



HAL
open science

Refined forms of Oppenheim and Cusa-Huygens type inequalities

Yogesh J. Bagul, Christophe Chesneau

► **To cite this version:**

Yogesh J. Bagul, Christophe Chesneau. Refined forms of Oppenheim and Cusa-Huygens type inequalities. 2019. hal-01972893v1

HAL Id: hal-01972893

<https://hal.science/hal-01972893v1>

Preprint submitted on 8 Jan 2019 (v1), last revised 6 May 2019 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Refined forms of Oppenheim and Cusa-Huygens type inequalities

Yogesh J. Bagul¹, Christophe Chesneau²,

¹Department of Mathematics, K. K. M. College Manwath,

Dist : Parbhani(M.S.) - 431505, India.

Email : yjbagul@gmail.com

²LMNO, University of Caen Normandie, France

Email : christophe.chesneau@unicaen.fr

Abstract : In this article we refine generalized Cusa-Huygens type inequalities established recently by some researchers. One of the results where the bounds of $\sin(x)/x$ are tractable will be used to obtain sharp version of Yang Le's inequality.

Keywords : Cusa-Huygens inequality; trigonometric-polynomial bounds; hyperbolic version; monotonicity.

Mathematics Subject Classification(2010): 26D05, 26D07, 26D15.

1 Introduction

The famous Cusa-Huygens inequality [4, 6, 7, 9, 11, 12, 16–18] is known as

$$\frac{\sin(x)}{x} < \frac{2 + \cos(x)}{3}; x \in (0, \pi/2) \quad (1.1)$$

and its hyperbolic version, sometimes called hyperbolic Cusa-Huygens inequality [12] is stated as follows:

$$\frac{\sinh(x)}{x} < \frac{2 + \cosh(x)}{3}; x \neq 0. \quad (1.2)$$

Obtaining the extended and generalized sharp versions of the above inequalities have been the interest among many researchers in recent years. In [6, 16], the following inequality has been obtained:

$$\left(\frac{2 + \cos(x)}{3}\right)^a < \frac{\sin(x)}{x} < \frac{2 + \cos(x)}{3}; x \in (0, \pi/2) \quad (1.3)$$

with the best positive constants $a \approx 1.11374$ and 1.

J. Sándor and R. Oláh-Gál [18, Theorems 1 and 2] proved the the following inequalities:

$$\frac{2 + \cos(x)}{\pi} < \frac{\sin(x)}{x} < \frac{2 + (\pi/2) \cos(x)}{\pi}; \quad x \in (0, \pi/2). \quad (1.4)$$

For $x \in (-\pi/2, \pi/2)$, the double inequality

$$\frac{p - 1 + \cos(x)}{p} < \frac{\sin(x)}{x} < \frac{2 + \cos(x)}{3}, \quad (1.5)$$

where $p \approx 2.75194$ is due to B. A. Bhayo, R. Klén and J. Sándor [4]. Left inequality in (1.5) is sharper than the corresponding inequalities in (1.3) and (1.4). In our main results, we will first present a very simple alternative proof of (1.5). Then another natural generalization of (1.4) and (1.5) will be established.

On the other side, as a natural approach Cristinel Mortici established trigonometric-polynomial bound for $\sin(x)/x$ as follows:

$$\frac{2 + \cos(x)}{3} - \frac{x^4}{180} < \frac{\sin(x)}{x} < \frac{2 + \cos(x)}{3} - \frac{x^4}{180} + \frac{x^6}{3780}; \quad x \in (0, \pi/2). \quad (1.6)$$

Recently, B. Malešević *et al.* [9] extended and generalized (1.6) as follows: If $x \in (0, \pi/2)$ and n is a natural number, then

$$\begin{aligned} \frac{2 + \cos(x)}{3} + \sum_{k=2}^{2n} (-1)^{k+1} B(k) x^{2k} &< \frac{\sin(x)}{x} \\ &< \frac{2 + \cos(x)}{3} + \sum_{k=2}^{2n+1} (-1)^{k+1} B(k) x^{2k}, \end{aligned} \quad (1.7)$$

where $B(k) = 2(k-1)/[3(2k+1)!]$.

Putting $n = 1$ in (1.7), the authors of the paper [9] obtain (1.6). Putting $n = 2, 3$ respectively, they obtain the inequalities listed below for $x \in (0, \pi/2)$.

$$\begin{aligned} \frac{2 + \cos(x)}{3} - \frac{x^4}{180} + \frac{x^6}{3780} - \frac{x^8}{181440} &< \frac{\sin(x)}{x} \\ &< \frac{2 + \cos(x)}{3} - \frac{x^4}{180} + \frac{x^6}{3780} - \frac{x^8}{181440} + \frac{x^{10}}{14968800} \end{aligned} \quad (1.8)$$

and

$$\begin{aligned} & \frac{2 + \cos(x)}{3} - \frac{x^4}{180} + \frac{x^6}{3780} - \frac{x^8}{181440} + \frac{x^{10}}{14968800} - \frac{x^{12}}{1868106240} < \frac{\sin(x)}{x} \\ & < \frac{2 + \cos(x)}{3} - \frac{x^4}{180} + \frac{x^6}{3780} - \frac{x^8}{181440} + \frac{x^{10}}{14968800} - \frac{x^{12}}{1868106240} \\ & + \frac{x^{14}}{326918592000}. \end{aligned} \quad (1.9)$$

Motivated by these results, our main aim is to refine the inequalities in (1.7).

