x - 1} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = \sinh x / x - \cosh x$ and $f_2(x) = \cosh x - 1$ with $f_1(0+) = 0$ and $f_2(0) = 0$. Then

$$\frac{f_1'(x)}{f_2'(x)} = \frac{x \cosh x - \sinh x}{x^2 \sinh x} - 1.$$

By Lemmas 1 and 3, $f(x)$ is strictly decreasing in $(0, \lambda)$. The desired result follows from l'Hôpital's rule, $f(\lambda) = \delta$ and $f(0+) = -2/3$. \square

We present another simple proof of (1.6) in the following theorem.

Theorem 3. *[3, 15] If $x \in (0, \pi/2)$ then the inequalities*

$$\frac{2 + (\pi - 2) \cos x}{\pi} < \frac{\sin x}{x} < \frac{2 + (4/\pi) \cos x}{\pi} \quad (3.3)$$

are true.

Proof. Let us set

$$f(x) = \frac{\frac{\sin x}{x} - \frac{2}{\pi}}{\cos x} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = \sin x / x - 2/\pi$ and $f_2(x) = \cos x$ with $f_1(\pi/2) = 0$ and $f_2(\pi/2) = 0$. Then the function

$$\frac{f_1'(x)}{f_2'(x)} = \frac{\sin x - x \cos x}{x^2 \sin x}$$

is strictly increasing in $(0, \pi/2)$ by Lemma 2. So $f(x)$ is also strictly increasing in $(0, \pi/2)$ by Lemma 1. Therefore, the proof of Theorem 3 follows with $f(0+) = (\pi - 2)/\pi$ and $f(\pi/2-) = 4/\pi^2$ by l'Hôpital's rule. \square

Motivated by (1.7), trigonometric-polynomial bounds for $\sin x / x$ with new approach are established in the theorem below.

Theorem 4. *If $x \in (0, \pi/2)$ then the following inequalities hold:*

$$\frac{2 + \cos x}{3} - \frac{x^4}{a} < \frac{\sin x}{x} < \frac{2 + \cos x}{3} - \frac{x^4}{b} \quad (3.4)$$

with the constants $a = 180$ and $b \approx 202.618886$.

Proof. Let us set

$$f(x) = \frac{-3x^5}{3 \sin x - 2x - x \cos x} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = -3x^5$ and $f_2(x) = 3 \sin x - 2x - x \cos x$ with $f_1(0) = 0$ and $f_2(0) = 0$. Then we have

$$\frac{f_1'(x)}{f_2'(x)} = \frac{-15x^4}{2 \cos x - 2 + x \sin x} = \frac{f_3(x)}{f_4(x)},$$

where $f_3(x) = -15x^4$ and $f_4(x) = 2 \cos x - 2 + x \sin x$ with $f_3(0) = 0$ and $f_4(0) = 0$. Differentiation gives

$$\frac{f_3'(x)}{f_4'(x)} = \frac{-60x^3}{-\sin x + x \cos x} = \frac{f_5(x)}{f_6(x)},$$

where $f_5(x) = -60x^3$ and $f_6(x) = -\sin x + x \cos x$ with $f_5(0) = 0$ and $f_6(0) = 0$. Again differentiating

$$\frac{f_5'(x)}{f_6'(x)} = 180 \frac{x}{\sin x}$$

which is clearly increasing in $(0, \pi/2)$. Hence, by l'Hôpital's rule with $a = f(0+) = 180$ and $b = f(\pi/2) \approx 202.618886$, we complete the proof. \square

The hyperbolic version of (3.4) is given in Theorem 5 below.

Theorem 5. *If $x \in (0, \lambda)$ where $\lambda > 0$ then the following inequalities hold:*

$$\frac{2 + \cosh x}{3} - \frac{x^4}{m} < \frac{\sinh x}{x} < \frac{2 + \cosh x}{3} - \frac{x^4}{n} \quad (3.5)$$

with the constants $m = -3\lambda^5/(3 \sinh \lambda - 2\lambda - \lambda \cosh \lambda)$ and $n = 180$.

