



HAL
open science

Trotter product formula and linear evolution equations on Hilbert spaces On the occasion of the 100th birthday of Tosio Kato

Hagen Neidhardt, Artur Stephan, Valentin A Zagrebnov

► **To cite this version:**

Hagen Neidhardt, Artur Stephan, Valentin A Zagrebnov. Trotter product formula and linear evolution equations on Hilbert spaces On the occasion of the 100th birthday of Tosio Kato. 2019. hal-01971610v1

HAL Id: hal-01971610

<https://hal.science/hal-01971610v1>

Preprint submitted on 7 Jan 2019 (v1), last revised 28 Jan 2020 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Trotter product formula and linear evolution equations on Hilbert spaces

On the occasion of the 100th birthday of Tosio Kato

Hagen Neidhardt, Artur Stephan, Valentin A. Zagrebnov

Abstract The paper is devoted to evolution equations of the form

$$\frac{\partial}{\partial t}u(t) = -(A + B(t))u(t), \quad t \in \mathcal{I} = [0, T],$$

on separable Hilbert spaces where A is a non-negative self-adjoint operator and $B(\cdot)$ is family of non-negative self-adjoint operators such that $\text{dom}(A^\alpha) \subseteq \text{dom}(B(t))$ for some $\alpha \in [0, 1)$ and the map $A^{-\alpha}B(\cdot)A^{-\alpha}$ is Hölder continuous with the Hölder exponent $\beta \in (0, 1)$. It is shown that the solution operator $U(t, s)$ of the evolution equation can be approximated in the operator norm by a combination of semigroups generated by A and $B(t)$ provided the condition $\beta > 2\alpha - 1$ is satisfied. The convergence rate for the approximation is given by the Hölder exponent β . The result is proved using the evolution semigroup approach.

H. Neidhardt
Weierstrass Institute for Applied Analysis and Stochastics,
Mohrenstr. 39, D-10117 Berlin, Germany
e-mail: hagen.neidhardt@wias-berlin.de

A. Stephan
Institut für Mathematik, Humboldt-Universität zu Berlin,
Unter den Linden 6, D-10099 Berlin, Germany
e-mail: stephan@math.hu-berlin.de

V. A. Zagrebnov
Institut de Mathématiques de Marseille (UMR 7373)
Université d'Aix-Marseille, CMI - Technopôle Château-Gombert
39 rue F. Joliot Curie, 13453 Marseille, France
e-mail: valentin.zagrebnov@univ-amu.fr

1 Introduction

A closer look to Kato's work shows that abstract evolution equations and Trotter product formula were topics of high interest for Kato. Already at the beginning of his scientific career Kato was interested in evolution equations [16, 17]. This interest has lasted a lifetime [18, 19, 20, 21, 26, 27, 29]. Another topic of great interest for him was the so-called Trotter product formula [23, 24, 22, 28]. Even the paper [23] has inspired further developments in this field [15].

The topic of the present paper is to link evolution equations with the Trotter product formula. To this end we consider an abstract evolution equation of type

$$\begin{aligned} \frac{\partial u(t)}{\partial t} &= -C(t)u(t), \quad u(s) = x_s, \quad s \in [0, T], \quad t \in \mathcal{I} := [0, T], \\ C(t) &= A + B(t), \end{aligned} \quad (1.1)$$

on the separable Hilbert space \mathfrak{H} . Evolution equations of that type on Hilbert or Banach spaces are widely investigated, cf. [1, 2, 4, 3, 7, 29, 31, 32, 43, 44, 45, 46, 47, 48, 51, 52, 53, 55, 56] or the books [5, 49, 54]. We consider the equation (1.1) under the following assumptions.

Assumption 1.1

- (S1) The operator A is self-adjoint in the Hilbert space \mathfrak{H} such that $A \geq I$. Let $\{B(t)\}_{t \in \mathcal{I}}$ be a family of non-negative self-adjoint operators in \mathfrak{H} such that the function $(I + B(\cdot))^{-1} : \mathcal{I} \rightarrow \mathcal{L}(\mathfrak{H})$ is strongly measurable.
- (S2) There is an $\alpha \in [0, 1)$ such that for a.e. $t \in \mathcal{I}$ the inclusion $\text{dom}(A^\alpha) \subseteq \text{dom}(B(t))$ holds. Moreover, the function $B(\cdot)A^{-\alpha} : \mathcal{I} \rightarrow \mathcal{L}(\mathfrak{H})$ is strongly measurable and essentially bounded, i.e.

$$C_\alpha := \text{ess sup}_{t \in \mathcal{I}} \|B(\cdot)A^{-\alpha}\| < \infty. \quad (1.2)$$

- (S3) The map $A^{-\alpha}B(\cdot)A^{-\alpha} : \mathcal{I} \rightarrow \mathcal{L}(\mathfrak{H})$ is Hölder continuous, i.e. for some $\beta \in (0, 1)$ there is a constant $L_{\alpha, \beta} > 0$ such that the estimate

$$\|A^{-\alpha}(B(t) - B(s))A^{-\alpha}\| \leq L_{\alpha, \beta} |t - s|^\beta, \quad (t, s) \in \mathcal{I} \times \mathcal{I}, \quad (1.3)$$

holds. △

Notice that under the assumption (S2) the operator $C(t)$ is also an invertible non-negative self-adjoint operator for each $t \in \mathcal{I}$. Assumptions of that type were made in [13, 14, 34, 35, 56]. One checks that the assumptions (S1)-(S3) and the additional assumption $\beta > \alpha$ imply the assumptions (I), (VI) and (VII) of [56] for the family $\{C(t)\}_{t \in \mathcal{I}}$. Hence, Proposition 3.1 and Theorem 3.2 of [56] yield the existence of a so-called *solution* (or *evolution*) operator for the evolution equation (1.1), i.e., a strongly continuous, uniformly bounded family of bounded operators $\{U(t, s)\}_{(t, s) \in \Delta}$, $\Delta := \{(t, s) \in \mathcal{I} \times \mathcal{I} : 0 \leq s \leq t \leq T\}$, such that the conditions

$$\begin{aligned} U(t,t) &= I, \quad \text{for } t \in \mathcal{I}, \\ U(t,r)U(r,s) &= U(t,s), \quad \text{for } t,r,s \in \mathcal{I} \quad \text{with } s \leq r \leq t, \end{aligned} \quad (1.4)$$

are satisfied and $u(t) = U(t,0)x$ is for every $x \in \mathfrak{X}$ a strict solution of (1.1), see Definition 1.1 of [56]. Because the involved operators are self-adjoint and non-negative one checks that the solution operator consists of contractions.

The aim of the present paper is to analyze the convergence of the following approximation to the solution operator $\{U(t,s)\}_{(t,s) \in \Delta}$. Let

$$s =: t_0 < t_1 < \dots < t_{n-1} < t_n := t, \quad t_j := s + j \frac{t-s}{n}, \quad (1.5)$$

$j = \{0, 1, 2, \dots, n\}$, $n \in \mathbb{N}$, be a partition of the interval $[s, t]$. Let

$$\begin{aligned} G_j(t,s;n) &:= e^{-\frac{t-s}{n}A} e^{-\frac{t-s}{n}B(t_j)}, \quad j = 0, 1, 2, \dots, n, \\ V_n(t,s) &:= G_{n-1}(t,s;n)G_{n-1}(t,s;n) \times \dots \times G_2(t,s;n)G_0(t,s;n), \end{aligned} \quad (1.6)$$

$n \in \mathbb{N}$. The main result in the paper is the following. If the assumptions (S1)-(S3) are satisfied and in addition the condition $\beta > \alpha$ holds, then the solution operator $\{U(t,s)\}_{(t,s) \in \Delta}$ of [56] admits the approximation

$$\operatorname{ess\,sup}_{(t,s) \in \Delta} \|V_n(t,s) - U(t,s)\| \leq \frac{R_\beta}{n^\beta}, \quad n \in \mathbb{N}, \quad (1.7)$$

with some constant $R_\beta > 0$. The result shows that the convergence of the approximation $\{V_n(t,s)\}_{(t,s) \in \Delta}$ is determined by the smoothness of the perturbation $B(\cdot)$.

If the map $A^{-\alpha}B(\cdot)A^{-\alpha} : \mathcal{I} \rightarrow \mathcal{L}(\mathfrak{X})$ is Lipschitz continuous, then the map is of course Hölder continuous with any exponent $\gamma \in (\alpha, 1)$. Hence from (1.7) it immediately follows that for any $\gamma \in (\alpha, 1)$ there is a constant R_γ such that

$$\operatorname{ess\,sup}_{(t,s) \in \Delta} \|V_n(t,s) - U(t,s)\| \leq \frac{R_\gamma}{n^\gamma}, \quad n \in \mathbb{N}. \quad (1.8)$$

In particular, for any γ close to one the estimate (1.8) holds.

In [14] the Lipschitz case was considered. It was shown that there is a constant $\Upsilon_0 > 0$ such that the estimate

$$\operatorname{ess\,sup}_{t \in \mathcal{I}} \|V_n(t,0) - U(t,0)\| \leq \Upsilon_0 \frac{\log(n)}{n}, \quad n = 2, 3, \dots \quad (1.9)$$

holds. It is obvious that the estimate (1.9) is stronger than

$$\operatorname{ess\,sup}_{t \in \mathcal{I}} \|V_n(t,0) - U(t,0)\| \leq \frac{R_\gamma}{n^\gamma}, \quad n \in \mathbb{N}.$$

(which follows from (1.8) for any γ independent of how close it is to one.

To prove (1.7) we use the so-called evolution semigroup approach which allows not only to verify the estimate (1.7) but also to generalise it. The approach is quite different from the technique used in [14, 56]. We have successfully applied this approach already in [34] and [35]. The key idea is to forget about the evolution equation (1.1) and to consider instead of it the operators \mathcal{K}_0 and \mathcal{K} on $\mathfrak{K} = L^2(\mathcal{I}, \mathfrak{H})$. The operator \mathcal{K}_0 is the generator of the contraction semigroup $\{\mathcal{U}_0(\tau)\}_{\tau \in \mathbb{R}_+}$,

$$(\mathcal{U}_0(\tau)f)(t) := e^{-\tau A} \chi_{\mathcal{I}}(t - \tau) f(t - \tau), \quad f \in L^2(\mathcal{I}, \mathfrak{H}), \quad (1.10)$$

and \mathcal{K} is given by

$$\mathcal{K} = \mathcal{K}_0 + \mathcal{B}, \quad \text{dom}(\mathcal{K}) = \text{dom}(\mathcal{K}_0) \cap \text{dom}(\mathcal{B}),$$

where \mathcal{B} is the multiplication operator induced by the family $\{\mathcal{B}(t)\}_{t \in \mathcal{I}}$ in $L^2(\mathcal{I}, \mathfrak{H})$ which is self-adjoint and non-negative, for more details see Section 2. It turns out that under the assumptions (S1) and (S2) the operator \mathcal{K} is the generator of a contraction semigroup $\{\mathcal{U}(\tau)\}_{\tau \in \mathbb{R}_+}$ on $L^2(\mathcal{I}, \mathfrak{H})$. For the pair $\{\mathcal{K}_0, \mathcal{B}\}$ we consider the Lie-Trotter product formula. From the original paper of Trotter [50] one gets that

$$s\text{-}\lim_{n \rightarrow \infty} \left(e^{-\frac{\tau}{n} \mathcal{K}_0} e^{-\frac{\tau}{n} \mathcal{B}} \right)^n = e^{-\tau \mathcal{K}}, \quad \tau \in \mathbb{R}_+ := [0, \infty), \quad (1.11)$$

holds uniformly in τ on any bounded interval of \mathbb{R}_+ . Since $e^{-\tau \mathcal{K}_0} = 0$ and $e^{-\tau \mathcal{K}} = 0$ for $\tau \geq T$ one gets even uniformly in $\tau \in \mathbb{R}_+$.

Previously it was shown that under certain assumptions the strong convergence can be improved to operator-norm convergence on Hilbert spaces, see [9, 10, 15, 38, 42] as well as on Banach spaces, see [11]. For an overview the reader is referred to [37]. To consider the Trotter product formula for evolution equations is relatively new and was firstly realized in [34, 35] for Banach spaces.

In the following we improve the convergence (1.11) to operator-norm convergence. We show that under the assumptions (S1)-(S3) and $\beta > 2\alpha - 1$ there is a constant $R_\beta > 0$ such that

$$\sup_{\tau \in \mathbb{R}_+} \left\| \left(e^{-\frac{\tau}{n} \mathcal{K}_0} e^{-\frac{\tau}{n} \mathcal{B}} \right)^n - e^{-\tau \mathcal{K}} \right\| \leq \frac{R_\beta}{n^\beta}, \quad n \in \mathbb{N}, \quad (1.12)$$

holds.

It turns out that \mathcal{K} is the generators of an evolution semigroup. This means, there is a propagator $\{U(t, s)\}_{(t, s) \in \Delta_0}$, $\Delta_0 := \{(t, s) \in \mathcal{I}_0 \times \mathcal{I}_0 : s \leq t\}$, $\mathcal{I}_0 = (0, T]$, such that the contraction semigroup $\{\mathcal{U}(\tau) = e^{-\tau \mathcal{K}}\}_{\tau \in \mathbb{R}_+}$ admits the representation

$$(\mathcal{U}(\tau)f)(t) = U(t, t - \tau) \chi_{\mathcal{I}}(t - \tau) f(t - \tau), \quad f \in L^2(\mathcal{I}, \mathfrak{H}). \quad (1.13)$$

We recall that a strongly continuous, uniformly bounded family of bounded operators $\{U(t, s)\}_{(t, s) \in \Delta_0}$ is called a propagator if (1.4) is satisfied for \mathcal{I}_0 and Δ_0 instead of \mathcal{I} and Δ , respectively. Roughly speaking, a propagator is a solution operator

restricted to Δ_0 where the assumption that $U(t,0)x$ should be a strict solution is dropped. Obviously, the notion of a propagator is weaker than that one of a solution operator. For its existence one needs only the assumptions (S1) and (S2), see Theorem 4.4 and 4.5 in [34] or Theorem 3.3 [35]. Of course, the propagator coincides with the solution operator of [56] if the assumptions (S1)-(S3) are satisfied and $\beta > \alpha$.