2 Preliminaries and Lemmas

We start by recalling the following power series expansions:

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}; \quad x \in \mathbb{R} \quad (2.1)$$

and

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}; \quad x \in \mathbb{R}. \quad (2.2)$$

The following lemmas will be applied to prove the main results of this paper.

Lemma 1. *(The l'Hôpital's monotonicity rule [1]) : Let $f, g : [p, q] \rightarrow \mathbb{R}$ be two continuous functions which are derivable on (p, q) and $g'(x) \neq 0$ for any $x \in (p, q)$. Let,*

$$A(x) = \frac{f(x) - f(p)}{g(x) - g(p)}$$

and

$$B(x) = \frac{f(x) - f(q)}{g(x) - g(q)}.$$

If f'/g' is increasing (or decreasing) on (p, q) , then the functions $A(x)$ and $B(x)$ are also increasing (or decreasing) on (p, q) . The strictness of the monotonicity in the conclusion depends on the strictness of the monotonicity of f'/g' .

Lemma 2. *[2, Lemma 2] The function $H(x) = (\sin(x) - x \cos(x))/(x^2 \sin(x))$ is strictly positive increasing in $(0, \pi/2)$.*

Lemma 3. *The function $\phi(x) = (\sinh(x) - x \cosh(x))/(x^2 \sinh(x))$ is strictly negative increasing for $x > 0$.*

Proof of Lemma 3. The function $\phi(x)$ is clearly negative as $\sinh(x)/x < \cosh(x)$ for $x > 0$ (see [10]). Let us now set

$$\phi(x) = \frac{\sinh(x) - x \cosh(x)}{x^2 \sinh(x)} = \frac{\phi_1(x)}{\phi_2(x)},$$

where $\phi_1(x) = \sinh(x) - x \cosh(x)$ and $\phi_2(x) = x^2 \sinh(x)$ are such that $\phi_1(0) = 0$ and $\phi_2(0) = 0$. By differentiating, we obtain

$$\frac{\phi_1'(x)}{\phi_2'(x)} = \frac{-\sinh(x)}{x \cosh(x) + 2\sinh(x)} = \frac{\phi_3(x)}{\phi_4(x)},$$

where $\phi_3(x) = -\sinh(x)$ and $\phi_4(x) = x \cosh(x) + 2 \sinh(x)$ with $\phi_3(0) = 0$ and $\phi_4(0) = 0$. By differentiating, we get

$$\frac{\phi_3'(x)}{\phi_4'(x)} = \frac{-\cosh(x)}{x \sinh(x) + 3\cosh(x)} = \frac{-1}{x \tanh(x) + 3}.$$

Now it is well known that $x \tanh(x)$ is increasing implying that $x \tanh(x) + 3$ is increasing. By Lemma 1, $\phi(x)$ is strictly increasing function for $x > 0$. \square

3 Main Results and Proofs

We first give a simpler alternative proof of (1.5) in the following theorem.

Theorem 1. [4, Theorem 1] *If $x \in (-\pi/2, \pi/2)$ then it holds that*

$$\frac{2 + (\pi - 2)\cos(x)}{\pi} < \frac{\sin(x)}{x} < \frac{2 + \cos(x)}{3}. \quad (3.1)$$

Proof of Theorem 1. It is enough to prove the result for $x \in (0, \pi/2)$. Let us set

$$f(x) = \frac{\frac{\sin(x)}{x} - \cos(x)}{\cos(x) - 1} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = \sin(x)/x - \cos(x)$ and $f_2(x) = \cos(x) - 1$ satisfy $f_1(0+) = 0$ and $f_2(0) = 0$. Then

$$\frac{f_1'(x)}{f_2'(x)} = \frac{\frac{x \cos(x) - \sin(x)}{x^2} + \sin(x)}{-\sin(x)} = \frac{\sin(x) - x \cos(x)}{x^2 \sin(x)} - 1.$$

By Lemmas 1 and 2, $f(x)$ is strictly increasing in $(0, \pi/2)$. We end the proof by applying l'Hôpital's rule with $f(0+) = -2/3$ and $f(\pi/2) = -2/\pi$. \square

The double inequality in the next theorem refines the inequalities (1.4). It gives an optimal answer to the Oppenheim's problem [13]. It is already discussed in [3] and proved by F. Qi *et al.* [15] with more general case. However we give another simple proof.