The proof of Theorem 5 is omitted since it is similar to the one of Theorem 4.

A refinement of upper bounds of (1.4) and (1.6) can be seen in the theorem below.

Theorem 6. For $x \in (0, \pi/2)$, one has

$$\frac{\sin x}{x} < \frac{2 + (\pi - 2) \cos x}{\pi} + \frac{\pi - 3}{3\pi} x^2. \quad (3.6)$$

Proof. Let us set

$$f(x) = \frac{\pi \sin x - 2x - (\pi - 2)x \cos x}{\pi x^3} = \frac{f_1(x)}{f_2(x)}$$

where $f_1(x) = \pi \sin x - 2x - (\pi - 2)x \cos x$ and $f_2(x) = \pi x^3$ satisfy $f_1(0) = 0$ and $f_2(0) = 0$. Then

$$\frac{f_1'(x)}{f_2'(x)} = \frac{\pi \cos x - 2 - (\pi - 2)[\cos x - x \sin x]}{3\pi x^2} = \frac{f_3(x)}{f_4(x)}$$

where $f_3(x) = \pi \cos x - 2 - (\pi - 2)(\cos x - x \sin x)$ and $f_4(x) = 3\pi x^2$ with $f_3(0) = 0$ and $f_4(0) = 0$. Differentiating

$$\frac{f_3'(x)}{f_4'(x)} = \frac{-\pi \sin x + (\pi - 2)(2 \sin x + x \cos x)}{6\pi x} = \frac{f_5(x)}{f_6(x)}$$

where $f_5(x) = -\pi \sin x + (\pi - 2)(2 \sin x + x \cos x)$ and $f_6(x) = 6\pi x$ satisfy $f_5(0) = 0$ and $f_6(0) = 0$. Again, by differentiating, we obtain

$$\frac{f_5'(x)}{f_6'(x)} = \frac{-\pi \cos x + (\pi - 2)(\cos x - x \sin x)}{6\pi} = \frac{f_7(x)}{6\pi}$$

where

$$f_7(x) = -\pi \cos x + (\pi - 2)(3 \cos x - x \sin x).$$

For $x \in (0, \pi/2)$, on differentiation, it gives us

$$f_7'(x) = (-3\pi + 8) \sin x - (\pi - 2)x \cos x < 0.$$

Therefore by Lemma 1, $f(x)$ is decreasing and hence

$$f(0+) > \frac{\pi \sin x - 2x - (\pi - 2)x \cos x}{\pi x^3}.$$

With $f(0+) = (\pi - 3)/(3\pi)$, the result is proved. \square

Remark 1. There is no strict comparison between the corresponding bounds of (1.6) and (3.6). However when x is close to zero, (3.6) is better since the inequality becomes an equality for $x = 0$.

Combining (1.6) and (3.6) we have for $x \in (0, \pi/2)$ that

$$\frac{2 + (\pi - 2) \cos x}{\pi} < \frac{\sin x}{x} < \frac{2 + (\pi - 2) \cos x}{\pi} + \frac{\pi - 3}{3\pi} x^2. \quad (3.7)$$

In the following theorem, we extend and refine the bounds of (3.7).

Theorem 7. *If $n \in \mathbb{N}$ (the set of natural numbers) and $x \in (0, \pi)$ then we have*

$$M(x) < \frac{\sin x}{x} < N(x) \quad (3.8)$$

where

$$M(x) = \frac{2 + (\pi - 2) \cos x}{\pi} + \frac{2}{\pi} \sum_{k=1}^{2n} (-1)^{k+1} \frac{(k\pi - 2k - 1)}{(2k + 1)!} x^{2k}$$

and

$$N(x) = \frac{2 + (\pi - 2) \cos x}{\pi} + \frac{2}{\pi} \sum_{k=1}^{2n+1} (-1)^{k+1} \frac{(k\pi - 2k - 1)}{(2k + 1)!} x^{2k}.$$