By Proposition 3.8 of [37] and (1.12) we immediately get that under the assumptions (S1)-(S3) and $\beta > 2\alpha - 1$ the estimate

$$\operatorname{ess\,sup}_{(t,s) \in \Delta_0} \|V_n(t,s) - U(t,s)\| \leq \frac{R_\beta}{n^\beta}, \quad n \in \mathbb{N}, \quad (1.14)$$

holds, where the constant R_β is that one of (1.12). Notice that the condition $\beta > 2\alpha - 1$ is weaker than $\beta > \alpha$, i.e., if $\beta > \alpha$, then $\beta > 2\alpha - 1$ holds. If α satisfies the condition $\frac{1+\beta}{2} > \alpha > \beta$, then the assumptions (I), (VI) and (VII) of [56] for the family $\{C(t)\}_{t \in \mathcal{I}}$ are not valid but nevertheless we get an approximation of the corresponding propagator $\{U(t,s)\}_{(t,s) \in \Delta_0}$.

The results are stronger than those in [34, 35] for Banach spaces. In [34] a convergence rate $O(n^{-(\beta-\alpha)})$ was found, whereas in [35] the Lipschitz case has been considered and the rate $O(n^{-(1-\alpha)})$ for $\alpha \in (\frac{1}{2}, 1)$ was proved.

It turns out that the result (1.7) can be hardly improved. Indeed in [36] the simple case $\mathfrak{H} := \mathbb{C}$ and $A = 1$ was considered. In that case the family $\{B(t)\}_{t \in \mathcal{I}}$ reduces to a non-negative bounded measurable function: $b(\cdot) : \mathcal{I} \rightarrow \mathbb{R}$ which has to be Hölder continuous with exponent $\beta \in (0, 1)$. For that case it was found in [36] that the convergence rate is $O(n^{-\beta})$ which coincides with (1.7). For the Lipschitz case it was found $O(n^{-1})$ which suggests that (1.8) and (1.9) might be not optimal.

The paper is organised as follows. In Section 2 we give a short introduction into evolution semigroups. For more details the reader is referred to [33, 34, 39, 40]. The results are proven in Section 3. In Section 3.1 we prove auxiliary results which are necessary to prove the main results of Section 3.2.

Notation: Spaces, in particular, Hilbert are denoted by Gothic capital letters like \mathfrak{H} , \mathfrak{K} , etc. Operators are denoted by Latin or italic capital letters. The Banach space of bounded operators on space is denoted by $\mathcal{L}(\cdot)$, like $\mathcal{L}(\mathfrak{H})$. We set $\mathbb{R}_+ := [0, \infty)$. If a function is called measurable, then it means Lebesgue measurable. The notation ‘‘a.e.’’ means that a statement or relation fails at most for a set of Lebesgue measure zero. In the following we use the notation $\operatorname{ess\,sup}_{(t,s) \in \Delta}$ or $\operatorname{ess\,sup}_{(t,s) \in \Delta_0}$. In that case the Lebesgue measure of \mathbb{R}^2 is meant.

We point out that we call operator \mathcal{K} to be generator of a semigroup $\{e^{-\tau\mathcal{K}}\}_{\tau \in \mathbb{R}_+}$, see e.g. [41], although in [12, 25] it is the operator $-\mathcal{K}$, which is called the generator.

2 Evolution semigroups

Below we consider the Hilbert space $\mathfrak{K} = L^2(\mathcal{I}, \mathfrak{H})$ consisting of all measurable functions $f(\cdot) : \mathcal{I} \rightarrow \mathfrak{H}$ such that the norm function $\|f(\cdot)\| : \mathcal{I} \rightarrow \mathbb{R}_+$ is square integrable. Further, let D_0 be the generator of the right-hand sift semigroup on $L^2(\mathcal{I}, \mathfrak{H})$, i.e.

$$(e^{-\tau D_0} f)(t) = \chi_{\mathcal{I}}(t - \tau) f(t - \tau), \quad t \in \mathcal{I}, \quad \tau \in \mathbb{R}_+, \quad f \in L^2(\mathcal{I}, \mathfrak{H}).$$

Notice that $e^{-\tau D_0} = 0$ for $\tau \geq T$. The generator D_0 is given by

$$\begin{aligned} (D_0 f)(t) &:= \frac{\partial}{\partial t} f(t), \\ f \in \text{dom}(D_0) &:= W_0^{1,2}(\mathcal{I}, \mathfrak{H}) = \{f \in W^{1,2}(\mathcal{I}, \mathfrak{H}) : f(0) = 0\}. \end{aligned} \quad (2.1)$$

We remark that D_0 is a closure of the maximal symmetric operator and its semigroup is contractive.

Further we consider the multiplication operator \mathcal{A} in $L^2(\mathcal{I}, \mathfrak{H})$,

$$\begin{aligned} (\mathcal{A}f)(t) &:= Af(t), \\ f \in \text{dom}(\mathcal{A}) &:= \left\{ f \in L^2(\mathcal{I}, \mathfrak{H}) : \begin{array}{l} f(t) \in \text{dom}(A) \text{ for a.e. } t \in \mathcal{I} \\ Af(t) \in L^2(\mathcal{I}, \mathfrak{H}). \end{array} \right\} \end{aligned}$$

If (S1) is satisfied, then \mathcal{A} is self-adjoint and $\mathcal{A} \geq I_{L^2(\mathcal{I}, \mathfrak{H})}$. For the resolvent one has the representation

$$((\mathcal{A} - z)^{-1} f)(t) = (A - z)^{-1} f(t), \quad t \in \mathcal{I}_0, \quad f \in L^2(\mathcal{I}, \mathfrak{H}), \quad z \in \rho(\mathcal{A}) = \rho(A),$$

and the corresponding semigroup $\{e^{-\tau \mathcal{A}}\}_{\tau \in \mathbb{R}_+}$ is given by

$$(e^{-\tau \mathcal{A}} f)(t) = e^{-\tau A} f(t), \quad t \in \mathcal{I}, \quad f \in L^2(\mathcal{I}, \mathfrak{H}), \quad \tau \in \mathbb{R}_+. \quad (2.2)$$

Notice that the operators $e^{-\tau D_0}$ and $e^{-\tau \mathcal{A}}$ commute. Let us consider the contraction semigroup

$$\mathcal{U}_0(\tau) := e^{-\tau D_0} e^{-\tau \mathcal{A}}, \quad \tau \in \mathbb{R}_+. \quad (2.3)$$

Obviously, the semigroup $\{\mathcal{U}_0(\tau)\}_{\tau \in \mathbb{R}_+}$ admits the representation (1.13). Due to the maximal L^2 -regularity of A , cf. [6], its generator \mathcal{K}_0 is given by

$$\mathcal{K}_0 := D_0 + \mathcal{A}, \quad \text{dom}(\mathcal{K}_0) := \text{dom}(D_0) \cap \text{dom}(\mathcal{A}).$$

Further we consider the multiplication operator \mathcal{B} , defined as

$$\begin{aligned} (\mathcal{B}f)(t) &:= B(t)f(t) \\ f \in \text{dom}(\mathcal{B}) &:= \left\{ f \in L^2(\mathcal{I}, \mathfrak{H}) : \begin{array}{l} f(t) \in \text{dom}(B(t)) \text{ for a.e. } t \in \mathcal{I} \\ B(t)f(t) \in L^2(\mathcal{I}, \mathfrak{H}) \end{array} \right\}. \end{aligned} \quad (2.4)$$

If (S1) is satisfied, then \mathcal{B} is self-adjoint and non-negative. For the resolvent we have the representation

$$((\mathcal{B} - z)^{-1}f)(t) = (\mathcal{B}(t) - z)^{-1}f(t), \quad f \in L^2(\mathcal{I}, \mathfrak{H}), \quad z \in \mathbb{C}_{\pm},$$

for a.e. $t \in \mathcal{I}$. The semigroup $\{e^{-\tau\mathcal{B}}\}_{\tau \in \mathbb{R}_+}$, admits the representation

$$(e^{-\tau\mathcal{B}}f)(t) = e^{-\tau\mathcal{B}(t)}f(t), \quad f \in L^2(\mathcal{I}, \mathfrak{H}),$$

for a.e. $t \in \mathcal{I}$.

By [34, Proposition 4.4] we get that under the assumptions (S1) and (S2) the operator

$$\mathcal{K} := \mathcal{K}_0 + \mathcal{B}, \quad \text{dom}(\mathcal{K}) := \text{dom}(\mathcal{K}_0) \cap \text{dom}(\mathcal{B}),$$

is a generator of a contraction semigroup on $L^2(\mathcal{I}, \mathfrak{H})$. From [34, Proposition 4.5] we obtain that \mathcal{K} is the generator of an evolution semigroup. Because \mathcal{K} is a generator of a contraction semigroup it turns out that the corresponding propagator consists of contractions.

If $\{U(t, s)\}_{(t, s) \in \Delta_0}$ is a propagator, then by virtue of (1.13) it defines a semigroup, which by definition is an evolution semigroup. It turns out that there is a one-to-one correspondence between the set of evolution semigroups on $L^2(\mathcal{I}, \mathfrak{H})$ and propagators. It is interesting to note that evolution generators can be characterize quite independent from a propagator, see [33, Theorem 2.8] or [34, Theorem 3.3].

3 Results

We start with a general observation concerning the conditions (S1)-(S3).

Remark 3.1 If the conditions (S1)-(S3) are satisfied for some $\alpha \in [0, 1)$, then they are also satisfied for each $\alpha' \in (\alpha, 1]$. Indeed, the condition (S1) is obviously satisfied. To show (S2) we note that $\text{dom}(A^{\alpha'}) \subseteq \text{dom}(A^\alpha) \subseteq \text{dom}(B(t))$ for a.e. $t \in \mathcal{I}$. Using the representation

$$B(t)A^{-\alpha'} = B(t)A^{-\alpha}A^{-(\alpha' - \alpha)} \tag{3.1}$$

for a.e. $t \in \mathcal{I}$ we get that the map $B(\cdot)A^{-\alpha'} : \mathcal{I} \rightarrow \mathcal{L}(\mathfrak{H})$ is strongly measurable. Further, from (3.1)

$$C_{\alpha'} := \text{ess sup}_{t \in \mathcal{I}} \|B(t)A^{-\alpha'}\| \leq \text{ess sup}_{t \in \mathcal{I}} \|B(t)A^{-\alpha}\| = C_\alpha < \infty.$$

Moreover we have

$$\|A^{-\alpha'}(B(t) - B(s))A^{-\alpha'}\| \leq \|A^{-\alpha}(B(t) - B(s))A^{-\alpha}\| \leq L_{\alpha, \beta} |t - s|^\beta,$$

$(t, s) \in \mathcal{I} \times \mathcal{I}$, which shows that there is a constant $L_{\alpha', \beta} \leq L_{\alpha, \beta}$ such that

$$\|A^{-\alpha'}(B(t) - B(s))A^{-\alpha'}\| \leq L_{\alpha',\beta}|t-s|^\beta \quad (t, s) \in \mathcal{I} \times \mathcal{I}.$$

holds for $(t, s) \in \mathcal{I} \times \mathcal{I}$. \triangle

Since A is self-adjoint and non-negative, one has $\|A^\gamma e^{-\tau A}\| \leq 1/\tau^\gamma$ for any $\tau \in \mathbb{R}_+$ and $\gamma \in [0, 1]$. Then by virtue of (2.2) and of (1.10), (2.3) one gets the estimates

$$\|\mathcal{A}^\gamma e^{-\tau \mathcal{A}}\| = \|\overline{e^{-\tau \mathcal{A}} \mathcal{A}^\gamma}\| \leq \frac{1}{\tau^\gamma} \quad \text{and} \quad \|\mathcal{A}^\gamma e^{-\tau \mathcal{K}_0}\| = \|\overline{e^{-\tau \mathcal{K}_0} \mathcal{A}^\gamma}\| \leq \frac{1}{\tau^\gamma}. \quad (3.2)$$

3.1 Auxiliary estimates

In this section we prove a series of estimates necessary to establish (1.12). The following lemma can be partially derived from [34, Lemma 7.4].

Lemma 3.2 *Let the assumptions (S1) and (S2) be satisfied. Then for any $\gamma \in [\alpha, 1]$ there is a constants $\Lambda_\gamma \geq 1$ such that*

$$\|\mathcal{A}^\gamma e^{-\tau \mathcal{K}}\| \leq \frac{\Lambda_\gamma}{\tau^\gamma} \quad \text{and} \quad \|\overline{e^{-\tau \mathcal{K}} \mathcal{A}^\gamma}\| \leq \frac{\Lambda_\gamma}{\tau^\gamma}, \quad \tau > 0, \quad (3.3)$$

holds.

Proof. The proof of the first estimate follows from Lemma 7.4 of [34] and Remark 3.1. The second estimate can be proved similarly as the first one. One has only to modify the proof of Lemma 7.4 of [34] in a suitable manner and to apply again Remark 3.1. \square

Remark 3.3 Lemma 2.1 of [14] claims that for the Lipschitz case the solution operator $\{U(t, s)\}_{(t,s) \in \Delta}$ of (1.1) admits the estimates

$$\sup_{(t,s) \in \Delta} (t-s)^\gamma \|A^\gamma U(t, s)\| < \infty \quad \text{and} \quad \sup_{(t,s) \in \Delta} (t-s)^\gamma \|U(t, s)A^\gamma\| < \infty$$

for $\gamma \in [0, 1]$. Proposition 2.1 of [36] immediately yields that the corresponding evolution semigroup $\{\mathcal{U}(\tau) = e^{-\tau \mathcal{K}}\}_{\tau \in \mathbb{R}_+}$ satisfies the estimates (3.3) for $\gamma = 1$. \triangle

Now we set

$$\mathcal{T}(\tau) = e^{-\tau \mathcal{K}_0} e^{-\tau \mathcal{B}}, \quad \tau \in \mathbb{R}_+. \quad (3.4)$$

Notice that $\mathcal{T}(\tau) = 0$ for $\tau \geq T$.