Theorem 2. *If $x \in (0, \pi/2)$ then the inequalities*

$$\frac{2 + (\pi - 2)\cos(x)}{\pi} < \frac{\sin(x)}{x} < \frac{2 + (4/\pi)\cos(x)}{\pi} \quad (3.2)$$

are true.

Proof of Theorem 2. Let us set

$$f(x) = \frac{\frac{\sin(x)}{x} - \frac{2}{\pi}}{\cos(x)} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = \sin(x)/x - 2/\pi$ and $f_2(x) = \cos(x)$ with $f_1(\pi/2) = 0$ and $f_2(\pi/2) = 0$. Then the function

$$\frac{f_1'(x)}{f_2'(x)} = \frac{\sin(x) - x \cos(x)}{x^2 \sin(x)}$$

is strictly increasing in $(0, \pi/2)$ by Lemma 2. So $f(x)$ is also strictly increasing in $(0, \pi/2)$ by Lemma 1. Therefore, the proof of Theorem 2 follows with $f(0+) = (\pi - 2)/\pi$ and $f(\pi/2-) = 4/\pi^2$ by l'Hôpital's rule. \square

A refinement of upper bounds of (1.4) and (3.2) can be seen in the theorem below.

Theorem 3. *For $x \in (0, \pi/2)$, one has*

$$\frac{\sin(x)}{x} < \frac{2 + (\pi - 2)\cos(x)}{\pi} + \frac{\pi - 3}{3\pi} x^2. \quad (3.3)$$

Proof of Theorem 3. Let us set

$$f(x) = \frac{\pi \sin(x) - 2x - (\pi - 2)x \cos(x)}{\pi x^3} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = \pi \sin(x) - 2x - (\pi - 2)x \cos(x)$ and $f_2(x) = \pi x^3$ satisfy $f_1(0) = 0$ and $f_2(0) = 0$. Then

$$\frac{f_1'(x)}{f_2'(x)} = \frac{\pi \cos(x) - 2 - (\pi - 2)[\cos(x) - x \sin(x)]}{3\pi x^2} = \frac{f_3(x)}{f_4(x)},$$

where $f_3(x) = \pi \cos(x) - 2 - (\pi - 2)[\cos(x) - x \sin(x)]$ and $f_4(x) = 3\pi x^2$ with $f_3(0) = 0$ and $f_4(0) = 0$. By differentiating we get

$$\frac{f_3'(x)}{f_4'(x)} = \frac{-\pi \sin(x) + (\pi - 2)(2\sin(x) + x \cos(x))}{6\pi x} = \frac{f_5(x)}{f_6(x)},$$

where $f_5(x) = -\pi \sin(x) + (\pi - 2)[2\sin(x) + x \cos(x)]$ and $f_6(x) = 6\pi x$ satisfy $f_5(0) = 0$ and $f_6(0) = 0$. Again, by differentiating, we obtain

$$\frac{f_5'(x)}{f_6'(x)} = \frac{-\pi \cos(x) + (\pi - 2)[3\cos(x) - x \sin(x)]}{6\pi} = \frac{f_7(x)}{6\pi},$$

where

$$f_7(x) = -\pi \cos(x) + (\pi - 2)[3\cos(x) - x \sin(x)].$$

For $x \in (0, \pi/2)$, on differentiation, it gives us

$$f_7'(x) = (-3\pi + 8)\sin(x) - (\pi - 2)x \cos(x) < 0.$$

Therefore by Lemma 1, $f(x)$ is decreasing and hence

$$f(0+) > \frac{\pi \sin(x) - 2x - (\pi - 2)x \cos(x)}{\pi x^3}.$$

With $f(0+) = (\pi - 3)/(3\pi)$, the result is proved. \square

Remark 1. *There is no strict comparison between the corresponding bounds of (3.2) and (3.3). However when x is close to zero, (3.3) is better since the inequality becomes an equality for $x = 0$.*

Combining (3.2) and (3.3) we have for $x \in (0, \pi/2)$ that

$$\frac{2 + (\pi - 2)\cos(x)}{\pi} < \frac{\sin(x)}{x} < \frac{2 + (\pi - 2)\cos(x)}{\pi} + \frac{\pi - 3}{3\pi} x^2. \quad (3.4)$$

In the following theorem, we extend and refine the bounds of (3.4).