Proof. Let us set

$$f(x) = \frac{\sin x}{x} - M(x).$$

Utilizing (2.1) and (2.2), we get

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k + 1)!} - \frac{2}{\pi} - \frac{\pi - 2}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \\ &\quad - \frac{2}{\pi} \sum_{k=1}^{2n} (-1)^{k+1} \frac{k\pi - 2k - 1}{(2k + 1)!} x^{2k} \\ &= \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^{k+1} \frac{k\pi - 2k - 1}{(2k + 1)!} x^{2k} - \frac{2}{\pi} - \frac{2}{\pi} \sum_{k=1}^{2n} (-1)^{k+1} \frac{k\pi - 2k - 1}{(2k + 1)!} x^{2k} \\ &= -\frac{2}{\pi} R, \end{aligned}$$

where

$$R = \sum_{k=2n+1}^{\infty} (-1)^k \frac{k\pi - 2k - 1}{(2k + 1)!} x^{2k}.$$

Hence R can be viewed as a rest of the alternating series S given by

$$S = \sum_{k=2}^{\infty} (-1)^k a_k,$$

with $a_k = [(k\pi - 2k - 1)/(2k + 1)!]x^{2k} > 0$. Now for $k \geq 2$ and $x \in (0, \pi)$ we have

$$(k\pi - 2k - 1)(4k^2 + 10k + 6 - x^2) \geq (2\pi - 5)(42 - \pi^2) > (\pi - 2)x^2.$$

Equivalently,

$$(k\pi - 2k - 1)(2k + 3)(2k + 2) > [(k\pi - 2k - 1) + (\pi - 2)]x^2$$

which gives

$$\frac{k\pi - 2k - 1}{(2k + 1)!}x^{2k} > \frac{(k + 1)\pi - 2(k + 1) - 1}{(2k + 3)!}x^{2(k+1)}.$$

Hence $|a_k| > |a_{k+1}|$. Moreover, for $x \in (0, \pi)$ we have

$$\lim_{k \rightarrow \infty} |a_k| = \lim_{k \rightarrow \infty} \frac{k\pi - 2k - 1}{(2k + 1)!}x^{2k} = \frac{\pi - 2}{2} \lim_{k \rightarrow \infty} \frac{x^{2k}}{(2k)!} = 0.$$

This implies that S is convergent and, by a special result on alternating series, R has the same sign to the first term of its sum, i.e. $(-1)^{2n+1}a_{2n+1} < 0$, so $R < 0$, implying that $f(x) > 0$, and

$$M(x) < \frac{\sin x}{x}. \quad (3.9)$$

Similarly setting

$$\begin{aligned} g(x) &= N(x) - \frac{\sin x}{x} \\ &= \frac{2 + (\pi - 2)\cos x}{\pi} + \frac{2}{\pi} \sum_{k=1}^{2n+1} (-1)^{k+1} \frac{k\pi - 2k - 1}{(2k + 1)!} x^{2k} - \frac{\sin x}{x}. \end{aligned}$$

Using again (2.1) and (2.2) and proceeding as in case of $f(x)$, we obtain

$$g(x) = \frac{2}{\pi} R_*,$$

where

$$R_* = \sum_{k=2n+2}^{\infty} (-1)^k \frac{k\pi - 2k - 1}{(2k + 1)!} x^{2k}.$$

Let us observe that R_* is a rest of the alternating series S . By applying similar arguments to R , we get $R_* > 0$, implying that $g(x) > 0$ and hence

$$\frac{\sin x}{x} < N(x). \quad (3.10)$$

The proof follows from (3.9) and (3.10). \square

Some particular cases of Theorem 7 are presented below. If $x \in (0, \pi)$ then by putting $n = 1$ in (3.8) we get

$$\begin{aligned} & \frac{2 + (\pi - 2) \cos x}{\pi} + \frac{2}{\pi} \left[\frac{\pi - 3}{3!} x^2 - \frac{2\pi - 5}{5!} x^4 \right] < \frac{\sin x}{x} \\ & < \frac{2 + (\pi - 2) \cos x}{\pi} + \frac{2}{\pi} \left[\frac{\pi - 3}{3!} x^2 - \frac{2\pi - 5}{5!} x^4 + \frac{3\pi - 7}{7!} x^6 \right]. \end{aligned} \quad (3.11)$$