Lemma 3.4 *Let the assumptions (S1) and (S2) be satisfied. Then for any $\gamma \in [\alpha, 1]$ the estimates*

$$\|\mathcal{A}^{-\gamma}(\mathcal{T}(\tau) - \mathcal{U}(\tau))\| \leq 2C_\gamma \tau \quad \text{and} \quad \|(\mathcal{T}(\tau) - \mathcal{U}(\tau))\mathcal{A}^{-\gamma}\| \leq 2C_\gamma \tau, \quad (3.5)$$

hold for $\tau \geq 0$, where

$$C_\gamma := \operatorname{ess\,sup}_{t \in \mathcal{I}} \|B(t)A^{-\gamma}\|. \quad (3.6)$$

Proof. The proof of the first estimate follows from Lemma 7.6 of [34] and Remark 3.1. The specific constant $2C_\gamma$ is obtained following carefully the proof of Lemma 7.6 of [34]. The second estimate can be proved modifying the proof of the first estimate in an obvious manner. \square

Lemma 3.5 *Let the assumptions (S1)-(S3) be satisfied. Then for any $\gamma \in [\alpha, 1)$ and $\beta \in (0, 1)$ there is a constant $Z_{\gamma, \beta} > 0$ such that*

$$\|\mathcal{A}^{-\gamma}(\mathcal{T}(\tau) - \mathcal{U}(\tau))\mathcal{A}^{-\gamma}\| \leq Z_{\gamma, \beta} \tau^{1+\varkappa}, \quad \tau \in \mathbb{R}_+, \quad (3.7)$$

holds where $\varkappa := \min\{\gamma, \beta\}$.

Proof. We use the representation:

$$\begin{aligned} \frac{d}{d\sigma} e^{-(\tau-\sigma)\mathcal{K}} e^{-\sigma\mathcal{K}_0} e^{-\sigma\mathcal{B}} &= e^{-(\tau-\sigma)\mathcal{K}} \left\{ \mathcal{K} e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{K}_0} \mathcal{K}_0 - e^{-\sigma\mathcal{K}_0} \mathcal{B} \right\} e^{-\sigma\mathcal{B}} \\ &= e^{-(\tau-\sigma)\mathcal{K}} \left\{ \mathcal{B} e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{K}_0} \mathcal{B} \right\} e^{-\sigma\mathcal{B}} \end{aligned}$$

which yields

$$\begin{aligned} &e^{-(\tau-\sigma)\mathcal{K}} \left\{ \mathcal{B} e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{K}_0} \mathcal{B} \right\} e^{-\sigma\mathcal{B}} \\ &= (e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0}) \left\{ \mathcal{B} e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{K}_0} \mathcal{B} \right\} (e^{-\sigma\mathcal{B}} - I) + \\ &e^{-(\tau-\sigma)\mathcal{K}_0} \left\{ \mathcal{B} e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{K}_0} \mathcal{B} \right\} (e^{-\sigma\mathcal{B}} - I) + \\ &(e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0}) \left\{ \mathcal{B} e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{K}_0} \mathcal{B} \right\} + \\ &e^{-(\tau-\sigma)\mathcal{K}_0} \left\{ \mathcal{B} e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{K}_0} \mathcal{B} \right\}. \end{aligned}$$

Hence, we obtain the identity

$$\begin{aligned} &\mathcal{A}^{-\gamma} e^{-(\tau-\sigma)\mathcal{K}} \left\{ \mathcal{B} e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{K}_0} \mathcal{B} \right\} e^{-\sigma\mathcal{B}} \mathcal{A}^{-\gamma} \\ &= \mathcal{A}^{-\gamma} (e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0}) \left\{ \mathcal{B} e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{K}_0} \mathcal{B} \right\} (e^{-\sigma\mathcal{B}} - I) \mathcal{A}^{-\gamma} + \\ &e^{-(\tau-\sigma)\mathcal{K}_0} \mathcal{A}^{-\gamma} \left\{ \mathcal{B} e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{K}_0} \mathcal{B} \right\} (e^{-\sigma\mathcal{B}} - I) \mathcal{A}^{-\gamma} + \\ &\mathcal{A}^{-\gamma} (e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0}) \left\{ \mathcal{B} e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{K}_0} \mathcal{B} \right\} \mathcal{A}^{-\gamma} + \\ &e^{-(\tau-\sigma)\mathcal{K}_0} \mathcal{A}^{-\gamma} \left\{ \mathcal{B} e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{K}_0} \mathcal{B} \right\} \mathcal{A}^{-\gamma} \end{aligned}$$

which leads to the estimate

$$\begin{aligned} & \left\| \mathcal{A}^{-\gamma} e^{-(\tau-\sigma)\mathcal{K}} \left\{ \mathcal{B} e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{K}_0} \mathcal{B} \right\} e^{-\sigma\mathcal{B}} \mathcal{A}^{-\gamma} \right\| \leq \\ & \left\| \mathcal{A}^{-\gamma} (e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0}) \left\{ \mathcal{B} e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{K}_0} \mathcal{B} \right\} (e^{-\sigma\mathcal{B}} - I) \mathcal{A}^{-\gamma} \right\| + \end{aligned} \quad (3.8)$$

$$\left\| e^{-(\tau-\sigma)\mathcal{K}_0} \mathcal{A}^{-\gamma} \left\{ \mathcal{B} e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{K}_0} \mathcal{B} \right\} (e^{-\sigma\mathcal{B}} - I) \mathcal{A}^{-\gamma} \right\| + \quad (3.9)$$

$$\left\| \mathcal{A}^{-\gamma} (e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0}) \left\{ \mathcal{B} e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{K}_0} \mathcal{B} \right\} \mathcal{A}^{-\gamma} \right\| + \quad (3.10)$$

$$\left\| e^{-(\tau-\sigma)\mathcal{K}_0} \mathcal{A}^{-\gamma} \left\{ \mathcal{B} e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{K}_0} \mathcal{B} \right\} \mathcal{A}^{-\gamma} \right\| \quad (3.11)$$

Note that by (3.2) and (3.6) one gets

$$\begin{aligned} \|e^{-\sigma\mathcal{K}_0} \mathcal{B}\| &= \|e^{-\sigma\mathcal{K}_0} \mathcal{A}^\gamma \overline{\mathcal{A}^{-\gamma} \mathcal{B}}\| \leq \sigma^{-\gamma} \|\overline{\mathcal{A}^{-\gamma} \mathcal{B}}\| = C_\gamma \sigma^{-\gamma}, \\ \|\mathcal{B} e^{-\sigma\mathcal{K}_0}\| &= \|\mathcal{B} \mathcal{A}^{-\gamma} \mathcal{A}^\gamma e^{-\sigma\mathcal{K}_0}\| \leq \sigma^{-\gamma} \|\mathcal{B} \mathcal{A}^{-\gamma}\| = C_\gamma \sigma^{-\gamma}, \end{aligned} \quad (3.12)$$

for $\sigma > 0$. Due to (3.12) one estimates (3.8) as

$$\begin{aligned} & \left\| \mathcal{A}^{-\gamma} (e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0}) \left\{ \mathcal{B} e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{K}_0} \mathcal{B} \right\} (e^{-\sigma\mathcal{B}} - I) \mathcal{A}^{-\gamma} \right\| \\ & \leq 2 C_\gamma \sigma^{-\gamma} \left\| \mathcal{A}^{-\gamma} (e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0}) \right\| \left\| (e^{-\sigma\mathcal{B}} - I) \mathcal{A}^{-\gamma} \right\|. \end{aligned}$$

Since the fundamental properties of semigroups and (3.6) yield

$$\left\| (e^{-\sigma\mathcal{B}} - I) \mathcal{A}^{-\gamma} \right\| \leq \|\mathcal{B} \mathcal{A}^{-\gamma}\| \sigma \leq C_\gamma \sigma, \quad \sigma \in \mathbb{R}_+, \quad (3.13)$$

and

$$\left\| \mathcal{A}^{-\gamma} (e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0}) \right\| \leq C_\gamma (\tau - \sigma), \quad \sigma \in \mathbb{R}_+,$$

we get for (3.8) the estimate

$$\begin{aligned} & \left\| \mathcal{A}^{-\gamma} (e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0}) \left\{ \mathcal{B} e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{K}_0} \mathcal{B} \right\} (e^{-\sigma\mathcal{B}} - I) \mathcal{A}^{-\gamma} \right\| \\ & \leq 2 C_\gamma^3 \sigma^{1-\gamma} (\tau - \sigma), \quad 0 \leq \sigma \leq \tau. \end{aligned} \quad (3.14)$$

To estimate (3.9) we recall that \mathcal{A} and \mathcal{K}_0 commute. Then by (3.6) one gets

$$\begin{aligned} & \left\| e^{-(\tau-\sigma)\mathcal{K}_0} \mathcal{A}^{-\gamma} \left\{ \mathcal{B} e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{K}_0} \mathcal{B} \right\} (e^{-\sigma\mathcal{B}} - I) \mathcal{A}^{-\gamma} \right\| \\ & \leq 2 C_\gamma \left\| (e^{-\sigma\mathcal{B}} - I) \mathcal{A}^{-\gamma} \right\| \leq 2 C_\gamma^2 \sigma, \quad 0 \leq \sigma \leq \tau, \end{aligned} \quad (3.15)$$

where (3.13) was used for the last inequality.

To estimate (3.10) we have

$$\begin{aligned} & \left\| \mathcal{A}^{-\gamma} (e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0}) \left\{ \mathcal{B} e^{-\sigma\mathcal{K}_0} - e^{-\sigma\mathcal{K}_0} \mathcal{B} \right\} \mathcal{A}^{-\gamma} \right\| \\ & \leq 2 C_\gamma \left\| \mathcal{A}^{-\gamma} (e^{-(\tau-\sigma)\mathcal{K}} - e^{-(\tau-\sigma)\mathcal{K}_0}) \right\| \leq 2 C_\gamma^2 (\tau - \sigma), \quad 0 \leq \sigma \leq \tau. \end{aligned} \quad (3.16)$$

To estimate (3.11) we use the representation

$$\begin{aligned} & \mathcal{A}^{-\gamma} \left\{ \mathcal{B} e^{-\sigma \mathcal{K}_0} - e^{-\sigma \mathcal{K}_0} \mathcal{B} \right\} \mathcal{A}^{-\gamma} \\ &= \mathcal{A}^{-\gamma} \mathcal{B} (e^{-\sigma \mathcal{A}} - I) \mathcal{A}^{-\gamma} e^{-\sigma \mathcal{D}_0} - e^{-\sigma \mathcal{D}_0} \mathcal{A}^{-\gamma} (e^{-\sigma \mathcal{A}} - I) \mathcal{B} \mathcal{A}^{-\gamma} + \\ & \quad \mathcal{A}^{-\gamma} \left\{ \mathcal{B} e^{-\sigma \mathcal{D}_0} - e^{-\sigma \mathcal{D}_0} \mathcal{B} \right\} \mathcal{A}^{-\gamma}, \end{aligned}$$

that yields

$$\begin{aligned} & \left\| \mathcal{A}^{-\gamma} \left\{ \mathcal{B} e^{-\sigma \mathcal{K}_0} - e^{-\sigma \mathcal{K}_0} \mathcal{B} \right\} \mathcal{A}^{-\gamma} \right\| \\ & \leq \left\| \mathcal{A}^{-\gamma} \mathcal{B} (e^{-\sigma \mathcal{A}} - I) \mathcal{A}^{-\gamma} \right\| + \left\| \mathcal{A}^{-\gamma} (e^{-\sigma \mathcal{A}} - I) \mathcal{B} \mathcal{A}^{-\gamma} \right\| + \\ & \quad \left\| \mathcal{A}^{-\gamma} \left\{ \mathcal{B} e^{-\sigma \mathcal{D}_0} - e^{-\sigma \mathcal{D}_0} \mathcal{B} \right\} \mathcal{A}^{-\gamma} \right\|. \end{aligned}$$

Then by (3.6) and by semigroup properties one gets

$$\left\| \mathcal{A}^{-\gamma} \mathcal{B} (e^{-\sigma \mathcal{A}} - I) \mathcal{A}^{-\gamma} \right\| \leq \frac{C_\gamma}{\gamma} \sigma^\gamma, \quad \sigma \in \mathbb{R}_+,$$

and

$$\left\| \mathcal{A}^{-\gamma} (e^{-\sigma \mathcal{A}} - I) \mathcal{B} \mathcal{A}^{-\gamma} \right\| \leq \frac{C_\gamma}{\gamma} \sigma^\gamma, \quad \sigma \in \mathbb{R}_+.$$

The last term is obtained by using (S3) (for α substituted by γ) and the definitions (2.1), (2.4):

$$\left\| \mathcal{A}^{-\gamma} \left\{ \mathcal{B} e^{-\sigma \mathcal{D}_0} - e^{-\sigma \mathcal{D}_0} \mathcal{B} \right\} \mathcal{A}^{-\gamma} \right\| \leq \sigma^\beta L_{\gamma, \beta}, \quad \sigma \in \mathbb{R}_+.$$