Theorem 4. *If $n \in \mathbb{N}$ (the set of natural numbers) and $x \in (0, \pi)$ then we have*

$$M(x) < \frac{\sin(x)}{x} < N(x), \quad (3.5)$$

where

$$M(x) = \frac{2 + (\pi - 2)\cos(x)}{\pi} + \frac{2}{\pi} \sum_{k=1}^{2n} (-1)^{k+1} \frac{k\pi - 2k - 1}{(2k + 1)!} x^{2k}$$

and

$$N(x) = \frac{2 + (\pi - 2)\cos(x)}{\pi} + \frac{2}{\pi} \sum_{k=1}^{2n+1} (-1)^{k+1} \frac{k\pi - 2k - 1}{(2k + 1)!} x^{2k}.$$

Proof of Theorem 4. Let us set

$$f(x) = \frac{\sin(x)}{x} - M(x).$$

Utilizing (2.1) and (2.2), we get

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k+1)!} - \frac{2}{\pi} - \frac{\pi-2}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \\ &\quad - \frac{2}{\pi} \sum_{k=1}^{2n} (-1)^{k+1} \frac{k\pi - 2k - 1}{(2k+1)!} x^{2k} \\ &= \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^{k+1} \frac{k\pi - 2k - 1}{(2k+1)!} x^{2k} - \frac{2}{\pi} - \frac{2}{\pi} \sum_{k=1}^{2n} (-1)^{k+1} \frac{k\pi - 2k - 1}{(2k+1)!} x^{2k} \\ &= -\frac{2}{\pi} R, \end{aligned}$$

where

$$R = \sum_{k=2n+1}^{\infty} (-1)^k \frac{k\pi - 2k - 1}{(2k+1)!} x^{2k}.$$

Hence R can be viewed as a rest of the alternating series S given by

$$S = \sum_{k=2}^{\infty} (-1)^k a_k,$$

with $a_k = [(k\pi - 2k - 1)/(2k+1)!]x^{2k} > 0$. In order to determine the sign of R , let us prove that S is convergent, i.e. $|a_k| > |a_{k+1}|$ and $\lim_{k \rightarrow +\infty} |a_k| = 0$.

For $k \geq 2$ and $x \in (0, \pi)$ we have

$$(k\pi - 2k - 1)(4k^2 + 10k + 6 - x^2) \geq (2\pi - 5)(42 - \pi^2) > (\pi - 2)x^2.$$

Equivalently,

$$(k\pi - 2k - 1)(2k + 3)(2k + 2) > [(k\pi - 2k - 1) + (\pi - 2)]x^2$$

which gives

$$\frac{k\pi - 2k - 1}{(2k+1)!} x^{2k} > \frac{(k+1)\pi - 2(k+1) - 1}{(2k+3)!} x^{2(k+1)}.$$

Hence $|a_k| > |a_{k+1}|$. Moreover, for $x \in (0, \pi)$ we have

$$\lim_{k \rightarrow \infty} |a_k| = \lim_{k \rightarrow \infty} \frac{k\pi - 2k - 1}{(2k+1)!} x^{2k} = \frac{\pi - 2}{2} \lim_{k \rightarrow \infty} \frac{x^{2k}}{(2k)!} = 0.$$

This implies that S is convergent and, by a special result on alternating series, R has the same sign to the first term of its sum, i.e. $(-1)^{2n+1}a_{2n+1} < 0$, so $R < 0$, implying that $f(x) > 0$, and

$$M(x) < \frac{\sin(x)}{x}. \quad (3.6)$$

Similarly setting

$$\begin{aligned} g(x) &= N(x) - \frac{\sin(x)}{x} \\ &= \frac{2 + (\pi - 2)\cos(x)}{\pi} + \frac{2}{\pi} \sum_{k=1}^{2n+1} (-1)^{k+1} \frac{k\pi - 2k - 1}{(2k+1)!} x^{2k} - \frac{\sin(x)}{x}. \end{aligned}$$

Using again (2.1) and (2.2) and proceeding as in case of $f(x)$, we obtain

$$g(x) = \frac{2}{\pi} R_*,$$

where

$$R_* = \sum_{k=2n+2}^{\infty} (-1)^k \frac{k\pi - 2k - 1}{(2k+1)!} x^{2k}.$$

Let us observe that R_* is a rest of the alternating series S . By applying similar arguments to R , we get $R_* > 0$, implying that $g(x) > 0$ and hence

$$\frac{\sin(x)}{x} < N(x). \quad (3.7)$$

The proof follows from (3.6) and (3.7). \square

Some particular cases of Theorem 4 are presented below. If $x \in (0, \pi)$ then by putting $n = 1$ in (3.5) we get

$$\begin{aligned} &\frac{2 + (\pi - 2)\cos(x)}{\pi} + \frac{2}{\pi} \left[\frac{\pi - 3}{3!} x^2 - \frac{2\pi - 5}{5!} x^4 \right] < \frac{\sin(x)}{x} \\ &< \frac{2 + (\pi - 2)\cos(x)}{\pi} + \frac{2}{\pi} \left[\frac{\pi - 3}{3!} x^2 - \frac{2\pi - 5}{5!} x^4 + \frac{3\pi - 7}{7!} x^6 \right]. \end{aligned} \quad (3.8)$$

By putting $n = 2$ in (3.5) we get

$$\begin{aligned} &< \frac{2 + (\pi - 2)\cos(x)}{\pi} + \frac{2}{\pi} \left[\frac{\pi - 3}{3!} x^2 - \frac{2\pi - 5}{5!} x^4 + \frac{3\pi - 7}{7!} x^6 - \frac{4\pi - 9}{9!} x^8 \right] \\ &< \frac{\sin(x)}{x} \\ &< \frac{2 + (\pi - 2)\cos(x)}{\pi} \\ &+ \frac{2}{\pi} \left[\frac{(\pi - 3)}{3!} x^2 - \frac{2\pi - 5}{5!} x^4 + \frac{3\pi - 7}{7!} x^6 - \frac{4\pi - 9}{9!} x^8 + \frac{5\pi - 11}{11!} x^{10} \right]. \end{aligned} \quad (3.9)$$

In the next result, we generalize the inequality (1.2).