By putting $n = 2$ in (3.8) we get

$$\begin{aligned} & < \frac{2 + (\pi - 2) \cos x}{\pi} + \frac{2}{\pi} \left[\frac{\pi - 3}{3!} x^2 - \frac{2\pi - 5}{5!} x^4 + \frac{3\pi - 7}{7!} x^6 - \frac{4\pi - 9}{9!} x^8 \right] \\ & < \frac{\sin x}{x} \\ & < \frac{2 + (\pi - 2) \cos x}{\pi} \\ & + \frac{2}{\pi} \left[\frac{(\pi - 3)}{3!} x^2 - \frac{2\pi - 5}{5!} x^4 + \frac{3\pi - 7}{7!} x^6 - \frac{4\pi - 9}{9!} x^8 + \frac{5\pi - 11}{11!} x^{10} \right]. \end{aligned} \quad (3.12)$$

Next we claim that, the generalized extended bounds for $\sin x/x$ in the following theorem are sharper than the corresponding bounds in (1.8).

Theorem 8. *If $m = 2n - 1$ where $n \in \mathbb{N}$ and $x \in (0, \pi)$, then we have*

$$F(x) < \frac{\sin x}{x} < G(x), \quad (3.13)$$

where

$$F(x) = \frac{2m + \cos x}{2m + 1} + \frac{2}{2m + 1} \sum_{k=1}^{m+1} \frac{k - m}{(2k + 1)!} (-1)^{k+1} x^{2k}$$

and

$$G(x) = \frac{(2m + 2) + \cos x}{2m + 3} + \frac{2}{2m + 3} \sum_{k=1}^{m+2} \frac{k - m - 1}{(2k + 1)!} (-1)^{k+1} x^{2k}.$$

Proof. Let us set

$$\begin{aligned} f(x) &= \frac{\sin x}{x} - F(x) \\ &= \frac{\sin x}{x} - \frac{2m}{2m+1} - \frac{1}{2m+1} \cos x - \frac{2}{2m+1} \sum_{k=1}^{m+1} \frac{k-m}{(2k+1)!} (-1)^{k+1} x^{2k}. \end{aligned}$$

Utilizing (2.1) and (2.2), after some calculus, we get

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k} - \frac{2m}{2m+1} - \frac{1}{2m+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \\ &\quad - \frac{2}{2m+1} \sum_{k=1}^{m+1} \frac{k-m}{(2k+1)!} (-1)^{k+1} x^{2k} \\ &= -\frac{2m}{2m+1} + \frac{2}{2m+1} \sum_{k=0}^{\infty} \frac{k-m}{(2k+1)!} (-1)^{k+1} x^{2k} \\ &\quad - \frac{2}{2m+1} \sum_{k=1}^{m+1} \frac{k-m}{(2k+1)!} (-1)^{k+1} x^{2k} \\ &= \frac{2}{2m+1} \sum_{k=m+2}^{\infty} \frac{k-m}{(2k+1)!} (-1)^{k+1} x^{2k} = \frac{2}{2m+1} x^{2m} T, \end{aligned}$$

where

$$T = \sum_{k=2}^{\infty} \frac{k}{(2k+2m+1)!} (-1)^k x^{2k}.$$

Then T can be viewed as a rest of the alternating series U given by

$$U = \sum_{k=1}^{\infty} (-1)^k b_k,$$

with $b_k = (k/(2k+2m+1)!)x^{2k}$. Now for $k \geq 1$ and $x \in (0, \pi)$, we have

$$k[4(k+m)^2 + 10(k+m) + 6] - kx^2 \geq k(42 - \pi^2) > x^2.$$

Equivalently,

$$k(2k+2m+3)(2k+2m+2) > (k+1)x^2$$

which gives

$$\frac{k}{(2k+2m+1)!} x^{2k} > \frac{k+1}{(2k+2m+3)!} x^{2k+2}.$$

Hence $|b_k| > |b_{k+1}|$. Moreover, for $x \in (0, \pi)$, we have

$$\lim_{k \rightarrow \infty} |b_k| = \lim_{k \rightarrow \infty} \frac{k}{(2k + 2m + 1)!} x^{2k} = \frac{1}{2} \lim_{k \rightarrow \infty} \frac{x^{2k}}{(2k + 2m)!} = 0.$$