Summing up one finds that

$$\left\| \mathcal{A}^{-\gamma} \left\{ \mathcal{B} e^{-\sigma \mathcal{K}_0} - e^{-\sigma \mathcal{K}_0} \mathcal{B} \right\} \mathcal{A}^{-\gamma} \right\| \leq \frac{2C_\gamma}{\gamma} \sigma^\gamma + L_{\gamma, \beta} \sigma^\beta, \quad \sigma \in \mathbb{R}_+. \quad (3.17)$$

Using the estimates (3.14), (3.15), (3.16) and (3.17) we get

$$\begin{aligned} & \left\| \mathcal{A}^{-\gamma} e^{-(\tau - \sigma) \mathcal{K}} \left\{ \mathcal{B} e^{-\sigma \mathcal{K}_0} - e^{-\sigma \mathcal{K}_0} \mathcal{B} \right\} e^{-\sigma \mathcal{B}} \mathcal{A}^{-\gamma} \right\| \\ & \leq 2C_\gamma^3 \sigma^{1-\gamma} (\tau - \sigma) + 2C_\gamma^2 \sigma + 2C_\gamma^2 (\tau - \sigma) + \frac{2C_\gamma}{\gamma} \sigma^\gamma + L_{\gamma, \beta} \sigma^\beta \\ & = 2C_\gamma^3 \sigma^{1-\gamma} (\tau - \sigma) + 2C_\gamma^2 \tau + \frac{2C_\gamma}{\gamma} \sigma^\gamma + L_{\gamma, \beta} \sigma^\beta, \end{aligned}$$

or returning back to its derivative

$$\begin{aligned} & \left\| \mathcal{A}^{-\gamma} \frac{d}{d\sigma} e^{-(\tau-\sigma)\mathcal{K}} e^{-\sigma\mathcal{K}_0} e^{-\sigma\mathcal{B}} \mathcal{A}^{-\gamma} \right\| \\ & \leq 2C_\gamma^3 \sigma^{1-\gamma} (\tau - \sigma) + 2C_\gamma^2 \tau + \frac{2C_\gamma}{\gamma} \sigma^\gamma + L_{\gamma,\beta} \sigma^\beta, \quad 0 \leq \sigma \leq \tau. \end{aligned}$$

Since

$$\mathcal{A}^{-\gamma} (e^{-\tau\mathcal{K}_0} e^{-\tau\mathcal{B}} - e^{-\tau\mathcal{K}}) \mathcal{A}^{-\gamma} = \int_0^\tau \mathcal{A}^{-\gamma} \frac{d}{d\sigma} e^{-(\tau-\sigma)\mathcal{K}} e^{-\sigma\mathcal{K}_0} e^{-\sigma\mathcal{B}} \mathcal{A}^{-\gamma} d\sigma,$$

we find the estimate

$$\left\| \mathcal{A}^{-\gamma} (e^{-\tau\mathcal{K}_0} e^{-\tau\mathcal{B}} - e^{-\tau\mathcal{K}}) \mathcal{A}^{-\gamma} \right\| \leq \int_0^\tau \left\| \mathcal{A}^{-\gamma} \frac{d}{d\sigma} e^{-(\tau-\sigma)\mathcal{K}} e^{-\sigma\mathcal{K}_0} e^{-\sigma\mathcal{B}} \mathcal{A}^{-\gamma} \right\| d\sigma,$$

which yields the estimate

$$\begin{aligned} & \left\| \mathcal{A}^{-\gamma} (e^{-\tau\mathcal{K}_0} e^{-\tau\mathcal{B}} - e^{-\tau\mathcal{K}}) \mathcal{A}^{-\gamma} \right\| \\ & \leq 2C_\gamma^3 \int_0^\tau \sigma^{1-\gamma} (\tau - \sigma) d\sigma + 2C_\gamma^2 \tau^2 + \frac{2C_\gamma}{(1+\gamma)\gamma} \tau^{1+\gamma} + \frac{L_{\gamma,\beta}}{1+\beta} \tau^{1+\beta} \end{aligned}$$

or after integration:

$$\begin{aligned} & \left\| \mathcal{A}^{-\gamma} (e^{-\tau\mathcal{K}_0} e^{-\tau\mathcal{B}} - e^{-\tau\mathcal{K}}) \mathcal{A}^{-\gamma} \right\| \\ & \leq \frac{2C_\gamma^3}{(2-\gamma)(3-\gamma)} \tau^{3-\gamma} + 2C_\gamma^2 \tau^2 + \frac{2C_\gamma}{(1+\gamma)\gamma} \tau^{1+\gamma} + \frac{L_{\gamma,\beta}}{1+\beta} \tau^{1+\beta}, \quad \tau \in \mathbb{R}_+. \end{aligned}$$

If $\gamma \in [\alpha, 1)$ and $\gamma \leq \beta < 1$, then one gets

$$\begin{aligned} & \left\| \mathcal{A}^{-\gamma} (e^{-\tau\mathcal{K}_0} e^{-\tau\mathcal{B}} - e^{-\tau\mathcal{K}}) \mathcal{A}^{-\gamma} \right\| \tag{3.18} \\ & \leq \left(\frac{2C_\gamma^3}{(2-\gamma)(3-\gamma)} \tau^{2-2\gamma} + 2C_\gamma^2 \tau^{1-\gamma} + \frac{2C_\gamma}{(1+\gamma)\gamma} \tau^{1+\gamma} + \frac{L_{\gamma,\beta}}{1+\beta} \tau^{\beta-\gamma} \right) \tau^{1+\gamma}, \end{aligned}$$

$\tau \in \mathbb{R}_+$, which immediately yields (3.7).

If $\gamma \in [\alpha, 1)$ and $0 < \beta < \gamma$, then one can rewrite it as

$$\begin{aligned} & \left\| \mathcal{A}^{-\gamma} (e^{-\tau\mathcal{K}_0} e^{-\tau\mathcal{B}} - e^{-\tau\mathcal{K}}) \mathcal{A}^{-\gamma} \right\| \tag{3.19} \\ & \leq \left(\frac{2C_\gamma^3}{(2-\gamma)(3-\gamma)} \tau^{2-\gamma-\beta} + 2C_\gamma^2 \tau^{1-\beta} + \frac{2C_\gamma}{(1+\gamma)\gamma} \tau^{\gamma-\beta} + \frac{L_{\gamma,\beta}}{1+\beta} \right) \tau^{1+\beta}, \end{aligned}$$

$\tau \in \mathbb{R}_+$, which shows (3.7) for this choice of γ and β . \square

Remark 3.6 For $\gamma \in [\alpha, 1)$ and $\gamma \leq \beta < 1$ we find from (3.18) that

$$Z_{\gamma,\beta} := \frac{2C_\gamma^3}{(2-\gamma)(3-\gamma)}T^{2-2\gamma} + 2C_\gamma^2T^{1-\gamma} + \frac{2C_\gamma}{(1+\gamma)\gamma} + \frac{L_{\gamma,\beta}}{1+\beta}T^{\beta-\gamma}. \quad (3.20)$$

For $\gamma \in [\alpha, 1)$ and $0 < \beta < \gamma$ we get from (3.19) that

$$Z_{\gamma,\beta} := \frac{2C_\gamma^3}{(2-\gamma)(3-\gamma)}T^{2-\gamma-\beta} + 2C_\gamma^2T^{1-\beta} + \frac{2C_\gamma}{(1+\gamma)\gamma}T^{\gamma-\beta} + \frac{L_{\gamma,\beta}}{1+\beta}.$$

Here $C_\gamma := \text{esssup}_{t \in \mathcal{I}} \|\mathcal{B}\mathcal{A}^{-\gamma}\|$, see (3.3), and $L_{\gamma,\beta}$ is the Hölder constant of the function $\mathcal{A}^{-\gamma}\mathcal{B}(\cdot)\mathcal{A}^{-\gamma} : \mathcal{I} \rightarrow \mathcal{L}(\mathcal{H})$, see (S3).

Lemma 3.7 *Let the assumptions (S1) and (S2) be satisfied. Then*

$$\|\mathcal{A}^\gamma(\mathcal{U}(\tau) - \mathcal{T}(\tau))\mathcal{A}^{-\gamma}\| \leq \left(\frac{\Lambda_\gamma}{1-\gamma} + 1\right)C_\gamma\tau^{1-\gamma}, \quad \tau \in \mathbb{R}_+, \quad (3.21)$$

for $\gamma \in [\alpha, 1)$.

Proof. We use the representation

$$\mathcal{U}(\tau) - \mathcal{T}(\tau) = e^{-\tau\mathcal{K}} - e^{-\tau\mathcal{K}_0} + e^{-\tau\mathcal{K}_0}(I - e^{-\tau\mathcal{B}})$$

which yields

$$\mathcal{A}^\gamma(\mathcal{U}(\tau) - \mathcal{T}(\tau))\mathcal{A}^{-\gamma} = \mathcal{A}^\gamma(e^{-\tau\mathcal{K}} - e^{-\tau\mathcal{K}_0})\mathcal{A}^{-\gamma} + \mathcal{A}^\gamma e^{-\tau\mathcal{K}_0}(I - e^{-\tau\mathcal{B}})\mathcal{A}^{-\gamma}$$

Using the semigroup property we obtain for the first term the representation:

$$\mathcal{A}^\gamma(e^{-\tau\mathcal{K}} - e^{-\tau\mathcal{K}_0})\mathcal{A}^{-\gamma} = - \int_0^\tau \mathcal{A}^\gamma e^{-(\tau-x)\mathcal{K}} \mathcal{B} \mathcal{A}^{-\gamma} e^{-x\mathcal{K}_0} dx.$$

Hence, by (3.3) and (3.6) one gets

$$\begin{aligned} \|\mathcal{A}^\gamma(e^{-\tau\mathcal{K}} - e^{-\tau\mathcal{K}_0})\mathcal{A}^{-\gamma}\| &\leq \int_0^\tau \|\mathcal{A}^\gamma e^{-(\tau-x)\mathcal{K}}\| \|\mathcal{B}\mathcal{A}^{-\gamma}\| dx \leq \\ &\Lambda_\gamma C_\gamma \int_0^\tau \frac{1}{(\tau-x)^\gamma} dx = \frac{\Lambda_\gamma C_\gamma}{1-\gamma} \tau^{1-\gamma}. \end{aligned} \quad (3.22)$$

To estimate the second term we use the inequality

$$\|\mathcal{A}^\gamma e^{-\tau\mathcal{K}_0}(I - e^{-\tau\mathcal{B}})\mathcal{A}^{-\gamma}\| \leq \|\mathcal{A}^\gamma e^{-\tau\mathcal{K}_0}\| \|(I - e^{-\tau\mathcal{B}})\mathcal{A}^{-\gamma}\|.$$

Using (3.2) and (3.13) we estimate the second term as

$$\|\mathcal{A}^\gamma e^{-\tau\mathcal{K}_0}(I - e^{-\tau\mathcal{B}})\mathcal{A}^{-\gamma}\| \leq C_\gamma \tau^{1-\gamma}. \quad (3.23)$$

Now the estimates (3.22) and (3.23) yield (3.21). \square

Lemma 3.8 *Let the assumption (S1) be satisfied. If for each $\gamma \in [\alpha, 1)$ there is a constant $M_\gamma > 0$ such that*

$$\|\mathcal{A}^\gamma \mathcal{T}(\tau)^m\| \leq \frac{M_\gamma}{(m\tau)^\gamma}, \quad m \in \mathbb{N}, \quad \tau \in \mathbb{R}_+, \quad (3.24)$$

holds for $\mathcal{T}(\tau)$ defined in (3.4), then

$$\|\mathcal{A}^\sigma \mathcal{T}(\tau)^m\| \leq \frac{M_\gamma^\delta}{(m\tau)^\sigma}, \quad m \in \mathbb{N}, \quad (3.25)$$

holds for $\sigma \in [0, \gamma]$ and $\delta := \sigma/\gamma$.

Proof. If (3.24) is satisfied, then

$$\|\overline{(\mathcal{T}(\tau)^*)^m \mathcal{A}^\gamma}\| \leq \frac{M_\gamma}{(m\tau)^\gamma}, \quad m \in \mathbb{N},$$

holds, which is equivalent to

$$\mathcal{A}^\gamma \mathcal{T}(\tau)^m \overline{(\mathcal{T}(\tau)^*)^m \mathcal{A}^\gamma} \leq \frac{M_\gamma^2}{(m\tau)^{2\gamma}}, \quad m \in \mathbb{N},$$

or

$$\mathcal{T}(\tau)^m (\mathcal{T}(\tau)^*)^m \leq \frac{M_\gamma^2}{(m\tau)^{2\gamma}} \mathcal{A}^{-2\gamma}, \quad m \in \mathbb{N}.$$

Let $\delta \in (0, 1)$. Using the Heinz inequality [8, Theorem X.4.2] we get

$$\left(\mathcal{T}(\tau)^m (\mathcal{T}(\tau)^*)^m \right)^\delta \leq \frac{M_\gamma^{2\delta}}{(m\tau)^{2\delta\gamma}} \mathcal{A}^{-2\delta\gamma}, \quad m \in \mathbb{N}.$$

Since $\mathcal{T}(\tau)^m (\mathcal{T}(\tau)^*)^m$ is a self-adjoint contraction we get

$$\mathcal{T}(\tau)^m (\mathcal{T}(\tau)^*)^m \leq \left(\mathcal{T}(\tau)^m (\mathcal{T}(\tau)^*)^m \right)^\delta, \quad m \in \mathbb{N},$$

which yields

$$\mathcal{T}(\tau)^m (\mathcal{T}(\tau)^*)^m \leq \frac{M_\gamma^{2\delta}}{(m\tau)^{2\delta\gamma}} \mathcal{A}^{-2\delta\gamma}, \quad m \in \mathbb{N},$$

or

$$\mathcal{A}^{\delta\gamma} \mathcal{T}(\tau)^m \overline{(\mathcal{T}(\tau)^*)^m \mathcal{A}^{\delta\gamma}} \leq \frac{M_\gamma^{2\delta}}{(m\tau)^{2\delta\gamma}}, \quad m \in \mathbb{N}.$$

Therefore, one gets

$$\|\overline{(\mathcal{T}(\tau)^*)^m \mathcal{A}^{\delta\gamma}}\| \leq \frac{M_\gamma^\delta}{(m\tau)^{\delta\gamma}}, \quad m \in \mathbb{N},$$

or

$$\|\mathcal{A}^{\delta\gamma} \mathcal{T}(\tau)^m\| \leq \frac{M_\gamma^\delta}{(m\tau)^{\delta\gamma}}, \quad m \in \mathbb{N}.$$

Setting $\delta = \sigma/\gamma$ we obtain the proof of (3.25). \square

Lemma 3.9 *Let the assumptions (S1) and (S2) be satisfied and let $\gamma \in (\alpha, 1)$. Then there is a constant $M_\gamma > 0$ such that*

$$\|\mathcal{A}^\gamma \mathcal{T}(\tau)^m\| \leq \frac{M_\gamma}{(m\tau)^\gamma}, \quad m = 1, 2, \dots, n, \quad (3.26)$$

holds for any $T > 0$ if $\tau \in (0, \frac{T}{n})$ and $n \geq n_0$ where $n_0 := \lfloor (2(\frac{\Lambda_\gamma}{1-\gamma} + 1)C_\gamma)^{\frac{1}{1-\gamma}} T \rfloor + 1$ and $\lfloor x \rfloor$ denotes the largest integer smaller than x .