Theorem 5. *If $x \in (0, \lambda)$ where $\lambda > 0$ then*

$$-\delta + (1 + \delta)\cosh(x) < \frac{\sinh(x)}{x} < \frac{2 + \cosh(x)}{3}, \quad (3.10)$$

where $\delta = (\sinh(\lambda)/\lambda - \cosh(\lambda))/(\cosh(\lambda) - 1)$.

Proof of Theorem 5. Let us consider

$$f(x) = \frac{\frac{\sinh(x)}{x} - \cosh(x)}{\cosh(x) - 1} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = \sinh(x)/x - \cosh(x)$ and $f_2(x) = \cosh(x) - 1$ with $f_1(0+) = 0$ and $f_2(0) = 0$. Then

$$\frac{f_1'(x)}{f_2'(x)} = \frac{x \cosh(x) - \sinh(x)}{x^2 \sinh(x)} - 1.$$

By Lemmas 1 and 3, $f(x)$ is strictly decreasing in $(0, \lambda)$. The desired result follows from l'Hôpital's rule, $f(\lambda) = \delta$ and $f(0+) = -2/3$. \square

Motivated by (1.6), trigonometric-polynomial bounds for $\sin(x)/x$ with new approach are established in the theorem below.

Theorem 6. *If $x \in (0, \pi/2)$ then the following inequalities hold:*

$$\frac{2 + \cos(x)}{3} - \frac{x^4}{a} < \frac{\sin(x)}{x} < \frac{2 + \cos(x)}{3} - \frac{x^4}{b} \quad (3.11)$$

with the constants $a = 180$ and $b \approx 202.618886$.

Proof of Theorem 6. Let us set

$$f(x) = \frac{-3x^5}{3\sin(x) - 2x - x\cos(x)} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = -3x^5$ and $f_2(x) = 3\sin(x) - 2x - x\cos(x)$ with $f_1(0) = 0$ and $f_2(0) = 0$. Then we have

$$\frac{f_1'(x)}{f_2'(x)} = \frac{-15x^4}{2\cos(x) - 2 + x\sin(x)} = \frac{f_3(x)}{f_4(x)},$$

where $f_3(x) = -15x^4$ and $f_4(x) = 2\cos(x) - 2 + x\sin(x)$ with $f_3(0) = 0$ and $f_4(0) = 0$. Differentiation gives

$$\frac{f_3'(x)}{f_4'(x)} = \frac{-60x^3}{-\sin(x) + x\cos(x)} = \frac{f_5(x)}{f_6(x)},$$

where $f_5(x) = -60x^3$ and $f_6(x) = -\sin(x) + x\cos(x)$ with $f_5(0) = 0$ and $f_6(0) = 0$. Again differentiating

$$\frac{f'_5(x)}{f'_6(x)} = 180 \frac{x}{\sin(x)}$$

which is clearly increasing in $(0, \pi/2)$. Hence, by l'Hôpital's rule with $a = f(0+) = 180$ and $b = f(\pi/2) \approx 202.618886$, we complete the proof. \square

The hyperbolic version of (3.4) is given in Theorem 5 below.

Theorem 7. *If $x \in (0, \lambda)$ where $\lambda > 0$ then the following inequalities hold:*

$$\frac{2 + \cosh(x)}{3} - \frac{x^4}{m} < \frac{\sinh(x)}{x} < \frac{2 + \cosh(x)}{3} - \frac{x^4}{n} \quad (3.12)$$

with the constants $m = -3\lambda^5/(3\sinh(\lambda) - 2\lambda - \lambda\cosh(\lambda))$ and $n = 180$.

The proof of Theorem 5 is omitted since it is similar to the one of Theorem 4.

Next we claim that, the generalized extended bounds for $\sin(x)/x$ in the following theorem are sharper than the corresponding bounds in (1.7).