This implies that U is convergent and $(-1)^2 b_2 > 0$, so $T > 0$. Thus, $f(x) > 0$ and

$$F(x) < \frac{\sin x}{x}. \quad (3.14)$$

Similarly, let us set

$$\begin{aligned} g(x) &= G(x) - \frac{\sin x}{x} \\ &= \frac{(2m + 2) + \cos x}{2m + 3} + \frac{2}{2m + 3} \sum_{k=1}^{m+2} \frac{k - m - 1}{(2k + 1)!} (-1)^{k+1} x^{2k} - \frac{\sin x}{x}. \end{aligned}$$

Using (2.1), (2.2) and proceeding as in case of $f(x)$ gives

$$\begin{aligned} g(x) &= \frac{2}{2m + 3} \left[\sum_{k=1}^{m+2} \frac{k - m - 1}{(2k + 1)!} (-1)^{k+1} x^{2k} - \sum_{k=1}^{\infty} \frac{k - m - 1}{(2k + 1)!} (-1)^{k+1} x^{2k} \right] \\ &= \frac{2}{2m + 3} x^{2m+2} T_*, \end{aligned}$$

where

$$T_* = \sum_{k=2}^{\infty} \frac{k}{(2k + 2m + 3)!} (-1)^k x^{2k}.$$

By applying the same arguments related to the alternating series used to show that $T > 0$, we prove that $T_* > 0$, implying that $g(x) > 0$ and

$$\frac{\sin x}{x} < G(x). \quad (3.15)$$

The proof follows from (3.14) and (3.15). \square

Some particular cases of Theorem 8 are presented below. If $x \in (0, \pi)$ then by putting $n = 1$ (so $m = 1$) in (3.13) we get

$$\frac{2 + \cos x}{3} - \frac{x^4}{180} < \frac{\sin x}{x} < \frac{4 + \cos x}{5} - \frac{x^2}{15} + \frac{x^6}{12600}. \quad (3.16)$$

By putting $n = 2$ (so $m = 3$) in (3.13) we get

$$\begin{aligned} & \frac{6 + \cos x}{7} - \frac{2x^2}{21} + \frac{x^4}{420} - \frac{x^8}{1270080} < \frac{\sin x}{x} \\ & < \frac{8 + \cos x}{9} - \frac{x^2}{9} + \frac{x^4}{270} - \frac{x^6}{22680} + \frac{x^{10}}{179625600}. \end{aligned} \quad (3.17)$$

By putting $n = 3$ (so $m = 5$) in (3.13) we get

$$\begin{aligned} & \frac{10 + \cos x}{11} - \frac{4x^2}{33} + \frac{x^4}{220} - \frac{x^6}{13860} + \frac{x^8}{1995840} - \frac{x^{12}}{34248614400} < \frac{\sin x}{x} \\ & < \frac{12 + \cos x}{13} - \frac{5x^2}{39} + \frac{x^4}{195} - \frac{x^6}{10920} + \frac{x^8}{1179360} - \frac{x^{10}}{259459200} \\ & + \frac{x^{14}}{8499883392000}. \end{aligned} \quad (3.18)$$

For comparison between particular cases, it can be verified by any mathematical software that the bounds of $\sin x/x$ in (3.16), (3.17) and (3.18) are sharper than the corresponding bounds in (1.7), (1.9) and (1.10) respectively. Moreover, all the bounds are trigonometric-polynomial. Therefore, considering the number of terms and the degrees of polynomials in respective bounds, we conclude that our bounds are better.

4 An Application

The Yang Le's inequality [19] can be stated as follows.