Proof. Let $M_\gamma > 0$ be a constant which satisfies the inequality

$$5\Lambda_\gamma + 2\left(\frac{\Lambda_\gamma}{1-\gamma} + 1\right)C_\gamma M_\gamma T^{1-\gamma} \frac{1}{n^{1-\gamma}} + 4\Lambda_\gamma C_\gamma M_\gamma^{\frac{\alpha}{\gamma}} B(1-\alpha, 1-\gamma) T^{1-\alpha} \leq M_\gamma \quad (3.27)$$

for $n \geq n_0$. Here constants Λ_γ and C_γ are defined by Lemma 3.2 and Lemma 3.4, respectively, while $B(\cdot, \cdot)$ denotes the Euler Beta-function. (Note that such $M_\gamma > 0$ always exists, see Remark 3.10 below.)

Let $m = 1$. Then by (3.2) and (3.4) we get

$$\|\mathcal{A}^\gamma \mathcal{T}(\tau)\| \leq \|\mathcal{A}^\gamma e^{-\tau\mathcal{K}_0}\| \leq \frac{1}{\tau^\gamma} \leq \frac{\Lambda_\gamma}{\tau^\gamma} \leq \frac{M_\gamma}{\tau^\gamma},$$

for $\tau > 0$ and, in particular, for $\tau \in (0, T/n)$. Hence (3.26) holds for $m = 1$.

Let us assume that (3.26) holds for $l = 1, 2, \dots, m-1$, with $m \leq n$, i.e.

$$\|\mathcal{A}^\gamma \mathcal{T}(\tau)^l\| \leq \frac{M_\gamma}{(l\tau)^\gamma}, \quad l = 1, 2, \dots, m-1, \quad (3.28)$$

for $\tau \in (0, T/n)$. We are going to show that (3.28) holds for $l = m$. To this aim we use the representation

$$\mathcal{U}(\tau)^m - \mathcal{T}(\tau)^m = \sum_{k=0}^{m-1} \mathcal{U}(\tau)^{m-1-k} (\mathcal{U}(\tau) - \mathcal{T}(\tau)) \mathcal{T}(\tau)^k, \quad m = 2, 3, \dots,$$

which implies

$$\mathcal{T}(\tau)^m = \mathcal{U}(\tau)^m - \sum_{k=0}^{m-1} \mathcal{U}(\tau)^{m-1-k} (\mathcal{U}(\tau) - \mathcal{T}(\tau)) \mathcal{T}(\tau)^k, \quad m = 2, 3, \dots$$

Hence

$$\mathcal{A}^\gamma \mathcal{T}(\tau)^m = \mathcal{A}^\gamma \mathcal{U}(\tau)^m - \sum_{k=0}^{m-1} \mathcal{A}^\gamma \mathcal{U}(\tau)^{m-1-k} (\mathcal{U}(\tau) - \mathcal{T}(\tau)) \mathcal{T}(\tau)^k$$

or

$$\begin{aligned} \mathcal{A}^\gamma \mathcal{T}(\tau)^m &= \mathcal{A}^\gamma \mathcal{U}(\tau)^m - \mathcal{A}^\gamma \mathcal{U}(\tau)^{m-1} (\mathcal{U}(\tau) - \mathcal{T}(\tau)) \\ &\quad - \mathcal{A}^\gamma (\mathcal{U}(\tau) - \mathcal{T}(\tau)) \mathcal{T}(\tau)^{m-1} - \sum_{k=1}^{m-2} \mathcal{A}^\gamma \mathcal{U}(\tau)^{m-1-k} (\mathcal{U}(\tau) - \mathcal{T}(\tau)) \mathcal{T}(\tau)^k. \end{aligned}$$

for $m = 3, 4, \dots$. This yields the inequality

$$\begin{aligned} \|\mathcal{A}^\gamma \mathcal{T}(\tau)^m\| &\leq \|\mathcal{A}^\gamma \mathcal{U}(\tau)^m\| + \|\mathcal{A}^\gamma \mathcal{U}(\tau)^{m-1} (\mathcal{U}(\tau) - \mathcal{T}(\tau))\| + \\ &\quad \|\mathcal{A}^\gamma (\mathcal{U}(\tau) - \mathcal{T}(\tau)) \mathcal{T}(\tau)^{m-1}\| + \sum_{k=1}^{m-2} \|\mathcal{A}^\gamma \mathcal{U}(\tau)^{m-1-k} (\mathcal{U}(\tau) - \mathcal{T}(\tau)) \mathcal{T}(\tau)^k\| \end{aligned} \quad (3.29)$$

for $m = 3, 4, \dots$. From Lemma 3.2 we get the estimates

$$\|\mathcal{A}^\gamma \mathcal{U}(\tau)^m\| \leq \frac{\Lambda_\gamma}{(m\tau)^\gamma}, \quad m = 2, 3, \dots,$$

and consequently:

$$\|\mathcal{A}^\gamma \mathcal{U}(\tau)^{m-1} (\mathcal{U}(\tau) - \mathcal{T}(\tau))\| \leq \frac{2\Lambda_\gamma}{((m-1)\tau)^\gamma} \leq \frac{4\Lambda_\gamma}{(m\tau)^\gamma}, \quad m = 2, 3, \dots$$

Then summing up estimates for the first two terms in the right-hand side of (3.29) we obtain

$$\|\mathcal{A}^\gamma \mathcal{U}(\tau)^m\| + \|\mathcal{A}^\gamma \mathcal{U}(\tau)^{m-1} (\mathcal{U}(\tau) - \mathcal{T}(\tau))\| \leq \frac{5\Lambda_\gamma}{(m\tau)^\gamma}, \quad m = 2, 3, \dots \quad (3.30)$$

Next we get for the third term in the right-hand side of (3.29) the estimate

$$\|\mathcal{A}^\gamma (\mathcal{U}(\tau) - \mathcal{T}(\tau)) \mathcal{T}(\tau)^{m-1}\| \leq \|\mathcal{A}^\gamma (\mathcal{U}(\tau) - \mathcal{T}(\tau)) \mathcal{A}^{-\gamma}\| \|\mathcal{A}^\gamma \mathcal{T}(\tau)^{m-1}\|,$$

$m = 2, 3, \dots$. Then using Lemma 3.7 we find that

$$\|\mathcal{A}^\gamma (\mathcal{U}(\tau) - \mathcal{T}(\tau)) \mathcal{T}(\tau)^{m-1}\| \leq \left(\frac{\Lambda_\gamma}{1-\gamma} + 1 \right) C_\gamma \tau^{1-\gamma} \|\mathcal{A}^\gamma \mathcal{T}(\tau)^{m-1}\|, \quad m = 2, 3, \dots$$

By assumption (3.28) this yields

$$\|\mathcal{A}^\gamma (\mathcal{U}(\tau) - \mathcal{T}(\tau)) \mathcal{T}(\tau)^{m-1}\| \leq \left(\frac{\Lambda_\gamma}{1-\gamma} + 1 \right) M_\gamma C_\gamma \frac{1}{((m-1)\tau)^\gamma} \tau^{1-\gamma}, \quad m = 2, 3, \dots,$$

for $\tau \in (0, T/n)$, which leads to

$$\|\mathcal{A}^\gamma (\mathcal{U}(\tau) - \mathcal{T}(\tau)) \mathcal{T}(\tau)^{m-1}\| \leq \left(\frac{\Lambda_\gamma}{1-\gamma} + 1 \right) M_\gamma C_\gamma \frac{2}{(m\tau)^\gamma} \tau^{1-\gamma}, \quad m = 2, 3, \dots \quad (3.31)$$

Finally one gets for the sum in (3.29)

$$\begin{aligned} & \sum_{k=1}^{m-2} \|\mathcal{A}^\gamma \mathcal{U}(\tau)^{m-1-k} (\mathcal{U}(\tau) - \mathcal{T}(\tau)) \mathcal{T}(\tau)^k\| \\ & \leq \sum_{k=1}^{m-2} \|\mathcal{A}^\gamma \mathcal{U}(\tau)^{m-1-k}\| \|(\mathcal{U}(\tau) - \mathcal{T}(\tau)) \mathcal{A}^{-\alpha}\| \|\mathcal{A}^\alpha \mathcal{T}(\tau)^k\|, \quad m = 2, 3, \dots \end{aligned}$$

Then by Lemma 3.2 this implies

$$\begin{aligned} & \sum_{k=1}^{m-2} \|\mathcal{A}^\gamma \mathcal{U}(\tau)^{m-1-k} (\mathcal{U}(\tau) - \mathcal{T}(\tau)) \mathcal{T}(\tau)^k\| \\ & \leq \Lambda_\gamma \sum_{k=1}^{m-2} \frac{1}{((m-1-k)\tau)^\gamma} \|(\mathcal{U}(\tau) - \mathcal{T}(\tau)) \mathcal{A}^{-\alpha}\| \|\mathcal{A}^\alpha \mathcal{T}(\tau)^k\|, \quad m = 2, 3, \dots \end{aligned}$$

Taking into account Lemma 3.4 we get

$$\begin{aligned} & \sum_{k=1}^{m-2} \|\mathcal{A}^\gamma \mathcal{U}(\tau)^{m-1-k} (\mathcal{U}(\tau) - \mathcal{T}(\tau)) \mathcal{T}(\tau)^k\| \\ & \leq 2\Lambda_\gamma C_\alpha \sum_{k=1}^{m-2} \frac{\tau}{((m-1-k)\tau)^\gamma} \|\mathcal{A}^\alpha \mathcal{T}(\tau)^k\|, \quad m = 2, 3, \dots \end{aligned}$$

Finally, using assumption (3.28) and Lemma 3.8 one obtains

$$\begin{aligned} & \sum_{k=1}^{m-2} \|\mathcal{A}^\gamma \mathcal{U}(\tau)^{m-1-k} (\mathcal{U}(\tau) - \mathcal{T}(\tau)) \mathcal{T}(\tau)^k\| \\ & \leq 2\Lambda_\gamma C_\alpha M_\gamma^{\frac{\alpha}{\gamma}} \sum_{k=1}^{m-2} \frac{\tau}{((m-1-k)\tau)^\gamma} \frac{1}{(k\tau)^\alpha}, \quad m = 2, 3, \dots, \end{aligned}$$

or

$$\begin{aligned} & \sum_{k=1}^{m-2} \|\mathcal{A}^\gamma \mathcal{U}(\tau)^{m-1-k} (\mathcal{U}(\tau) - \mathcal{T}(\tau)) \mathcal{T}(\tau)^k\| \\ & \leq 2\Lambda_\gamma C_\gamma M_\gamma^{\frac{\alpha}{\gamma}} \left(\sum_{k=1}^{m-2} \frac{1}{(m-1-k)^\gamma} \frac{1}{k^\alpha} \right) \tau^{1-\gamma-\alpha}, \quad m = 2, 3, \dots, \end{aligned}$$

for $\tau \in (0, T/n)$. Since Lemma 3.11 below yields

$$\sum_{k=1}^{m-2} \frac{1}{(m-1-k)^\gamma} \frac{1}{k^\alpha} \leq B(1-\alpha, 1-\gamma) (m-1)^{1-\gamma-\alpha}, \quad m = 2, 3, \dots, \quad (3.32)$$

where $B(\cdot, \cdot)$ is the Euler Beta-function, we get

$$\begin{aligned} & \sum_{k=1}^{m-2} \|\mathcal{A}^\gamma \mathcal{U}(\tau)^{m-1-k} (\mathcal{U}(\tau) - \mathcal{T}(\tau)) \mathcal{T}(\tau)^k\| \\ & \leq 2\Lambda_\gamma C_\gamma M_\gamma^{\frac{\alpha}{\gamma}} B(1-\alpha, 1-\gamma) \tau^{1-\gamma-\alpha} (m-1)^{1-\gamma-\alpha}, \quad m = 2, 3, \dots, \end{aligned}$$

which in turn leads to

$$\begin{aligned} & \sum_{k=1}^{m-2} \|\mathcal{A}^\gamma \mathcal{U}(\tau)^{m-1-k} (\mathcal{U}(\tau) - \mathcal{T}(\tau)) \mathcal{T}(\tau)^k\| \\ & \leq \frac{4\Lambda_\gamma C_\gamma M_\gamma^{\frac{\alpha}{\gamma}} B(1-\alpha, 1-\gamma)}{(m\tau)^\gamma} \tau^{1-\alpha} m^{1-\alpha}, \end{aligned} \tag{3.33}$$

for $m = 2, 3, \dots$ and any $\tau \in (0, T/n)$.