Theorem 8. *If $m = 2n - 1$ where $n \in \mathbb{N}$ and $x \in (0, \pi)$, then we have*

$$F(x) < \frac{\sin(x)}{x} < G(x), \quad (3.13)$$

where

$$F(x) = \frac{2m + \cos(x)}{2m + 1} + \frac{2}{2m + 1} \sum_{k=1}^{m+1} \frac{k - m}{(2k + 1)!} (-1)^{k+1} x^{2k}$$

and

$$G(x) = \frac{(2m + 2) + \cos(x)}{2m + 3} + \frac{2}{2m + 3} \sum_{k=1}^{m+2} \frac{k - m - 1}{(2k + 1)!} (-1)^{k+1} x^{2k}.$$

Proof of Theorem 8. Let us set

$$\begin{aligned} f(x) &= \frac{\sin(x)}{x} - F(x) \\ &= \frac{\sin(x)}{x} - \frac{2m}{2m + 1} - \frac{1}{2m + 1} \cos(x) - \frac{2}{2m + 1} \sum_{k=1}^{m+1} \frac{k - m}{(2k + 1)!} (-1)^{k+1} x^{2k}. \end{aligned}$$

Utilizing (2.1) and (2.2), after some calculus, we get

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k} - \frac{2m}{2m+1} - \frac{1}{2m+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \\
&\quad - \frac{2}{2m+1} \sum_{k=1}^{m+1} \frac{k-m}{(2k+1)!} (-1)^{k+1} x^{2k} \\
&= -\frac{2m}{2m+1} + \frac{2}{2m+1} \sum_{k=0}^{\infty} \frac{k-m}{(2k+1)!} (-1)^{k+1} x^{2k} \\
&\quad - \frac{2}{2m+1} \sum_{k=1}^{m+1} \frac{k-m}{(2k+1)!} (-1)^{k+1} x^{2k} \\
&= \frac{2}{2m+1} \sum_{k=m+2}^{\infty} \frac{k-m}{(2k+1)!} (-1)^{k+1} x^{2k} = \frac{2}{2m+1} x^{2m} T,
\end{aligned}$$

where

$$T = \sum_{k=2}^{\infty} \frac{k}{(2k+2m+1)!} (-1)^k x^{2k}$$

(for the last line, we used a change of index (k is changed by $k-m$) and the fact that $(-1)^{m+1} = 1$). Then T can be viewed as a rest of the alternating series U given by

$$U = \sum_{k=1}^{\infty} (-1)^k b_k,$$

with $b_k = (k/(2k+2m+1)!)x^{2k}$. In order to determine the sign of T , let us prove that U is convergent.

For $k \geq 1$ and $x \in (0, \pi)$, we have

$$k[4(k+m)^2 + 10(k+m) + 6] - kx^2 \geq k(42 - \pi^2) > x^2.$$

Equivalently,

$$k(2k+2m+3)(2k+2m+2) > (k+1)x^2$$

which gives

$$\frac{k}{(2k+2m+1)!} x^{2k} > \frac{k+1}{(2k+2m+3)!} x^{2k+2}.$$

Hence $|b_k| > |b_{k+1}|$. Moreover, for $x \in (0, \pi)$, we have

$$\lim_{k \rightarrow \infty} |b_k| = \lim_{k \rightarrow \infty} \frac{k}{(2k+2m+1)!} x^{2k} = \frac{1}{2} \lim_{k \rightarrow \infty} \frac{x^{2k}}{(2k+2m)!} = 0.$$

This implies that U is convergent and, by a special result on alternating series, T has the same sign to the first term of its sum, i.e. $(-1)^2 b_2 > 0$, so $T > 0$. Thus, $f(x) > 0$ and

$$F(x) < \frac{\sin(x)}{x}. \quad (3.14)$$

Similarly, let us set

$$\begin{aligned} g(x) &= G(x) - \frac{\sin(x)}{x} \\ &= \frac{(2m+2) + \cos(x)}{2m+3} + \frac{2}{2m+3} \sum_{k=1}^{m+2} \frac{k-m-1}{(2k+1)!} (-1)^{k+1} x^{2k} - \frac{\sin(x)}{x}. \end{aligned}$$

Using (2.1), (2.2) and proceeding as in case of $f(x)$ gives

$$\begin{aligned} g(x) &= \frac{2}{2m+3} \left[\sum_{k=1}^{m+2} \frac{k-m-1}{(2k+1)!} (-1)^{k+1} x^{2k} - \sum_{k=1}^{\infty} \frac{k-m-1}{(2k+1)!} (-1)^{k+1} x^{2k} \right] \\ &= \frac{2}{2m+3} x^{2m+2} T_*, \end{aligned}$$

where

$$T_* = \sum_{k=2}^{\infty} \frac{k}{(2k+2m+3)!} (-1)^k x^{2k}.$$

By applying the same arguments related to the alternating series used to show that $T > 0$, we prove that $T_* > 0$, implying that $g(x) > 0$ and

$$\frac{\sin(x)}{x} < G(x). \quad (3.15)$$

The proof follows from (3.14) and (3.15). \square

Some particular cases of Theorem 6 are presented below. If $x \in (0, \pi)$ then by putting $n = 1$ (so $m = 1$) in (3.13) we get

$$\frac{2 + \cos(x)}{3} - \frac{x^4}{180} < \frac{\sin(x)}{x} < \frac{4 + \cos(x)}{5} - \frac{x^2}{15} + \frac{x^6}{12600}. \quad (3.16)$$