If $A_1 > 0, A_2 > 0, A_1 + A_2 \leq \pi$ and $0 \leq \lambda \leq 1$, then

$$\cos^2 \lambda A_1 + \cos^2 \lambda A_2 - 2 \cos \lambda A_1 \cos \lambda A_2 \cos \lambda \pi \geq \sin^2 \lambda \pi \quad (4.1)$$

Inequality (4.1) and its generalizations play an important role in the theory of distribution of values. Therefore many refinements of (4.1) are established so far. For more details we refer reader to [5, 8, 14, 20] and the references therein.

Recently C.-P. Chen and N. Elezović [5] obtained an improved form of Yang Le's inequality which is given below:

Let $A_i > 0$ with $\sum_{i=1}^n A_i \leq \pi$ and $n \in \mathbb{N}$, $n > 1$, $0 \leq \mu \leq 1$. Then

$$N(\mu) \leq \sum_{1 \leq i < j \leq n} H_{ij} \leq M(\mu), \quad (4.2)$$

where

$$H_{ij} = \cos^2 \mu A_i + \cos^2 \mu A_j - 2 \cos \mu A_i \cos \mu A_j \cos \mu \pi,$$

$$N(\mu) = \frac{n(n-1)}{2} \pi^2 \left[\frac{1 - (\mu/2)^2}{1 + (\mu/2)^2} \right]^2 \mu^2 \cos^2 \frac{\mu \pi}{2}$$

and

$$M(\mu) = \frac{n(n-1)}{2} \pi^2 \left[\frac{1 - (\mu/2)^3}{1 + 2(\mu/2)^3} \right]^2 \mu^2.$$

Here we refine the Yang Le's inequality by using the tractable bounds of $\sin(x)/x$.

Theorem 9. *Let, $A_i > 0$ with $\sum_{i=1}^n A_i \leq \pi$ and $n \in \mathbb{N}$, $n > 1$; $0 \leq \mu \leq 1$. Then*

$$P(\mu) \leq \sum_{1 \leq i < j \leq n} H_{ij} \leq Q(\mu), \quad (4.3)$$

where

$$H_{ij} = \cos^2 \mu A_i + \cos^2 \mu A_j - 2 \cos \mu A_i \cos \mu A_j \cos \mu \pi,$$

$$P(\mu) = 2n(n-1) \left[1 + \frac{(\pi-2)}{2} \cos \frac{\mu \pi}{2} \right]^2 \mu^2 \cos^2 \frac{\mu \pi}{2}$$

and

$$Q(\mu) = 2n(n-1) \left\{ \left[1 + \frac{(\pi-2)}{2} \cos \frac{\mu \pi}{2} \right] + \frac{\mu^2 \pi^2 (\pi-3)}{24} \right\}^2 \mu^2.$$

Proof. We substitute $x = \mu\pi/2$ in (3.7) to get

$$\mu \left[1 + \frac{(\pi-2)}{2} \cos \frac{\mu \pi}{2} \right] < \sin \frac{\mu \pi}{2}$$

and

$$\sin \frac{\mu \pi}{2} < \mu \left[1 + \frac{(\pi-2)}{2} \cos \frac{\mu \pi}{2} \right] + \frac{\mu^3 \pi^2 (\pi-3)}{24}.$$

Using the inequality [20], for $1 \leq i < j \leq n$, we have

$$4 \sin^2 \frac{\mu \pi}{2} \cos^2 \frac{\mu \pi}{2} \leq H_{ij} \leq 4 \sin^2 \frac{\mu \pi}{2}.$$

We deduce that

$$\begin{aligned}
& 4 \left[1 + \frac{(\pi - 2)}{2} \cos \frac{\mu\pi}{2} \right]^2 \mu^2 \cos^2 \frac{\mu\pi}{2} \leq H_{ij} \\
& \leq 4 \left\{ \left[1 + \frac{(\pi - 2)}{2} \cos \frac{\mu\pi}{2} \right] + \frac{\mu^2 \pi^2 (\pi - 3)}{24} \right\}^2 \mu^2. \tag{4.4}
\end{aligned}$$

Summing up all the inequalities in (4.4), we get (4.3). \square

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