Now we take into account (3.29), (3.30), (3.31) and (3.33) to conclude that

$$\begin{aligned} & \|\mathcal{A}^\gamma \mathcal{T}(\tau)^m\| \leq \\ & \left\{ 5\Lambda_\gamma + 2 \left(\frac{\Lambda_\gamma}{1-\gamma} + 1 \right) M_\gamma C_\gamma \tau^{1-\gamma} + 4\Lambda_\gamma C_\gamma M_\gamma^{\frac{\alpha}{\gamma}} B(1-\alpha, 1-\gamma) \tau^{1-\alpha} m^{1-\alpha} \right\} \frac{1}{(m\tau)^\gamma}, \end{aligned}$$

for $m = 2, 3, \dots$ and $\tau \in (0, T/n)$. Then

$$\begin{aligned} & \|\mathcal{A}^\gamma \mathcal{T}(\tau)^m\| \leq \\ & \left\{ 5\Lambda_\gamma + 2 \left(\frac{\Lambda_\gamma}{1-\gamma} + 1 \right) M_\gamma C_\gamma T^{1-\gamma} \frac{1}{n^{1-\gamma}} + 4\Lambda_\gamma C_\gamma M_\gamma^{\frac{\alpha}{\gamma}} B(1-\alpha, 1-\gamma) T^{1-\alpha} \right\} \frac{1}{(m\tau)^\gamma}. \end{aligned}$$

From assumption (3.27) we get

$$5\Lambda_\gamma + 2 \left(\frac{\Lambda_\gamma}{1-\gamma} + 1 \right) M_\gamma C_\gamma T^{1-\gamma} \frac{1}{n^{1-\gamma}} + 4\Lambda_\gamma C_\gamma M_\gamma^{\frac{\alpha}{\gamma}} B(1-\alpha, 1-\gamma) T^{1-\alpha} \leq M_\gamma$$

for $n \geq n_0$, which shows that (3.28) holds for $l = 1, 2, 3, \dots, n$ and $n \geq n_0$ which proves (3.26). \square

Remark 3.10 One checks that condition (3.27) is always satisfied for sufficiently large $M = M_\gamma$ and $n \geq n_0$. Indeed, after setting

$$c_0 := 5\Lambda_\gamma, \quad c_1 := 2 \left(\frac{\Lambda_\gamma}{1-\gamma} + 1 \right) C_\gamma T^{1-\gamma}, \quad c_2 := 4\Lambda_\gamma C_\gamma B(1-\alpha, 1-\gamma) T^{1-\alpha}$$

we get the condition

$$c_0 + \frac{c_1}{n^{1-\gamma}} M + c_2 M^{\frac{\alpha}{\gamma}} \leq M$$

which yields

$$c_0 + c_2 M^{\frac{\alpha}{\gamma}} \leq \left(1 - \frac{c_1}{n^{1-\gamma}}\right) M$$

or

$$\frac{c_0}{M} + \frac{c_2}{M^{1-\frac{\alpha}{\gamma}}} \leq 1 - \frac{c_1}{n^{1-\gamma}}$$

Since $n > c_1^{\frac{1}{1-\gamma}}$ we have $1 - c_1/n^{1-\gamma} > 0$. The left-hand side tends to zero if $M \rightarrow \infty$. Hence, choosing M sufficiently large we guarantee the existence of M_γ such that condition (3.27) is satisfied for any $n \geq n_0$. \triangle

It remains only to verify the following statement.

Lemma 3.11 *Let $\alpha \in [0, 1)$ and $\gamma \in [\alpha, 1)$. Then*

$$\sum_{k=1}^{n-1} \frac{1}{(n-k)^\gamma} \frac{1}{k^\alpha} \leq B(1-\alpha, 1-\gamma) n^{1-\gamma-\alpha}, \quad n \in 2, 3, \dots$$

the estimate holds where $B(\cdot, \cdot)$ is the Euler Beta-function.

$$B(1-\alpha, 1-\gamma) := \int_0^1 \frac{1}{x^\alpha(1-x)^\gamma} dx$$

Proof. If $x \in (k-1, k]$, then

$$\frac{1}{k^\alpha} \leq \frac{1}{x^\alpha} \quad \text{and} \quad \frac{1}{(n-k)^\gamma} \leq \frac{1}{(n-1-x)^\gamma}$$

for $k = 1, 2, \dots, n-1$. Hence

$$\frac{1}{(n-k)^\gamma k^\alpha} \leq \frac{1}{(n-1-x)^\gamma x^\alpha}, \quad x \in (k-1, k].$$

Therefore

$$\frac{1}{(n-k)^\gamma k^\alpha} = \int_{k-1}^k \frac{1}{(n-k)^\gamma k^\alpha} dx \leq \int_{k-1}^k \frac{1}{(n-1-x)^\gamma x^\alpha} dx, \quad x \in (k-1, k],$$

or

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{1}{(n-k)^\gamma k^\alpha} &= \sum_{k=1}^{n-1} \int_{k-1}^k \frac{1}{(n-k)^\gamma k^\alpha} dx \leq \sum_{k=1}^{n-1} \int_{k-1}^k \frac{1}{(n-1-x)^\gamma x^\alpha} dx \\ &= \int_0^{n-1} \frac{1}{(n-1-x)^\gamma x^\alpha} dx = B(1-\alpha, 1-\gamma) n^{1-\alpha-\gamma} \end{aligned}$$

\square

3.2 Main Results

In this section we collect our main results and their proofs. They are based on preliminaries established in Section 3.1.

Theorem 3.12 *Let the assumptions (S1)–(S3) be satisfied and let $\beta > 2\alpha - 1$. Then there is a constant $R_\beta > 0$ such that*

$$\sup_{\tau \in \mathbb{R}_+} \|\mathcal{U}(\tau) - \mathcal{T}(\tau/n)^n\| \leq \frac{R_\beta}{n^\beta} \quad (3.34)$$

holds for $n \in \mathbb{N}$ and $\tau \in \mathbb{R}_+$.

Proof. Taking into account the representation

$$\mathcal{U}(\tau/n)^n - \mathcal{T}(\tau/n)^n = \sum_{m=0}^{n-1} \mathcal{U}(\tau/n)^{n-m-1} (\mathcal{U}(\tau/n) - \mathcal{T}(\tau/n)) \mathcal{T}(\tau/n)^m, \quad n \in \mathbb{N},$$

or, identically,

$$\begin{aligned} & \mathcal{U}(\tau/n)^n - \mathcal{T}(\tau/n)^n \\ &= \mathcal{U}(\tau/n)^{n-1} (\mathcal{U}(\tau/n) - \mathcal{T}(\tau/n)) + (\mathcal{U}(\tau/n) - \mathcal{T}(\tau/n)) \mathcal{T}(\tau/n)^{n-1} + \\ & \quad \sum_{m=1}^{n-2} \mathcal{U}(\tau/n)^{n-m-1} (\mathcal{U}(\tau/n) - \mathcal{T}(\tau/n)) \mathcal{T}(\tau/n)^m, \quad n = 3, 4, \dots, \end{aligned}$$

we obtain the estimate

$$\begin{aligned} & \|\mathcal{U}(\tau/n)^n - \mathcal{T}(\tau/n)^n\| \\ & \leq \|\mathcal{U}(\tau/n)^{n-1} \mathcal{A}^\gamma\| \|\mathcal{A}^{-\gamma} (\mathcal{U}(\tau/n) - \mathcal{T}(\tau/n))\| \\ & \quad + \|(\mathcal{U}(\tau/n) - \mathcal{T}(\tau/n)) \mathcal{A}^{-\gamma}\| \|\mathcal{A}^\gamma \mathcal{T}(\tau/n)^{n-1}\| \\ & \quad + \sum_{m=1}^{n-2} \|\mathcal{U}(\tau/n)^{n-m-1} \mathcal{A}^\gamma\| \|\mathcal{A}^{-\gamma} (\mathcal{U}(\tau/n) - \mathcal{T}(\tau/n)) \mathcal{A}^{-\gamma}\| \|\mathcal{A}^\gamma \mathcal{T}(\tau/n)^m\|, \end{aligned} \quad (3.35)$$

$n = 3, 4, \dots$

Note that using Lemma 3.2 and Lemma 3.4 one gets

$$\|\mathcal{U}(\tau/n)^{n-1} \mathcal{A}^\gamma\| \|\mathcal{A}^{-\gamma} (\mathcal{U}(\tau/n) - \mathcal{T}(\tau/n))\| \leq 2 \frac{\Lambda_\gamma C_\gamma}{(\tau(n-1)/n)^\gamma} \frac{\tau}{n},$$

which yields

$$\|\mathcal{U}(\tau/n)^{n-1} \mathcal{A}^\gamma\| \|\mathcal{A}^{-\gamma} (\mathcal{U}(\tau/n) - \mathcal{T}(\tau/n))\| \leq 2^{1+\gamma} \Lambda_\gamma C_\gamma T^{1-\gamma} \frac{1}{n}. \quad (3.36)$$

for $n = 3, 4, \dots$ and $\tau \in [0, T]$.

Now using Lemma 3.4 and Lemma 3.9 for $m = n - 1$ we find

$$\|(\mathcal{U}(\tau/n) - \mathcal{T}(\tau/n))\mathcal{A}^{-\gamma}\| \|\mathcal{A}^\gamma \mathcal{T}(\tau/n)^{n-1}\| \leq 2 C_\gamma \frac{\tau}{n} \frac{M_\gamma}{(\tau(n-1)/n)^\gamma},$$

for $n \geq n_0$, where n_0 is defined in Lemma 3.9 and $\tau \in [0, T]$. Hence,

$$\|(\mathcal{U}(\tau/n) - \mathcal{T}(\tau/n))\mathcal{A}^{-\gamma}\| \|\mathcal{A}^\gamma \mathcal{T}(\tau/n)^{n-1}\| \leq 2^{1+\gamma} C_\gamma M_\gamma T^{1-\gamma} \frac{1}{n}. \quad (3.37)$$

Taking into account Lemma 3.2, Lemma 3.5 and Lemma 3.9 (for $\varkappa = \min\{\gamma, \beta\}$) one gets

$$\begin{aligned} & \sum_{m=1}^{n-2} \|\mathcal{U}(\tau/n)^{n-m-1} \mathcal{A}^\gamma\| \|\mathcal{A}^{-\gamma}(\mathcal{U}(\tau/n) - \mathcal{T}(\tau/n))\mathcal{A}^{-\gamma}\| \|\mathcal{A}^\gamma \mathcal{T}(\tau/n)^m\| \\ & \leq \sum_{m=1}^{n-2} \frac{\Lambda_\gamma Z_{\gamma,\beta}}{((n-m-1)\tau/n)^\gamma} \left(\frac{\tau}{n}\right)^{1+\varkappa} \frac{M_\gamma}{(m\tau/n)^\gamma} \\ & = \frac{\Lambda_\gamma Z_{\gamma,\beta} M_\gamma \tau^{1+\varkappa-2\gamma}}{n^{1+\varkappa-2\gamma}} \sum_{m=1}^{n-2} \frac{1}{(n-m-1)^\gamma} \frac{1}{m^\gamma}, \end{aligned}$$

for $n > \max\{2, n_0\}$ and $\tau \in [0, T]$. Then by (3.32) we obtain

$$\begin{aligned} & \sum_{m=1}^{n-2} \|\mathcal{U}(\tau/n)^{n-m-1} \mathcal{A}^\gamma\| \|\mathcal{A}^{-\gamma}(\mathcal{U}(\tau/n) - \mathcal{T}(\tau/n))\mathcal{A}^{-\gamma}\| \|\mathcal{A}^\gamma \mathcal{T}(\tau/n)^m\| \\ & \leq \frac{\Lambda_\gamma Z_{\gamma,\beta} M_\gamma \tau^{1+\varkappa-2\gamma}}{n^{1+\varkappa-2\gamma}} B(1-\gamma, 1-\gamma) n^{1-2\gamma}, \end{aligned}$$

or

$$\begin{aligned} & \sum_{m=1}^{n-2} \|\mathcal{U}(\tau/n)^{n-m-1} \mathcal{A}^\gamma\| \|\mathcal{A}^{-\gamma}(\mathcal{U}(\tau/n) - \mathcal{T}(\tau/n))\mathcal{A}^{-\gamma}\| \|\mathcal{A}^\gamma \mathcal{T}(\tau/n)^m\| \\ & \leq \Lambda_\gamma Z_{\gamma,\beta} M_\gamma B(1-\gamma, 1-\gamma) T^{1+\varkappa-2\gamma} \frac{1}{n^\varkappa}. \end{aligned} \quad (3.38)$$

Therefore, by virtue of (3.35), (3.36), (3.37) and (3.38) we get for $n > \max\{2, n_0\}$ and $\tau \in [0, T]$ the estimate

$$\begin{aligned} & \|\mathcal{U}(\tau)^n - \mathcal{T}(\tau/n)^n\| = \|\mathcal{U}(\tau/n)^n - \mathcal{T}(\tau/n)^n\| \\ & \leq 2^{1+\gamma} \Lambda_\gamma C_\gamma T^{1-\gamma} \frac{1}{n} + 2^{1+\gamma} C_\gamma M_\gamma T^{1-\gamma} \frac{1}{n} + \Lambda_\gamma Z_{\gamma,\beta} M_\gamma B(1-\gamma, 1-\gamma) T^{1+\varkappa-2\gamma} \frac{1}{n^\varkappa} \\ & \leq \{2^{1+\gamma} \Lambda_\gamma C_\gamma T^{1-\gamma} + 2^{1+\gamma} C_\gamma M_\gamma T^{1-\gamma} + \Lambda_\gamma Z_{\gamma,\beta} M_\gamma B(1-\gamma, 1-\gamma) T^{1+\varkappa-2\gamma}\} \frac{1}{n^\varkappa}. \end{aligned}$$

If $\alpha < \beta < 1$, then we choose $\gamma = \beta$, i.e., $\varkappa = \beta$ and $1 + \varkappa - 2\gamma = 1 - \beta \geq 0$. Setting

$$R'_\beta := 2^{1+\beta} \Lambda_\beta C_\beta T^{1-\beta} + 2^{1+\beta} C_\beta M_\beta T^{1-\beta} + \Lambda_\beta Z_{\beta,\beta} M_\beta B(1-\beta, 1-\beta) T^{1-\beta}$$

one obtains the estimate

$$\|\mathcal{U}(\tau)^n - \mathcal{T}(\tau/n)^n\| \leq \frac{R'_\beta}{n^\beta}, \quad (3.39)$$

for $n > \max\{2, n_0\}$ and $\tau \in [0, T]$.