By putting $n = 2$ (so $m = 3$) in (3.13) we get

$$\begin{aligned} &\frac{6 + \cos(x)}{7} - \frac{2x^2}{21} + \frac{x^4}{420} - \frac{x^8}{1270080} < \frac{\sin(x)}{x} \\ &< \frac{8 + \cos(x)}{9} - \frac{x^2}{9} + \frac{x^4}{270} - \frac{x^6}{22680} + \frac{x^{10}}{179625600}. \end{aligned} \quad (3.17)$$

By putting $n = 3$ (so $m = 5$) in (3.13) we get

$$\begin{aligned}
& \frac{10 + \cos(x)}{11} - \frac{4x^2}{33} + \frac{x^4}{220} - \frac{x^6}{13860} + \frac{x^8}{1995840} - \frac{x^{12}}{34248614400} < \frac{\sin(x)}{x} \\
& < \frac{12 + \cos(x)}{13} - \frac{5x^2}{39} + \frac{x^4}{195} - \frac{x^6}{8190} + \frac{x^8}{1179360} - \frac{x^{10}}{259459200} \\
& + \frac{x^{14}}{8499883392000}. \tag{3.18}
\end{aligned}$$

For comparison between the particular cases, it can be verified by any mathematical software that the bounds of $\sin(x)/x$ in (3.16), (3.17) and (3.18) are sharper than the corresponding bounds in (1.6), (1.8) and (1.9) respectively. Moreover, all the bounds are trigonometric-polynomial. Therefore, considering the number of terms and the degrees of polynomials in respective bounds, we conclude that our bounds are better.

4 An Application

The Yang Le's inequality [19] can be stated as follows.

If $A_1 > 0, A_2 > 0, A_1 + A_2 \leq \pi$ and $0 \leq \lambda \leq 1$, then

$$\cos^2(\lambda A_1) + \cos^2(\lambda A_2) - 2\cos(\lambda A_1)\cos(\lambda A_2)\cos(\lambda\pi) \geq \sin^2(\lambda\pi) \tag{4.1}$$

Inequality (4.1) and its generalizations play an important role in the theory of distribution of values. Therefore many refinements of (4.1) are established so far. For more details we refer reader to [5, 8, 14, 20] and the references therein.

Recently C.-P. Chen and N. Elezović [5] obtained an improved form of Yang Le's inequality which is given below:

Let $A_i > 0$ with $\sum_{i=1}^n A_i \leq \pi$ and $n \in \mathbb{N}, n > 1, 0 \leq \mu \leq 1$. Then

$$N(\mu) \leq \sum_{1 \leq i < j \leq n} H_{ij} \leq M(\mu), \tag{4.2}$$

where

$$H_{ij} = \cos^2(\mu A_i) + \cos^2(\mu A_j) - 2\cos(\mu A_i)\cos(\mu A_j)\cos(\mu\pi),$$

$$N(\mu) = \frac{n(n-1)}{2} \pi^2 \left[\frac{1 - (\mu/2)^2}{1 + (\mu/2)^2} \right]^2 \mu^2 \cos^2 \left(\frac{\mu\pi}{2} \right)$$

and

$$M(\mu) = \frac{n(n-1)}{2} \pi^2 \left[\frac{1 - (\mu/2)^3}{1 + 2(\mu/2)^3} \right]^2 \mu^2.$$

Here we refine the Yang Le's inequality by using the tractable bounds of $\sin(x)/x$.

Theorem 9. *Let, $A_i > 0$ with $\sum_{i=1}^n A_i \leq \pi$ and $n \in \mathbb{N}$, $n > 1$; $0 \leq \mu \leq 1$. Then*

$$P(\mu) \leq \sum_{1 \leq i < j \leq n} H_{ij} \leq Q(\mu), \quad (4.3)$$

where

$$H_{ij} = \cos^2(\mu A_i) + \cos^2(\mu A_j) - 2\cos(\mu A_i) \cos(\mu A_j) \cos(\mu\pi),$$

$$P(\mu) = 2n(n-1) \left[1 + \frac{\pi-2}{2} \cos \left(\frac{\mu\pi}{2} \right) \right]^2 \mu^2 \cos^2 \left(\frac{\mu\pi}{2} \right)$$

and

$$Q(\mu) = 2n(n-1) \left\{ \left[1 + \frac{\pi-2}{2} \cos \left(\frac{\mu\pi}{2} \right) \right] + \frac{\mu^2 \pi^2 (\pi-3)}{24} \right\}^2 \mu^2.$$