Now let $0 < \beta \leq \alpha$. Since $1 + \beta - 2\alpha > 0$, there exists $\gamma \in (\alpha, 1)$ such that $1 + \beta - 2\gamma \geq 0$. Indeed, there is a $\varepsilon > 0$ verifying $1 + \beta - 2\alpha > 2\varepsilon$. Setting $\gamma = \alpha + \varepsilon$ we get $1 + \beta - 2\gamma > 0$. Notice that $\varkappa = \beta$. Then setting

$$R'_\beta := 2^{1+\gamma} \Lambda_\gamma C_\gamma T^{1-\gamma} + 2^{1+\gamma} C_\gamma M_\gamma T^{1-\gamma} + \Lambda_\gamma Z_{\gamma,\beta} M_\gamma B(1-\gamma, 1-\gamma) T^{1+\beta-2\gamma},$$

we obtain (3.39) for $n > \max\{2, n_0\}$.

Both results immediately imply that there is a constant R_γ such that (3.34) holds for $\tau \in [0, T]$ and $n \in \mathbb{N}$. Finally, using $\mathcal{U}(\tau) = 0$ and $\mathcal{T}(\tau/n)^n = 0$ for $\tau \geq T$ we obtain (3.28) for any $\tau \in \mathbb{R}_+$. \square

Now we set

$$\tilde{\mathcal{T}}(\tau) := e^{-\tau\mathcal{B}} e^{-\tau\mathcal{K}_0}, \quad \tau \in \mathbb{R}_+.$$

Corollary 3.13 *Let the assumptions (S1)–(S3) be satisfied and $\beta > 2\alpha - 1$. Then there exists $\tilde{R}_\beta > 0$ such that estimate*

$$\sup_{\tau \in \mathbb{R}_+} \|\mathcal{U}(\tau) - \tilde{\mathcal{T}}(\tau/n)^n\| \leq \frac{\tilde{R}_\beta}{n^\beta} \quad (3.40)$$

holds for $n \in \mathbb{N}$ and $\tau \in \mathbb{R}_+$.

Proof. Notice that

$$\tilde{\mathcal{T}}(\tau/n)^{n+1} = e^{-\tau\mathcal{B}/n} \mathcal{T}(\tau/n)^n e^{-\tau\mathcal{K}_0/n}, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

Hence

$$\begin{aligned} \mathcal{U}((n+1)\tau/n) - \tilde{\mathcal{T}}(\tau/n)^{n+1} &= e^{-(n+1)\tau\mathcal{K}/n} - e^{-\tau\mathcal{B}/n} \mathcal{T}(\tau/n)^n e^{-\tau\mathcal{K}_0/n} \\ &= e^{-(n+1)\tau\mathcal{K}/n} - e^{-\tau\mathcal{B}/n} e^{-\tau\mathcal{K}} e^{-\tau\mathcal{K}_0/n} + e^{-\tau\mathcal{B}/n} (\mathcal{U}(\tau) - \mathcal{T}(\tau/n)^n) e^{-\tau\mathcal{K}_0/n} \\ &= (I - e^{-\tau\mathcal{B}/n}) e^{-\tau\mathcal{K}} e^{-\tau\mathcal{K}_0/n} + e^{-\tau\mathcal{K}} (e^{-\tau\mathcal{K}/n} - e^{-\tau\mathcal{K}_0/n}) + \\ &\quad e^{-\tau\mathcal{B}/n} (\mathcal{U}(\tau) - \mathcal{T}(\tau/n)^n) e^{-\tau\mathcal{K}_0/n}, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}, \end{aligned}$$

which yields the estimate

$$\begin{aligned} &\|\mathcal{U}((n+1)\frac{\tau}{n}) - \tilde{\mathcal{T}}(\frac{\tau}{n})^{n+1}\| \\ &\leq \|(I - e^{-\frac{\tau}{n}\mathcal{B}}) e^{-\tau\mathcal{K}}\| + \|e^{-\tau\mathcal{K}} (e^{-\frac{\tau}{n}\mathcal{K}} - e^{-\frac{\tau}{n}\mathcal{K}_0})\| + \\ &\quad \|\mathcal{U}(\tau) - \mathcal{T}(\frac{\tau}{n})^n\|, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}. \end{aligned} \quad (3.41)$$

Obviously, one has

$$\|(I - e^{-\frac{\tau}{n}\mathcal{B}})e^{-\tau\mathcal{K}}\| \leq \|(I - e^{-\frac{\tau}{n}\mathcal{B}})\mathcal{A}^{-\alpha}\| \|\mathcal{A}^\alpha e^{-\tau\mathcal{K}}\|, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

Using

$$(I - e^{-\frac{\tau}{n}\mathcal{B}})\mathcal{A}^{-\alpha} = \int_0^{\frac{\tau}{n}} e^{-\sigma\mathcal{B}}\mathcal{B}\mathcal{A}^{-\alpha}d\sigma, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N},$$

we get the estimate

$$\|(I - e^{-\frac{\tau}{n}\mathcal{B}})\mathcal{A}^{-\alpha}\| \leq C_\alpha \frac{\tau}{n}, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

Taking into account condition (S2) and Lemma 3.2 we find

$$\|(I - e^{-\frac{\tau}{n}\mathcal{B}})e^{-\tau\mathcal{K}}\| \leq C_\alpha \Lambda_\alpha \frac{\tau^{1-\alpha}}{n} \leq C_\alpha \Lambda_\alpha T^{1-\alpha} \frac{1}{n}, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}, \quad (3.42)$$

where we have used that $e^{-\tau\mathcal{K}} = 0$ for $\tau \geq T$.

Further, we have

$$\|e^{-\tau\mathcal{K}}(e^{-\frac{\tau}{n}\mathcal{K}} - e^{-\frac{\tau}{n}\mathcal{K}_0})\| \leq \|e^{-\tau\mathcal{K}}\mathcal{A}^\alpha\| \|\mathcal{A}^{-\alpha}(e^{-\frac{\tau}{n}\mathcal{K}} - e^{-\frac{\tau}{n}\mathcal{K}_0})\|,$$

$\tau \in \mathbb{R}_+, n \in \mathbb{N}$. Then using

$$\mathcal{A}^{-\alpha}(e^{-\frac{\tau}{n}\mathcal{K}} - e^{-\frac{\tau}{n}\mathcal{K}_0}) = - \int_0^{\frac{\tau}{n}} e^{-\sigma\mathcal{K}_0} \overline{\mathcal{A}^{-\alpha}\mathcal{B}} e^{-(\tau-\sigma)\mathcal{K}} d\sigma,$$

$\tau \in \mathbb{R}_+, n \in \mathbb{N}$, we find the estimate

$$\|\mathcal{A}^{-\alpha}(e^{-\frac{\tau}{n}\mathcal{K}} - e^{-\frac{\tau}{n}\mathcal{K}_0})\| \leq C_\alpha \frac{\tau}{n}, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

Applying again Lemma 3.2 one gets

$$\|e^{-\tau\mathcal{K}}(e^{-\frac{\tau}{n}\mathcal{K}} - e^{-\frac{\tau}{n}\mathcal{K}_0})\| \leq C_\alpha \Lambda_\alpha T^{1-\alpha} \frac{1}{n}, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}. \quad (3.43)$$

The insertion of (3.42) and (3.43) into (3.41) yields

$$\|\mathcal{U}((n+1)\frac{\tau}{n}) - \tilde{\mathcal{T}}(\frac{\tau}{n})^{n+1}\| \leq 2C_\alpha \Lambda_\alpha \frac{1}{n} + \|\mathcal{U}(\tau) - \mathcal{T}(\frac{\tau}{n})^n\|, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

Then by Theorem 3.12 we obtain

$$\|\mathcal{U}((n+1)\frac{\tau}{n}) - \tilde{\mathcal{T}}(\frac{\tau}{n})^{n+1}\| \leq 2C_\alpha \Lambda_\alpha \frac{1}{n} + R_\gamma \frac{1}{n^\gamma}, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

Therefore, by setting $R'_\gamma := 2C_\alpha \Lambda_\alpha + R_\gamma$ we obtain

$$\|\mathcal{U}((n+1)\frac{\tau}{n}) - \tilde{\mathcal{T}}(\frac{\tau}{n})^{n+1}\| \leq \frac{R'_\gamma}{n^\gamma}, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

which yields

$$\sup_{\tau \in \mathbb{R}_+} \|\mathcal{U}((n+1)\frac{\tau}{n}) - \tilde{\mathcal{T}}(\frac{\tau}{n})^{n+1}\| \leq \frac{R'_\gamma}{n^\gamma}, \quad \tau \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

Let $\tau = \tau' n / (n+1)$ for $\tau' \in \mathbb{R}_+$. Then

$$\sup_{\tau \in \mathbb{R}_+} \|\mathcal{U}((n+1)\frac{\tau}{n}) - \tilde{\mathcal{T}}(\frac{\tau}{n})^{n+1}\| = \sup_{\tau' \in \mathbb{R}_+} \|\mathcal{U}(\tau') - \tilde{\mathcal{T}}(\frac{\tau'}{n+1})^{n+1}\| \leq \frac{R'_\gamma}{n^\gamma},$$

or

$$\sup_{\tau' \in \mathbb{R}_+} \|\mathcal{U}(\tau') - \tilde{\mathcal{T}}(\frac{\tau'}{n+1})^{n+1}\| \leq 2^\gamma \frac{R'_\gamma}{(n+1)^\gamma},$$

$\tau \in \mathbb{R}_+, n \in \mathbb{N}$. Setting $\tilde{R}_\gamma := \max\{2, 2^\gamma R'_\gamma\}$ we prove (3.40). □

These results can be immediately extended to propagators. To this end we set

$$\begin{aligned} \tilde{G}_j(t, s; n) &:= e^{-\frac{t-s}{n}B(t_j)} e^{-\frac{t-s}{n}A}, \quad j = 0, 1, 2, \dots, n, \\ \tilde{V}_n(t, s) &:= \tilde{G}_n(t, s; n) \tilde{G}_{n-1}(t, s; n) \times \dots \times \tilde{G}_2(t, s; n) \tilde{G}_1(t, s; n), \end{aligned} \quad (3.44)$$

$t_j := s + j\frac{t-s}{n}$, $j = 0, 1, 2, \dots, n$, in analogy to (1.6).

Theorem 3.14 *Let the assumptions (S1)-(S3) be satisfied. Further, let $\{U(t, s)\}_{(t,s) \in \Delta_0}$ be the propagator corresponding to the evolution generator \mathcal{K} and let $\{V_n(t, s)\}_{(t,s) \in \Delta_0}$ and $\{\tilde{V}_n(t, s)\}_{(t,s) \in \Delta_0}$ be defined by (1.6) and (3.44), respectively. If $\beta > 2\alpha - 1$, then the estimates*

$$\operatorname{ess\,sup}_{(t,s) \in \Delta_0} \|U(t, s) - V_n(t, s)\| \leq \frac{R_\beta}{n^\beta} \quad \text{and} \quad \operatorname{ess\,sup}_{(t,s) \in \Delta_0} \|U(t, s) - \tilde{V}_n(t, s)\| \leq \frac{\tilde{R}_\beta}{n^\beta} \quad (3.45)$$

hold for $n \in \mathbb{N}$, where the constants R_γ and \tilde{R}_γ are those of Theorem 3.12 and Corollary 3.13.

Proof. Note that Proposition 2.1 of [36] yields

$$\sup_{\tau \in \mathbb{R}_+} \|\mathcal{U}(\tau) - \mathcal{T}(\frac{\tau}{n})^n\| = \operatorname{ess\,sup}_{(t,s) \in \Delta_0} \|U(t, s) - V_n(t, s)\|, \quad n \in \mathbb{N}.$$

Then applying Theorem 3.12 we prove (3.45).

To proof the second estimate we use Proposition 3.8 of [37] where the relation

$$\sup_{\tau \in \mathbb{R}_+} \|\mathcal{U}(\tau) - \tilde{\mathcal{T}}(\frac{\tau}{n})^n\| = \operatorname{ess\,sup}_{(t,s) \in \Delta_0} \|U(t, s) - \tilde{V}_n(t, s)\|, \quad n \in \mathbb{N}.$$

was shown. Applying Corollary 3.13 we complete the proof. \square

4 Example

As an example we consider the diffusion equation perturbed by a time-dependent scalar potential. For this aim let $\mathfrak{H} = L^2(\Omega)$, where $\Omega \subset \mathbb{R}^3$ is a bounded domain with sufficiently smooth boundary. Domains in higher dimension can be treated analogously. The equation reads as

$$\dot{u}(t) = \Delta u(t) - B(t)u(t), \quad u(s) = u_s \in \mathfrak{H}, \quad t, s \in [0, T], \quad (4.1)$$

where Δ denotes the Laplace operator in $L^2(\Omega)$ with Dirichlet boundary conditions, i.e. $\Delta : \text{dom}(\Delta) = H^2(\Omega) \cap H_0^1(\Omega) \rightarrow L^2(\Omega)$ and $H_0^1(\Omega)$ denotes the subset of functions that vanish at the boundary. Then operator $-\Delta$ is self-adjoint on \mathfrak{H} and positive. For any $\alpha \in (0, 1)$ the fractional power of operator $-\Delta$ is defined on the domain $\text{dom}((-\Delta)^\alpha)$, i.e. $(-\Delta)^\alpha : \text{dom}((-\Delta)^\alpha) \rightarrow L^2(\Omega)$. The domain is given by a fractional Sobolev space and for $\alpha > 1/2$, we have $\text{dom}((-\Delta)^\alpha) = H_0^{2\alpha}(\Omega) \subset H^{2\alpha}(\Omega)$ (see [30] for more information).