Proof of Theorem 9. We substitute $x = \mu\pi/2$ in (3.3) to get

$$\mu \left[1 + \frac{\pi-2}{2} \cos \left(\frac{\mu\pi}{2} \right) \right] < \sin \left(\frac{\mu\pi}{2} \right)$$

and

$$\sin \left(\frac{\mu\pi}{2} \right) < \mu \left[1 + \frac{\pi-2}{2} \cos \left(\frac{\mu\pi}{2} \right) \right] + \frac{\mu^3 \pi^2 (\pi-3)}{24}.$$

Using the inequality [20], for $1 \leq i < j \leq n$, we have

$$4\sin^2 \left(\frac{\mu\pi}{2} \right) \cos^2 \left(\frac{\mu\pi}{2} \right) \leq H_{ij} \leq 4\sin^2 \left(\frac{\mu\pi}{2} \right).$$

We deduce that

$$\begin{aligned} & 4 \left[1 + \frac{\pi-2}{2} \cos \left(\frac{\mu\pi}{2} \right) \right]^2 \mu^2 \cos^2 \left(\frac{\mu\pi}{2} \right) \leq H_{ij} \\ & \leq 4 \left\{ \left[1 + \frac{\pi-2}{2} \cos \left(\frac{\mu\pi}{2} \right) \right] + \frac{\mu^2 \pi^2 (\pi-3)}{24} \right\}^2 \mu^2. \end{aligned} \quad (4.4)$$

Summing up all the inequalities in (4.4), we get (4.3). \square

References

- [1] G. D. Anderson, M. K. Vamanamurthy, M. Vuorinen, *Conformal Invariants, Inequalities and Quasiconformal maps*, John Wiley and Sons, New York, 1997.
- [2] Y. J. Bagul, C. Chesneau, *Some new simple inequalities involving exponential, trigonometric and hyperbolic functions*, Preprint, hal-01930521, 2018.
- [3] B. A. Bhayo, J. Sándor, *On Carlson's and Shafer's inequalities*, Issues of Analysis, Volume **3**(21), Number 1, 2014, pp. 3-15.
- [4] B. A. Bhayo, R. Klén and J. Sándor, *New Trigonometric and Hyperbolic Inequalities*, Miskolc Mathematical Notes, Volume **18**, Number 1, 2017, pp. 125-137.
- [5] C.-P. Chen and N. Elezović, *Sharp Redheffer-type and Becker-Stark-type inequalities with an application*, Math. Inequal. Appl., Volume **21**, Number 4, 2018, Pages 1059-1078.
- [6] C.-P. Chen and W.-S. Cheung, *Sharp Cusa and Becker-Stark inequalities*, J. Inequal. Appl., Volume 2011, article 136, 2011.
- [7] C. Huygens, *Oeuvres Completes*, Société Hollondaise des Sciences, Haga, 1888-1940.
- [8] W. D. Jiang and H. Yun, *Sharpening of Jordan's inequality and its applications*, Journal of Inequalities in Pure and Applied Mathematics, Volume **7**, Number 3, Article 102, 2006, Pages 4.
- [9] B. Malešević, T. Lutovac, M. Rašajski and C. Mortici, *Extensions of the natural approach to refinements and generalizations of some trigonometric inequalities*, Advances in Difference Equations, 2018, 2018:90.
- [10] D. S. Mitrinovic, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.
- [11] C. Mortici, *The natural approach of Wilker-Cusa-Huygens Inequalities*, Math. Inequal. Appl., Volume **14**, Number 3, 2011, pp. 535-541.
- [12] E. Neuman and J. Sándor, *On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker and Huygens inequalities*, Math. Inequal. Appl., Volume **13**, Number 4, 2010, pp. 715-723.

- [13] A. Oppenheim, *E1277*, Amer. Math. Monthly, Volume **64**, Number 6, 1957, page no. 504.
- [14] A. Y. Özban, *A new refined form of Jordan's inequality and its applications*, Applied Mathematics Letters, Volume **19**, Number 2, February 2006, pp. 155-160.
- [15] F. Qi, Q.-M. Luo and B.-N. Guo, *A Simple proof of Oppenheim's double inequality relating to the cosine and sine functions*, J. Math. Inequal., Volume **6**, Number 4, 2012, pp. 645-654.
- [16] J. Sándor, *Sharp Cusa-Huygens and related inequalities*, Notes on Number Theory and Discrete Mathematics, Volume **19**, Number 1, 2013, pp. 50-54.
- [17] J. Sándor and M. Bencze, *On Huygen's trigonometric inequality*, RGMIA Res. Rep. Coll., Volume **8**, Number 3, 2005, Art. 14.
- [18] J. Sándor and Róbert Oláh-Gal, *On Cusa-Huygens type trigonometric and hyperbolic inequalities*, Acta. Univ. Sapientiae Mathematica, Volume **4**, Number 2, 2012, pp. 145-153.
- [19] L. Yang, *Distribution of Values and New Research*, Science Press, Beijing (in Chinese), 1982.
- [20] C. J. Zhao, *Generalization and strengthen of Yang Le inequality*, Mathematics Practice Theory, Volume **30**, 2000 (in Chinese), pp. 493-497.