Moreover let $B(t)$ denote a time-dependent scalar-valued multiplication operator given by

$$\begin{aligned} (B(t)f)(x) &= V(t, x)f(x), \\ \text{dom}(B(t)) &= \{f \in L^2(\mathcal{I}, \mathfrak{H}) : V(\cdot, x)f(x) \in L^2(\mathcal{I}, \mathfrak{H})\} \end{aligned} \quad (4.2)$$

where $V : \mathcal{I} \times \Omega \rightarrow \mathbb{R}$ is measurable. We assume that the potential $V(\cdot, \cdot)$ is real and non-negative. Then $B(t)$ is obviously self-adjoint and non-negative on \mathfrak{H} .

Theorem 4.1 *Let A be the Laplacian operator $-\Delta$ with Dirichlet boundary conditions in $L^2(\Omega)$, see above. Further, let $\{B(t)\}_{t \in \mathcal{I}}$ be the family of multiplication operators defined by (4.2). If $V(\cdot, \cdot) : \mathcal{I} \times \Omega \rightarrow \mathbb{R}$ is measurable, real, non-negative with regularity $V \in L^\infty(\mathcal{I}, L^{2+\varepsilon}(\Omega)) \cap C^\beta(\mathcal{I}, L^{1+\varepsilon}(\Omega))$ for $\beta \in (0, 1)$ and some $\varepsilon > 0$, then the assumptions (S1)-(S3) are satisfied with $\alpha \in [3/4, 1)$. Moreover, if $\beta > 2\alpha - 1$ then the converging rates of Theorem 3.12, Corollary 3.13 and Theorem 3.14 hold.*

Proof. Since Ω is bounded there one has $\inf \sigma(A) > 0$ which does not satisfy $A \geq I$ in general and, hence, assumption (S1) is not satisfied. Nevertheless $\inf \sigma(A) > 0$ is sufficient to prove the converging results. So we can believe that (S1) is satisfied.

Let $\alpha \geq 3/4$. Using the Sobolev space embeddings, we get that $H^{2\alpha}(\Omega) \subset L^\gamma(\Omega)$ for any $\gamma \in [2, \infty[$. Hence, if $V \in L^\infty(\mathcal{I}, L^{2+\varepsilon}(\Omega))$, we conclude that the function $[0, T] \ni t \mapsto B(t)(-\Delta)^{-\alpha}$ is essentially operator-norm bounded in $t \in \mathcal{I}$ and thus, (S2) is satisfied. Now, we consider

$$F(t) := (-\Delta)^{-\alpha} B(t) (-\Delta)^{-\alpha} : L^2(\Omega) \rightarrow H^{2\alpha}(\Omega) \subset L^2(\Omega).$$

The function $F(\cdot) : \mathcal{I} \rightarrow \mathcal{L}(\mathfrak{H})$ is bounded for fixed $t \in [0, T]$ if for any $f, g \in H^{2\alpha}(\Omega)$ the function $\langle f, B(t)g \rangle$ is bounded. This holds since $V(t, \cdot) \in L^{1+\varepsilon}(\Omega)$ and $H^{2\alpha}(\Omega) \subset L^\gamma(\Omega)$ for any $\gamma \in [2, \infty[$. Hence we conclude that (S3) is satisfied and the claim is proved. \square

Theorem 4.1 provides a convergence rate of an approximation of the solution of (4.1) by the time-ordered product

$$\tilde{V}_n(t, s) = \prod_{j=1}^n e^{-\frac{t-s}{n} V(\frac{j+(n-j)s}{n}, \cdot)} e^{\frac{t-s}{n} \Delta} \quad (4.3)$$

This looks elaborate, but is indeed simple. There are strategies to compute the semi-group of the Laplace operator for bounded domains and there are also explicit formulas on special domains like disks etc. The factors $e^{-\tau V(t_j)}$, $j = 1, 2, \dots, n$ are scalar valued and can be easily computed.

Acknowledgment

We thank Takashi Ichinose and Hideo Tamura for the explanation of details of the proof of Theorem 1.1 of [14], which makes possible to prove Lemma 3.8 and Lemma 3.9.

References

1. P. Acquistapace and B. Terreni. On the abstract Cauchy problem in the case of constant domains. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8), 76(1):7–13, 1984.
2. P. Acquistapace and B. Terreni. Maximal space regularity for abstract linear nonautonomous parabolic equations. *J. Funct. Anal.*, 60(2):168–210, 1985.
3. H. Amann. On abstract parabolic fundamental solutions. *J. Math. Soc. Japan*, 39(1):93–116, 1987.
4. H. Amann. Parabolic evolution equations in interpolation and extrapolation spaces. *J. Funct. Anal.*, 78(2):233–270, 1988.
5. H. Amann. *Linear and quasilinear parabolic problems. Vol. I*, volume 89 of *Monographs in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1995. Abstract linear theory.
6. W. Arendt, R. Chill, S. Fornaro, and C. Poupaud. L^p -maximal regularity for non-autonomous evolution equations. *J. Differential Equations*, 237(1):1–26, 2007.
7. W. Arendt, D. Dier, and E.-M. Ouhabaz. Invariance of convex sets for non-autonomous evolution equations governed by forms. *J. Lond. Math. Soc.* (2), 89(3):903–916, 2014.
8. M. Sh. Birman and M. Z. Solomjak. *Spectral theory of selfadjoint operators in Hilbert space*. Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht, 1987.
9. V. Cachia, H. Neidhardt, and V. A. Zagrebnov. Accretive perturbations and error estimates for the Trotter product formula. *Integral Equations Operator Theory*, 39(4):396–412, 2001.

10. V. Cachia, H. Neidhardt, and V. A. Zagrebnov. Comments on the Trotter product formula error-bound estimates for nonself-adjoint semigroups. *Integral Equations Operator Theory*, 42(4):425–448, 2002.
11. V. Cachia and V. A. Zagrebnov. Operator-norm approximation of semigroups by quasi-sectorial contractions. *J. Funct. Anal.*, 180(1):176–194, 2001.
12. K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.
13. Y. Fujie and H. Tanabe. On some parabolic equations of evolution in Hilbert space. *Osaka J. Math.*, 10:115–130, 1973.
14. T. Ichinose and H. Tamura. Error estimate in operator norm of exponential product formulas for propagators of parabolic evolution equations. *Osaka J. Math.*, 35(4):751–770, 1998.
15. T. Ichinose, Hideo Tamura, Hiroshi Tamura, and V. A. Zagrebnov. Note on the paper: “The norm convergence of the Trotter-Kato product formula with error bound” by T. Ichinose and H. Tamura. *Comm. Math. Phys.*, 221(3):499–510, 2001.
16. T. Kato. Integration of the equation of evolution in a Banach space. *J. Math. Soc. Japan*, 5:208–234, 1953.
17. T. Kato. On linear differential equations in Banach spaces. *Comm. Pure Appl. Math.*, 9:479–486, 1956.
18. T. Kato. Abstract evolution equations of parabolic type in Banach and Hilbert spaces. *Nagoya Math. J.*, 19:93–125, 1961.
19. T. Kato. Nonlinear evolution equations in Banach spaces. In *Proc. Sympos. Appl. Math.*, Vol. XVII, pages 50–67. Amer. Math. Soc., Providence, R.I., 1965.
20. T. Kato. Linear evolution equations of “hyperbolic” type. *J. Fac. Sci. Univ. Tokyo Sect. I*, 17:241–258, 1970.
21. T. Kato. Linear evolution equations of “hyperbolic” type. II. *J. Math. Soc. Japan*, 25:648–666, 1973.
22. T. Kato. On the Trotter-Lie product formula. *Proc. Japan Acad.*, 50:694–698, 1974.
23. T. Kato. Trotter’s product formula for an arbitrary pair of self-adjoint contraction semigroups. In *Topics in functional analysis*, volume 3 of *Adv. in Math. Suppl. Stud.*, pages 185–195. Academic Press, New York-London, 1978.
24. T. Kato. Trotter’s product formula for some nonlinear semigroups. In *Nonlinear evolution equations (Proc. Sympos., Univ. Wisconsin, Madison, Wis., 1977)*, volume 40 of *Publ. Math. Res. Center Univ. Wisconsin*, pages 155–162. Academic Press, New York-London, 1978.
25. T. Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995.
26. T. Kato. Linear and quasi-linear equations of evolution of hyperbolic type. In *Hyperbolicity*, volume 72 of *C.I.M.E. Summer Sch.*, pages 125–191. Springer, Heidelberg, 2011.
27. T. Kato. Semi-groups and temporally inhomogenous evolution equations. In *Abstract differential equations (Italian)*, volume 29 of *C.I.M.E. Summer Sch.*, pages 1–43. Springer, Heidelberg, 2011.
28. T. Kato and K. Masuda. Trotter’s product formula for nonlinear semigroups generated by the subdifferentials of convex functionals. *J. Math. Soc. Japan*, 30(1):169–178, 1978.
29. T. Kato and H. Tanabe. On the abstract evolution equation. *Osaka Math. J.*, 14:107–133, 1962.
30. J. L- Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I* Springer Verlag, New York 1972.
31. A. Lunardi. On the evolution operator for abstract parabolic equations. *Israel J. Math.*, 60(3):281–314, 1987.
32. S. Monniaux and A. Rhandi. Semigroup methods to solve non-autonomous evolution equations. *Semigroup Forum*, 60(1):122–134, 2000.
33. H. Neidhardt. On abstract linear evolution equations. I. *Math. Nachr.*, 103:283–298, 1981.
34. H. Neidhardt, A. Stephan, and V. A. Zagrebnov. Convergence rate estimates for Trotter product approximations of solution operators for non-autonomous Cauchy problems. *arXiv: 1612.06147 [math.FA]*, December 2016. (to appear in Publications of RIMS, Kyoto University)

35. H. Neidhardt, A. Stephan, and V. A. Zagrebnov. On convergence rate estimates for approximations of solution operators for linear nonautonomous evolution equations. *Nanosystems: Physics, Chemistry, Mathematics*, 8:202–215, 2017.
36. H. Neidhardt, A. Stephan, and V. A. Zagrebnov. Remarks on the operator-norm convergence of the Trotter product formula. *Integral Equations Operator Theory*, 90(2), 2018.
37. H. Neidhardt, A. Stephan, and V. A. Zagrebnov. *Operator-Norm Convergence of the Trotter Product Formula on Hilbert and Banach Spaces: A Short Survey*, pages 229–247. Springer International Publishing, 2018.
38. H. Neidhardt and V. A. Zagrebnov. On error estimates for the Trotter-Kato product formula. *Lett. Math. Phys.*, 44(3):169–186, 1998.
39. H. Neidhardt and V. A. Zagrebnov. Linear non-autonomous Cauchy problems and evolution semigroups. *Adv. Differential Equations*, 14(3-4):289–340, 2009.
40. G. Nickel. On evolution semigroups and nonautonomous Cauchy problems. *Diss. Summ. Math.*, 1(1-2):195–202, 1996.
41. M. Reed and B. Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press, New York-London, 1975.
42. Dzh. L. Rogava. On the error of Trotter-type formulas in the case of selfadjoint operators. *Funktsional. Anal. i Prilozhen.*, 27(3):84–86, 1993.
43. H. Tanabe. A class of the equations of evolution in a Banach space. *Osaka Math. J.*, 11:121–145, 1959.
44. H. Tanabe. On the equations of evolution in a Banach space. *Osaka Math. J.*, 12:363–376, 1960.
45. H. Tanabe. Remarks on the equations of evolution in a Banach space. *Osaka Math. J.*, 12:145–166, 1960.
46. H. Tanabe. Evolutional equations of parabolic type. *Proc. Japan Acad.*, 37:610–613, 1961.
47. H. Tanabe. On regularity of solutions of abstract differential equations in Banach space. *Proc. Japan Acad.*, 43:305–307, 1967.
48. H. Tanabe. On regularity of solutions of abstract differential equations of parabolic type in Banach space. *J. Math. Soc. Japan*, 19:521–542, 1967.
49. H. Tanabe. *Equations of evolution*, volume 6 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1979.
50. H. F. Trotter. On the product of semi-groups of operators. *Proc. Amer. Math. Soc.*, 10:545–551, 1959.
51. A. Yagi. On the abstract linear evolution equations in Banach spaces. *J. Math. Soc. Japan*, 28(2):290–303, 1976.
52. A. Yagi. On the abstract evolution equation of parabolic type. *Osaka J. Math.*, 14(3):557–568, 1977.
53. A. Yagi. Fractional powers of operators and evolution equations of parabolic type. *Proc. Japan Acad. Ser. A Math. Sci.*, 64(7):227–230, 1988.
54. A. Yagi. *Abstract parabolic evolution equations and their applications*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010.
55. A. Yagi. Parabolic evolution equations in which the coefficients are the generators of infinitely differentiable semigroups. *Funkcial. Ekvac.*, 32(1):107–124, 1989.
56. A. Yagi. Parabolic evolution equations in which the coefficients are the generators of infinitely differentiable semigroups. II. *Funkcial. Ekvac.*, 33(1):139–150, 1990.